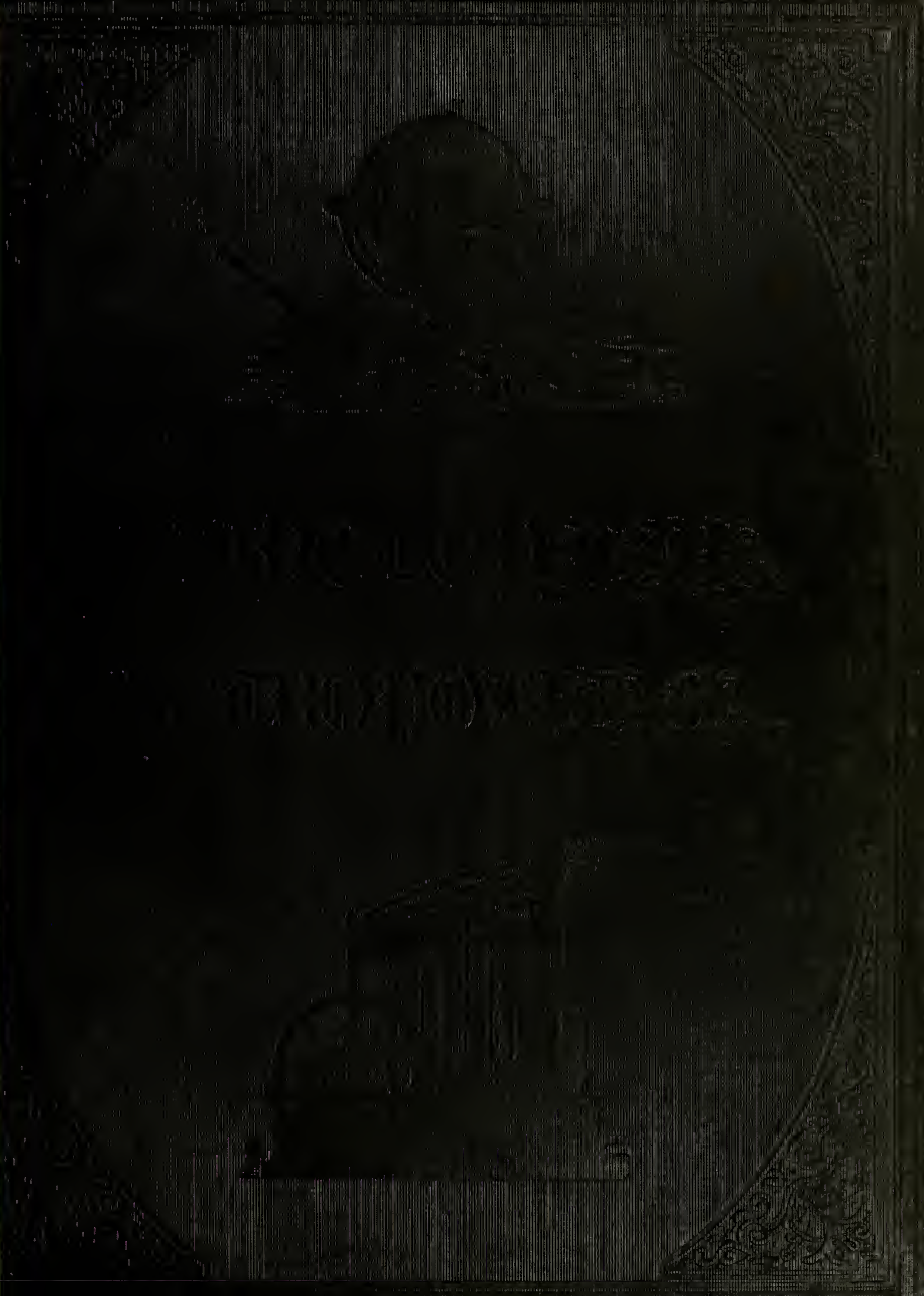

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THE

ENCYCLOPÆDIA OF ASTRONOMY:

COMPRISING

PLANE ASTRONOMY.
By Peter Barlow, Esq., F.R.S.,
Professor at the Royal Military Academy, Woolwich.

NAUTICAL ASTRONOMY.
By Captain Kater, F.R.S.,
Member of the Board of Longitude.

PHYSICAL ASTRONOMY.
By Sir John F. W. Herschel, Bart., M.A., F.R.S., F.R.S.E.

Essay
ON THE FIGURE OF THE EARTH
By George Biddell Airy, Esq., M.A., F.R.S.,
Astronomer Royal.

Treatise
ON TIDES AND WAVES.
By George Biddell Airy, Esq., M.A., F.R.S.,
Astronomer Royal.

ILLUSTRATED BY NUMEROUS ENGRAVINGS.

FORMING A PORTION OF THE ENCYCLOPÆDIA METROPOLITANA.

LONDON:
PUBLISHED BY JOHN JOSEPH GRIFFIN AND COMPANY,
CHEMICAL MUSEUM, 53, BAKER STREET, PORTMAN SQUARE;
AND RICHARD GRIFFIN AND COMPANY, GLASGOW.
1848.

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THE Contents of the present work form part of Volumes 3 and 5 of the Encyclopædia Metropolitana. The Treatises are described in the following extracts from the General Preface to the Encyclopædia:—

" From Pure Mathematics we proceed in natural order to their application to physical phenomena. Of these sciences, some belong to the more elementary branches of physical knowledge, and others to a higher and more advanced stage. Now the treatises on—

HYDRODYNAMICS, MECHANICS, PNEUMATICS, OPTICS, PLANE ASTRONOMY,*

have been written by Professor Barlow with an express view to this distinction. They are elementary enough to enable any student, with a competent knowledge of Pure Mathematics, to overcome their difficulties; and yet they are so based on scientific principles, that they will also prepare him to enter readily on the higher branches of Mixed Mathematics. In Mechanics, more especially, a foundation is laid for the succeeding investigations of Physical Astronomy, which is, in fact, only one of the higher branches of Analytical Physics.

" While, however, for these portions of the work we claim only that high share of approbation due to the presentation of ascertained results and knowledge already acquired, in an elegant and useful form, there are some treatises in the volumes devoted to the Mixed Sciences which demand a separate notice, as enlarging the boundaries of our scientific knowledge."

" The simple mention of Sir J. F. W. Herschel's name is a sufficient recommendation to the Treatise on PHYSICAL ASTRONOMY. It proves at once that it must be an Essay of the highest order of merit, and worthy of the present state of the Science. Indeed the name of Sir J. F. W. HERSCHEL stands so confessedly at the head of Physical Science in England, that the conductors of this Encyclopædia may justly be proud that he has contributed so largely to its pages.

" But although Plane and Physical Astronomy had been thus ably treated, it was considered that something more was required; and the late Captain Kater kindly furnished the very useful and able Treatise on NAUTICAL ASTRONOMY, a subject with which his acquaintance was at once profound and practical."

" The third volume of Mixed Sciences is chiefly devoted to the Fine Arts; but there are two or three essays in the early part of the volume which belong to the more exact sciences, viz., the Essay on the FIGURE OF THE EARTH, by the present Astronomer Royal, and his Treatise on the TIDES. With regard to the former, much novelty was hardly to be expected; but the Editor believes he is justified in stating that this

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* Of the articles here enumerated, only that on Astronomy is published in the present volume. The others form part of the Encyclopædia of Mechanical Philosophy.

Treatise contains the most complete combination and discussion of observations relating to the subject which has yet appeared in England. But the treatise into which this great mathematician has thrown all his power is the Theory of the Tides. It was remarked in 1833, by Mr. Lubbock, on the subject of the tides, that 'there is no branch of physical astronomy in which so much remains to be accomplished.' ('REPORT ON THE TIDES,' *published in the Report of the First and Second Meetings of the British Association for the Advancement of Science.*) The Astronomer Royal, in this treatise, has made a large step in advance in this science; he has, at all events, demonstrated the unsoundness of the equilibrium theory and the inapplicability of the theory of Laplace. The latter he has explained in such a manner as to bring it within the reach of good mathematicians; whereas, in the manner it was presented by its author in the *Mécanique Céleste*, none but persons of very high mathematical ability and undaunted perseverance would venture to encounter its difficulties. Still the theory was inapplicable; and the Astronomer Royal gave all the leisure he could command, for some years, to the consideration of these questions, and to an endeavour to place this great problem on a firm foundation. The Editor does not pretend to speak on this point from his own knowledge; but the terms in which some of the most distinguished mathematicians of Cambridge have spoken to him of this treatise prove that they consider it to have advanced the knowledge of this difficult subject in no ordinary degree. Indeed, the Editor believes that he may confidently assert that every previous treatise on the subject is entirely superseded by this theory, and that it will prove, for many years to come, the only sound foundation of our knowledge of the Theory of the Tides."

London, January, 1848.

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ASTRONOMY.

Astronomy (formed of *αστρον, a star*, and *νομος, law*,) is a mixed mathematical science, which treats of the heavenly bodies, their motions, periods, eclipses, magnitudes, &c. and of the causes on which they depend. That part of the science which relates to the motions, magnitudes, and periods of revolutions, is denominated *pure astronomy;* and that which investigates the causes of those motions, and the laws by which they are regulated, is called *physical astronomy.*

§ I. *History of Astronomy.*

General View.

It would be useless, if even the nature of our work would admit of it, to attempt to trace the history of this science from its earliest state of infancy, which is probably nearly coeval with that of society itself; at least if we regard the rude observations of shepherds and herdsmen as exhibiting the first dawn of astronomy. A man must be strangely divested of the curiosity peculiar to his species, who, while exposed to the varying canopy of the heavens, through successive nights and seasons, could suffer such a brilliant spectacle to pass repeatedly before him, without noticing the fixed or variable objects there presented to his view; and his attention, once drawn to a contemplation of the firmament, he would remark the invariable position of the greater number of those bodies with regard to each other; the irregular motion of others; and hence, by some denomination or other, we should have a distinction made between what we now call the *fixed stars* and the *planets;* while the sun and moon are in their appearances sufficiently distinct from the rest of the heavenly bodies, to have called for a farther distinguishing appellation, and to have claimed the particular regard of these rude observers.

Such was probably the origin of astronomy; and in this state, in all likelihood, it might remain for many ages, and in many countries unknown to and unconnected with each other. The length of the year, the duration of a lunar revolution, the particular rising of certain stars at certain seasons, and a few other common and obvious phenomena, might therefore be predicted with a certain degree of accuracy, long before those observations assumed any thing like a scientific form, and long anterior to that time from which we date the origin of astronomy as a science, properly so called.

Claims of the Chaldeans, &c.

The honour of being the first inventors of this sublime study has been attributed to various nations; the Chaldeans, the Egyptians, the Chinese, the Indians, have each had their advocates amongst our astronomical historians; and even a certain unknown people have been created by the enthusiasm of some writers, of whom all traces are supposed to have been long lost; but to whom all original knowledge of astronomy has been attributed. The more closely, however, we examine the claims of these actual or imaginary people, the more we shall be convinced that their astronomy consisted of little more than we have indicated above; *viz.* a tolerable approximation to certain periods, and to the re-appearance of certain phenomena, that required nothing more than a continued and patient observation of stated occurrences, which as we have observed, could not long remain unnoticed even in the most infant state of society.

Egyptians.

We may judge of the state of Egyptian astronomy from the circumstance of Thales having first taught them how to find the heights of the pyramids from the length of their shadows. It is true that they had some idea of the length of the year, and had, in a certain measure, approximated towards a determination of the obliquity of the ecliptic, or of the path of the sun, which they stated to be 24°. The Chaldeans appear to have made some rude observations on eclipses, but still little scientific knowledge can be attributed to this people; who after observing these phenomena, were contented to explain them by teaching that the two great luminaries of the heavens were only on fire on one side, and that eclipses were occasioned by the accidental turning of their dark sides towards us. And again, that these bodies were carried round the heavens in chariots, close on all sides except one, in which there was a round hole, and that a total or partial eclipse was occasioned by the complete or partial shutting of this aperture. Similar absurd and extravagant notions will be found amongst all the early pretenders to the study of astronomy; but we cannot concede to such knowledge and pretences the term science; they had in fact no science, they had amassed together a number of rude observations, and had been thus enabled to determine certain periods, and to predict some few phenomena; but we have no proof, nor even any reason whatever to imagine, from any facts that have been handed down to us, that these predictions rested upon any other basis than that of simply observing, the repeated returns of these appearances within certain periods.

Astronomy as it was received by the early Greeks.

If to the knowledge above indicated, we add an arbitrary collection of certain clusters or groups of stars into constellations; the division of the zodiac into twelve signs, corresponding to the twelve months of the year; into twenty-seven or twenty-eight hours, answering to the daily motion of the moon; an obscure idea of the revolution of the earth upon its axis, which was afterwards lost; a knowledge of five planets; and some contradictory notions respecting the nature and motion of comets, we shall have a pretty correct picture of the state of astronomy as it was received amongst the Greeks; and from whom it first derived its scientific character. It is therefore only from this period that we shall commence our historical sketch, and attempt to trace the rise and progress of astronomy.

Successive periods in the history of astronomy.

We shall follow it, from the state above described, when every thing depended upon observation only, unaided either by a calculus or instruments; through that in which the latter began to be employed, and some assistance was derived from the more elementary positions of geometry. We shall next examine it from the period when astronomical science was enriched and extended by the invention of the telescope, but while the principles of computations were still founded on the elements of pure geometry; and lastly, we shall exhibit the science as it now exists, supported by every aid that can be derived from the present high state of practical and theoretical mechanics and optics; when the effect of every celestial motion, and every disturbing force is made to depend upon one universal law; and the amount of each investigated and submitted to computation, by means of the powerful assistance derived from the modern analysis.

The first period above alluded to, comprehends a long series of ages, during which the science of astronomy passed into the hands of different people and nations. First, to the Greeks, then to the Arabs; from which latter it seems probable that it found its way to India and China, about the same time that it was also brought into Spain; whence it afterwards spread throughout all parts of civilized Europe. We shall therefore divide this period into the following minor sections. The astronomy of the Greeks; of the Arabs; of the Indians and Chinese; and of modern Europe; which latter will bring us up to the time of Copernicus and Galileo; including, in all, about twenty-one centuries.

Of the Astronomy of the Greeks.

Thales. B. C. 600.

Thales is generally considered as the founder of astronomy amongst the Greeks. This philosopher, who must have flourished about 600 years before the commencement of the Christian æra, is said to have taught that the stars were fire, or that they shone by means of their own light; the moon received her light from the sun, and that she became invisible in her conjunctions, in consequence of being hidden or absorbed in the solar rays, which it must be acknowledged is but an obscure way of saying that she then turned towards us her unenlightened hemisphere. He taught farther that the earth is spherical, and placed in the centre of the world; he divided the heavens, or rather found them divided into five circles, the equator, the two tropics, and the arctic and antarctic circles. The year he made to consist of 365 days; and determined "the motion of the sun in declination." What is meant by this expression is not very easy to comprehend; if it only means that he discovered such a motion, it can scarcely be considered as correct, as it must have been known prior to his time; viz. to the first observers; and it cannot mean that he laid down rules for computing it, as we have every reason to know that the most simple principles of trigonometry were not propagated till many centuries after his time.

Predict an eclipse.

Thales is also said to have first observed an eclipse, and to have predicted that celebrated one which terminated the war between the Medes and the Lydians; an eclipse on which much has been written, but from which very little satisfactory information has been obtained. Herodotus says, "it happened that the day was changed suddenly into night, a change which Thales the Milesian had announced to the people of Ionia, assigning for the limit of his prediction, the year in which the change actually took place." Thales had therefore neither predicted the day nor the month; and in all probability he had no other principle to proceed upon, than the Chaldean period of eclipses already alluded to in the preceding part of this article.

The pointed declaration of the historian, that the limits assigned by the astronomer for the appearance of this phenomenon, was the year in which it happened, is a pretty obvious proof of the low state of astronomical science at this time, and it would be of little importance whether the eclipse was itself partial or total; but as there is little doubt that such an event actually took place, it becomes a matter of high importance in chronology, to ascertain whether it was such as it is described, viz. a total eclipse; for no partial obscuration of the sun's light would accord with the description of Herodotus, *of the day being suddenly changed into night*; and such a phenomenon in any particular place being an extremely rare occurrence, it would, if correct, enable us to determine not only the year, but the very day and hour at which it happened, and thus furnish at least one indisputable period in chronology and history.

Dates assigned to this eclipse.

Various dates have been assigned to this eclipse. Pliny places it in the fourth year of the forty-eighth Olympiad which answers to the year 585 B. C. *(Hist. Nat.* lib. 2. cap. 12.), a similar opinion has been advanced by Cicero *(De Divinat.* lib. 1. § 49.) and probably by Eudemus *(Clement. Alex. Strom.* lib. 1. p. 354.); by Newton *(Chron. of Anc. Kings amended)*; Riccioli *(Chron. Reform,* vol. 1. p. 228.); Desvignoles *(Chronol.* lib. 4. cap. 5. § 7, &c.); and by Brosses (*Mém. de l'Acad. des Belles Lettres,* tom. 21. Mém. p. 33.)

Scaliger, in two of his writings, *(Animad. ad Euseb.* p. 89.) and in (Ολυμ. ἀναγραφη) has adopted also the opinion of Pliny; but in another work *(De Emen. Temp. in Can. Isag.* p. 321.) he fixes the date of this eclipse to the 1st of October, 583, B. C. Calvisius states it in his *(Opus Chron.)* to have taken place in 607 B. C. Petavius says it happened July 9th, 597 B. C. *(De Doct. Temp.* lib. 10. cap. 1.) which date has likewise been adopted by Marsham, Bouhier, Corsini; and by M. Larcher the French translator of Herodotus, (tom. i. p. 335.) Usher is of opinion that it happened 601 B. C.; and Bayer, May 18, 603, B. C.; which latter opinion has been supported by two English astronomers, Costard and Stukeley, *(Phil. Trans.* for 1753.) But Volney attempts to show in his *(Chronologie d'Herodote,)* that it could be no other than the eclipse which happened February 3d, 626 B. C.

Mr. F. Bailly has examined with great care and labour the probability of these several statements, from which it appears, that most of the eclipses above alluded to happened under circumstances which render it absolutely impossible any of them should be that alluded to by Herodotus; most of them were not even visible in that country, which must necessarily have been the scene of action between the Medes and the Lydians, and none of them was total in those places. He has therefore with great perseverance, by means of the last new astronomical tables of the *Bureau des Longitudes,* computed backward to find whether any eclipse of the sun actually happened within the probable limits of the event recorded by the historian, and the result of his research is, that on the 10th of September, 610 B. C., there was a solar eclipse, which was total in some parts of Asia Minor; and which, he therefore concludes, with great probability, was the identical one referred to by Herodotus. Admitting, therefore, the conclusion, we have one decided point of time to which we are enabled to refer with confidence, and at which time, the state of astronomy is known to have been such as we have described. See *Phil. Trans. for* 1811.

Anaximander, Anaximanes, Anaxagoras. B.C. 530.

The successors of Thales, Anaximander, Anaximanes, and Anaxagoras, contributed considerably to the advancement of astronomy. The first is said to have invented or introduced the gnomon into Greece; to have observed the obliquity of the ecliptic; and taught that the earth was spherical, and the centre of the universe, and that the sun was not less than it. He is also said to have made the first globe, and to have set up a sun-dial at Lacedæmon, which is the first we hear of among the Greeks; though some are of opinion that these pieces of knowledge were brought from Babylon by Pherecydes, a contemporary of Anaximander. Anaxagoras also predicted an eclipse which happened in the fifth year of the Peloponnesian war; and taught that the moon was habitable, consisting of hills, valleys, and waters, like the earth. His contemporary Pythagoras, however, greatly improved not only astronomy and mathematics, but every other branch of philosophy. He taught that the universe was composed of four elements, and that it had the sun in the centre; that the earth was round, that we had antipodes; and that the moon reflected the rays of the sun; that the stars were worlds, containing earth, air, and ether; that the moon was inhabited like the earth; and that the comets were a kind of wandering stars, disappearing in the superior parts of their orbits, and becoming visible only in the lower parts of them. The white colour of the milky-way he ascribed to the brightness of a great number of small stars; and he supposed the distances of the moon and planets from the earth to be in certain harmonic proportion to one another. He is said also to have exhibited the oblique course of the sun in the ecliptic and the tropical circles, by means of an artificial sphere; and he first taught that the planet Venus is both the evening and morning star. This philosopher is said to have been taken prisoner by Cambyses, and thus to have become acquainted with all the mysteries of the Persian magi; after which he settled at Crotona in Italy, and founded the Italian sect.

Pythagoras. B.C. 540.

Philolaus. B. C. 440-432.

About 440 years before the Christian æra, Philolaus, a celebrated Pythagorean, asserted the annual motion of the earth round the sun; and soon after Hicetas, a Syracusan, taught its diurnal motion on its own axis. About this time also flourished Meton and Euctemon at Athens, who took an exact observation of the summer solstice 432 years before Christ; which is the oldest observation of the kind we have, excepting some doubtful ones of the Chinese. Meton is said to have composed a cycle of 19 years, which still bears his name; and he marked the risings and settings of the stars, and what seasons they pointed out: in all of which he was assisted by his companion Euctemon. The science, however, was obscured by Plato and Aristotle, who embraced the system afterwards called the *Ptolemaic,* which places the earth in the centre of the universe.

Eudoxus. B.C. 370.

After Philolaus, the next astronomer we meet with of great reputation is Eudoxus, who flourished 370 B.C. He was a contemporary with Aristotle though considerably older, and is greatly celebrated for his skill in this science. He is said to have been the first to apply geometry to astronomy, and is supposed to be the inventor of many of the propositions attributed to Euclid. Having travelled into Egypt in the early part of his life, he obtained a recommendation from Agesilaus to Nectanebus, king of Egypt, and by his means got access to the priests, who were then held to have great knowledge of astronomy; after which he taught in Asia and Italy. Seneca tells us, that he brought the knowledge of astronomy, i. e. of the planetary motions, from Egypt into Greece: and according to Archimedes, his opinion was, that the diameter of the sun was nine times that of the moon. He was also acquainted with the method of drawing a sun dial on a plane.

Callippus. B.C. 330.

Soon after Eudoxus, we meet with Callippus, whose system of the celestial sphere is mentioned by Aristotle; but he is better known for a period of 76 years, containing four corrected Metonic periods, and which had its beginning at the summer solstice, in the year 330 B. C. And it was about this time, or rather earlier, that the Greeks having begun to plant colonies in Italy, Gaul, and Egypt, became acquainted with the Pythagorean system, and the notions of the ancient Druids concerning astronomy.

Autolycus B.C. 300.

Passing over the names of various other astronomers of this period, who appear to have done very little towards the advancement of the science, we come to Autolycus, the most ancient writer whose works have been handed down to our time. He wrote two books, viz. *Of the Sphere which moves,* and the other, *On the Risings and Settings of the Stars.* These works were composed about 300 B. C.

We have now passed over a period of 300 years from the time of Thales, and therefore, by making a few extracts from these works of Autolycus, we shall be enabled to form some idea of the progress of astronomy during this period. In the work on the moveable sphere, we have several propositions, of which the following are the most important.

Earliest Work extant on Astronomy.

1. If a sphere move uniformly about its axis, all the points on its surface which are not in its axis, will describe parallel circles, having for their common poles, those of the sphere itself, and of which all the planes will be perpendicular to the axis.

2. All these points will describe, of their respective circles, similar arcs in equal times.

3. Reciprocally, similar arcs will indicate equal time.

4. If a great fixed circle, perpendicular to the axis, divide the sphere into two hemispheres, the one visible, the other invisible, and that the sphere turns about its axis; those points on the surface that are

hidden, will never rise; and those that are visible will never set. This is what we now denominate a parallel sphere; the great fixed circle corresponding with our equator.

5. If a great circle pass through the poles, all the points of the surface will rise and set alternately. This corresponds to our horizon, and to our right sphere.

6. If the great circle be oblique to the axis, it will touch two equal parallel circles; of which, that adjacent to the one pole will be always apparent, the other always invisible.

The first of these circles was called by the Greeks, (although not by this author,) as we still denominate it, the *arctic* circle, and the other the *antarctic* circle.

7. If the horizon be oblique, the circles, perpendicular to the axis, will always have their points of rising and setting in the same points of the horizon, to which they are all equally inclined.

8. The great circles which touch the arctic and antarctic circle, will, during the complete revolution of the sphere, twice coincide with the horizon.

9. In the oblique sphere, of all the points which rise at the same instant, those which are nearest to the visible pole will set last; and of the points which set at the same instant, those that are nearest the same pole will rise first.

10. In the oblique sphere, every circle which passes through the poles, will be perpendicular to the horizon twice in the course of one complete revolution.

Remarks on this author.

We omit some other propositions of this author, which are of less importance than the above; and even those which we have given, are such as one would imagine could not have escaped the observation of any one who would think of employing an artificial sphere to represent the celestial motions; yet, from the tenor of the work in question, it would seem, that if they were known, they were never before, at least, embodied in the form of a regular treatise.

Here then we may begin to date the first scientific form of astronomy; because in this work, however low and elementary, we have an application of geometry to illustrate the motions of the heavenly bodies; but we shall still find two other centuries pass away, before the same principles were applied to actual computation.

Euclid.
B. C. 300.

Contemporary with Autolycus, was Euclid; whose elements of geometry, after so many ages, still maintain their pre-eminence; and in which we find all the propositions that are necessary for establishing every useful theorem in trigonometry; yet it is perfectly evident that no ideas were yet conceived of the latter science. Neither Euclid nor Archimedes, great as were their skill and talents in geometry, had any idea of the method of estimating the measure of any angle by the arc, which, the two lines forming it, intercepted; nor does it appear that they knew of any instrument whatever for taking angles; a very convincing proof of which appears in the process adopted by the latter justly celebrated philosopher, in order to determine the apparent diameter of the sun.

Aristarchus
B. C. 264.

Passing over the poet Aratus, who is supposed to have embodied in his poem all the astronomical knowledge of the time in which he wrote, viz. 270 B. C.; but who had not himself made any observations, we come to Aristarchus, who has left us a work, entitled *Of Magnitudes and Distances;* in which he teaches, that the moon receives her light from the sun, and that the earth is only a point in comparison with the sphere of the moon. He likewise added, that when the moon is dichotomized, we are in the plane of the circle which separates the enlightened part from the unenlightened, which is the most curious and original remark of this author: in this state of the moon, he also observes, that the angle subtended by the sun and moon, is one-thirtieth less than a right angle; which, in other words, is saying, that the angle is 87°, whereas we now know that this angle exceeds 89° 50′. In another proposition he asserts, that the breadth of the shadow of the earth is equal to two semi-diameters of the moon, whereas these are to each other as 83 to 64. In his sixth proposition, he states the apparent diameter of the moon to be one-fifteenth part of a sign, or 2°; whereas we know that it is only about half a degree. Again, the distance of the earth from the moon being assumed as unity, its distance from the sun was said to be 17·107, and the distance of the earth from the sun 19·081. Such was the astronomical knowledge in the time of Aristarchus, who lived about 264 years before the Christian æra.

Eratosthenes.
B. C. 230.

In order of time we pass now to Eratosthenes, who may, perhaps, with more propriety than Autolycus, be considered as the founder of astronomical science; particularly if it be true that he placed in the portico of Alexandria certain armillary spheres; of which so much use was afterwards made, and which, it is said, he owed to the munificence of Ptolemy Euergetes, who called him to Alexandria, and gave him the charge and direction of his library.

Ancient armillary sphere.

According to the description given of these armillaries by Ptolemy, they were assemblages of different circles; the principal one of which served as a meridian; the equator, the ecliptic, and the two colures, constituted an interior assemblage, which turned on the poles of the equator. There was another circle, which turned on the poles of the ecliptic, and carried an index to point out the division at which it stopped. The instrument of which the above appears to be the general construction, was applied to various uses; amongst others, it served to determine the equinoxes, after the following manner:—The equator of the instrument being pointed with great care in the plane of the celestial equator, the observer ascertained, by watching the moment when neither the upper nor the lower surface was enlightened by the sun; or rather, which was less liable to error, when the shadow of the anterior convex position of the circle completely covered the concave part on which it was projected. This instant of time was evidently that of the equinox. And if this did not happen, although the sun shone, two observations were selected, in which the shadow was projected on the concave part of the circle in opposite directions; and the mean of the interval between these observations was accounted the time of the equinox. At this time we find enumerated five planets, viz. *Φαινων, Φαεθων, Πυροειδηρ,* which appear to indicate Jupiter, Saturn, and Mars; and to which were added Venus and Mercury.

Determination of the equinoxes.

Magnitude of the earth.

Eratosthenes not only taught the spherical figure of the earth, but attempted to ascertain its actual circumference, by measuring, as exactly as could be done in his time, the length of a certain terrestrial arc, and then finding the astronomical arc in degrees intercepted between the zeniths of the two places. The segment of the meridian which he fixed upon for this purpose, was that between Alexandria and Syene; the measured distance of which was found to be 5,000 stadia; and the angle of the shadow upon the scaphia, which was observed at Alexandria, was equal to the fiftieth part of the circle; and at Syene there was no shadow from this gnomon at noonday of the summer solstice; and that this might be the more accurately taken, they dug a deep well, which, being perpendicular, was completely illuminated at the bottom when the sun was on the meridian. The exact quantity which this philosopher assigned to the circumference of the earth is not known; at least, different opinions have been advanced: some state it at 250,000, and others at 252,000 stadia; the length of this unit of measure is also somewhat uncertain. It is, however, of small importance, as we may be pretty well convinced, that by such means as he employed, no very accurate conclusion could be expected; it is sufficient that he attempted the solution of the problem in a very rational manner, to entitle him to the honour of being one of the most celebrated of the Grecian astronomers.

Obliquity of the ecliptic.

Eratosthenes also observed the obliquity of the ecliptic, and made it to consist of $\frac{11}{166}$th of a circumference, which answers to about 23° 51′ 19·5″. This observation is commonly stated to have been made in the year 230 B. C.

Archimedes. B. C. 227.

Archimedes, the justly celebrated geometer of Syracuse, was contemporary with Eratosthenes; and although most conspicuous as a mechanic and geometrician, the great impulse which he gave to the sciences generally, will not admit of our passing him over in silence in this history. All that we have of this author with reference to astronomy, is found in his *Arenarius*, a work which has been translated into most modern languages; where he undertakes to prove, that the numerical denominations which he has indicated in his books to Zeuxippes, are more than sufficient to express the grains of sand, that would compose a globe, not only as large as our earth, but as the whole universe. He supposes that the circumference of the earth is not more than three million stadia; that the diameter of the earth is greater than that of the moon, and less than that of the sun; that the diameter of the sun is 300 times greater than that of the moon, and moreover, that the diameter of the sun is greater than the side of the inscribed chiliagon, that is greater than $\frac{360}{1000}$, or 21′ 36″.

Archimedes ascertains the apparent diameter of the sun.

The manner in which he arrives at his conclusion is very interesting, as showing the state of the sciences at this time, even in the hands of this great master:—"I have used," says he, "every effort to determine by means of instruments, the angle which comprehends the sun, and has its summit at the eye of the observer; but this is not easy; for neither our eyes nor our hands, nor any of the means which it is possible for us to employ, have the requisite precision to obtain this measure. This, however, is not the place to enlarge upon such a subject. It will suffice to demonstrate that which I have advanced, to measure an angle which is not greater than that which includes the sun's apparent diameter, and has its summit in our eyes; and then to take another angle which is not less than that of the sun, and which equally has its summit in our eyes. Having, therefore, directed a long ruler on a horizontal plane towards the point of the horizon where the sun ought to rise, I place a small cylinder perpendicularly on this ruler. When the sun is on the horizon, and we look at it without injury, I direct the ruler towards the sun, the eye being at one of its extremities, and the cylinder is placed between the sun and the eye in such a manner, that it entirely conceals the sun from view. I then remove the cylinder farther from the eye, until the sun begins to be perceived by a thin stream of light on each side of the cylinder. Now, if the eye perceived the sun from a single point, it would suffice to draw from that point tangential lines to the two sides of the cylinder. The angle included between these lines would be a little less than the apparent diameter of the sun; because there is a ray of light on each side. But as our eyes are not a single point, I have taken another round body, not less than the interval between the two pupils; and placing this body at the point of sight at the end of the ruler, and drawing tangents to the two bodies, of which one is cylindric, I obtained the angle subtended by the sun's (apparent) diameter. Now the body, which is not less than the preceding distance (between the pupils), I determine thus: I take two equal cylinders, one white the other black, and place them before me; the white further off, the other near, so near indeed as to touch my face. If these two cylinders are less than the distance between the eyes, the nearer cylinder will not entirely cover the one that is more remote, and there will appear on both sides some white part of that remote cylinder. By different trials, we may find cylinders of such magnitude, that the one shall completely conceal the other: we then have the measure of our view (the distance between the pupils), and an angle, which is not smaller than that in which the sun appears. Now, having applied these angles successively to a quarter of a circle, I have found that one of them has less than its 164th part, and the other greater than its 200th part. It is therefore evident, that the angle which includes the sun, and has its summit at our eye, is greater than the 164th part of a right angle, and less than the 200th part of a right angle."

By this process, Archimedes found the sun's apparent diameter to be between 27′ and 32′ 56″.

Singular accuracy in the result.

It is not a little remarkable, considering the obvious inaccuracy of the method, that the maximum limit thus obtained, differs only $\frac{1}{3}$ of a minute from 32′ 35″·6, which is the largest angle actually subtended by the sun's diameter, and which is observed about the time of the winter solstice, when the sun is nearest to the earth. But this quotation from the *Arenarius* is extremely curious also on other accounts. We may learn from it, first, that Archimedes, with all his fecundity of genius, and with all the variety of his inventions, had no means of diminishing the effect of the sun's rays upon his eyes, and therefore performed this interesting experiment when the sun was in the horizon, that the optic organ might sustain its light without inconvenience. It also proves to us, that there was not then any instrument known to Archimedes, which he thought capable of giving the diameter of the sun, to within four or six minutes;

since he found it necessary to devise means at which he stopped, after an attempt not very satisfactory. We see, further, that he carried his angles, or their chords, over a quarter of a circle; but he does not say expressly that his arc had been divided; to render his language accurately, it is simply requisite to say, having carried one of the chords 200 times over upon the arc, he found it exhausted; and that the other chords could only be applied 164 times upon the quadrant.

We see, also, that Archimedes had not the means of computing the angle at the vertex of an isosceles triangle, of which he knew the base and the two equal sides. He was obliged to recur to a graphical operation as uncertain as the observation itself. Thus he was entirely ignorant even of rectilinear trigonometry, and he had not any notion of computing the chords of circular arcs.

Hipparchus B. C. 135.

Finds the length of the year.

We come now to the great father of true astronomy, Hipparchus; but our limits will not admit of our entering very deeply into his discoveries and improvements. One of his first cares was to rectify the length of the year, which before his time we have seen had been made to consist of 365 days and 6 hours. By comparing one of his own observations at the summer solstice with a similar observation made 145 years before by Aristarchus, he shortened the year about 7 minutes; making it to consist of 365 days, 5 hours, 53 minutes; which, however, was not sufficient: but the cause of the mistake is said to have rested principally with Aristarchus, and not with Hipparchus; for the observations of the latter, compared with those of modern times, give 365 days, 5 hours, 48 min. 49½ sec. for the duration of the year; a result, which exceeds the truth very little more than a second. It is to be observed, however, that this is no very exact criterion, unless the same be compared with the observation of the more ancient observer; for supposing all the error on the side of Hipparchus, it is more divided by comparing it with others at the distance of 19 or 20 centuries, than in comparing it with one, where the distance of time is only 145 years.

Introduction of trigonometry by chords.

Establishes the theory of the sun's motion,

and the first lunar inequality.

Hour of the night found by the stars.

One of the greatest benefits, which astronomy derived from this philosopher was his enunciation and demonstration of the method of computing triangles, whether plane or spherical. He constructed a table of chords, which he applied nearly in the same manner as we now do our tables of sines. As an observer, however, he rendered great service to the doctrine of astronomy, having made much more numerous observations than any of his predecessors, and upon far more accurate principles. He established the theory of the sun's motion in such a manner, that Ptolemy 130 years afterwards, found no essential alteration requisite; he determined also the first lunar inequality, and gave to the motions of the moon those of the apogee and of its nodes, which Ptolemy afterwards very slightly modified. Hipparchus also prepared the way for the discovery of the second lunar inequality, and from his observation it was, that the fact of the precession of the equinoxes was first inferred. He employed the transit of the stars over the meridian to find the hour of the night, and invented the planisphere, or the means of representing the concave sphere of the stars, on a plane, and thence deduced the solution of problems in spherical astronomy, with considerable exactness and facility. To him also we owe the happy idea of making the position of towns and cities, as we do those of the stars, by circles drawn through the poles perpendicularly to the equator; that is, by latitudes and by circles parallel to the equator, corresponding to our longitudes. From his projection it is, that our maps and nautical charts are now principally constructed; and his rules for the computation of eclipses were long the only ones employed for determining the differences of meridian.

Catalogue of the stars.

Another most important work of Hipparchus, was his formation of a catalogue of the stars. The appearance of a new star in his time, caused him to form the grand project of enabling future astronomers to ascertain, whether the general picture of the heavens were always the same. This he aimed to effect, by attempting the actual enumeration of the stars. The magnitude and difficulty of the undertaking did not deter this indefatigable astronomer; he prepared and arranged an extensive catalogue of the fixed stars, which subsequently served as the basis of that of Ptolemy. So great, indeed, is the merit of this prince of Grecian astronomy, that the enthusiastic language in which Pliny speaks of him in his Hist. Nat. (lib. ii. cap. 26.) may rather be admired than censured.

Ptolemy. A. D. 120.

After Hipparchus, we meet with no astronomer of eminence amongst the Greeks till the time of Ptolemy, who flourished between the years 125 and 140 of the Christian æra; which, therefore, includes a space of nearly three hundred years. There were, however, some astronomical writers, both Greeks and Romans in the course of this time, whom it may not be amiss to enumerate, although the little progress that the science made in their hands will exempt us from the necessity of entering minutely into an analysis of their several works: these were, Geminus, who lived about 70 years B. C., whose book is entitled *Introduction to the Phenomena*; Achilles Tatius, of about the same period; Cleomedes, who lived in the time of Augustus; Theodosius, Menalaus, and Hypsicles, who are supposed to have written about the year 50 B. C.; Manilius, Strabo, Posidonius, and Cicero, who were about half a century later; after which, we meet with no one to whom it is at all necessary even to refer, till we come to Ptolemy, who was born in the year of Christ 70; and who made, as we have stated above, most of his observations between the years 125 and 140 of our æra.

Various labours of Ptolemy.

Ptolemy has rendered all succeeding astronomers indebted to him, both for his own observations, which were very numerous, and his construction of various tables; but most of all for the important collection which he made of all astronomical knowledge prior to his time, and which he entitled, or the Arabs after him, the *Almagest, or Great Collection.* Of his own labours, we may mention his theory and calculation of tables of the planets, and his determination, with a precision little to be expected in his time, of the ratio of their epicycles to their mean distances; that is to say in other terms, the ratio of their mean distances to the distance of the earth from the sun. This theory, imperfect as it was, was adopted and generally admitted, for the space of fourteen centuries; during which time, it was transmitted to the Arabs, the Persians, and the Indians, and with whom it is still held sacred. The equi-distant centres of the earth, from the excentric and the equant, an hypothesis of Ptolemy, led, in all probability, Kepler to the idea of an ellipse and its foci. Ptolemy thus preparing the way for Kepler, as the laws of the latter may be considered as the precursors of the theory of Newton.

Sines introduced into trigonometry.

To this celebrated Grecian we also owe the substitution of the sines of arcs instead of their chords; as also the first enumeration of some important theorems in trigonometry.

Ptolemy's arguments to prove the immobility of the earth.

Ptolemy was the author of that system of astronomy which still bears his name; or if he did not entirely invent it, (as there is great reason to suppose he did not,) he enforced it by such arguments as led to its establishment; and it was afterwards rendered sacred through the stupid bigotry and intolerance of the Romish church. He endeavours to prove the absolute immobility of the earth, by observing, "If the earth had a motion of translation common to other heavy bodies, it would, in consequence of its superior mass, precede them in space, and pass even beyond the bounds of the heavens, leaving all the animals and other bodies without any support but air; which are consequences to the last degree ridiculous and absurd." In the same place he adds, "Some persons pretend, that there is nothing to prevent us from supposing that the heavens remain immovable, while the earth turns on its own axis from west to east, making this revolution in a day nearly; or, that if the heavens and the earth both turn, it is in a ratio corresponding with the relations we have observed between them. It is true, that as to the stars themselves, and considering only their phenomena, there is nothing to prevent us, for the sake of simplicity, from making such a supposition. But these people are not aware how ridiculous their opinion is, when considered with reference to events which take place about us; for if we concede to them that the lightest bodies, consisting of parts the most subtle, are not possessed of levity, (which is contrary to nature,) or that they move not differently from bodies of a contrary kind, (although we daily witness the reverse); or, if we concede to them that the most compact and heaviest bodies possess a rapid and constant motion of their own (while, it is well known, that they yield only with difficulty to the impulses we give to them), still, they would be obliged to acknowledge, that the earth, by its revolution, would have a motion more rapid than any of those bodies which encompass it, in consequence of the great circuit through which it must pass in so short a period; wherefore such bodies as are not supported on it, would always appear to possess a motion contrary to itself; and neither clouds, nor any projected bodies, nor birds in flight, would ever appear to move towards the east; since the earth, always preceding them in this direction, would anticipate them in their motion; and every thing, except the earth itself, would constantly appear to be retiring towards the west."

If we did not feel convinced that, in certain cases, even the errors and false reasoning of such a man as Ptolemy, possess a greater interest than the more correct and refined arguments of minor philosophers, we should certainly not have laid before our readers this extract from the introduction to the *Almagest*; but considering it as the defence of an hypothesis, which maintained its ascendancy for fourteen centuries amongst all nations, and which is still held sacred throughout every part of Asia, it is impossible to divest it of its interest and importance.

Analysis of the *Almagest*.

The other part of this great work is more worthy of the talents of its author, and is more deserving of our attention; but the limits of this article will not admit of our giving more than a very concise abstract of its contents. The first book, beside what we have hitherto mentioned, exhibits a highly interesting specimen of the ancient trigonometry; and the method of computing the chords of arcs, which, in fact, involves our fundamental theorems of trigonometry, though expressed in a manner totally different.

Theorems in trigonometry.

Ptolemy first shows, how to find the sides of a pentagon, decagon, hexagon, square and equilateral triangle, inscribed in a circle, which he exhibits in parts of the diameter, this being supposed divided into 120. He next demonstrates a theorem equivalent to our expression $\sin(a-b) = \sin a \cos b - \sin b \cos a$; by means of which he finds the chords of the difference of any two arcs, whose chords are known. He then finds the chord of any half arc, that of the whole arc being given, and then demonstrates what is equivalent to our formula for the sine of two arcs; that is, $\sin. (a+b) = \sin a \cos b + \sin b \cos a$; and by means of this he computes the chord to every half degree of the semicircle. These theorems it may be said belong rather to the history of trigonometry than to that of astronomy; but we trust that the obvious dependence of the latter science upon the former, will be found to justify us in introducing them to the reader in this place.

Climates.

We are next presented with a table of climates nearly equivalent to our nonagesimal tables, and it is not a little singular, that amongst them, we find none appertaining to the latitude of Alexandria; because, without such an auxiliary, Ptolemy must have contented himself with interpolations, which were not only difficult to make, but attended at the same time with great inaccuracy; a circumstance from which it has been concluded, that Ptolemy himself made few observations, or that he was not very particular concerning the accuracy of his calculations. The examination of this question would carry us too far out of our track to admit of our entering upon it in this place; but the reader may see it developed in all requisite detail, in the learned *History of Astronomy*, lately published by Delambre.

Length of the year, &c.

Having passed over the above preliminary details, the author treats of the length of the year, the motion of the sun, the mean and apparent anomaly, &c. &c. The length of the year, according to the sexagesimal notation, he makes 365d. $14' \, 48''$, which answers to 365d. 5h. $55' \, 12''$; the diurnal motion of the sun is stated to be $0^{\circ} \, 59' \, 8'' \, 7''' \, 13^{iv} \, 12^{v} \, 31^{vi}$, and the horary motion $2' \, 27'' \, 50''' \, 43^{iv} \, 3^{v} \, 1^{vi}$. To this is also added two tables, one of the mean motion of the sun, and the other of the solar anomaly. The fourth book of the *Almagest* is employed in treating of the motion of the moon, being prefaced by a few remarks respecting the observations which are most useful for that purpose: he then gives an abstract of all the lunar motions, with a table of them; in the first of which the motion is exhibited for periods of eighteen years;

in the second for years and hours; and in the third for Egyptian months and days. Four other columns of the same table present the number of degrees which belong to each of the times indicated in the first column; viz. the second, the longitude; the third, the anomaly; the fourth, the latitude; and the fifth, the elongation.

Lunar motion.

The author next treats of various subjects connected with the lunar motion; as, for instance, its general anomaly; its eccentricity; the lunar parallax; the construction of instruments for observing the parallax; the distance of the moon from the earth, which he states at $38{\cdot}4\frac{1}{3}$ terrestrial radii, when in the quadratures; the apparent diameters of the sun and moon; the distance of the sun from the earth, which is stated at 1210 radii of the latter; and the relative magnitudes of the sun, moon, and earth. The diameters of these are stated to be to each other, as the numbers 18·8, 1, and $3\frac{2}{3}$; also their masses as $6644\frac{1}{2}$, 1 and $39\frac{1}{4}$.

The next book is entirely occupied with the doctrine of eclipses of the sun and the moon; the determination of their limits and durations; tables of conjunctions; and methods of computation and construction, &c.

Particular deductions of Ptolemy.

We cannot extend the analysis of this important work to a greater length; but must content ourselves with a few remarks relative to some of the deductions to which we have referred. We have seen that Ptolemy made the length of the year to be more than 365 days, 5h. 55m., which is about 6 minutes longer than it really is; but considering that the observations before his time, with the exception of those of Hipparchus, were very imperfect; and that the distance of time between these two celebrated astronomers, was not sufficient to determine such a question, with the means they possessed, to the greatest nicety, we may rather admire the near approximation to the truth, than be astonished at the difference between his result and that deduced from numerous and long continued observations.

The evection discovered

His researches on the theory of the sun and moon were however attended with better success. Hipparchus had shown that these two bodies were not placed in the centre of their orbits; and Ptolemy demonstrated the same truths by new observations. He moreover made another important discovery, which belongs exclusively to him, except so far as relates to the observations of Hipparchus, by a comparison of which with his own, his conclusion was deduced,—we allude here to the second lunar inequality, at present distinguished by the term *evection.* It is known, generally, that the velocity of the moon in its orbit, is not always the same, and that it augments or diminishes, as the diameter of this satellite appears to increase or decrease; we know, also, that it is greatest and least at the extremities of the line of the apsides of the lunar orbit. Ptolemy observed that from one revolution to another; the absolute quantities of these two extreme velocities varied, and that the more distant the sun was from the line of the apsides of the moon, the more the difference between these two velocities augmented; whence he concluded that the first inequality of the moon, which depends on the eccentricity of its orbit, is itself subject to an annual inequality, depending on the position of the line of the apsides of the lunar orbit with regard to the sun.

When we consider Ptolemy's system of astronomy, as founded upon a false hypothesis, the complication of his various epicycles, in order to account for the several phenomena of the heavenly bodies; and the rude state of the ancient astronomy, it is impossible to withhold our admiration of the persevering industry and penetrating genius of this justly celebrated philosopher; who, with such means, was enabled to discover an irregularity which would seem to require the most delicate and refined aid of modern mechanics to be rendered perceptible.

The work of this author to which we have hitherto confined our remarks, is the *Almagest*;* but Ptolemy also composed other treatises; which, if not equal to the above in importance, are still such as to be highly honourable to his memory and talents, particularly his geography.

Ptolemy's geography.

This work, although imperfect as to its detail, is notwithstanding founded upon correct principles; the places being marked by their latitude and longitude agreeably to the method of Hipparchus. As to the inaccuracies of their position, although they cannot be denied, they will readily be pardoned, when we consider that he had for the determination of the situation of cities and places of which he speaks, only a small number of observations subject to considerable errors; and the mere report of travellers, whose observations we may readily grant were still more erroneous than those of his own. It requires many years to give great perfection to geography: even in the present time, when observations with accurate instruments have been made in every part of the globe, we are still finding corrections necessary; a remarkable instance of which seems to have occurred lately (1818) to Captain Ross, in his voyage into Baffin's Bay, where he is said to have found some parts of the land laid down nearly a degree and a half out of their proper places. Many other minor pieces on astronomy and optics are also attributed to this author; but we have already extended our accounts of his works to a greater length than we had intended, and must now therefore pass on to his successors.

Greeks posterior to Ptolemy.

After the time of Ptolemy we find no Greek authors of eminence, although we have some few writers on this subject. The science of astronomy had now obviously passed its zenith, and began rapidly to decline. The Alexandrian school, it is true, still subsisted; but during the long period of 500 years, all that can be said is, that the taste for, and the tradition of the science was preserved, by various commentators on Hipparchus and Ptolemy; of whom the most distinguished were Theon and the unfortunate Hypatia, his daughter. The latter is said to have herself computed certain astronomical tables, which are lost.

Destruction of the Alexandrian school.

We now arrive at that period, so fatal to the Grecian sciences. These had for a long time taken refuge in the school of Alexandria; where, destitute of support and encouragement, they could not fail to degenerate. Still, however, they preserved, as we have said above, at least by tradition or imitation, some resemblance of the original; but about the middle of the seventh century, a tremendous storm arose which threatened their total destruction. Filled with all the enthusiasm a military government is calculated to inspire, the successor of Mahomet ravaged that vast extent of country, which stretches from the east to the southern confines of Europe. All the cultivators of the arts and sciences who had from every nation assembled at Alexandria, were driven away with ignominy: some fell beneath the swords of their conquerors, while others fled into remote countries, to drag out the remainder of their lives in obscurity and distress. The places and the instruments which had been so useful in making an immense number of astronomical observations, were involved with the records of them, in one common ruin. The entire library, containing the works of so many eminent authors, which was the general depository of all human knowledge, was devoted to the devouring flames, by the Arabs; the caliph Omar observing, " that if they agreed with the Koran, they were useless; and if they did not they ought to be destroyed:" a sentiment worthy of such a leader, and of the cause in which he was engaged. In the midst of this conflagration, the sun of Grecian science, which had long been declining from its meridian, finally set; never perhaps again to rise in those regions once so celebrated for the cultivation of every art and science that does honour to the human mind.

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* The first printed edition of this celebrated performance, was a Latin translation from the Arabic version of Cremoneus; which, however, abounds so much in the idiom of that language, as to render it nearly unintelligible, without a constant reference to the Greek text. This was published at Venice in 1515; and in 1538 the collection appeared in its original language, under the superintendence of Simon Grynaeus, at Basil, together with the eleven books of the Commentaries of Theon. The Greek text was again republished at the same place, with a Latin version, in 1541; and again, with all the works of Ptolemy, in 1551; and lastly, a splendid French edition with the Greek text, by M. Halma, in three beautiful volumes, royal quarto, Paris, 1813.

Astronomy of the Chinese and Indians.

Astronomy of the Chinese.

If we were to adopt the opinions of some authors who have written on the subject of the Chinese astronomical knowledge, we should have now to commence at a much earlier period than we did in giving our account of this science amongst the Greeks; as it is stated that the former possess records of eclipses and other celestial phenomena, so far back as the year 2159 B.C., and that even in the year 2857 B.C. the study of astronomy and the desire of propagating a knowledge of that science amongst his people, were objects of great moment with the emperor Ion Hi; such at least is the doctrine supported in the *Histoire Générale de la Chine, ou Annales de cet Empire*, translated into French from Tong-Kien-Kang-Mou, by the Pere De-Mailla, a French Jesuit, one of the missionaries to Pekin.

We cannot attempt to enter here upon a refutation of the ideas supported in this work, relative to the antiquity of the Chinese astronomy; it will perhaps be sufficient to observe, that of 460 eclipses reported to have been predicted and recorded in the Chinese annals, the first is dated 2159 B.C., and the second, 776 B.C., leaving a great blank of 1383 years, during which no such phenomenon is noticed. That from the latter date to the year 1699 of our æra, they run on in pretty regular succession; but that of this number, the Pere Gaubil, who was at the labour of computing them, found only twelve which answered to the year, month, and day stated in the annals; and that of these twelve, only one of them was anterior to the time of Ptolemy, and even this one is doubtful.

The antiquity of it inadmissible.

It appears then that very little credit is to be given to the assumed antiquity of Chinese science; but perhaps the following abstract, which Delambre has made from the annals above referred to, will give the reader a better idea of the state of astronomy amongst this singular people, and place their pretensions in a more tangible form than any thing we can advance respecting the improbability of the notions advanced by Mailla.

In the year 687 B.C. we find recorded a night without stars, and without clouds; and that towards midnight, there fell a shower of stars, which vanished before they approached the earth.

141 B.C. The sun and moon appeared of a deep red colour, which produced great alarm among the people.

74 B.C. There appeared a star as big as the moon, followed by many other stars of the ordinary magnitude.

38 B.C. A shower of stones as big as nuts.

88 A.D. Another shower of stars.

321. Spots in the sun visible to the naked eye.

522. The astronomy of Hiuen-Chi-Ly was replaced by the astronomy of Tching-Kouange-Ly. (We do not understand these to be the names of astronomers but of systems.) In the year 892, another change is recorded; and again in 956.

In 949, in the fourth moon, there appeared a star in the mid-day, which was regarded as such a dreadful omen, that the people were forbidden to look at it; and many were put to death for disregarding the injunction.

These facts alone, independent of the superstitious fears and ceremonies, which even to this day are observed during the time of an eclipse, would alone give us a sufficient contempt for the astronomy of the Chinese; and lead us to reject as mere fables their pretensions to ancient observations. All that we can concede to them, and to the Indians, whose claims rest upon nearly the same grounds, is what we have already attributed to the Chaldeans, viz. that they had become, very early, observers of the motions and phenomena of the heavenly bodies; that they registered certain events, and thence were able to discover periods at which these phenomena would return again; sufficiently approximate to admit of their predicting an eclipse or occultation, within certain limits; but frequently, their prediction has been belied by the non-appearance of the expected phenomenon, while others have happened that were not foretold; which omissions, as well as a false prediction, have cost some few unfortunate astronomers the forfeiture of their heads: of which a particular instance is recorded in the case of Hi and Ho.

Upon the whole, therefore, we may conclude, that however ancient may be the rude observations of the Chinese and Indians, they possessed no science, properly so called, but what they obtained from the Greeks, through the medium of the Arabs; which people, after deriving it from the former source, carried it to Persia, whence it was transmitted to India and China. Such at least is the conclusion drawn by M. Delambre from a dispassionate examination of all the claims of these nations. We are aware that this notion is totally at variance with that of Bailly, who has also examined the case in point with great labour and attention, but certainly not without great enthusiasm and prejudice

Astronomy of the Arabs.

Astronomy of the Arabs

In our last mention of this people, we saw them acting the part of the most savage barbarians, burning and destroying every thing, the most distantly connected with scientific research; we have now to exhibit them in a more honourable and dignified point of view. We stated, in the passage referred to, that some of the philosophers of Alexandria, escaped the vengeance of their barbarous conquerors, and these of course carried with them some remnant of that general learning, for which the school was so deservedly celebrated. Still, however, destitute of books, of instruments, and probably also of the means of subsistence without manual labour, very little farther knowledge could be accumulated, and still less propagated; so that in a few years, every species of knowledge connected with astronomy and mathematics, must have become extinct, had not the Arabians themselves within less than two centuries of the dreadful conflagration of the Alexandrian library, become the admirers and supporters of those very sciences they had before so nearly annihilated. They studied the works of the Greek authors which had escaped the general wreck, with great assiduity; and if they added little to the stock of knowledge these works contained, they became sufficient masters of many of the subjects to enable them to comment upon them, and to set a due estimation upon these valuable relics of ancient science.

Almansor. A.D. 754.

Haroun. A.D 786

Almamon. A.D. 813.

Translate the Almagest.

The destruction of the Alexandrian school occurred in the year 640; and it is about a century after, before we find any author worthy of particular notice amongst the Arabs; this, therefore, brings us to the middle of the eighth century; and from Ptolemy to this date there had passed away at least six hundred years, during which time the science had rather retrograded than advanced. The caliph Almansor, A. D. 754, is the first to claim our attention; but rather, perhaps, for the impulse, which, as a philosophical prince, he gave to the science amongst his people, than for any actual improvements which he had personally made. This impulse was, however, so great, that most of his successors seem to have thought it their duty to support and to study the different sciences, particularly astronomy. Haroun, the grandson of Almansor, is particularly noticed as following in the steps of his great predecessor; and one of his sons, Almamon, pursued the same path with still greater enthusiasm. He caused to be translated all the Greek works that he could procure; and in particular the *Almagest* of Ptolemy. He is even said to have made the delivery of certain manuscripts deposited at Constantinople, one of the conditions of the peace which he concluded with the Greek emperor Michael III. He himself made numerous observations, and employed and instructed others to supply his place, when public business prevented him from his favourite pursuit. He ordered the obliquity of the ecliptic to be observed at Bagdad and at Dumas; whence it was found to be 23° 35′, which is less than some preceding observations had indicated.

Measures a degree of latitude.

Another important operation performed under the orders of Almamon, was the measure of a degree of the terrestrial meridian; but the unit of the Arabic measure in which it is expressed, is too uncertain for us to attempt to form from the reported result any decisive conclusion; nor are we to expect that any very great accuracy could be expected, seeing the great discrepancy between some modern measures made with the assistance of the most perfect instruments. Almamon also directed certain of his philosophers to compose a work for the purpose of facilitating the study of astronomy amongst his people, entitled, according to the Latin translation, *Astronomia elaborata a compluribus D. D. jussu regis Maimon.* which is still preserved in many libraries.

Alfraganus. A.D. 850.

In the reign of this prince, there were many other celebrated Arabian astronomers, particularly Alfraganus, Thebit-Ibn-Chora, and Albatenius. The former composed a work, many editions of which have been made since the invention of printing, besides some other works more or less connected with this science.

Thebit. A.D. 860.

Thebit was an annalyst, a geometer, and astronomer. He observed the obliquity of the ecliptic and reduced it to 23° 33′ 30″. He also determined the length of the year, very nearly the same as it is now established by modern observations.

Albatenius. A.D. 879.

Albatenius was one of the greatest promoters of Arabian astronomy. His numerous observations and important knowledge in all the sciences of his time, were the cause of his being surnamed the Ptolemy of the Arabs; an honour by no means ill merited. By a comparison of many of his own observations with those of Ptolemy and others, he corrected the determination of the latter respecting the motion of the stars in longitude, stating it to be one degree in 70 years instead of 100 years: modern observations make it one degree in 72 years. He determined very exactly the eccentricity of the ecliptic, and corrected the length of the year, making it consist of 365 days 5 hours 46 minutes 24 seconds, which is about 2 minutes too short, but 4 minutes nearer the truth than had been given by Ptolemy. He also discovered the motion of the apogee; and rectified the theories of Ptolemy respecting the motion of the planets, and formed new tables of them.

The works of this author have been collected, and published in two volumes 4to., under the title of *De scientia stellarum*, of which there are two editions, one in 1537, and the other in 1646.

Montucla, in his valuable history of mathematics, enumerates a long list of Arabian astronomers which followed Albatenius; but we meet with none deserving of particular notice till we arrive at Ebn Iounis, who wrote in the year 1004, and even he is rather celebrated for his having collected and embodied the knowledge of his time, than for his discoveries, although he made numerous observations. The work of this author is still extant, a concise notice of which Delambre has given in the *Mém. de l'Institut*, vol. 2. p. 5, where we learn, that it contains 28 observations of eclipses of the sun and moon, made between the years 829 and 1004; seven observations on the equinoxes; one on the obliquity of the ecliptic: and some others highly important in the determination of certain data, particularly as regards the acceleration of the moon.

Let us now turn our attention to Spain, where the Arabs, who had long been masters of that country, pursued the sciences with the same ardour as in the east. The most distinguished, however, of the astro-

Arsachel.
A.D. 1020.

nomers in this country were Arsachel and Alhazen; the former of whom is celebrated for having added greater accuracy to the theory of the sun, by employing a principle different to that of Ptolemy and Hipparchus, and susceptible of more accuracy. He made some fortunate changes in the dimensions of the solar orbit, and discovered certain inequalities in the sun's motion, which have since been confirmed by the Newtonian theory of gravitation.

Alhazen.

Alhazen is also esteemed a philosopher and astronomer of high reputation; he is said to have first discovered the laws of refraction, and the effect of it in astronomical observations. He explained the phenomenon of the horizontal moon, and indicated the true cause of the crepuscula in the morning and evening; beside various other minor discoveries highly honourable to his memory.

From this time the science of the Arabians seems to have begun to decline; we meet with very little after this period deserving of particular notice; a general shade appears to have been cast over every species of human knowledge, and nearly four hundred years are again lost in darkness and obscurity. We then find the Greeks making some feeble efforts to re-establish astronomy in its original empire, where some faint glimmerings of the genius which animated Archimedes, Hipparchus, and Ptolemy, once more began to discover itself; but which, alas! like the lustre of a passing meteor, was soon extinguished, and darkness and barbarity once more assumed their reign.

Astronomy of modern Europe.

Copernicus
died 1543.

We may without impropriety refer the revival of modern astronomy to the time of Copernicus, although he was preceded by some others, who prevented all traces of the Grecian and Arabic science from being lost and forgotten, by their reading and studying such works as had been preserved, during what are commonly denominated the dark ages. Copernicus was born at Thorn, in Poland, in the year 1473, but he did not commence his studies till about the year 1507; when after having well "fathomed every depth and shoal" of the ancient doctrine of astronomy; made numerous observations, and various comparisons; he became at first a convert, and afterwards the most strenuous advocate of a system of astronomy, commonly attributed to Pythagoras, and which we have seen Ptolemy using so many ingenious but false arguments to refute.

System of
Copernicus

According to Copernicus, the sun is placed in the centre of the planetary world, about which the several bodies of our system revolve from west to east; viz. 1st, Mercury, 2d, Venus, 3d, the Earth, 4th, Mars, 5th, Jupiter, and 6th, Saturn; the moon revolves about the earth in the same direction, while the latter body itself is carried in its orbit round the sun. In the next place, he taught that the earth turns on its own axis from west to east, in a little less than 24 hours, and that this axis is always preserved parallel to itself, making an angle of about 23½° with the ecliptic.

The orbits of the several planets he supposed to be circular; but he did not make the sun their common centre. We have seen that what was anciently called the solar orbit had been long known to be eccentric, as well as those of the planets; and to account for the phenomena produced by these eccentricities, Copernicus rendered them still greater, by giving to each a different centre, and to the sun such a position with regard to them all, as with the addition of certain epicycles, best agreed with the appearances already observed. This part of the Copernican system is not often well illustrated in our elementary works on astronomy, where it is generally asserted, that the sun was placed in the common centre, and the several planetary orbits were circles concentric with it; by which means, the discoveries of Kepler are rendered more astonishing than they actually are; for he had at least this much to proceed upon; too little, it is true, for any man possessed of less genius and perseverance than himself. This is by no means the only service which Copernicus bestowed upon astronomy and trigonometry, but it is the principal one, and that which has added most to the celebrity of his name; we shall therefore not stop to report his other discoveries and improvements, as we shall find numerous important points to refer to before we conclude this sketch, already considerably extended. The work containing his new doctrine was composed in the year 1530; but it was not printed till the year 1543; the author having, it is said, received the last sheet a few hours before he expired.

Copernicus was followed by a great number of excellent astronomers, some of whom were firm supporters of his system; others endeavoured to refute it, as Ptolemy had formerly done a similar doctrine; while some few who saw its beauties and advantages, but who by giving too literal a signification to some scriptural passages, wished to modify it so as to retain as many as possible of its advantages, while the system should still be such as to correspond with the passages in question.

Tycho
Brache died
1601.

His system.

Of this number was Tycho Brache, a noble Dane, one of the greatest observers perhaps, if we except Kepler, that ever lived. His system consisted in depriving the earth of its orbicular and diurnal motion; he places the earth in the centre, and made the moon and sun revolve about it, agreeably to the doctrine of Ptolemy; but he made the sun the centre of the other planets, which, therefore, he supposed to revolve with the former about the earth. By this means, the different motions and phases of the planets may be reconciled, the latter of which could not be by the Ptolemaic system; and he was not obliged to retain the epicycles, in order to account for their retrograde and stationary appearances. This theory was extremely complicated, and did not long survive its author.

Other discoveries of
Tycho

Many of his other labours were attended with much more important consequences; particularly his discovery of the variation and annual equation of the moon; the greater and less inclination of the lunar orbit; the correction for refraction in astronomical observations; his astronomy of comets; his account of the appearance and disappearance of a great star which happened in his time: his reformation of the calendar, and some other subjects which might be enumerated: these have contributed most to establish his name as a great astronomer, and will not fail to hand it down to the latest posterity.

Kepler died
1631.

Kepler was about twenty years younger than Tycho, but by no means inferior to him in genius and perseverance; the latter quality is proved by the numerous

His observations on Mars.

observations that he made on all the planets, particularly on Mars; and the former from the memorable consequences which he drew from them. Kepler first determined the line of the apsides by a method independent of the form of the orbit of Mars, and ascertained the ratio of the aphelion and perihelion distances; the former he found to be to the latter, as 166,780 to 138,500; or calling the distance of the earth from the sun 100,000; the above numbers will express the actual distances. Hence the mean distance of Mars was 152,640, and the eccentricity of its orbit 14,140. He then determined, in like manner, three other distances, and found them to be 147,750; 163,100; 166,255. He next computed the same three distances upon a supposition that the orbit was a circle, and found them to be 148,539; 163,883; 166,605; the errors, therefore, of a circular hypothesis, were 789, 783, 350. But he had too good an opinion of Tycho's observations, (upon which he founded all these calculations), to suppose, that the differences arose from their inaccuracies; and as the distance between the aphelion and perihelion was too great according to the hypothesis in question, he conceived that the orbit must be an oval. And, as of all ovals the ellipse is the most simple, he naturally made trial of this figure, placing the sun in one of its foci; and upon making the requisite calculation, he found the agreement complete. He did the same for other points of the orbit, and still found the same accurate agreement; and hence he pronounced the orbit of Mars to be an ellipse, and that the sun occupied one of its foci. Having established this important point for the orbit of Mars, he conjectured the same to have place in the other planets; and upon trial he found his conjecture fully verified: and, hence, he concluded, that the six primary planets revolve about the sun in elliptic orbits, that body occupying one of the foci.

Kepler's first law.

Kepler's second law.

Having thus discovered the relative mean distances of the planets from the sun, and knowing their periodic times, he next endeavoured to find, whether there were any relations between them; and having naturally a strong turn for numerical analogies, he began by comparing the powers of those quantities with each other; and even in the first instance (March 8th, 1616,) he assumed the correct law; viz. *that the squares of the times of revolution are proportional to the cubes of the mean distances*; but, in consequence of some error in his calculation, his comparison did not appear to be complete. Nor did he discover it till the following May. He then found his first conjecture to be correct, and thus established this celebrated law, which Newton afterwards demonstrated to be the necessary consequence of a body revolving in an orbit about a central attracting point coinciding with one of its foci. *Prin. Phil.* lib. i. sec. ii. pr. 15.

Kepler's third law.

Kepler also discovered from observation, that the velocities of the planets, when in their apsides, are inversely as their distances from the sun; whence it followed, that they described at these points equal areas in equal times; and although he could not prove the same for every point of their orbits, he had still no doubt that it was so. He therefore applied this principle to determine the equation of the orbit; and finding that his calculations agreed with observation, he concluded that it was true in general; "that the planets describe about the sun equal areas in equal times." This discovery was perhaps the foundation of the *Principia*, probably suggesting to Newton the idea, that all the planets of our system were governed by one general law! and that the sun was the general focus of action; a proposition which he afterwards succeeded in demonstrating, and thus founded the basis of physical astronomy.

Kepler also speaks of *gravity* as a power which is mutual between all bodies; and tells us that the earth and moon would move towards each other, and meet at a point so much nearer the earth than the moon, as the former is greater than the latter, if these motions did not prevent it. And he farther adds, that the tides rise from the gravity of the waters towards the moon.

Invention of the telescope.

The beginning of the 17th century was distinguished by two of the most important events for astronomy that we have yet recorded; viz. the invention of the telescope and logarithms; by means of the one, we are enabled to penetrate into the remotest parts of space, and to bring under our immediate view phenomena, which the most sanguine minds could never, if they had suspected their existence, have hoped to bring within the limits of human observation: by means of the other, all the laborious calculations of former astronomers were reduced to mere operations in addition and subtraction; and what was before the labour of a month, became, as it were, the amusement of an hour. With two such powerful auxiliaries, the progress of astronomy could not but be extremely rapid; but still the extent to which it has been since carried, must certainly far have exceeded what the boldest of the astronomers and philosophers of that day could have dared to hope for. It never could have been contemplated that this science, apparently so far beyond the reach of all human power, would become the most perfect of all the physical sciences; that every disturbing force, and every celestial phenomenon of our system, should be submitted, with the greatest precision, to one general simple principle, at that time wholly unknown; and that an analysis would be discovered, to enable us to investigate and compute the motion of bodies of immense magnitudes, and at almost immeasurable distances, with greater accuracy than we can calculate the motion of a projectile from a piece of ordnance. Yet such is actually the case. We know, to a greater nicety, the moment when a planet will arrive at a certain point in the heavens, than we can tell the time that a cannon ball will employ in passing from the gun to the extremity of its destined range, the moment of explosion being given.

Origin of physical astronomy.

As far as we have hitherto traced the progress of astronomy, all our knowledge has been supposed to consist in observations on the motions of the heavenly bodies; and every species of computation relating to it, rested on no other foundation. We are now arrived nearly at that period, when our illustrious Newton, by his discovery of the law of universal gravitation, added an entire new branch to this important science, commonly denominated physical astronomy. From this time a greater range was given to celestial researches; our knowledge was not confined to the mere observation of phenomena, and the return of certain bodies to certain parts of space: the causes of these phenomena, and of these motions, were now to be investigated, and the amount of them computed from the pure principles of celestial mechanics. Every minute inequality was henceforward to be traced to the same source—the action of gravitation: while, on the other hand, every minute variation indicated by this theory, ought to be observable in the heavens. And nothing, perhaps, is better calculated to demonstrate the generality of this law, and the beauty and profundity of the modern analysis, than the fact, that certain small inequalities were indicated by the theory, before the accuracy of instruments, and the delicacy of observation, were sufficiently attained to render them appreciable; but the existence of which, the present high state of practical mechanics, and a corresponding improvement in optics, have confirmed in the most satisfactory manner. We shall, therefore, now divide the remaining part of this historical sketch under two distinct heads; viz. practical and physical astronomy; and slightly glance at the successive steps that have been made in each; but a simple indication of them is all that can be expected, and indeed it is all that is requisite, since we must necessarily meet the same subjects again in the course of the following treatise; where we shall enter upon them as much at length, as is consistent with the limits allotted to this department of our work.

Discoveries depending upon the telescope.

Our business here, is neither to give the history of the invention of the telescope nor of logarithms; the former has already been treated of in our treatise of optics, and the other will be found in its appropriate place in this work. We have seen in the article above referred to, that the telescope is perhaps an earlier invention than it is commonly supposed to be; and that Harriot, in a very early part of the 17th century, viz. between the years 1610 and 1613, had actually observed the spots on the sun's disc with telescopes of various magnifying powers; a fact, which has been but lately ascertained by Baron Zach, of Saxe Gotha, who, in a visit to this country, had access to certain MSS. of this English author, in the possession of the Earl of Egremont, a descendant of the Earl of Northumberland, who was Harriot's patron. This, after Galileo's, is one of the earliest well attested facts of the application of this instrument to astronomical purposes.

Galileo discovers the satellites of Jupiter. A.D. 1610.

Galileo, as we have seen in our history of optics, was informed of the accidental discovery of Jansen's children, in the year 1609; and on the 8th of January, 1610, he perceived, by directing his new instrument to the heavens, three small stars near the planet Jupiter; two on the one side, and the third on the opposite side; which he took at first for fixed bodies. But having continued to observe them on the next and the following nights, he found that they changed their places and positions; and that they performed their revolutions about that planet, and were, in fact, satellites to it, as the moon is to the earth; and some days after he discovered a fourth. These four satellites he named, in honour of the house of Medici, the Medicean stars; but the appellation was soon lost in the more general and appropriate name of Jupiter's satellites. This discovery was published in the month of March following, in a writing, entitled *Nuncius Sydereus;* he undertook even to investigate the theory of their motion; and, in the year 1613, to predict their configurations for two consecutive months.

Irregularity in the form of Saturn.

Directing his telescope to Saturn, a new surprize, and new pleasure presented itself. After a few imperfect observations on this planet, he was led to believe that Saturn was not a simple globe like the other bodies of our system, but was compounded of three stars touching each other, immovable with regard to themselves, and so disposed, that the largest occupied the centre, with a smaller one on each side of it; but it was not long before he discovered this idea to be erroneous, so far at least as regarded their immobility, or invariable appearance: he found that the figure was changeable, but his telescope had not sufficient power for him fully to unravel the mystery; he found that Saturn appeared irregularly formed, and supposed the extreme parts, of what was afterwards found to be a ring encompassing the body of the planet, were attached to it, forming *ansæ,* or handles; and it was some years after, that Huygens discovered the actual conformation of this beautiful telescopic object. Galileo also first observed the phases of Venus, predicted by Copernicus; as he did also the spots on the sun's disc, unless indeed our countryman Harriot had the advantage of him, in point of time, in this observation. There are others who contest with Galileo for the honour of the important discovery of Jupiter's satellites, as Simon Marius, of Brandebourg, who asserts, that he observed them in 1609; but the truth of this seems to rest on no better foundation than the mere declaration of the author, in his *Nuncius Jovialis, anno* 1609, *detectus,* &c. published in 1614. The spots on the sun were also observed by Fabricius and Scheiner; and it is perhaps doubtful by whom they were first seen. The telescope soon became well known in all parts of Europe; the sun was an object which would naturally claim the regard of astronomers, and it is not unlikely these spots might be observed by many at nearly the same time.

Phases of Venus, and spots on the sun.

Other claims for these discoveries.

Galileo defends the Copernican system.

The system of Copernicus, even before these discoveries, had made considerable progress; and what had now occurred, had rendered the truth of it still more certain. Galileo, therefore, in 1615, had the courage openly to support it; but it caused him a reprimand from the *Holy Inquisition;* he was imprisoned, but afterwards liberated upon certain conditions of silence. Twenty years afterwards, greater advances being made in the progress of this science, he again ventured to assert his opinion, but still with the same effect: he convinced all unprejudiced people; but the holy fathers were not in this class, and he was once more obliged to abjure upon his knees his heretical doctrine. Such are the fruits of ignorance blended with bigotry and superstition.

Huygens discovers Saturn's 6th satellite. A.D. 1635.

We shall deviate a little in the order of time, in order to bring together those discoveries which have added to the number of the bodies of the solar system. Huygens, therefore, according to this arrangement, is the first to claim our attention. This eminent philosopher, having in 1635 constructed for his own use two excellent telescopes, the one of twelve, and the other of twenty-four feet, discovered a satellite to Saturn, which is now the sixth in the order of distances, and determined the dimensions of its orbit, the duration of its revolution, &c. with an astonishing degree of exactness; but falling into a metaphysical error, concerning the number of the bodies of our system, he sought to find no more, considering that

Discovers Saturn's ring.

this one was all that was wanting to render the whole planetary scheme complete. He was drawn, however, to a minute examination of Saturn itself, and was the first to announce the actual conformation of it. He stated, that this planet was encircled by a flat broad circular ring, detached from it on all sides, but which, according to the position under which it was observed from the earth, would assume the several appearances of a circle, an ellipse, and a right line; he never, however, observed it under its latter form. He discovered that the diameter of the exterior edge of the ring, was to the diameter of the planet as 9 to 4; and that its breadth was equal to the space contained between the globe and its interior circumference; results which, with some slight modifications, have been confirmed by more modern observations.

Cassini discovers four other satellites, A.D. 1684. Herschel two others.

Four other satellites to this planet were discovered by Dominique Cassini: which, in the order of distances, are now the 3d, 4th, 5th, and 7th; the 5th and 7th in 1671, and the 4th and 5th in 1684; two others have since been added by Sir W. Herschel in 1789; making in all seven attendant luminaries to this remote and otherwise solitary planet.

Herschel discovers the Georgium Sidus.

Discovers six of its satellites.

Eight years prior to the above discovery, viz. in 1781, Sir W. Herschel had observed a small star, which after a little attention, he found changed its place; having well ascertained this fact, he communicated a knowledge of the circumstance to M. Laxel, a celebrated astronomer of the academy of St. Petersburgh, who was then in London; the same information was also transmitted to other eminent astronomers, who observed it with great care, and soon afterwards it was announced as a new planet, the most remote in our system, circulating about the sun at the astonishing distance of nearly *eighteen hundred million miles,* and performing its orbicular revolution in about 80 of our years. This new planet was first named by foreign astronomers after its observer, *the Herschel;* but Herschel himself, in imitation of Galileo, dedicated it to his late Majesty, under the name of the *Georgium Sidus;* both these appellations are, however, now nearly become extinct, that of *Uranus* being almost universally adopted. The same indefatigable observer has since discovered six satellites to this planet, which revolve about him under very peculiar circumstances; but the particulars must be reserved for the proper place in the subsequent treatise.

Four other newplanets. Ceres, by Piazzi.

Pallas, by Olbers.

Juno, by Harding.

Vesta, by Olbers.

In order to complete this part of our historical sketch, we must now pass to the beginning of the 19th century; the *first day* of which is remarkable for the discovery of another new planet, between the orbits of Mars and Jupiter; this we owe to the observation of Piazzi. This planet has received the name of Ceres. Another new planet was discovered by Dr. Olbers, on the 28th of March, in the same year, 1801, which is called *Pallas;* its distance and periodic revolution being nearly the same as that of the former. A third, having likewise nearly the same mean distance, was discovered by M. Harding, at Lilienthal, near Bremen, September the 4th, 1804; and a fourth by Dr. Olbers, on the 19th of March, 1807, being the second we owe to this indefatigable observer. Of these two, the former has been named *Juno,* and the latter *Vesta.*

For the elements, and other particulars respecting these new planets, we must refer the reader to the respective articles in the following treatise.

Uniting together these several discoveries, it will appear, that in less than two centuries, there have been added to the known bodies of our system, no less than five planets, and seventeen satellites, about three times as many as were known at the time of the promulgation of our present system by its venerable author Copernicus.

In the paragraphs immediately preceding, we have in some degree disregarded the order of time, for the purpose of bringing under one point of view those discoveries which have increased the number of the bodies in the solar system; we must now, therefore, retrace our steps, in order to glance at some other facts connected with the history of this science; but the bare enumeration of them is all that our limits will allow us to attempt.

Bayer's catalogue of the stars, A.D. 1603.

In 1603, Bayer formed a catalogue of the stars, which he published under the title of *Uranometria,* a highly important and useful work.

Transit of Mercury. A.D. 1632.

Nov. 7, 1631, Gassendi observed the passage of Mercury over the solar disc, agreeably to the prediction of Kepler; and published his account of it in 1632, in a work entitled *Mercurius in Sole visus,* &c.

Transit of Venus. A.D. 1638.

1638. The transit of Venus was observed over the sun by Horrox, a young English astronomer, which is described in his *Venus in Sole visus,* &c.

These are the two first observations of this kind that had ever been made, and much importance was in consequence attached to them, although their great utility in establishing certain astronomical data was not then foreseen.

Hevelius died. A.D. 1638.

In 1638, the sciences had to deplore the loss of Hevelius, a celebrated astronomer and senator of Dantzic, to whose indefatigable labours we owe many valuable observations; but our limits will not permit of our entering into particulars.

Royal Society and Academy of Sciences.

Royal Observatories of London and Paris.

We pass now to that period when the Royal Society of London, and the Academy of Sciences at Paris, were first instituted, and the observatories of Paris and Greenwich were erected. The advantage of these institutions to the science of astronomy, is unbounded; instruments of the best kind that could be obtained, were immediately constructed: the members of the two academies communicated with each other; various experiments were proposed and executed; and an impulse was thus given to science in general, and to astronomy in particular, which it would have been in vain to have expected from individual perseverance and talent, however conspicuous.

English astronomers royal.

The charter of the Royal Society was granted by Charles II. in 1660; that of the Academy of Sciences by Louis XIV. in 1666. In 1667, the observatory of Paris was erected; and the first stone of the observatory of Greenwich, was laid by Flamstead (who was appointed astronomer royal) on the 10th of August, 1675, at the recommendation of Sir Jonas Moore, to whose influence we are indebted even for the institution itself. Flamstead continued to fill this situation in a manner equally honourable to himself and his country, for 43 years; he was succeeded by Dr. Halley, who continued in it for 23 years; Dr. Bradley followed Dr. Halley, and held the office for twenty years: Mr. Bliss only remained two years, and was followed by Dr. Maskelyne, who died in 1811, after holding it for 46 years.

Never has any appointment been more honourably filled than that of the astronomer royal of England. Of the names which we have above enumerated, four of them will be handed down to the latest posterity; but we shall only have occasion to refer to Halley and Bradley, because we can only notice the more prominent features of the history of this science; and the observations and results of the first and last of those great men, although of infinite advantage to the attainment of accurate knowledge, and would therefore form an important part in a complete history of astronomy, are still not of that striking nature to claim more than a general notice in this place.

Dr. Halley.

Dr. Halley, who was equally celebrated as an analyst, a geometer, and astronomer, very early distinguished himself by a geometrical method of determining the apsides, the eccentricities and dimensions of the principal planetary orbits; he afterwards undertook a voyage to the island of St. Helena, for the purpose of forming a catalogue of the stars in the southern hemisphere; he projected the method which was afterwards put in practice for observing the transit of Venus in 1761 and 1769, for determining the parallax of the sun; and predicted the return of a comet in 1759, fixing the period of its revolution at 75 years; this comet, which is the only one whose orbit is known, and which is expected to reappear in 1834, bears his name, as an honourable testimony of the truth of his prediction, and the profundity of his knowledge. Many other valuable works of this author we must necessarily pass over.

Predicts the return of a comet.

Roemer.
A.D. 1667.

In 1667, a highly important discovery was made by Roemer, a Danish mathematician, at that time resident at Paris. He had long been engaged in making very accurate observations on the motion and eclipses of Jupiter's satellites; in the course of which he noticed that, at certain times, these bodies emerged from the shadow of the planet some minutes later, and at others, as much before the time given by the most accurate tables. By comparing these variations with each other, it appeared, that the satellite emerged too late from the shadow, when the distance between the earth and Jupiter was the greatest, and too soon when that distance was the least; whence, after some conjecture, he hit upon the happy idea, that as the apparent emersion happened too late or too soon, according as the planet was more remote or nearer to us, it must proceed from the time that light employed in passing over the difference in the two distances; in fact, that the propagation of light is not instantaneous; but that it requires a certain time to pass from one point to another; and submitting his ideas to calculation, it appeared, that a luminous ray employs about eleven minutes in describing a distance equal to that of the earth from the sun. This bold idea he communicated to the Academy of Sciences on the 16th of November, 1667: it has since been confirmed with some modification, reducing the time to 7½ minutes, and has immortalized the name of Roemer.

Determines the velocity of light.

Aberration of the fixed stars.

Although the system of Copernicus had now gained a complete ascendancy, yet many persons were inclined to imagine, that if it were true, some parallax ought to be observed in the fixed stars: that these bodies should be at such an immense distance, that a base of nearly *two hundred millions of miles*, the diameter of the earth's orbit, should not sensibly alter their positions, was, it must be allowed, a doctrine sufficient to startle those who could not expand their minds to a contemplation of the infinitude of celestial space; and, accordingly, various observations were made with a view of ascertaining whether or not such a parallax had place; and instruments had now arrived at that degree of perfection, that it was thought by many eminent astronomers such an effect ought to be rendered appreciable.

The minuteness and accuracy of these observations, did, indeed, show a small change in the relative position of some stars, but it was, generally speaking, directly contrary to that which ought to result from a parallax. This motion the cause of which was unknown, was denominated *aberration;* and it was this which Dr. Bradley undertook to examine, and endeavoured to reduce to a general law. In the prosecution of this design, he found that certain stars appeared to have, in the course of a year, a sort of vibration, in longitude, without changing their latitude; some varied only in latitude, while others, and that the greater number, appeared to describe in the heavens, in the course of the year, a small ellipse, more or less elongated. This period of a year to which these variations answered, although so different from each other, was a certain indication that they were connected with the annual motion of the earth in its orbit about the sun; and after a time, he fortunately perceived the cause of all the irregularities he had observed. He attributed the apparent aberration of the fixed stars to the combined motion of the earth, and that with which light is propagated.

Dr. Bradley explains the cause of it.
A.D. 1736.

Roemer had shown that the velocity of light is about 10,000 times greater than that of the earth in its orbit, therefore, a ray of light issuing from a star, will not carry the impression of this star to the eye, till after the earth has sensibly changed its place; and consequently, when the eye receives the impression, it ought necessarily to refer the object to a different point in the heavens to that in which it is actually placed, or to that in which it would appear, if the earth were at rest. This explanation satisfied every doubt on the subject, and amounted to nearly a mathematical demonstration of the truth of the Copernican system.

Nutation of the earth's axis.

We owe to this celebrated astronomer, another discovery no less important than the above, viz. the nutation of the earth's axis. The celestial mechanics of our illustrious Newton had shown, that the unequal attractions of the sun and moon on the different parts of the terrestrial spheroid, ought to produce a variation in the position of its axis as referred to the plane of the ecliptic. Bradley undertook to examine the effect of this motion by means of a long series of delicate observations, made in those positions of the sun and moon most proper to render them manifest. The result of his researches were,—1. That the axis of the earth has a conical motion, by which its extremities describe about the pole of the ecliptic, and contrary to the order of the signs, an entire circle in 25,900 years, or an arc of about 50″ annually. This explains the cause of the precession of the equinoxes. 2dly, That this axis has, with reference to the plane of the ecliptic, a libration, by which it inclines about 18″ in the course of one revolution of the nodes, and is also made contrary to the order of the signs; after which it returns to its first position, inclines again, and so on, in each successive period of nineteen years. Such are the discoveries which render the name of Bradley immortal in the history of this science. We are also highly indebted to him for numerous other important results, to which we shall have occasion to refer in the course of the following treatise.

There still remains one department of practical astronomy, which requires to be briefly noticed in this sketch, viz. the measurement of the terrestrial circumference; which is important, because the terrestrial radius, in all our actual determination, is the only unit of measure to which we can have recourse, in order to reduce the several distances of the heavenly bodies to known measures: but as this is also an important datum in the physical branch of astronomy, we shall defer the consideration of those measurements, till we have to consider it under the latter point of view.

Progress of physical astronomy.

Dr. Hook's ideas of gravitation. A.D. 1668.

We have seen in our account of Kepler, that he had formed some general ideas of the nature of universal gravity; but it was too vague to become the foundation of any mechanical principle. Dr. Hook, an English philosopher of a most extraordinary genius, had made a much nearer approximation to a development of this great law of the universe. In one of his communications to the Royal Society, May 3d, 1668, he expressed himself as follows:—"I will explain a a system of the world very different from any yet received. It is founded on the three following positions:

"1. That all the heavenly bodies have not only a gravitation of their parts to their own proper centres, but that they also mutually attract each other within their spheres of action.

"2. That all bodies having a simple motion will continue to move in a straight line, unless continually deflected from it by some extraneous force, causing them to describe a circle, an ellipse, or some other curve.

"3. That this extraction is so much the greater as the bodies are nearer. As to the proportion in which those forces diminish by an increase of distance, I own I have not yet discovered it, although I have made some experiments to this purpose. I leave this to others who have time and knowledge sufficient for the task."

This is a very precise enunciation of a proper philosophical theory. The phenomenon of the change of motion, is considered as the mark and measure of a change of force, and his audience is referred to experience for the nature of this force: he having before exhibited to the society a very neat experiment, contrived to shew the nature of it. A ball, suspended by a long thread from the ceiling, was made to swing round another ball, laid on a table immediately below the point of suspension. When the impulse given to the pendulum was nicely adjusted to its deviation from the perpendicular, it described a perfect circle round the ball on the table; but when the impulse was very great, or very small, it described an ellipse, having the other ball in its centre.

Hook shewed that this was the operation of a deflecting force proportioned to the distance from the other ball; and he added, that although this illustrated the planetary motions in some degree, yet it was not suitable to their case; for the planets describe ellipses, having the sun, not in their centre, but in their focus. Therefore, they are not retained by a force proportional to the distance from the sun.

The exalted genius of Newton can suffer no diminution by the enumeration of the above opinions; for though the idea of such a principle as gravitation was not suggested first by Newton, yet so very obscure were the notions of even the most enlightened philosophers on this subject, that it had never been successfully applied to the explanation of a single astronomical phenomenon.

Newton's discoveries. A.D. 1666.

The important discovery of the law of universal gravitation is so intimately connected with the history of philosophical science, that every circumstance relating to it has been recorded with the greatest care. Dr. Pemberton relates, that Newton, in the year 1666, having retired from Cambridge to the country on account of the plague, was led to meditate on the probable cause of the planetary motions, and upon the nature of that central force which retained them in their orbits; when it occurred to him, that the same force, or some modification of the same force, which caused a heavy body with us to descend to the earth, might likewise retain the moon in her orbit, by causing a constant deflection from her rectilinear path. But before this could be submitted to the test of computation, it was necessary that some hypothesis should be formed relative to the modification of its action with respect to distance; and probably, that which has actually place, namely, that it is reciprocally as the square of the distance, almost immediately suggested itself to his mind, as being observed in all kinds of emanations with which we are acquainted.

When Newton first attempted to verify this conjecture, the requisite data with regard to the distance of the moon in terrestrial radii, and the measure of the radius itself were but imperfectly known; the result therefore which he obtained, though nearly, did not accurately agree with observed phenomena; and he in consequence abandoned his theory as untenable: a remarkable instance of the cool and dispassionate frame of mind which this great philosopher observed, even at the moment when he flattered himself with the idea of having discovered one of the most important secrets of nature.

Sometime afterwards, however, he was induced to renew his calculations; as, in the interval, more correct data had been obtained by the measurement of Picard, in France. This attempt succeeded; and he is stated to have been extremely agitated towards the conclusion of his calculation. Whether this were the case or not we cannot attempt to settle: at all events, a moment of greater interest will never be recorded in the annals of science.

Principles of calculation.

Let us briefly illustrate the nature of the calculation to which we have referred. Our author having established as a datum, that the distance of the moon from the earth, was about 60 semi-diameters of the latter, that is, 60 times as far from the earth's centre as a heavy body placed at its surface; and assuming the power of gravity to decrease inversely as the square of the distance, it would follow, that at the moon, this force would only be one 3600th part of what it is at the surface of the earth; and, consequently, the space passed over by a body at this distance, when submitted to the terrestrial gravitation, would be only $\frac{1}{3600}$ of that which is actually described here; or since the spaces are as the squares of the times, and the force inversely as the square of the distance; the moon ought to fall towards the earth through the same space in a minute, as a heavy body here describes in a second, viz. about $16\frac{1}{12}$ feet.

Hence it follows, supposing the moon to be retained in her orbit by this force, that her deflection from the tangent to her orbit at any point, ought to be 16·083 feet in one minute, or $\frac{16·083}{3600}$ in one second. Now the distance of the moon from the earth in semi-diameters being known, and the radius of the earth itself being supposed determined by actual measurement of terrestrial arcs, as also the exact time of one lunar revolution, it would be very easy to find the circumference of the lunar orbit, and the measure of the arc which the moon describes in one second, whence again the versed sine of that arc, or the difference between the secant and radius, which is nearly the same thing, may also readily be obtained in feet; and this, as above observed, ought to be $\frac{16·083}{3600}$. This measure Newton verified by his calculation, and thus established this most important law with regard to the earth and moon, which was afterwards readily extended to the whole planetary system.

Huygens's central forces.

Several years before this discovery of Newton's, Huygens had given in thirteen propositions the properties of centrifugal and centripetal forces in a circle; but he did not think of applying his theory to the motion of the earth on its axis, and to the moon about the earth; had he done this, it is highly probable he would have arrived at the same conclusion as Newton; this grand step in his process was, however, wanting, and the entire honour of the discovery justly devolved upon our illustrious countryman.

Law of the planetary motions.

In order to extend the same principles to the other planetary motions, Newton was led to consider, that when two bodies act on each other by attraction, the action is reciprocal; moreover, that the attraction of each body is equal to the sum of the attraction of all its parts, and therefore proportional to the mass; consequently, some other data were requisite for demonstrating that the whole system was regulated by this one general principle;—the motion of the moon appertaining only to the earth, while the earth and all the other planets revolved about the sun.

Newton demonstrated, generally, that if a body projected into space be continually turned from its direction, by any force which urges it towards a fixed centre, and which causes it to describe a curve, the areas of the sectors comprised between the arc of the curve and the right lines joining the body and fixed centre, are proportional to the times of description, and *vice versa*, if the areas are proportional to the times the revolving body is necessarily urged towards a fixed centre. Now the primary planets observe this law in revolving about the sun; therefore regarding this body as fixed, it follows, that each planet is continually attracted towards the sun as a centre. But from this condition alone he would have been able to determine nothing respecting the nature of this force. Kepler, however, furnished the law necessary for this determination; for if to the condition of equal areas being described in equal times, we add the other condition, that the orbit is an ellipse, then it will follow, as demonstrated by Newton, that the force must vary reciprocally as the square of the distance. Still, however, this only proved that the law obtained for each planet individually in different parts of its orbit, and it still remained to be shown that the same had place for every planet in the system; viz. that each was attracted by a force which was reciprocally as the square of its respective distance. Here again another datum of Kepler's was of the highest importance; he had shown that the squares of the periodic times were as the cubes of the distances; and this was sufficient for demonstrating the universality of the law in question; namely, that every material particle in nature attracts with a force proportional to its mass, and reciprocally as the square of its distance from the body on which it acts. It is this law which regulates all the planetary motions, and to which we must have recourse for explaining any irregularities observable in them.

Masses of certain planets determined.

Our planetary world is compounded of different systems: thus the sun and the primary planets may be considered as one system, the earth and moon as another, Jupiter and his satellites as a third, and so on; and it will be necessary to distinguish between these, when we are comparing the motions of bodies in one of those systems with those in another; the mass of the central body being a necessary datum in determining the actual motion of the circulating body; and conversely, the motion of two or more revolving bodies being known, the masses of the attracting bodies may be determined. It was thus Newton found, that, denoting the mass of the sun by unity, that of Jupiter would be expressed by the fraction $\frac{1}{1067}$, Saturn by $\frac{1}{3021}$, and the earth by $\frac{1}{169282}$.

We cannot follow Newton through the numerous inferences, calculations, and deductions, which resulted from this universal law; it will be sufficient to observe, that it accounted for the flux and reflux of the tides, the nutation of the earth's axis, the precession of the equinoxes, the spheroidal figure of the earth, and various irregular motions in the planetary system; but some of these have been illustrated and submitted to calculations within the last few years, in a manner more exact and conclusive than in the time of our author. Analysis has taken a much greater range, and every advance that it has made, has furnished some further confirmation of the truth and generality of the great law of universal gravitation.

Difficulty of the general problem.

If there were but two bodies in our system, as, for example,—the sun and earth; or but three,—the sun, earth, and moon, the determination of their actual and disturbing forces on each other would be comparatively a problem of easy solution; but when we consider the influence of several bodies, whose position with respect to each other is perpetually changing, and then endeavour to estimate their effects on any one in particular, we shall find the question involved in the greatest perplexity, and apparently far beyond the reach of human intellect to comprehend; yet such has been the progress of analysis within the last half century in the hands of D'Alembert, Euler, Lagrange, La Place, and some others, that not a single phenomenon now remains, nor the smallest irregularity in the motions of the principal bodies of our system, that is not accounted for; its amount computed, and its origin traced to that law which it is the glory of Newton to have first discovered.

As these subjects must necessarily come before us in our treatise on physical astronomy, we shall not detain the reader with the nature of the particular solutions and investigations of the authors above alluded to; but shall pass on to the only remaining topic—the figure and magnitude of the earth.

Magnitude and figure of the earth.

We have seen that this problem had been attempted by Eratosthenes, and again by the Arabian mathematician Almamon, but we know very little of their determinations; we may however easily imagine, knowing how great the difficulty of the operation is, even with the most perfect instruments, that their approximations must necessarily have been very remote from the truth; and probably the early attempts of this kind in Europe were not attended with much better success. We shall briefly notice those which seem to be of the greatest importance.

Different measures.

In 1525, M. Fernelius measured a degree of the meridian northward from Paris, which he made 68·7634 English miles. Snellius, professor of mathematics at Leyden, and our countryman, Norwood, measured each of them a meridional degree; the former in Holland in 1620, and the latter in England between London and York, in 1635. Snellius' measure, when reduced to English miles, is 66·91; and that of Norwood 69·545 miles. In 1644, Riccioli also undertook a similar task, and performed it according to three different measures, between mount Parderno and the tower of Modena in Italy, and obtained a mean length of 75·066 English miles to a degree.

Figure of the earth.

These results differed too much from each other to inspire the least confidence in any of them, and consequently nothing could thence be deduced respecting the figure of the earth; nor was this indeed, at that time, suspected to be any other than a perfect sphere, abstracting from the irregularities on its surface. But after the construction of the telescope had received considerable improvement, and when observations had been reduced to greater nicety, it was found that the planet Jupiter was considerably flattened at his poles; and the pendulum experiment of Richer, 1671, had shown that there was a difference in the action of gravity at the equator and in the latitude of Paris; and these two circumstances probably first suggested to Huygens that the earth was not spherical; and its rotatory motion about its axis naturally led him to conclude that it was flattened at its poles; from the combination of its centrifugal force with that of gravity, he calculated that its polar axes were to its equatorial diameter as 578 to 579. But as he regarded all the power of terrestrial attraction to be collected in the centre of the earth only, his solution was obviously defective; and Newton soon after undertook the same determination, on more correct principles, by supposing every particle in the whole mass to have a reciprocal attraction towards all the others; from which he found the figure to be an ellipsoid, having its polar and equatorial diameters to each other at 229 to 230.

Determined by Newton and Huygens.

Determined by practical measurements.

In this state the question remained for many years, till M. Picard undertook the measurement of a degree in France, in 1669, which was afterwards revised by Cassini in 1718; the result of which tended to show, that the earth was not an oblate but a prolate spheroid. Picard obtained for the length of a degree 68·945 English miles, and Cassini 69·119; whereas the latter, being the most southern, ought to have been the shortest. A circumstance so unexpected as this, naturally produced a great curiosity, and a considerable degree of inquiry and controversy between the astronomers and mathematicians of that period; and the French government, at the recommendation of the Academy of Sciences, in 1735, sent out two companies of mathematicians to determine the point in question, by measuring two degrees, one at the equator, and the other in as high a northern latitude as possible. Accordingly, MM. Godin, Bouguer, and Condamine, from France, with Dons Juan and Ulloa, from Spain, proceeded to Peru; while Maupertuis, Clairaut, Camus, La Monnier, &c. accompanied by Celsus, a Swedish astronomer, proceeded to Lapland.

After experiencing many unforeseen difficulties and delays, both parties accomplished the object of their missions, and returned to France; those from Lapland in 1737, and the other division in 1744. The former made the length of their degree, the middle point of which was in latitude 66° 20′, equal to 69·403 English miles; while at the equator the length, as determined by the other party, was found to be 68·724 English miles, taking the mean of three different results, deduced from the same operation. In the interval between the outfitting of the above expeditions, and the publication of their results, the degrees of Picard and Cassini were examined, recomputed, and found now to be, the first, whose middle point was in latitude 49° 22′, 69·121 English miles, and other 69·092 miles, its middle point being 45°.

Oblate figure established.

The results therefore of all these measures confirmed the earth to be an oblate spheroid; but as they were compared together in pairs, they gave very different degrees of ellipticity; nor has all the accuracy that has since been introduced into these operations been sufficient to establish this point. Colonel Mudge in England, has measured an arc extending from the southernmost point in the kingdom to the northernmost of the Shetland islands; and Messrs. Delambre, Mechain, Biot, Arago, &c. have carried another from Dunkirk to Fomentara, one of the Balearic isles. It is perhaps never to be expected that greater accuracy can be introduced into any operations, nor to see greater talents employed in conducting them, than in the cases to which we have last referred; and yet these two measures, compared with each other, in different portions, give very different ellipticities; different parts of the same arcs give very discordant results; even some of those of the English survey, are such as to lead again to the idea of the prolate figure of the earth. What are we then to conclude from the whole of these deductions, but that the irregularities of local attraction, or some other causes which we cannot discover, produce a certain influence or disturbing power, which renders useless the comparisons of small arcs, and that those results only can be depended upon that are drawn from measurements that extend through several degrees of latitude. Adopting this principle, we may state, that the mean length of a degree in the latitude 45° is 68·769 English miles, and that the ellipticity of the earth is between $\frac{1}{307}$ and $\frac{1}{314}$.

Ellipticity still uncertain

Medium results.

One other point still requires to be alluded to before we conclude our history. It was observed by Bouguer, in his operations in Peru, that the high mountains in some parts of that country, very sensibly disturbed the verticality of his plumb-line; that is, the lateral attraction drew the line out of its per-

Density of the earth.

pendicular direction, a certain quantity which he determined. This suggested to the Royal Society the idea of employing this deflection, in order to ascertain the actual density of the earth as compared with water, or any other known substance. In order to this, it was necessary to select some isolated mountain, whose density and magnitude might be ascertained: when by observing how much a pendulum or plumb-line was deflected on opposite sides of it, at given distances, the proportional forces between the earth and mountain would become known, and hence from the established laws of attractions, the relative masses of the two bodies would be determined, and hence the density of the earth, its magnitude being supposed already ascertained.

The mountain selected for this purpose was Schehallian, in Scotland, and Dr. Maskelyne, in 1742, was requested to direct the operations. The mean height of Schehallian above the surrounding valley is about 2000 feet, and its direction is nearly east and west. Two stations were chosen for observation, one on the north, and the other on the south side of the mountain. Every circumstance that could contribute to the accuracy of the experiment, was attended to with particular care; and from the observations on ten stars near the zenith, Dr. Maskelyne found the apparent difference of the latitudes of the stations to be 54′ 6″: and from a measurement by triangles from the two bases, on different sides, he found the actual difference of their parallels to be 4364 feet, which, in the latitude of Schehallian, 56° 40′, answers to an arc of the meridian of 43′, which is less, by 11′ 6″, than that found by the sector. These data being submitted to the calculation indicated above, a task which was performed by Dr. Hutton, it appeared that the density of the whole mass of the earth as compared with water, was nearly in the ratio of $4\frac{1}{2}$ to 1. *Phil. Trans.* vol. 65, and vol. 68; but from more accurate observations on the geology of this mountain by Playfair, it appears that there was, in the first instance, some error in assuming its density; and the corresponding reduction being made in Dr. Hutton's determination, gives the density of the earth 4·95, that of water being 1; which more nearly agrees with the determination of Mr. Cavendish, who on other principles, which are explained in the *Phil. Trans.* for 1798, makes it 5·48.*

We have now given, it is presumed, a sketch of all the more prominent points connected with the history of astronomy; much highly important matter is, we are aware, also either wholly passed over, or very slightly alluded to; but to have noticed all the circumstances worthy of record would have carried us far beyond our proposed limits; moreover, many of the minutiæ we shall have occasion to refer to in our illustration of the various topics in the following treatise; where we shall also take the opportunity of introducing the titles of some of the more important and useful works on this subject, and to which we would refer our readers for the attainment of a perfect knowledge of the ancient and present state of astronomical science.

PLANE ASTRONOMY.

§ II. *Introduction.—Containing a popular view and illustration of the most remarkable phenomena of the heavens.*

1. *General remarks.*

Introduction.

1. Astronomy differs very essentially from all other mathematical sciences, with regard to the connection of its propositions, and the force of its demonstrations. In geometry, for example, after the requisite definitions have been laid down, and certain axioms and postulates granted, the reader finds no farther claims made upon him for the admission of this or that hypothesis; he judges for himself at every step; and every step furnishes him with some new truth, as certain and incontestable as his own existence. So also, in theoretical mechanics: our investigations, if they do not always preserve the same concatenation, as in geometry, furnish an equal conviction to the mind of the reader, because he is never required to give his assent to a proposition, till the means are prepared to rest its demonstration upon something previously established.

In astronomy, on the contrary, a student is called upon for the admission of an hypothesis which is contrary to the evidence of his senses. While he and every thing about him are in a state of apparent permanent rest, he must admit that they are moving with an inconceivable velocity; and on the other hand, that those bodies, which, judging from his senses only, he supposes to be in rapid motion, are actually at rest; moreover, he must wait till the whole chain of reasoning is established before he will be able to judge, and to be convinced of the truth of the hypothesis which he has previously admitted. He will then at least feel this conviction, viz. if the law of gravitation, and the constitution of the solar system were such as he has assumed, that every phenomenon, the most important as well as the most minute, would happen exactly in the same manner as they actually do; and hence, he will be able to judge of the high degree of probability, that the supposition he has made is strictly and positively correct.

Facts on which the truth of the modern astronomy is founded.

To mention a few of the most prominent facts, we may observe, that the law of gravitation, and the hypothesis of the earth's rotation on its axis, render it highly probable, that the terrestrial globe is not a perfect sphere, but an oblate spheroid; and the various geodetic operations that have been carried on in different countries, leave no longer any doubt that such is its actual figure. Again, the same hypotheses indicate, that the intensity of gravity ought to be different in different latitudes; not merely in consequence of the figure of the earth, but on account of the difference in the centrifugal force in different latitudes; and therefore, that the seconds pendulum ought to be longer at or near the poles, than at the equator; and that such is truly the case, has been demonstrated by various experiments. The aberration of the stars, the nutation of the earth's axis; the precession of the equinoxes: phenomena, whose existence have been ascertained by observation, are inexplicable upon any other supposition; and to this we may add, that of all the minute inequalities which the accuracy of modern instruments, and a corresponding delicacy of observation have rendered appreciable in the motion of the planetary bodies, there is not one of them but what is immediately shown to be the necessary consequence of the law of universal gravitation, and the assumed constitution of the solar system.

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* In the last volume of the Phil. Trans. Dr. Hutton has given a paper on this subject, pointing out certain errors of calculation in Mr. Cavendish's Memoir; and which, when corrected, make the result more approximative to the last determination of this venerable mathematician.

These facts, and many others which we might have advanced, if they do not amount to actual demonstration, necessarily involve in them such a high degree of probability as falls very little short of it; leaving on the mind a conviction little inferior to that which we derive from absolute certainty.

This conviction, however, as we have before observed, cannot be felt till the entire chain of reasoning has been established; till this is effected, some doubt may be allowed to remain on the mind of the student; he will, it is true, as he advances, find greater and more substantial reasons for admitting the truth of his hypothesis, but he will not feel complete satisfaction till he is able to compare and weigh the whole.

Sketch of the arrangement of the several subjects.

2. In the following sketch, therefore, which we have given of the constitution of the solar system, we do not require the implicit assent of the reader; we wish him only to consider what we have stated, as the enunciation of a proposition, or of a chain of propositions, of which we shall endeavour, step by step, to demonstrate the truth; nor shall we attempt, in the first instance, to give the particulars with the utmost nicety; because this is unnecessary, where we wish simply to indicate a general view of the planetary motions; but in the subsequent part of our treatise, we shall exhibit only the most modern and authentic results. We shall enter upon our subject, on the supposition that our reader is wholly unacquainted with even the first principles of the science, and therefore that he requires an explanation of every phenomenon. This explanation we propose to render in the first instance merely popular, viz. without attempting any but the most simple computations; for example, we shall first illustrate the most remarkable celestial phenomena, the changes of season, the alternation of night and day, the phases of the moon and planets, the general principles of eclipses, &c. &c. Having thus indicated to the student, the first principles of the science, and given him a concise view of what we propose to establish; we shall then describe to him the circles of the sphere, the construction of some of the most indispensable astronomical instruments, introduce him to the observatory, and point out to him the nature of the observations that are necessary as the groundwork of his computations. These observations, however, will stand in need of certain corrections, the cause and quantity of which he must therefore be next instructed in; after which, he will find no difficulty in following us through the remaining part of this division of our subject.

Having said thus much with reference to the plan we propose to follow in this treatise, we shall begin by exhibiting a brief description of the stars, and of the constellations into which they have been divided.

2. *Of the fixed stars.*

Of the fixed stars.

3. It is impossible to conceive a more magnificent spectacle, nor one more worthy of the contemplation of an intelligent being, than that which is presented to our view by the starry firmament in an unclouded sky; a few hours after the setting of the sun. We then observe innumerable brilliant points spread in every direction over the azure canopy of the heavens; various in magnitude and lustre; differently disposed with regard to each other, and arranged in groups to which the imagination will readily attribute numerous ideal forms and characters.

If this scene be attentively observed for some successive nights, it will be perceived that the greater number of those bodies preserve constantly the same relative position with regard to each other; to these therefore has been given the name of *fixed stars.* Others of them are perpetually changing their places, varying in their lustre, and apparently traversing different circles of the celestial space; these are, therefore denominated, *wandering stars* or *planets*; while a third class, much more rare in their appearance, and totally distinct from all others in their figure and motion, are sometimes observed, surrounded by a faint sphere of light, and followed by a like luminous train, which at times extend over a considerable arc of the celestial vault. This sphere of light, from its supposed resemblance to hair, has given to these celestial visitants the name of *comets.* These three classes of bodies will form the subject of distinct chapters in the course of this treatise; at present our business is to take a popular view of the fixed stars, in order to illustrate the important uses to which they are applied, both in practical and physical astronomy.

Divisions of the stars into constellations.

4. One of the first objects of a student in this science should be to make himself acquainted with the names and situations of the most conspicuous stars and constellations; the latter being a term used to denote those groups of stars which are supposed to have some resemblance in their form to the animal or other objects, whose names they bear. The division of the stars into constellations is of very remote antiquity; and though it may be useless, and sometimes even inconvenient, for the purpose of minute observation; yet for a general recollection of the great features of the heavens, these arbitrary names and associations cannot but greatly assist the memory. It is also usual to describe particular stars by their situation with respect to the imaginary figure, to which they belong, as *Spica Virginis, Cor Hydræ,* &c. or more commonly at present, by the letters of the Greek alphabet, which were first applied by Bayer in 1603, and in addition to these by the Roman letters, and by the numbers of particular catalogues.

Apparent magnitudes.

5. The stars are again divided into different classes or magnitudes, according to the degrees of their apparent brightness. The largest or most vivid of which, are said to be of the first magnitude; the next in order of brightness of the second, and so on to the sixth, which latter class includes all those that are barely perceptible to the naked eye; all of a smaller kind, are generally called *telescopic stars,* being invisible without the assistance of the telescope.

Method of acquiring a knowledge of the stars.

In order to become familiar with the names and positions of the principal stars and constellations, the student must place himself in an open situation, where he can observe the whole of the celestial hemisphere; he will then perceive, after an attentive observation of their motion, that certain of the stars exposed to his view, will, after attaining their greatest altitude, decline and sink below the horizon; others will attain to the meridian, descend again, but will not sink below the horizon; and by directing his view with attention towards that part of the heavens where this occurs, he will perceive some few stars, and one in particular, which have no perceptible motion: this latter is called the pole star, because it is about an axis passing very near to this, that the apparent motion of the heavens takes place. The place of this star referred to the horizon is called the *north*, the opposite point the *south*, that to the right hand the *east*, and to the left the *west*.

Different methods of representing the stars.

6. There are two principal modes of representing the stars; the one by delineating them on a globe, where each star occupies the spot in which it would appear to an eye placed in the centre of the globe, and where the situations are reversed when we look down upon them; the other mode is by a chart, where the stars are generally so arranged as to represent them in positions similar to their natural ones, or as they would appear on the internal concave surface of the globe.

In acquiring a knowledge of the particular stars, it will be most convenient to begin with such as never set in our climates; after which, we may readily refer the situations of others to their position with respect to these.

Different constellations.

7. The *Great Bear* is the most conspicuous of the constellations, which never set in our latitude; it consists of seven principal stars, placed like the four wheels of a waggon, and its three horses, except that the horses are fixed to one of the wheels.

Their connections, &c.

The hind wheels are called the pointers, because they direct us to the pole star, in the extremity of the tail of the Little Bear; and further on to the constellation Cassiopeia, which is situated in the milky way (See Plate I. Astronomy,) which consists of several stars nearly in the form of the letter W; or we may easily imagine them to represent a *chair*; whence these few in particular form what is more commonly called Cassiopeia's Chair.

The two northernmost wheels of the Great Bear, or Wain, point at the bright star Capella, in Auriga. Descending along the milky way from Cassiopeia, if we go towards Capella, we come to Algenib, in Perseus, and a little further from the pole Algol, or Medusa's Head; but if we take the opposite direction, we arrive at Cygnus, the Swan, and beyond it, a little out of the milky way, is the bright star Lyra. The Dragon consists of a chain of stars, partly surrounding the Little Bear; and between Cassiopeia and the Swan, is the constellation Cepheus.

Near Algenib, and pointing directly towards it, are two stars of Andromeda, and a third is a little beyond them. A line drawn through the Great Bear and Capella, passes to the Pleiades, and then turning at a right angle towards the milky way, reaches Aldebaran, or the bull's eye, and the shoulder of Orion, who is known by his belt, consisting of three stars placed in the middle of a quadrangle. Aldebaran, the Pleiades, and Algol, make the upper end, Menkar, or the whale's jaw, with Aries, the lower point of a W. In Aries we observe two principal stars, one of them with a smaller attendant.

A line from the pole, midway between the Great Bear and Capella, passes to Gemini, the twins, and to Procyon; and then in order to reach Sirius, it must bend across the milky way. Algol and the Twins point at Regulus, the lion's heart, which is situated at one end of an arch, with Denabola at the other end.

The pole star, and the middle horse of the wain, direct us to Spica Virginis, considerably distant: the pole and the first horse, nearly to Arcturus in the waggoner, or Bootes. Much further southward, and towards the milky way, is Antares in the Scorpion; forming, with Arcturus and Spica, a triangle, within which are the two stars of Libra. The northern crown is nearly in a line between Lyra and Arcturus; and the heads of Hercules and Serpentarius are between Lyra and Scorpio. In the milky way, below the part nearest to Lyra, and on a line drawn from Arcturus, through the head of Hercules, is Aquila, making, with Lyra and Cygnus, a conspicuous triangle. The last of the three principal stars in Andromeda, make, with three of Pegasus, a square, of which one of the sides points to Fomalhaut, situated at a considerable distance in the southern fish, and in the vicinity of the whale, which has been already mentioned. By means of these supposititious lines, all the principal stars that are ever visible in our climates may be easily recognised. Of those which never rise above the horizon, there are several of the first magnitude; Canopus in the ship Argo, and Achernar in the river Eridanus, are the most brilliant of them; the feet of the Centaur, and the Crosier are the next; and, according to Humboldt's observations, some others perhaps may require to be admitted into the same class. See Plate II.

The constellations to which we have above referred, are divided into three principal classes; those on the northern side of the equator are called northern constellations, and those on the opposite, southern constellations; while those which are situated in that part of the concave hemisphere where the principal planets and other bodies of our system are observed to move, forming a zone, which crosses the equator in two points, are called zodiacal constellations. The names of these, with the number of stars belonging to each, in four of the principal catalogues; the names of the principal stars; and the characters of the zodiacal constellations, are as follow:—

Table of the Constellations.

Zodiacal.

Names and Characters of the Constellations.	Number of Stars in different Catalogues.				Principal Stars.	
	Ptolemy.	Tycho.	Hevelius.	Flamstead.		Mag.
Aries ...... ♈ the Ram	18	21	27	66		
Taurus...... ♉ the Bull	44	43	51	141	Aldebaran	1
Gemini ...... ♊ the Twins	25	25	38	85	Castor and Pollux	1—2
Cancer...... ♋ the Crab	13	15	29	83		
Leo .... ♌ the Lion, with Coma Berenices......	35	30	49	95	Regulus	1
Virgo ...... ♍ the Virgin	32	33	50	110	Spica Virginis ..	1
Libra ...... ♎ the Scales	17	10	20	51	Zubenich Meli ..	2
Scorpio ...... ♏ the Scorpion	24	10	20	44	Antares	1
Sagittarius .. ♐ the Archer	31	14	22	69		
Capricornus.. ♑ the Goat..........	28	28	29	51		
Aquarius .... ♒ the Water Bearer ..	45	41	47	108	Scheat	3
Pisces ...... ♓ the Fishes	38	36	39	113		

Northern Constellations.

Names of the Constellations.	Number of Stars in different Catalogues.				Principal Stars.	
	Ptolemy.	Tycho.	Hevelius.	Flamstead.		Mag.
Ursa Minor, the Little Bear..........	8	7	12	24	Pole-star......	2
Ursa Major, the Great Bear	35	29	73	87	Dubhe..........	1
Perseus	29	29	46	59	Algenib	2
Auriga, the Waggoner	14	9	40	66	Capella..........	1
Bootes..........................	23	18	52	54	Arcturus	1
Draco, the Dragon..................	31	32	40	80	Rastaber	3
Cepheus	13	4	51	35	Alderamin	3
* Canes Venatici, viz. Asterion et Chara, the Greyhounds	..	..	23	25		
* Cor Caroli	..	..	..	3		
Triangulum, the Triangle............	4	4	12	16		
* Triangulum Minus	..	..	..	10		
* Musca	..	..	..	6		
* Lynx..........................	..	..	19	44		
* Leo Minor, the Little Lion	..	..	..	53		
* Coma Berenices, Berenice's Hair	..	14	21	43		
* Camelopardalus	..	..	32	58		
* Mons Menelaus	..	..	..	11		
Corona Borealis, the Northern Crown. .	8	8	8	21		
Serpens, the Serpent	13	13	22	64		
* Scutum Sobieski, Sobieski's Shield ..	..	..	7	8		
Hercules, cum Ramo et Cerbero						
Hercules, since called Engonasia......	29	28	45	113	Ras Algiatha	3
Serpentarius, sive Ophiuchus..........	29	15	40	74	Ras Alhagus	3
* Taurus Poniatowski	..	..	..	7		
Lyra, the Harp	10	11	17	22	Vega	1
*Vulpeculus et Anser, the Fox and Goose	..	..	27	37		
Sagitta, the Arrow	5	5	5	18		
Aquila, the Eagle, with Antinous......	..	12—3	23—19	71	Altair	1
Delphinus, the Dolphin..............	10	10	14	18		
Cygnus, the Swan	19	18	47	81	Deneb Adige	1
Cassiopeia, the Lady in her Chair......	13	26	37	55		
Equulus, the Horse's Head	4	4	6	10		
* Lacerta, the Lizard................	..	..	..	16		
Pegasus, the Flying Horse	20	19	38	89	Markab	2
Andromeda..	23	23	47	66	Almaac	2

Those constellations distinguished * are new constellations.

Southern Constellations.

Names of the Constellations.	Number of Stars in different Catalogues.				Principal Stars.	
	Ptolemy.	Tycho.	Hevelius.	Flamstead.		Mag.
* Phœnix	..	..	..	13		
* Officina Sculptoria	..	..	..	12		
Eridanus	34	10	27	84	Achernar	1
* Hydrus, the Water Snake	..	..	..	10		
Cetus, the Whale	22	21	45	97	Menkar	2
* Fornax Chemica	..	..	..	14		
* Horologium.	..	..	..	12		
* Reticulus Rhomboidalis.	..	..	..	10		
* Xiphias, Dorado, the Sword Fish. . . .	..	..	..	7		
* Celapraxitellis	..	..	..	16		
Lepus, the Hare	12	13	16	19		
* Columba Noachi, Noah's Dove	..	..	..	10	Betelgense	1
Orion	38	42	62	78	Canopus	1
Argo Navis, the Ship	45	3	4	64		
Canis Major, the Great Dog.	29	13	21	31	Sirius	1
* Eqûuleus Pictorius	..	..	..	8		
* Monoceros, the Unicorn	..	..	19	31		
Canis Minor, the Little Dog	2	2	13	14	Procyon	1
* Chamæleon	..	..	..	10		
* Pyxis Nautica	..	..	..	4		
* Piscis Volans, the Flying Fish	..	..	..	8	Cor Hydræ . .	1
Hydra	27	19	31	60		
* Sextans	..	..	11	41		
* Robur Carolinum, the Royal Oak. . . .	..	..	..	12		
* Machina Pneumatica	..	..	..	3		
Crater, the Cup	7	3	10	31	Alkes	3
Corvus, the Crow	7	4	..	9	Algorab	3
* Crosiers, et Cruzero	..	..	..	6		
* Apis Musca, the Bee or Fly	..	..	..	4		
*Apus, or Avis Indica, the Bird of Paradise	..	..	..	11		
* Circinus, the Compass	..	..	..	4		
Centaurus, the Centaur	37	..	..	35		
Lupus, the Wolf	19	..	..	24		
* Quadra Euclidis	..	..	..	12		
* Triangulum Australe, the Southern Triangle	..	..	..	5		
Ara, the Altar	7	..	..	9		
* Telescopium	..	..	..	9		
Corona Australis, the Southern Crown	13	..	..	12		
* Pavo, the Peacock	..	..	..	14		
* Indus, the Indian	..	..	..	12		
* Microscopium	..	..	..	10		
* Octans Hadleianus	..	..	..	43		
* Grus, the Crane	..	..	..	14		
Toucan, the American Goose	..	..	..	9		
Piscis Australis, the Southern Fish . . .	18	..	..	24	Fomalhaut	1

Total Number of Constellations.		Number of Stars in each Magnitude, British Catalogue.						Total No.
		1st Mag.	2d Mag.	3d Mag.	4th Mag.	5th Mag.	6th Mag.	
Zodiacal	12	5	16	44	120	183	646	1014
Northern	34	6	24	95	200	291	635	1251
Southern	45	9	36	84	190	221	323	863
	91	20	76	223	510	695	1604	3128

Number of stars visible to the naked eye.

8. It appears from the preceding summary of the stars in the British catalogue, that the number of them in the entire celestial sphere, including all those of the sixth magnitude, does not much exceed 3000; and it is generally stated, that not more than 1000 are ever visible at one time to the naked eye: but when a telescope is employed, their number appears to increase without any other limit than that of the perfection of the instrument. Sir W. Herschel has observed in the *milky way* above 10,000 stars in the space of one square degree. This luminous track

Milky way.

> "Which nightly, as a circling zone, thou seest
> Powder'd with stars,"

encompasses the heavens, and forms nearly a great circle of the celestial sphere. It traverses the constellations Cassiopeia, Perseus, Auriga, Orion, Gemini, Canis Major, and the Ship, where it appears most brilliant; it then passes through the feet of the Centaur, the Cross, the southern Triangle, and returns towards the north by the Altar, the tail of the Scorpion, and the arc of Sagittarius, where it divides into two branches, passing through Aquila, Sagitta, the Swan, Serpentarius, the head of Cepheus, and returns to Cassiopeia. The ancients had many singular ideas respecting the cause of this phenomenon, but modern astronomers have long attributed it to an immense assemblage of stars too feeble to make distinct impressions; and Herschel has shown, by the observation above referred to, that these conjectures were well founded.

Nebulæ.

Besides the milky way, there are numerous other parts of the heavens, which exhibit an appearance of very nearly the same kind, which are called *Nebulæ;* the most considerable of which is that midway between the two stars in the blade of Orion's sword. This was first observed by Huygens; it contains only seven stars; the other part appearing like a luminous spot upon a dark ground, or like a bright opening into regions beyond.

Proper motion of some stars.

9. We have hitherto spoken of the fixed stars, as preserving actually the same relative situations with regard to each other; this, however, is not strictly true. The delicacy of modern observations has shown that some of these bodies have a progressive motion: Arcturus, for example, has a proper motion, amounting to about two seconds annually; and Dr. Maskelyne found, that out of 36 stars, of which he ascertained the places with great accuracy, 35 of them had a progressive motion.

Mr. Michell and Sir W. Herschel have conjectured that some of the stars revolve round others which are apparently situated near them; and perhaps even all the stars may in reality change their places more or less, although their relative situations, and the direction of their paths may render their motions imperceptible.

Variable stars.

That some of the stars have a periodical change of brightness has been well ascertained from repeated observations; and different hypotheses have been advanced to account for these singular phenomena. New stars have also appeared at certain times, remained stationary like the others, and have afterwards disappeared. Such a temporary star was observed by Hipparchus; and it was this circumstance which suggested to him the idea of forming a catalogue of them, and of delineating their situations. A new star was also discovered in Cassiopeia in 1572, which was so bright as to be seen in the day-time, but it gradually disappeared in 16 months. Another was observed by Kepler in 1604, more brilliant than any other star or planet, and changing perpetually into all the colours of the rainbow, except when it was near the horizon. This star remained visible about a year; and several other cases of the like kind are recorded.

Distance and magnitude unknown.

As to the actual magnitude and distance of the stars, we may be said to be almost wholly unacquainted with either; all that we are able to state with certainty is, that their distance is immense. We shall show, in a subsequent chapter, that the distance of the sun from the earth is nearly 100 million miles; and, consequently, the diameter of our orbit is nearly 200 million miles; we therefore view the stars, or any one of them in particular, from two points at different times of the year, which are distant from each other 200 million miles, and yet we are not able to detect any difference in their apparent places. If a change of place, amounting only to one second, actually obtained, there is no doubt that it would be detected by the accuracy of modern observations;* we know, therefore, that an isosceles triangle having the diameter of the earth's orbit for its base, and having its vertex in the nearest fixed star, does not subtend an angle of one second; the nearest star must therefore be distant from us more than 20 *billions of miles.* How much their distance may exceed this, it is impossible for us to say; and much less are we able to offer even the most distant conjecture as to their actual magnitude; judging from analogy, we can only suppose them to be bodies resembling our sun, some of greater, and some of less magnitude; that they shine like the sun by their own light, each forming the centre of its particular system, dispensing light, heat, and animation to thousands of worlds, and to myriads of beings. What a wonderful idea do these reflections give us of the immensity of the universe, and of the power and omniscience of the great Creator and Director of so stupendous a machine!

3. *Of the Solar System.*

The solar system.

10. The system of Copernicus has been already referred to in our historical chapter. We have seen, that, according to this astronomer, all the principal planets revolve about the sun as a general, but not as a common centre; the several planets being supposed by him to describe circular orbits, each however having its particular point of circulation. The Copernican and the modern system are not therefore the same, although these terms are frequently employed synonimously. We shall here confine our description to the modern or true system, without, however, deducing any facts in proof of its accuracy: we shall be content

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* It is proper to observe, that since this was written, Dr. Brinkley, of Trinity College, Dublin, has been led to think, that he has detected a sensible parallax in several stars; but as Mr. Pond, the Astronomer Royal, has not yet been able to verify Dr. Brinkley's observations, we do not feel ourselves justified in stating the existence of a parallax; although, from the confidence we feel in the talents and accuracy of the observer, we have no doubt that his deductions will be hereafter verified by other astronomers.

that the reader receives it at present merely as an hypothesis, or rather as the enunciation of a proposition; and that he suspends his decision, till, in the course of the following chapters we advance such proofs as can leave him no longer in any doubt respecting the constitution of the solar system, nor of the mechanical principles on which its motions depend.

The sun is the central body.

The sun has probably, as well as every body in the universe, a progressive motion in space; but if such a motion has place, he is accompanied in it with the whole system of which he forms the general centre, as well as the source of its light, heat, and motion; we may therefore consider this body at rest, as far as regards the relative motion of the other constituents of our system.

The sun, then, according to the modern hypothesis occupies a fixed centre, about which the several principal planets revolve, in elliptic orbits, one foci of each of which is found in the same point, coinciding very nearly with the solar centre. Several of these planets are again accompanied with attendant luminaries, which observe the same laws in their revolutions about their primaries, as these primaries do with respect to the sun.

Inclination of the orbits of the planets.

11. The orbits of the several planets are not all situated in the same plane, but are variously inclined to each other, and to a fixed plane passing through the sun, called the plane of the ecliptic. Even this plane is not actually fixed, but is subject to a certain annual motion, to which, however, we shall not refer in this description. The several planets of our system, with the exception of the four small ones lately discovered, have their inclinations very small; the particular measures of which will be stated below; and their motions are all made in one direction, viz. from west to east. Several of these bodies are also known to have a rotatory motion on their respective axes, but these axes are differently inclined with respect to the plane of the ecliptic, each, however, always preserving its own direction parallel to itself. This rotatory motion is likewise performed from west to east, at least in all those bodies in which it has been hitherto observed; but some, either from their immense distance, their inconsiderable magnitude, or from their proximity to the sun, have not yet had their diurnal revolution decidedly ascertained.

Motion of the planets.

Inclination of their axes.

Eccentricities of the orbits.

12. The orbits of the planets we have seen are all elliptical, but these ellipses have very different degrees of eccentricity; that is, the ratio of the major and minor axes vary very considerably in the orbits of the different planets. The orbits of the satellites have also various eccentricities, and are differently inclined to the ecliptic, and to each other; in some of the secondaries this inclination is extremely great, while in all the principal bodies of our system it is very inconsiderable.

These general remarks being premised, we shall proceed concisely to describe the particular circumstances attending the motion of each planet, beginning with the sun, and proceeding orderly with the other bodies, according to their respective distances from him.

Of the sun.

13. The sun, which is known to be 882,270 English miles in diameter, performs its diurnal revolution in 25 days 14h. 8′, about an axis, which is inclined 82° 30′ from the plane of the ecliptic.

Mercury.

14. Mercury is the nearest planet to the sun; its mean distance is about 36 millions English miles; the time of its sideral revolution 87 days 23h.; its diameter 3123 miles; the inclination of its orbit to the ecliptic, 7° 0′ 9″; and the ratio of the eccentricity of its orbit to that of its semi major axis ·2055149. The inclination of its axis is unknown, and the time of its diurnal revolution doubtful; it has been stated at 24h. 5′.

Venus.

15. Venus is the second planet in order from the sun; her mean distance being 68 millions of miles, and the time of her sideral revolution 224 days 16 hours. The diameter of this planet is 7702 English miles; the inclination of its orbit 3° 23′; and the ratio of its eccentricity to the semi major axis of its orbit ·0068529. Its diurnal revolution is performed in 23h. 21′, about an axis, which, like that of Mercury, is of unknown inclination.

The Earth.

16. The Earth is the third planet in the system; its distance from the sun is 93 million miles, and the time of its sideral revolution 365 days 6h. 9′; which must not be confounded with the length of the solar year, this being only 365 days 5h. 48′ 48″; the diameter of this planet is 7916 miles. The orbit of the earth being that to which the plane of the other planetary orbits are referred, its inclination is nothing; the eccentricity of its orbit is ·0168531; its diurnal revolution is performed in 23h. 56m. 4″, (which is called a sideral day), about an axis which inclines to the ecliptic in an angle of 66° 32′ 3″, the complement of which, viz. 23 27′ 57″, is the obliquity of the ecliptic as referred to the equator.

Mars.

17. The next primary planet in our system is Mars; its mean distance from the sun is 142 millions of miles, and its sideral revolution is performed in 686 days 23½h.; its diameter is little more than half that of the earth, being only 4398 miles.

Its orbit is inclined to the ecliptic in an angle of 1° 51′ 3″; the eccentricity of its orbit is ·0931340, and its diurnal revolution is performed in 24h. 39m., about an axis, which is inclined to the plane of the ecliptic at an angle of 59° 41′ 49″.

Vesta, Juno, Ceres, Pallas.

18. The next planets in order from the sun, are the four new ones; Vesta, Juno, Ceres, Pallas, of which little is known, except their times of revolution, and the inclination of their orbits; and these may even hereafter be found to stand in need of some corrections: they are at present stated as follows:—

Vesta; mean distance, 225 million miles; time of sideral revolution, 1335 days 4h.; inclination of orbit, 7° 8¾′; eccentricity, ·0932200.

Juno; mean distance, 253 million miles; period of sideral revolution, 1590 days 23h.; inclination of orbit, 21°; eccentricity, ·254944.

Ceres; mean distance, 263 million miles; period of revolution, 1681 days 12h.; inclination of orbit, 10° 37′; eccentricity, ·0783486.

Pallas; mean distance, 265 million miles; period of revolution, 1681 days 17h.; inclination of orbit, 34° 50′; eccentricity, ·245384.

The diameters of these planets (which must, however, be considered as doubtful,) have been given as follow:—Vesta, 238 miles; Juno, 1425 miles; Ceres, 1024 miles; and Pallas, 2099 miles.

Jupiter.

19. Jupiter, the largest of all the planets of our system, is the next after Pallas; its mean distance from the sun is 485 million miles, and its diameter is 91522

miles, more than eleven times that of the earth : the inclination of his orbit to the ecliptic is 1° 18′ 51″, and his sideral revolution 4,332 days 14 hours, the ratio of its eccentricity to the semi major axis ·0481784. The diurnal revolution of Jupiter is performed in 9 hours 55′ 49″, about an axis which is inclined to the ecliptic at an angle of 86° 54½′.

Saturn.

20. Saturn is next to Jupiter, as well in magnitude as in distance ; the latter being 890 million miles, and its diameter 76,068 miles ; his sideral revolution is performed in 10,758 days, or about thirty of our years, and the eccentricity of his orbit ·0561683. The diurnal revolution of Saturn is made in 10 hours 16 19″, about an axis which is inclined to the ecliptic at an angle of 58° 41′.

Uranus, or Georgium Sidus.

21. Last in the solar system is the Georgium Sidus, or Uranus ; whose mean distance is more than double that of Saturn, being no less than 1800 million miles ; its diameter is 35,112 miles, and the eccentricity of its orbit ·0446703. Its sideral revolution is performed in about eighty of our years : but its diurnal revolution, and the inclination of its axis, are not at present determined.

22. We have endeavoured to represent, the planetary distances, eccentricities, and proportional magnitude in Plate III., with the exception of the sun, which is there represented nearly as a point, whereas, in comparison with the magnitude we have given to the planets, its diameter ought to have exceeded even that of the orbit of Saturn. The several orbits in the plate are all represented as if situated in one plane ; it is necessary therefore for the reader to bear in mind the measures of the several inclinations above indicated.

Characters of the planets.

23. The following are the characters or symbols employed by astronomers for denoting the several planets :—

The Sun	⊙	Ceres	⚳
Mercury	☿	Pallas	⚴
Venus	♀	Jupiter	♃
The Earth	⊕	Saturn	♄
Mars	♂	Uranus	♅
Vesta	⚶	The Moon	☽
Juno	⚵		

24. At present we have spoken only of the primary planets, it now remains for us to say a few words with reference to the satellites by which some of them are attended.

The moon.

The first, and that which in this place claims our particular attention, is the moon, which revolves about the earth, as the earth itself does about the sun. Her mean distance from the terrestrial centre is 237,000 miles ; her diameter is 2,160 miles, and the eccentricity of her orbit is ·0548553. The period of her orbicular and diurnal revolutions, are exactly alike, being completed in 27 days 7 hours 43′ 4″·7. The inclination of her orbit is 5° 9′, and of her axis 88° 29′ 49″.

Satellites of Jupiter, Saturn, &c.

It is unnecessary in this place to enter into the same minutiæ with respect to the other satellites ; it will be sufficient to observe, that Jupiter has four, Saturn seven, and Uranus six, as represented in the plate above referred to.

Saturn s ring.

Besides the seven satellites which accompany Saturn in his dreary path, he is also encompassed by a double ring, by which he is distinguished from all the other planets of our system. This ring, which is very thin, not exceeding 4,500 miles, is inclined to the plane of the ecliptic at an angle of 31° 19′ 12″, and revolves from west to east in 10 hours 29′ 16″·8 ; this rotation is performed about an axis perpendicular to the plane of the ring, passing through the centre of the planet.

We shall not enter more particularly in this place into a description of this singularly beautiful telescopic object ; as we shall of course have to treat of it at length in a subsequent chapter, in which we shall state its several dimensions as determined from the best observations ; it will be sufficient here to observe, that its outside diameter is 204,883 miles, and its inside 146,345, consequently, its mean breadth is about 30,000 miles.

Comets.

25. We come now to the third class of bodies, which are only visible to us for a short time. The planets perform their revolutions about the sun within the limits of our observation, and at distances from him which vary comparatively very little in the different parts of their orbits ; but those to which we now refer, if they all actually revolve about the sun, are, for a very considerable time far beyond the known limits of the solar system. These are called comets : they generally appear attended with a nebulous light, either surrounding them as a coma, or stretched out to a considerable length as a tail, and sometimes they appear to consist of such light only. Their orbits are so eccentric that in the remoter parts of them they are invisible to us, although at other times they approach much nearer to the sun than any of the planets : the comet of 1680, for example, when nearest the sun, was at the distance of only one sixth of the sun's diameter from its surface. Their tails are frequently of great extent, appearing as a faint light directed towards a point always opposite to the sun. It is quite uncertain of what matter they consist, and it is difficult to say which of the conjectures concerning them is the least improbable. Nearly 500 comets are recorded as having been seen at different times, and certain particulars relative to the orbits of about a hundred of them, have been accurately ascertained : but commonly, we have no opportunity of observing a sufficient portion of the cometary path, to determine with accuracy the entire dimensions of the ellipse or other conic section to which the observed part appertains ; on which account little can be known of the periods and other circumstances of these wandering bodies. Only one comet has been recognized in its return to our system, which is that of 1759. Dr. Halley, by comparing together the elements of the several comets that had been observed up to his time, conjectured that those recorded to have appeared in 1531, in 1607, and in 1682, were, in fact, one and the same comet, and, consequently, that its return might be expected about the year 1758 or 1759, and its actual appearance in the last of those years, verified the conjecture. Another comet, which appeared in 1770, was suspected to move in an elliptic orbit ; and if so, its period ought, by Mr. Lexel's computation (which has been since remade by Burckhardt), to be about 5 years and 7 months ; it has never, however, been since observed ; but this circumstance must not lead us to discredit either the observations made on it, or the calculations founded on them, for it has

been satisfactorily shown, that supposing all the data correct, this comet must have passed so near to Jupiter, that its orbit would be deranged, and the body rendered in future invisible to us.

We shall not enter further on the subject of comets in this place, except to observe, that whether their orbits be all ellipses, or some of them parabolas or hyperbolas, a very small portion of them fall within the limits of our system; if they are all ellipses, they are of very great eccentricity, and are only for a short period visible in these regions of celestial space. The orbits of two of these bodies are shown in our Plate III.

4. *Phenomena in the heavens, due to the motion of the earth and planets.*

Diurnal motion of the earth.

26. We have already in our second section given a general view of the celestial sphere, with an enumeration of the several constellations; we propose now to illustrate a few particular phenomena, and to describe certain circles which astronomers have imagined for the better comprehension of the celestial motions.

We have seen that a person being situated in an open plane, in a star light evening, and watching attentively the motion of the fixed stars, will perceive them rise or emerge above the earth, continue to ascend, till they have attained a certain height, and then descend and disappear in the opposite side of the heavens to that in which they first rose. He will perceive, that accordingly as these stars are nearer the northern or southern points of the heaven, so they will appear visible to him for a greater or less time; and that certain stars very near the north point, never either rise or set, but would be always visible were not their light rendered imperceptible by the more refulgent rays of the sun.

One of the stars in this quarter of the heavens, to which we have already alluded, has scarcely any sensible motion, but, as far as the naked eye can distinguish, retains constantly the same situation; this is called the *pole star*, and those to which we have referred above, that appear to be constantly circulating about it, are, for that reason, called, *circumpolar stars.*

Apparent motion of the heavens.

The first impression that this apparent motion of the stars would make on the mind of an uninformed observer would be, that the entire celestial vault was uniformly revolving from east to west, about an imaginary axis, which passes through or near the pole star, in a direction perpendicular to the planes of the circles described by the several stars which are alternately observed to rise and set; and consequently, that this line produced would again meet the celestial sphere in an opposite point, which may thence be denominated the *south pole.* This, as we have observed, would doubtless be the first impression made on an uninformed observer; but at the same time, if he possess the requisite intelligence, it would not be difficult to convince him that the very same appearances would be produced by supposing the earth to be of a globular form, and that it performed a motion of rotation about an axis corresponding with the supposed axis of the heavens, but in an opposite direction, that is, from west to east; numerous instances might be recalled to his recollection in which he had appeared at rest, when he was actually in rapid motion, and when objects were apparently seen to move with great velocity, although they were in an absolute state of rest. This phenomenon must have presented itself to every one who has travelled in a close carriage in a narrow road, when at times it is difficult to be persuaded but that the trees, gates, &c. which we pass are not moving in an opposite direction to that of the vehicle; and the same appearances are observed still more strikingly in the cabin of a ship when sailing with a moderate gale near the land. In fact, every intelligent observer, whether he admits the actual motion of the earth or not, will not for a moment deny, that if it had such a motion as we have supposed, the appearances would be exactly such as he observes in the heavens. This then will be one step towards his conviction, and various others will afterwards suggest themselves to his mind; which, however, we shall not at present insist upon, because from what has been stated, it is obvious, that we may at any rate be allowed to advance such an hypothesis as the diurnal revolution of the earth, without in any respect changing the appearance of any observed phenomenon. We shall therefore proceed upon the supposition of the earth being a sphere or a spheroid of small ellipticity, and that it performs a motion of rotation from west to east in about 24 hours; or rather, as we shall see in a subsequent chapter, in 23 hours 56′ 4″·1.

Apparent motion of the sun.

27. By observing attentively the stars which first appear visible in that part of the heavens where the sun sets, and continuing to observe them for several successive nights, we shall soon perceive that those stars which in the first instance we had observed to set immediately after the sun, are no longer to be seen. but that their places are supplied by certain others, which in their turn will also be lost in the solar beams. If now we observe the heavens in the morning before the rising of the sun, we shall find that those stars which in the first instance we had observed to set just after the sun, and which in the course of a few nights were absorbed by his rays, are now rising before him; the sun therefore has made an apparent motion in the heavens contrary to the general motion of the stars, that is, from west to east; and by following his progress in this manner during the course of a year, we shall find that he has described a complete circle of the heavens, and now rises and sets with the same star as we had observed exactly a year before. This circle which the sun thus appears to describe in the heavens is called the *ecliptic*, it is not directly east and west, but deviates nearly 24° from these points of the heaven, as shown in Plates I. and II., and to his obliquity it is, that we owe the variations in the seasons, and various other phenomena, as we shall show in a future chapter.

This apparent motion of the sun is due to the proper motion of the earth. Fig. 1.

28. This progressive motion of the sun in the heavens, which is only apparent, is due to the actual proper motion of the earth in its orbit about the sun. For let AB (Plate IV., fig. 1) represent two positions of the earth in its orbit ABC about the sun S, and let ♈, ♉, ♊, ♋, &c. represent the ecliptic or the apparent path of the sun. Then when the earth is at A, a spectator will refer that body to that part of the heavens marked ♈; but when the earth is arrived at B, he will then see it in ♊; and being in the mean time insensible of his own motion, the sun will appear to him to have described the arc ♈ ♊,

just the same as if it had actually passed over the arc SS′, and the earth had, during that time, remained quiescent in its first position A. This fact will explain why certain remarkable stars and constellations are seen in the south in different seasons of the year, and at different hours of the night. For the hour depends wholly on the sun; it is noon when the sun is south, and midnight when it is north; the stars directly opposed to him will therefore, by the rotation, appear in the south about midnight; and as the sun, from one day to another, shifts its place in the heavens, so, of necessity, will different stars be opposed to him, and become south at midnight at different seasons of the year.

Proper motion of the planets.

29. Similar phenomena may be observed with regard to the planets; by tracing their motions in the heavens, and comparing them with the stars near to which they appear, they will also be seen to have a motion of their own, but it will be more irregular than that of the sun; for we shall sometimes observe them moving like that body from west to east, then become stationary, maintaining the same position for several nights; then moving in a contrary direction, or from east to west; again become stationary, and again assume their direct motion.

These phenomena, which the ancients spent so much labour and ingenuity to account for, by epicycle upon epicycle, are perfectly consistent with the system as we have described it in our third plate, being due to the proper motions of the earth and planets in their respective orbits. In order to illustrate this, let $v\ v'\ v''$ represent the orbit of the planet Venus; and suppose her to be in the point v when the earth is at A; then it is obvious that a spectator will refer her place in the heavens to ♈, and as her motion is from v towards v', while the earth is moving from A to B, her apparent motion will be direct; or from ♈ towards ♉: if, on the other hand, Venus had been at v'' while the earth was at A, as their motions now are made in the same way, we may suppose Venus to have arrived at v''', while the earth had passed from A to A′; and during this time, it is manifest her place in the heavens will appear to retrograde, or go backward, contrary to the order of the signs: in like manner it will appear, that for a very short time before and after the Earth and Venus attained their positions A and v'', their motions would so agree with each other, that the planet, during this period, would appear stationary. We shall enter more at length upon these phenomena in a subsequent chapter; we have merely referred to the subject here, to show that these appearances are strictly conformable with the constitution which we have supposed of the solar system.

Notwithstanding these irregularities in the apparent motion of the planets, they each, respectively, are observed to describe great circles of the sphere, but more or less inclined to the plane of the ecliptic; they deviate, however, but little from it; all their motions, with the exception of the new planets, being performed in a zone, whose breadth does not exceed 16 degrees, and which we have already spoken of and described in our first and second plates, under the denomination of the zodiac.

Similar observations show that the moon also describes her particular path amongst the stars; but we shall reserve this subject for a subsequent chapter.

5. *Of the Seasons.*

Of the seasons.

30. It will appear sufficiently obvious from what has been shown in the preceding sections, that the alternations of day and night are attributable generally to the rotation of the earth on its axis; but the difference in the length of the days in different seasons of the year still remains to be illustrated.

Equator defined. Fig. 2, 3.

We have seen that the two extremities of the terrestrial axis about which the diurnal rotation is performed, are called its poles, as N S, (fig. 2 and 3); and if at a quadrant distance from these we conceive a circle QEQ′ to be described, dividing the earth into two equal hemispheres, that circle is called the equator; the hemisphere towards the north pole N is called the northern hemisphere; and that towards the south pole, the southern.

Consequences that would follow if the equator and ecliptic coincided.

Now if the path of the earth in its orbit, or, which is the same, the apparent motion of the sun in the heavens coincided with the plane of the terrestrial equator, then it is obvious, that, at all times during this revolution, we should have the same equal alternations of day and night, each 12 hours in duration. For example: let NQSQ′ represent the earth; NS, its axis; and QEQ′, its equator: and let the plane of this circle produced pass through the sun S; then it is obvious, that, if we suppose the earth to revolve round the sun at the extremity of the line SE as a radius; and that during this revolution, it performed uniformly its rotatory motion about its axis NS: that line, NS, or the circle of which it is the projection, would terminate the limits of day and night; and the rotatory motion being uniform, every point of the globe, except the two poles, would have an equal succession of light and darkness during the entire revolution. We should then have no spring, no summer, no winter; these changes so pleasing in themselves, and so necessary for the production and re-production of the fruits of the earth would be wanting, and nature would thus be divested of a great portion of its charms.

Consequences of the obliquity of the ecliptic.

This, however, is not the case; we have seen that the sun appears to describe an oblique motion in the heavens; it rises higher in the summer than in the winter, and by thus darting upon us more perpendicularly its refulgent beams, produces a greater portion of heat, and describes a larger circuit in the heavens, lengthens out the days, and thus gives time for that heat to become more effective.

The actual motion of the earth is therefore as represented in (fig. 3,) where we still suppose it to describe its annual motion at the extremity of the radius S′E; but such, notwithstanding, that the axis NS preserves its inclination and parallelism, whereby it is always directed to the same point in the heavens. The axis NS being now inclined to the plane of the earth's motion, it is obvious, that, as it revolves on its axis, some parts of the earth will experience perpetual day for a certain portion of the year, while other parts will have to contend with an equal duration of night. In the position of the earth, as shown in the figure, the parts about the north pole will be in continued darkness, and those near the southern pole in perpetual light; while, from the nature of the annual motion, it is clear that it will be exactly the reverse when the earth shall have attained the opposite point of its orbit; the regions of the south pole will then be involved in darkness, and those of the northern will enjoy their return of light, as will appear obvious by supposing the sun to be transferred to the other side of the earth, which is exactly equivalent to the change of place in the earth.

31. Dr. Long, in his astronomy, gives us a pleasing illustration of this change of the seasons, with the variable lengths of the days and nights, by means of the following experiment:--

Illustrated by experiment. Fig. 4.

Take about seven feet of a strong wire, and bend it into a circular form, as *b c d*, (fig. 4,) which being viewed obliquely, appears elliptical, as in the figure. Place a lighted candle on a table, and having fixed one end of a thread K to the north pole of a small terrestrial globe H, about three inches in diameter; cause another person to hold the wire circle, or fix it so that it may be parallel to the table, and as high as the flame of the candle I, which should be in or near the centre; then having twisted the thread as towards the left hand, it will, by untwisting, turn the globe round eastward, or contrary to the way which the hands of a watch move; hang the globe by the thread within the circle, nearly contiguous to it; and, as the thread untwists, the globe, which is enlightened half round by the candle as the earth is by the sun, will turn round its axis, and the different places upon it will be carried through the light and dark hemisphere, and have the appearance of a regular succession of days and nights, as our earth has in reality by such a motion. As the globe turns, move the hand slowly, so as to carry the miniature earth round the candle according to the orders of the letters *a*, *b*, *c*, *d*, &c., keeping its centre even with the wire circle, and it will be perceived, that the candle, being still perpendicular to the equator, or rather in the plane of the equator produced, it will enlighten the globe from pole to pole, as we have shown in fig. 2, and that during the whole of its orbicular revolution; consequently every place on the globe goes equally through the light and dark, as it turns round by the untwisting of the thread. The motion of the globe turning in this way represents the earth revolving on its axis; and the motion of the same in the circle of wire, its revolution round the sun; and shows, that if the orbit of the earth had no inclination to its equator, all the days and nights in every part of the globe would be of equal duration throughout the year, as described in the first of the preceding figures.

Now desire the person who holds the wire to incline it obliquely, in the position ABCD, raising the side ♋ just as much as he depresses the side ♑, that the flame may be still in the plane of the circle; and twisting the thread as before, that the globe may turn round its axis the same way as you carry it round the candle; that is, from west to east. Let the globe down into the lowermost part of the circle ♑; and if the latter be properly inclined, the candle will shine perpendicularly on the globe at *a*, and the region about the north pole will be all in the light, as shown in the figure, and will still keep in the light while the globe revolves on its axis.

From the equator to the north polar circle, all the places have longer days and shorter nights; but from the equator to the north pole just the reverse takes place—the nights being longer than the days. The sun does not set to any part of the northern frigid zone, as is shown by the candle constantly shining upon it while the globe is in this position: and on the same principles, it will appear that the southern arctic regions are, during this time, involved in constant darkness; the revolution of the globe never bringing any part of it within the illuminated hemisphere; and, therefore, if the earth remained constantly in this part of its orbit, the sun would never set in the northern frigid zone, nor rise in the southern. In fact, we should thus have perpetual summer in the former, and perpetual winter in the latter; and the same would be the case with the hemispheres to which these zones appertain.

But as the globe moves round its axis, move your hand slowly forward, so as to carry it from H towards E, and the boundary of light and darkness will approach towards the north pole, and recede from the south; the northern places will pass through less and less light, while the southern will be more and more involved in it; whence is shewn the decrease in the length of the days in the northern hemisphere, and the increase of the same in the southern. When the globe is at E, it is in a mean state between the highest and lowest points of its orbit; the candle is directly over the equator, the boundary of light and darkness just reaches both the poles, and all places on the globe pass equally through the light and dark hemispheres, thereby showing that the days and nights are then equal in all parts of the earth excepting only the poles, the sun there appearing as setting to the northern and rising to the southern.

The globe being still moved forward, as it passes towards A, the north pole is more and more involved in the dark hemisphere, and the south pole advances more into the enlightened part; till, when it arrives at F, the candle is directly over the circle *b b*, and the days are then the shortest and the nights the longest throughout the northern hemisphere: and the reverse in the southern, the days there being the longest and the nights the shortest. The southern polar zone is now in perpetual light, and the northern in continual darkness.

If the motion of the globe be still continued, as it moves through the quarter B, the north pole advances towards the light, and the south pole recedes from it; the days lengthen in the northern hemisphere and shorten in the southern; till, when the globe arrives at G, the candle will be again over the equator (as when it was at E); the days and nights will be again equal in all parts of the earth; the north pole will be just emerging out of darkness, and the southern pole beginning to be involved in it, as the northern pole was in the former instance when the earth was at E.

Hence is shown the reason of the days lengthening and shortening from the equator to the polar circle every year, and why there is sometimes no day or no night for several revolutions of the earth within the two frigid zones; and why there is but one day and one night in a year at the poles themselves. We see also that the days and nights at the equator are equal all the year round, this being always equally bisected by the circle bounding the light and darkness.

A similar representation of these phenomena is exhibited in fig. 5; and in a subsequent chapter we shall

Fig. 5.

enter more particularly on the subject; showing the method of computing the length of the day in any given latitude; and the time of the rising, setting, and southing of the sun on any proposed day, and in any given place.

6. *Of the Phases of the Moon.*

Phases of the moon.

32. The moon, of all the celestial bodies, is that which perhaps attracts most the attention, both of illiterate and scientific observers. The former class are drawn to their observations by the remarkable beauty and serenity of her light, her numerous changes or phases, the inconstancy of her illumination, the enjoyment of her light at certain seasons of the year, when the sun sinks early below the horizon, and the want of it when both luminaries rise and set at nearly the same hour.

The philosophic observer examines her varying phases through his telescope; sees the shadows of the hills on her surface projected to a considerable distance through plains and vallies, rendered by the power of optics nearly as distinct as those of a terrestrial plain at the distance of only a few miles; he examines what he considers to be lunar volcanoes, and marks the progress of the lava from the crater to the vallies below; he looks to determine some indication of eruptions in present action; endeavours to distinguish her seas and continents, to measure the heights of her mountains, and to ascertain the existence or non-existence of a lunar atmosphere.

Again, the practical astronomer is watching carefully her deviating course in the heavens, and is endeavouring to submit her oscillating motion to the general principles of celestial mechanics; and to construct formulæ and tables, for the purpose of predicting her place at any appointed day and hour.

These subjects will each engage our attention in the course of the following chapters of this treatise; but at present we only propose to present to the reader an explanation of her most obvious phenomena, and to show their complete accordance with the motions and constitution of the solar system as described generally in the third chapter of this introduction.

Phases of the moon.

33. The first lunar phenomena to which we shall call the readers' attention, is the continual change of figure, or the phases which she exhibits to a terrestrial spectator. At one time perfectly full or globular, at others half or a quarter illuminated, and at others again exhibiting only a fine arched line, barely perceptible to the naked eye. These appearances are doubtless due to the revolution of the moon in her orbit; and the reflection of her light (which she receives from the sun) towards the earth. The lunar globe is necessarily always one half illuminated as we have shown the earth to be in the last section; and therefore, to a spectator placed in a line between the moon and sun, she would always present a full illuminated circle or hemisphere; but out of that line a greater or less part of the enlightened surface will be perceived, and which will obviously entirely vanish in certain positions. This may be illustrated by means of an ivory ball, as in the experiment described in the last section; for the ball being held before a candle in various positions, will present a greater or less portion of the illuminated hemisphere to the view of the observer, according to his situation with regard to the illuminated axis.

Phases illustrated. Fig. 6.

34. The same may be otherwise exhibited by means of our figure 6; where T is the earth, S the sun, and A, B, C, &c. the moon in different parts of her orbit. When the moon is at A, in conjunction with the sun S, her dark hemisphere being entirely turned towards the earth, she will disappear as at *a*, there being no light on that side to render her visible. When she comes to her first octant at B, or has gone an eighth part of her orbit from her conjunction, a quarter of her enlightened side is towards the earth, and she appears horned, as at *b*. When she has gone a quarter of her orbit from between the earth and sun to C, she shows us one half of her enlightened side, as at *c*, and we say she is a quarter old. At D, she is in her second octant; and by showing us more of her enlightened side she appears gibbous, as at *d*. At E, her whole enlightened side is towards the earth; and therefore she appears round, as at *e*; when we say it is full moon. In her third octant at F, part of her dark side being towards the earth, she again appears gibbous, and is on the decrease, as at *f*. At G, we see just one half of her enlightened side; and she appears half decreased, or in her third quarter, as at *g*. At H, we only see a quarter of her enlightened side, being in her fourth octant; where she appears horned, as at *h*. And at A, having completed her course from the sun to the sun again, she disappears; and we say it is new moon. Thus, in going from A to E, the moon seems continually to increase; and in going from E to A, to decrease in the same proportion; having like phases at equal distances from A to E. But as seen from the sun S she is always full.

The moon appears not perfectly round when she is full in the highest or lowest part of her orbit, because we have not a full view of her enlightened side at that time. When full in the highest point of her orbit, a small deficiency appears at her lower edge, and the contrary when full in the lower point of her orbit.

The earth is a moon to the moon.

35. The moon, as we have seen, shines by her reflected light; in the same manner, the earth, by throwing back the light it receives from the sun, becomes in its turn a moon to the moon; being full to the inhabitants of the lunar sphere when our moon changes, and *vice versa*. For, when the moon is at A, new to the earth, the whole enlightened side of the earth is turned towards the moon; and when the moon is at *d*, full to the earth, the dark side of the latter is turned towards the former. Hence a new moon answers to a full earth, and a full moon to a new earth. The quarters are also reversed with respect to each other.

Position of the cusps of the moon.

36. The position of the moon's cusps, or a right line touching the points of her horns, is very differently inclined to the horizon, at different hours of the same day of age. Sometimes she stands as it were upright on her lower horn, which is then necessarily perpendicular to the horizon; when this happens, she is said to be in her *nonagesimal degree*, which is the highest point of the ecliptic above the horizon; the ecliptic at that time being 90° from each side of the horizon, reckoning from the point where it is then cut by the former circle. But this never happens when the moon is on the meridian, except when she is in the beginning of Cancer or Capricorn.

The moon always presents the same face to the earth.

37. The moon turns on an axis which is nearly perpendicular to the plane of the ecliptic, and in such a way as to make one complete diurnal rotation during one lunation, and therefore always presents to us the same face or hemisphere, as is demonstrable by observations made on her by the telescope; both these revolutions are performed in 27 days 7 hours 43′, viz. from any star to the same star again, but from one lunation to another is about 29½ days. In consequence of this remarkable coincidence, which is still unaccounted for on physical principles, the earth must appear to a spectator on the moon to be permanently at rest. It will never quit any position in which it is once found, at whatever height it may be above the lunar horizon, or in whatever quarter it may appear; it will go through all its changes, but retain its position; consequently to one half of the moon the earth is always invisible, while the other half enjoys a constant illumination from its reflected rays; but each side has an equal participation of the solar light.

Phases of the planets.

38. The phases of the inferior planets, Mercury and Venus, strongly resemble those of the moon, but they are invisible except by means of the telescope. Copernicus, after having laid down his system of celestial motions, predicted that future astronomers would find that Venus underwent the same changes, and exhibited similar phases to the moon; which prediction was first fulfilled by Galileo, who directing his telescope to this planet, observed the phases foretold by the father of modern astronomy; he observed her to be sometimes full, sometimes horned, and sometimes gibbous.

Mercury also presents similar appearances. All the difference being, that when these are full, the sun is between them and us, whereas, when the moon is full, we are between her and the sun. Mars appears sometimes gibbous, but never horned, its orbit being exterior to that of the earth.

We shall have again to recur to this subject, in a subsequent chapter, our purpose here being merely a popular illustration of the most striking lunar phenomena.

7. *Of lunar and solar eclipses.*

Of eclipses.

39. The phenomena of eclipses are amongst the number of those which have most engaged the attention of mankind; the learned and the unlearned have found an equal interest in them; the one to observe their appearances, to discover the cause of the deprivation of light which we then encounter, to predict their return, &c.; and we have seen, in our historical sketch, the various absurd ideas that have been entertained on this subject by many of the most eminent philosophers of antiquity. Amongst the common people, the interest excited by such phenomena, arose from the fear which they inspired; they were considered by them as alarming deviations from the regular course of nature, and as the forerunners of some portentous event; hence, actuated either by curiosity or timidity, the subject of eclipses has probably from the earliest times engaged the serious attention of mankind, and they are still amongst the most interesting of the celestial phenomena. In illustrating the cause of eclipses, we shall follow still the method we have hitherto pursued in this introduction; that is, we shall render our explanation as popular and simple as possible, leaving all the minute particulars for another place, where we shall enter more fully upon the subject, and illustrate the principles of those calculations on which the prediction of eclipses depends.

Eclipse of the moon.

40. The earth being an opaque body enlightened by the sun, it necessarily projects a shadow into the regions of space in a contrary direction; and when it so happens that the moon in the course of her revolution about the earth, falls into this shadow, she loses the sun's light by which alone she is visible, and appears to us eclipsed. Let us suppose two straight lines drawn from the opposite parts of the solar disc tangents to the surface of the earth as AB, *a b*, (fig. 7) these lines will represent the limits of the shadow, and as the sun is much larger than the earth, these lines will meet at a point, and cross each other behind the earth; and the shadow will thus assume the figure of a right cone. When the moon enters this shadow, and a part of her disc is still enlightened by the sun, this part is not terminated by a straight line, but has the form of a luminous crescent, the concave part being turned towards the shade. The same circumstance happens again, when the moon begins to quit the shadow.

Fig. 7.

When the moon approaches the terrestrial shadow, she does not lose her light suddenly, but it gradually becomes more and more faint till the obscurity arrives at its greatest intensity. In order the better to comprehend this phenomenon, we have only to attend to the figure, and observe, that an opaque body may be so placed between an object and the sun as only to intercept a part of its light; let us suppose this object to be M, it will then be less illuminated than if it received the whole of the light, but more of it than if it were placed at *m* in total obscurity.

Penumbra.

41. This intermediate state comprehended between the angular space EBC on one side of the umbra or shadow, and FBC on the other, is called the *penumbra*; and it is the entrance of the moon into this partial shade, which produces the faint obscurity observed immediately before the eclipse commences, and after it is over.

The limits of this penumbra may be found by drawing two lines as A *e*, *a* E touching the surface of the sun and the earth, so as to cross at a point C between them. The angles EBC, *e b c* will determine the space occupied by the penumbra; for at a point situated beyond this space, the whole disc of the sun will be visible, and the visible portion will diminish from the line EB to CB, when it will entirely disappear, and consequently the penumbra will gradually increase in its intensity from its first limit EB, to its second BC, where it ceases, or is confounded with the shadow itself.

Faint light observed in a total lunar eclipse.

42. When the moon enters completely into the shadow of the earth, we still do not entirely lose sight of it; its surface is still faintly illuminated with a reddish light, something similar to that reflected by the clouds after the setting of the sun. This effect arises from the solar rays that have been refracted by our atmosphere, and afterwards inflected behind the earth; for those rays which are not enough refracted to reach the surface of the earth, continue their course through the atmosphere, and if not entirely absorbed by it, are inflected towards a focus or point in the same manner as in a convex lens.

The light thus refracted behind the earth is very considerable: regarding only one luminous point of the solar disc, it can only project one ray to every point of the surrounding space, but through the medium of the terrestrial atmosphere, a course of luminous points is collected behind the earth, an object placed in the focus or vertex of this cone, would be more strongly illuminated than by the direct light of the sun; every point of the sun producing a similar effect, the length and extension of the terrestrial shadow are much diminished; and if the atmosphere did not absorb a very great portion of the solar rays, the light reflected from the disc of the moon would be very great; it is, however, so much modified by the circumstances alluded to, as to exhibit only that faint red light above described.

It may be proper to observe, that this faint illumination of the lunar disc at the time of a total eclipse, has been accounted for upon different principles, but the above appears to us the most satisfactory.

Solar eclipse.

43. An eclipse of the sun is an occultation of the sun's body, occasioned by the interposition of the moon between the earth and sun. On this account it is by some considered rather as an eclipse of the earth, because the light of the sun is hidden from the earth by the moon, whose shadow involves a part of the terrestrial surface. The cause of a solar eclipse, and the circumstances attending it are represented in fig. 8, where S is the sun, *m* the moon, and CD the earth, *rmso* the moon's conical shadow traversing a part of the earth CoD, and thus producing an eclipse to all the inhabitants residing in that track, but no where else; excepting that for a large space around it, there is a fainter shade included within all the space *r*CD*s*, which, as in the lunar eclipse, is called the penumbra.

Fig. 8.

Hence, solar eclipses happen when the moon and sun are in conjunction, whereas the lunar eclipses only take place when they are in opposition; that is, the former happen at the time of the new moon, and the latter at the full.

Different kind.

44. Notwithstanding the moon is very considerably less than the sun, yet from its proximity to us, it so happens, that its apparent diameter differs very little from that of the latter body, and even sometimes exceeds it. Suppose an observer situated in a right line which joins the continuation of the sun and moon, he will see the former of these bodies eclipsed, as we have above stated. If the apparent diameter surpasses that of the sun, the eclipse will be total, and the observer will be entirely immerged in the conical shadow which is projected behind the moon: if the diameters are equal, the point of the cone will terminate at the earth's surface, and there will be a momentary total eclipse. If the diameter of the moon be less than that of the sun, the observer will see a zone of the sun surrounding the moon like a ring, and the eclipse will be central and annular. And lastly, if the observer be not exactly in the line joining the centres, the eclipse may be partial; that is, a part of the solar disc may be hid while the remaining part continues perfectly visible. Total eclipses, which are very rare occurrences in any particular place, are remarkable for the darkness which accompanies them, and which they spread over different parts of the surface of the earth, in the same manner as the shadow of a dense cloud, carried along by the wind, sweeps over the mountains and the plains, depriving them for some instants of the light of the sun. This total darkness under the most favourable circumstances may last about five minutes. The smallest apparent diameter of the sun is 31′ 30″; the diameter of the moon at its mean distance 31′ 25″, that is less than that of the sun; consequently there cannot be a total eclipse when the moon is beyond its mean distance. Eclipses of the sun are also modified as to quantity by the height of the moon above the horizon which increases her diameter; other circumstances also contribute to produce certain changes which must be considered in the computations relative to these phenomena, but which it would be useless to detail in this place. From what has been already stated we may draw the following general conclusions.

Total.

Annular.

Partial.

General deductions.

1. That no solar eclipse is universal; that is, none can be visible to the whole hemisphere to which the sun is risen: the moon's disc being too small and too near the earth to hide the sun from a whole terrestrial hemisphere. Commonly, the moon's dark shadow covers only a spot on the earth's surface, about 180 miles broad, when the sun's distance is greatest, and the moon's least. But her partial shadow or penumbra, may then cover a circular space of 4,900 miles in diameter, within which the sun is more or less eclipsed, as the places are nearer to or farther from the centre of the penumbra. In this case, the axis of the shade passes through the centre of the earth, or the new moon happens exactly in the *node*, and then it is evident that the section of the shadow is circular; but in every other case the conical shadow is cut obliquely by the surface of the earth, and the section will be oval, and very nearly a true ellipsis.

2. Nor does the eclipse appear the same in all parts of the earth, where it is seen; but when in one place it is total, in another it is only partial. Moreover, when the apparent diameter of the moon is less than that of the sun, as happens when the former is in apogee and the latter in perigee, the lunar shadow is then too short to reach the earth's surface; in which case, although the conjunction be central, yet the sun will be to no place totally eclipsed, but to certain observers, a bright rim of light will be seen surrounding the moon while the latter is on the solar disc; and the eclipse is then said to be annular.

3. A solar eclipse does not happen at the same time in all places where it is seen; but appears earlier to the western parts, and later to the eastern; as the motion of the moon, and consequently of her shadow, is from east to west.

4. In most solar eclipses, the moon's disc is covered with a faint light; which is attributed to the reflection of the rays from the illuminated part of the earth.

Lastly. In total eclipses of the sun, the moon's limb is seen surrounded by a pale circle of light, which has been considered as indicative of a lunar atmosphere; others, however, doubt this explanation, and offer different conjectures as to the cause of the phenomenon; but this is not the place for discussing this question.

Having thus given a brief description of the constitution of the solar system, with an illustration of the most remarkable phenomena which it presents to naked vision, as far as they can be illustrated independent of astronomical computation, we shall now conclude our introduction by describing certain astronomical machines, constructed for the purpose of exhibiting, in a simple and popular manner, all the most remarkable celestial movements and phenomena.

8. *Description of astronomical machines.*

Astronomical machines.

45. By astronomical machines is here to be understood any pieces of mechanism constructed for exhibiting the motions and phenomena of the heavenly bodies, being thus distinguished from astronomical instruments, which include all such constructions as are employed for the purpose of measuring altitudes, angles, &c. necessary for astronomical computation. The one, in fact, are merely employed for explanation; the other, for the purpose of research and calculation. Various improvements have been made by different artists in the construction of planetary machines, and that which is now exhibited in the lectures of the Royal Institution, is perhaps the most perfect of its kind; but it would carry us too far to describe this instrument, with all its apparatus, wheel work, &c. Beside, we cannot help observing, that much time, ingenuity, and expence are frequently wasted in these kind of constructions; because, after all, they are only, as we have before observed, explanatory; for which purpose the same degree of accuracy is not required as in instruments employed in astronomical observation. That student who stands in need of the assistance of such machines, will never become a great proficient in astronomy; to pursue this study with effect, a beginner must acquire the habit of constructing his own planetarium in his mind's eye, and of soaring with it into the regions of celestial space; he ought to conceive the orbits of the heavenly bodies in a free non-resisting medium, their nodes, their inclinations, and eccentricities unimpeded by the intervention of brass rings or ebony frames; which have always the effect of giving a stiffness and unnatural representation extremely offensive to the eye of the professed astronomer. We must, however, acknowledge, that to children or mere novices, these machines may be of some assistance, and shall therefore describe one or two of the most simple of them in this place.

Planetarium.

Planetarium, by Jones. Fig. 9.

46. The machine exhibited in fig. 9, is a planetarium, constructed by the late Mr. Jones, of Holborn. It represents, in a general manner, by various parts of it, all the principal motions and phenomena of the heavenly bodies.

The sun occupies the centre, with the planets Mercury, Venus, the Earth with its moon, Mars, Jupiter, with his four satellites, Saturn with his seven, and an occasional long arm may be attached for exhibiting the Herschel or Uranus, with his several attendants. To the earth and moon is applied a frame, CD, containing only four wheels and two pinions, which serve to preserve the earth's axis in its proper parallelism in its motion round the sun, and to give the moon her due revolution round the earth at the same time. These wheels are connected with the wheel work in the round box below, and the whole is set in motion by the winch H. The arm M that carries round the moon, points out on the plate C her age and phases for any situation in her orbit, which are engraved upon it. In the same manner, the arm points out her place in the ecliptic B, in signs and degrees, called her geocentric place; that is, as seen from the earth. The moon's orbit is represented by the flat rim A; this orbit is made to incline to any desired angle. The earth of this instrument is usually made of a 3-inch, or $1\frac{1}{2}$-inch globe, papered, &c. for the purpose; and by means of the terminating wire, that goes over it, points out the changes of the seasons, and the different length of days and nights. It may also be made to represent the Ptolemaic system, which places the earth in the centre, and the planets and sun revolving about it. This is done by an auxiliary small sun and earth, which change their places in the instrument; but at the same time it affords a most manifest confutation of it. For it is obviously perceived in this construction, first, that the planets Mercury and Venus being both within the orbit of the sun, cannot at any time be seen to go behind it, whereas, in nature, we see them as often go behind as before the sun in the heavens. It shows that as the planets move in circular orbits about the central earth, they ought at all times to be of the same apparent magnitude; whereas, on the contrary, we observe their apparent magnitude in the heavens to be very variable; and so far different, that Mars, for instance, will sometimes appear nearly as large as Jupiter; while, at others, he will scarcely be distinguishable from a fixed star.

Refutation of Ptolemaic system.

The planetarium, when thus adjusted, shows also that the motions of the planets ought always to be regular and uniform; that they ought always to move in the same direction; whereas, we find them, sometimes direct, at others stationary, and even retrograde; which plainly shows the fallacy of the Ptolemaic hypothesis, at the same time that the modern system is thus clearly represented. Let us, for example, take the earth from the centre, and replacing it by the ball representing the sun, also restoring the earth to its proper situation amongst the planets, and every phenomena will then correspond and agree exactly with celestial observations. For turning the handle H, we shall see the planets Mercury and Venus go both before and behind the sun, or have two conjunctions; we shall perceive also that Mercury can never have more than a certain angular distance 21° from that body, nor Venus a greater than 47°. It will likewise be seen, that the superior planets, particularly Mars, will sometimes be much nearer to the earth than at others; and, consequently, must vary considerably in their apparent magnitude; we shall see that these planets cannot appear from the earth to move with equal velocities; but that this will appear greater when they are nearest, and less as they are more remote; that their apparent motions ought sometimes to be direct, sometimes retrograde, while in particular positions they will seem to be stationary; all which are consistent with the actually observed phenomena.

Fig. 10.

These particulars are shown somewhat more minutely in fig. 10, where a hollow wire, with a slit at top, is placed over the arm of the planet Mercury or Venus, at E. The arm DG represents a ray of light proceeding from the planet at D to the earth, and is put over the centre which carries the earth at F. The machine being then put in motion, the planet D, as seen in the heavens from the earth at F, will undergo the several changes of position as above described; and a similar application may be made to the superior planets.

This apparatus serves also to illustrate the diurnal rotation of the earth on its axis; the cause of the different seasons, the difference in the lengths of the days and nights, &c. For as the earth is placed on an axis inclined to the plane of the ecliptic at an angle of $23\frac{1}{2}^\circ$, we shall have, when the machine is in motion, the most satisfactory illustration of the different inclination of the sun's rays upon the earth. The different quantities which fall on a given space, the unequal quantities of the atmosphere they pass through, and the unequal duration of the sun above the horizon at the same place at different times of the year; which circumstances constitute the primary causes of all that change of seasons and variable lengths of days and nights which we experience.

The globe representing the earth being moveable about an axis, if we draw upon it a circle to denote our own horizon, we may, by means of the terminating wire going over it, very naturally exhibit the cause of the different lengths of the days and nights in our particular latitude, by simply turning the artificial earth with the hand to imitate its diurnal rotation; but in some of the more modern instruments of this kind, this rotatory motion is communicated to the globe by the wheel work of the machine itself.

Fig. 11.

The eclipses of the sun and moon are still more perfectly shown by this machine than the phenomena to which we have above alluded; for by placing a light in the centre instead of the brass ball, denoting the sun (fig. 11), and turning the handle till the moon comes into a right line, between the centres of the light, or sun, and the earth, the shadow of the moon will fall upon the latter, and all the inhabitants of those parts over which the shadow passes, will see more or less of the eclipse; and on the other side the moon passes through the shadow of the earth, and is by that means eclipsed to the inhabitants of those parts to which the lunar disc is at that time visible.

All the phenomena of the satellites of Jupiter, Saturn, &c., might also, with equal facilities, be exhibited by this machine, and are actually so exhibited in some of the larger apparatus, denominated orreries; in the machine we are at present describing, these satellites are only moveable by hand.

Orreries.

Of the Orrery.

47. The term Orrery, to denote such a machine as that we are about to describe, appears rather singular, and is one of those derivations, which, if the history were lost, would involve future etymologists in inexplicable difficulties. The first machine of this kind appears to have been made by the celebrated instrument maker, Graham, by whom it was probably considered only as an improved planetarium: but Rowley, an artist of reputation in his time, copied Graham's machine, and the first of his construction was made for the Earl of Orrery; whence Sir R. Steel, who knew nothing of Graham's original claim, called the instrument after the name of the supposed first purchaser an *Orrery*, which designation it still bears. One of the most usual constructions of this kind is shown in (fig. 12,) which may be briefly described as follows :—

Fig. 12.

The frame of it, which contains the wheel work, &c. and regulates the whole machine, is made of ebony, and about four feet in diameter. Above the frame is a broad ring supported by twelve pillars, which ring represents the plane of the ecliptic. Upon it are two circles divided into degrees with the names and characters of the twelve signs of the zodiac. Near the outside is a circle of months and days, exactly corresponding to the sun's place at noon each day throughout the year. Above the ecliptic stand some of the principal circles of the sphere corresponding with their respective situations in the heavens; viz. *a a* are the two colures, divided into degrees and half degrees; *b* is one half of the equinoctial circle, making an angle of $23\frac{1}{2}$ degrees with the ecliptic. The tropic of Cancer and the arctic circle are each fixed parallel at their proper distances from the equinoctial. On the northern half of the ecliptic is a brass semicircle moveable upon two fixed points in ♈ and ♎.

This semicircle serves as a moveable horizon, to be put to any degree of latitude on the north part of the meridian, and the whole machine may be set to any latitude, without disturbing any of the internal motions, by means of two strong hinges fixed to the bottom frame upon which the instrument moves, and a strong brass arch, having holes at every degree, through which a pin may be passed at any required elevation. These hinges, with the arch, support the whole machine when set to the proposed latitude.

When the Orrery is thus adjusted, set the moveable horizon to any degree upon the meridian, whence may be formed a pretty correct idea of the respective altitude or depressions of the several planets, both primary and secondary. The sun S stands in the centre of the system on a wire making an angle with the ecliptic of about 82°; next in their order follow the planets Mercury, Venus, and the Earth, the axis of the latter being inclined to the plane of the ecliptic, at an angle of $66\frac{1}{2}^\circ$, which is the measure of the inclination of the earth's axis.

Near the bottom of this axis is a dial plate, having an index pointing to the hours of the day, as the Earth revolves; and about the latter is a small ring, supported by two small pillars, representing the orbit of the moon, with divisions answering to the moon's latitude. The motion of this ring represents the motion of the lunar orbit according to that of the nodes; and within it is a small ball with a black cup, or case, by which are exhibited all the phases of this celestial body.

Beyond the orbit of the earth are those of Mars, Jupiter, and Saturn, and in some instruments the Georgium Sidus, or Uranus. Jupiter is attended by his four satellites, and Saturn by his seven satellites and ring.

The machine is put in motion by turning a handle, or winch; and by pushing in, and pulling out a small pin above the handle. When it is in, all the planets, both primary and secondary, will move according to their respective periods. When it is out, the motion of the satellites of Jupiter and Saturn are stopped, while all the rest move without interruption. There is also a brass lamp, having two convex glasses to put in the place of the sun; and also a smaller earth and moon made somewhat in proportion to their distance from each other, and which may be put on or removed at pleasure. The lamp turns round at the same time with the earth, and the glasses of it cast a strong light upon her. When it is intended to use the machine, the planets must be first placed each in its respective position by means of an astronomical ephemeris, and a black patch or wafer may be placed on the middle of the sun; against the first degree of ♈ (Aries); patches may also be placed upon Venus, Mars, and Jupiter. Now turn the handle, one revolution of which corresponds to one diurnal revolution of the earth about its axis, and consequently answers to 24 hours upon the dial plate, placed at the foot of the wire on which the ball is fixed.

Again, when the index has moved over the space of 10 hours, Jupiter will have made one revolution on his axis, and so of the rest according to their respective periods of diurnal rotation. By these means the revolution of the planets, and their motion round their axis, will be represented to the eye, if not exactly, yet in nearly their due intervals of time.

48. We might have entered into the description of orreries on other and more correct principles than the above, but the explanation must have been proportionally longer; and we have already observed, that such machines are, in our opinion, rather calculated to show the ingenuity of their constructors, than to offer any advantages to the student. It is true, that they may convey to the uninformed reader some ideas of the planetary motions, but we think it is extremely probable, that the idea thus given, if not actually false, may, in many cases, be rather injurious than useful; and as instruments for computation, the most perfect of them are wholly incompetent, we shall therefore make no apology for not having extended our description of orreries to a greater length. What has been said, and a reference to the plate, will be quite sufficient for showing the general principle of their construction and operation, which, we conceive, is all that is requisite to be introduced in this place.

Cometarium.

Cometarium. 49. This machine, which must also be considered rather as an object of curiosity than utility, shows the motion of a comet, or very eccentric body, moving round the sun, and describing equal areas in equal times; and may be so adjusted, as to show such a motion for any degree of eccentricity. The first projection of it we owe to Desaguliers.

Fig. 13, 14. The dark elliptical groove, *a b c d* &c. (fig. 13,) is the orbit of the comet Y; which is carried round in this groove according to the order of those letters, by the wire W fixed to the sun S, and slides on the wire as it approaches nearer or recedes further from, the sun; being nearest, or in its perihelion in *a*, and most distant in the aphelion *g*. The areas *a* S *b*, *b* S *c*, *c* S *d*, &c. or the contents of these several trilaterals, are all equal; and in every turn of the winch, or handle, N, the comet Y is carried over one of these spaces; consequently, in the same time as it moves from *f* to *g*, or from *g* to *h*, it will also move from *m* to *a*, or from *a* to *b*, and so of the rest, its motion being quickest at *a*, and slowest at *g*. Thus the comet's velocity in its orbit continually decreases from the perihelion to the aphelion, and increases in the same proportion from *g* to *a*.

The ecliptic orbit is divided into 12 equal parts or signs, with their respective degrees, as is also the circle *n o p q*, &c., which represents a great circle in the heavens, and to which the comet's motion is referred by a small knob on the point of the wire W. While the comet moves from *f* to *g* in its orbit, it appears to move only about five degrees in this circle, as is shown by the small knob on the end of the wire W; but in as short a time as the comet moves from *m* to *a*, or from *a* to *b*, it appears to describe the large space in the heavens *t n*, or *n o*, either of which spaces contains 120°, or four signs. If the eccentricity of the orbit were greater, the greater also would be the difference in the cometary motion.

The circular orbit ABC, &c. is for showing the equable motion of a body about the sun S, describing equal areas in equal times, with those of a body Y in its elliptic orbit above referred to, but with this difference, that the circular areas ASB, BSC, &c. or the equal arcs AB, BC, &c. are described in the same times as the unequal elliptic arcs *a b*, *b c*, &c.

If we conceive the two bodies Y and R, to move from the points *a*, A, at the same moment of time, and each to go round its respective orbit, and to arrive at the same points again at the same instant, the body Y will be more forward in its orbit than the body R all the way from *a* to *g*, and from A to G: but R will be forwarder than Y through all the other half of the orbit; and the difference is equal to the equation of the body Y in its orbit. At the points *a* A, and *g* G, that is, in the perihelion and aphelion they will be equal; and then the equation vanishes. This shows why the equation of a body moving in an elliptic orbit, is added to the mean or supposed circular motion from the perihelion to the aphelion, and subtracted from the aphelion to the perihelion, in bodies moving round the sun, or from the perigee to the apogee, and from the apogee to the perigee in the moon's motion round the earth.

This motion is performed in the following manner by the machine, (fig. 15.) ABC is a wooden bar (in the box containing the wheel-work), above which are the wheels D and E, and below it the elliptic plates FF and GG; each plate being fixed on an axis in one of its foci, at E and K; and the wheel E is fixed on the same axis with the plate FF. These plates have grooves round their edges precisely of equal diameters to one another, and in these grooves is the cat-gut string *g g*, *g g* crossing between the plates at *h*. On H, the axis of the handle or winch N in fig. 13, is an endless screw in fig. 15, working in the wheels D and E, whose numbers of teeth being equal, and equal to the number of lines *a* S, *b* S, *c* S, &c. in fig. 14, they turn round their axis in equal times to one another, as do likewise the elliptic plates. For, the wheels D and E having equal numbers of teeth, the plate FF being fixed on the same axis with the wheel E, and turning the plate GG of equal size by a cat-gut string round them both, they must all go round their axis in as many turns of the handle N as either of the wheels has teeth. Fig. 15.

It is easy to see, that the end *h* of the elliptical plate FF being farther from its axis E than the opposite end I is, must describe a circle so much the larger in proportion, and must therefore move through so much more space in the same time; and for that reason the end *h* moves so much faster than the end I, although it goes no sooner round the centre E: at the same time the quick-moving end *h* of the plate FF leads about the short end *h* K of the plate GG with the same velocity; and the slow-moving end I of the plate FF coming half round as to B, must then lead the long end *k* of the plate GG about, with a corresponding slow motion: so that the elliptical plate FF and its axis E move uniformly and equally quick in every part of its revolution; but the elliptical plate GG, together with its axis K, must move very unequally in different parts of its revolution; the difference being always inversely as the distance of any point of the circumference of GG from its axis at K; or in other words, if the distance K *k*, be four, five, or six times as great as the distance K *h*, the point *h* will move in that position, four, five, or six times as fast as the point *k* does when the plate GG has gone half round; and so on for any other eccentricity or difference of the distances K *k*, K *h*. The I on the plate EF, falls in between the two teeth at *k* on the plate GG; by which means the revolution of the latter is adjusted to that of the former, so that they can never vary the one from the other.

On the top of the axis of the equally moving wheel D (fig. 15) is the sun S (fig. 14) which by means of the wire attached to it, carries the ball R round the circle, ABC, &c. with an equable motion, according to the order of the letters; and on the top of the axis K of the unequally moving ellipse GG in (fig. 15) is the sun S (fig. 14) carrying the ball Y unequally round the elliptic groove *a b c d*, &c. which elliptic groove must be exactly equal and similar to the verge of the plate GG, which again is also equal to that of EF.

Eclipsarean.

Eclipsarean

Fig. 16, 17.

50. The eclipsarean is an instrument invented by Mr. Ferguson for exhibiting the time, quantity, duration, and progress of solar eclipses, in all parts of the earth. This machine consists of a terrestrial globe A, (fig. 17) turned by a winch M, round its axis B, inclining $23\frac{1}{2}$°, and carrying an index round the hour circle D; a circular plate E, on which the months and days of the year are inserted, and which supports the globe in such a manner, that when the given day of the month is turned to the annual index G, the axis has the same position with the earth's axis at that time; a crooked wire F, which points to the middle of the earth's enlightened disc, and shows to what place of the earth the sun is vertical at any given time; a penumbra or thin circular plate of brass I, divided into twelve digits by twelve concentric circles and so proportioned to the size of the globe, that its shadow, formed by the sun, or a candle, placed at a convenient distance, with its rays transmitted through a convex lens, to make them fall parallel on the globe, may cover those parts of the globe which the shadow and penumbra of the moon cover on the earth; an upright frame HHHH, on the sides of which are scales of the moon's latitude, with two sliders K and K, fitted to them, by means of which the centre of the penumbra may be always adjusted to the moon's latitude; a solar horizon C, dividing the enlightened from the darkened hemisphere, and showing the places where the general eclipse begins and ends with the rising or setting sun, and a handle M, which turns the globe round its axis by the wheel work, and moves the penumbra across the frames by threads over the pulleys LLL, with a velocity duly proportioned to that of the moon's shadow over the earth as the earth turns round its axis.

Rectification.

51. If the moon's latitude at any conjunction exceeds the number of divisions on the scales, there can be no eclipse; if not, the sun will be eclipsed to some parts of the earth; the appearance of which may be represented by the machine, either with the light of the sun, or of a candle. For this purpose, let the indexes of the sliders KK, point to the moon's latitude, and let the plate E be turned till the day of the given new moon comes to G, and the penumbra be moved till its centre comes to the perpendicular thread in the middle of the frame, which thread represents the axis of the ecliptic; then turn the handle till the meridian of London on the globe comes under the point of the wire F, and turn the hour circle D till 12 at noon comes to its index; also turn the handle till the hour index points to the time of new moon in the circle D, and then screw fast the collar N. Lastly, elevate the machine till the sun shines through the sight holes in the small upright plates OO, on the pedestal, or place a candle before the machine, at the distance of about four yards, so that the shadow of the intersection of the cross thread in the middle of the frame, may fall precisely on that part of the globe to which the wire F points; with a pair of compasses take the distance between the centre of the penumbra and the intersection of the threads, and set the candle higher or lower, according to that distance; and place a large convex lens between the machine and candle, so that the candle may be in the focus of the lens; and thus the machine is rectified for use.

Phenomena shown by the machine.

52. Let the candle be turned backward till the penumbra almost touches the side, HF, of the frame, and then turning it forward, the following phenomena may be observed.

1. Where the eastern edge of the shadow of the penumbral plate I, first touches the globe at the solar horizon, those who inhabit the corresponding part of the earth, see the eclipse begin on the uppermost edge of the sun, just at the time of its rising.

2. In that place where the penumbra's centre first touches the globe, the inhabitants have the sun rising upon them centrally eclipsed.

3. When the whole penumbra just falls upon the globe, its western edge at the solar horizon touches, and leaves the place where the eclipse ends at sunrise on his lowermost edge.

4. By continued turning, the cross lines in the centre of the penumbra will go over all those places on the globe where the sun is centrally eclipsed.

5. When the eastern edge of the shadow touches any place of the globe, the eclipse begins there; when the vertical line in the penumbra comes to any place, then is the greatest obscuration at that place; and when the western edge of the penumbra leaves the place, the eclipse ends there, and the times are shown on the hour circle; and from the beginning to the end, the shadows of the concentric penumbral circles show the number of digits eclipsed at all the intermediate times.

6. When the eastern edge of the penumbra leaves

the globe at the solar horizon C, the inhabitants see the sun beginning to be eclipsed on its lowermost edge, at its setting.

7. Where the centre of the penumbra leaves the globe, the inhabitants see the sun centrally eclipsed; and lastly, where the penumbra is wholly departing from the globe, the inhabitants see the eclipse ending on the uppermost part of the sun's edge, at the time of its disappearing in the horizon.

This instrument will likewise serve for exhibiting the time of sun rising and setting; and of morning and evening twilight, as well as the places to which the sun is vertical on any day, by setting the day on the plate E to the index G, turning the handle till the meridian of the place comes under the point of the crooked wire F, and bringing XII on the hour circle D to the index: then if the globe be turned till the place touches the eastern edge of the horizon C, the index shows the time of sun setting; and when the place comes out from below the other edge of C, the index shows the time when evening twilight ends; morning twilight and sun rising are shown in the same manner on the other side of the globe. And the places under the point of the wire F are those to which the sun passes vertically on that day. *Ferguson's Astronomy*, by Brewster, or *Phil. Trans.* vol. xlviii.

53. The celestial and terrestrial globes may also be considered as astronomical machines of the kind we have been describing; but it would too much interrupt the order of our treatise to enter upon a description of these instruments in this place; they will, therefore be described under the proper head in our alphabetical arrangement; and we shall now proceed, having given the foregoing succinct view of the more popular celestial phenomena, to treat the subject under a more scientific point of view, in the following sections.

PART II.

PLANE ASTRONOMY.

§ III. *Containing the principles of astronomical computation.*

1. *Definitions.*

Of the sphere definitions.

54. Previous to our entering upon this subject, it will be requisite for the reader to render himself familiar with the following definitions. Some of them have been already given, but the convenience of having one decided place of reference will compensate for the few repetitions that occur.

A great circle. Fig. 18.

1. A *great circle* of a sphere is any circle QRST (fig. 18) whose plane passes through the centre of the sphere; and a *small circle* is any circle, BHK, whose plane does not pass through the centre. All great circles bisect each other.

A diameter

2. The *diameter* of a sphere is any line, PE, passing through the centre and terminated on both sides by the circumference; this diameter is said to be the *axis* of that great circle to which it is perpendicular; and the extremity of the axis PE are called the poles of that circle.

Hence it follows, that the pole of a great circle is 90° distant from every point of it upon the sphere; and that the arcs subtending any angles at the centre of a sphere are those of great circles. Consequently, all the triangles formed on the surface of a sphere for the solution of spherical problems must be formed by the arcs of great circles.

Secondaries.

3. *Secondaries* to a great circle are great circles, as PQE, PRE, which pass through its poles, and whose planes are therefore perpendicular to the plane of the latter. Hence every secondary bisects its great circle; a secondary also bisects every small circle that is parallel to the great circle to which it is secondary. Since every secondary passes through the pole of its great circle, and is perpendicular to it; it follows, that if a secondary passes through the poles of two great circles, it is perpendicular to each of them. And conversely, if one circle be perpendicular to two others, it must pass through their poles.

Poles of the celestial and terrestrial sphere.

55. The above definitions belong wholly to the sphere considered abstractedly as a geometrical solid; the following appertain to the sphere considered with reference to astronomy.

Fig. 19.

1. Let pep' of (fig. 19) represent the earth which at present we shall consider as a perfect sphere, and let pp' be the line about which it performs its diurnal rotation; then pp' are called its *poles*, and the line pp' its *axis*. And if we assume the circle PEP′Q to denote the circle of the celestial sphere, and conceive pp' to be produced to the heavens meeting them in PP′, these will be the *poles* of the celestial sphere.

2. The *terrestrial equator* is a great circle $erqs$, of the earth perpendicular to its axis; and if we conceive the plane of this circle to be produced to the sphere of the fixed stars, it will mark out the great circle ERQS, which is called the *celestial equator*.

Hence it follows, that the poles of the terrestrial and celestial spheres are the same as the poles of the respective equators.

Equator.

The equator divides either spheres into two equal portions, called the *northern* and *southern hemispheres*, and the corresponding poles are in like manner denominated the *north* and *south* poles. The northern hemisphere is the part of the earth which lies on the side of the equator which we inhabit, and which in the figure we may assume to be epq.

Latitude terrestrial.

2. The *latitude* of a place on the earth's surface is its angular distance from the equator, measured upon a secondary to it; thus the arc eb, measures the latitude of the point b. Any circle on which we measure the latitude of a place is called a *terrestrial meridian;* and when produced to the heavens a *celestial meridian*.

Meridian.

3. The small circles parallel to the terrestrial equator are called *parallels of latitude*.

4. The secondaries to the celestial equator are called circles of *declination;* and the small circles parallel to the equator on the earth's surface parallels of *declination*. Declination, therefore, in the celestial sphere, corresponds to latitude on the terrestrial sphere, thus the arc eb which measures the latitude of a place on the earth, corresponds to EZ, the declination in the heavens.

Longitude terrestrial

5. The longitude of a place on the earth's surface is an arc of the equator, intercepted between the meridian passing through the place, and another called a first meridian, passing through that place from which you begin to measure; which latter is different in different countries. Most nations account their first

meridian that passing over their capital. The English, for example, take the meridian of London, or rather that of the Royal Observatory, Greenwich; the French, that of Paris, &c.

Horizon. 6. If *b* be supposed to denote any place on the earth, and a tangent plane be supposed to be drawn to that place, and produced to the heavens, meeting them in the points *a c*, the circle *a b c*, which is here projected into a right line, is called the *sensible horizon.*

And the great circle HOR, which is drawn parallel to it, passing through the centre of the earth O, is called the *rational horizon.* It is in the former of those circles, which all the heavenly bodies are observed to rise and set.

Small circles parallel to the horizon are called *almucantars.*

Zenith and nadir. 7. If the radius O *b* of the earth at the place *b* of a spectator be produced both ways to the heavens, that point Z vertical to him is called the *zenith,* and the opposite point N the *nadir.* Consequently, the zenith and nadir are the poles of the rational horizon.

Vertical circles. 8. *Vertical circles* are those secondaries which are perpendicular to the horizon, and which therefore pass through the zenith: it is in these circles the altitude of the heavenly bodies are taken. The celestial meridian of a place is therefore a vertical circle passing through the pole and zenith of that place; as PEHP′ in the figure above referred to.

The two points in the horizon R H, which are cut by the meridian of any place, are called the *north* and *south* points, according as they are towards the north or south poles.

Prime vertical. 9. That vertical circle which cuts the meridian of any place at right angles, dividing it into two equal hemispheres, and which cuts the meridian in the east and west points, is called the *prime vertical,* as ZN, which is projected into the right line ZN.

Azimuth and amplitude. 10. When a body is referred to the horizon by a vertical circle, the distance of that point of the horizon from the north or south points, is called the *azimuth,* and its distance from the east or west points its amplitude.

Ecliptic. 11. The *ecliptic* is that great circle of the heavens which the sun appears to describe in the course of the year.

Obliquity of the ecliptic. 12. The angle which the ecliptic forms with the celestial equator, is called the *obliquity of the ecliptic;* and the two points in which these circles intersect and bisect each other, are called the *equinoctial points.* The times when the sun comes to these points are called the *equinoxes.*

For the signs, order, and characters of the twelve signs of the ecliptic, or zodiac, see our table of constellations, p. 506.

Of these signs, the first six which lie on the northern side of the equator, are called northern signs; viz. Aries, Taurus, Gemini, Cancer, Leo, Virgo: and the other six, Libra, Scorpio, Sagittarius, Capricornus, Aquarius, Pisces, the southern signs.

The equinoctial points correspond to the first points of Aries and Libra.

The six signs, Capricorn, Aquarius, Pisces, Aries, Taurus, and Gemini, are called ascending signs, the sun approaching our's, or the north pole, while it passes through them; the others, Cancer, Leo, &c., are, for a corresponding reason, called descending signs; the sun, while passing through them, being receding from our pole.

Zodiac. 14. The *zodiac* is a zone of the celestial sphere, extending 8° on each side of the ecliptic, within which the motion of all the principal planets are performed.

The signs of the ecliptic and of the zodiac are the same.

Direct and retrograde. 15. When any of the heavenly bodies appear to move according to the order of the signs, viz. through Aries, Taurus, &c. their motion is said to be *direct,* or *in consequentia;* when contrary to that order, *retrograde,* or *in antecedentia.* (See art. 29)

The real motion of all the planets is according to the order of the signs.

Right ascension. 16. We have seen, that the declination of any heavenly body is measured by the arc of a declination circle, or secondary to the equator, and the distance of that secondary on the equator from the first point of Aries, estimated according to the order of the signs, is called its right ascension.

Hence the right ascension of a heavenly body corresponds with the longitude of a terrestrial one, except as to the point whence it is measured.

Latitude and longitude in the heavens. Fig. 20. 17. The *latitude* of a heavenly body is measured by a secondary to the ecliptic passing through that body; that is, by the angular distance between it and the ecliptic, as by the arc *m s*, fig. 20: and the *longitude* is measured by the arc of the ecliptic, intercepted between the first point of Aries and the point where the secondary meets the ecliptic, estimated according to the order of the signs, as *o m*, considering *o* as the first point of Aries.

Hence the latitude and longitude of a heavenly body are the same with reference to the ecliptic and its secondaries; as those of a terrestrial body or place with reference to the equator and meridians.

Oblique ascension. 18. The *oblique ascension* is an arc of the equator, intercepted between the first point of Aries, and that point of the equator which rises with any body; and the difference between the right and oblique ascension is called the *ascensional difference.*

Tropics. 19. The tropics are two celestial circles parallel to the equator, and touching the ecliptic; the one at the beginning of Cancer, called the tropic of Cancer; the other at the beginning of Capricorn, called the tropic of Capricorn. The two points where the tropics touch the ecliptic, are called the *solstitial* points.

Arctic and antarctic circles. 20. The *arctic* and *antarctic* circles are two parallels of declination; the former about the north, and the latter about the south pole: the distance of each from the pole is equal to the distance of the tropics from the equator.

Zones. These two circles, and those of the tropics, when referred to the earth, divides it into five zones; two called the *frigid zones,* which are those towards the two poles; the temperate zones being those between each tropic and its corresponding polar circle; and one torrid zone, including all the space between the two tropics, extending therefore about 23½° on each side of the equator.

Nodes. 21. The *nodes* are the points where the orbit of a planet cuts the plane of the ecliptic; and the nodes of a satellite, are the points where its orbit cuts the plane of the orbit of its primary, or that about which it revolves.

The *ascending* node is that where the body passes from the south to the north side of the ecliptic, and the other is called the *descending* node.

22. The *aphelion* is that point of the orbit of a planet which is farthest from the sun; and the *perihelion* that point where it is nearest.

The *apogee* and *perigee* have the same signification with reference to the earth. The moon, for instance, is said to be in apogee when farthest from us; and in perigee, when nearest.

The above definitions will be sufficient for our present purpose; others, which require a previous knowledge of certain subjects not yet discussed, are reserved for those places in the course of the subsequent sections in which they naturally occur.

2. *Illustration of certain celestial phenomena.*

General remarks on the celestial sphere.

56. Let us now proceed to show the application of the doctrine of the sphere, to the illustration of certain celestial phenomena, as the rising, setting, southing, &c. of the heavenly bodies.

57. In art. 55. we have defined the sensible and rational horizon; but with reference to the sphere of the fixed stars, these may be considered as coinciding, the angle which the arc H a (fig. 19) subtends at the earth, becoming then insensible in consequence of the immense distance of these bodies. Now, if we suppose, as we have hitherto done, the earth to revolve daily about its axis, all the heavenly bodies must successively appear to rise and set, or revolve about the pole, in circles, whose planes are perpendicular to the earth's axis, and consequently parallel to each other; and will, to every appearance, be the same as if the spectator were at rest in the centre of a concave sphere, which revolved uniformly about him; or that the stars each revolved in parallel circles on such a sphere. We may therefore consider the earth but as a point with reference to the radius of the sphere of the fixed stars, and leave it out of the consideration in our farther inquiries upon this subject, and only employ the zenith, equator, poles, horizon, &c. of the celestial sphere, and such circles of declinations, as corresponding with the motion of the given bodies.

Oblique sphere constructions. Fig. 21.

58. Let then fig. 21. represent the position of the heavens to an observer, whose zenith is Z in north latitude; EQ the equator, PP′ the poles, HOR the rational horizon, PZHP′R the meridian of the spectator.

Draw the great circle ZON perpendicular to the meridian, and passing through the zenith Z, which from our definition will be the prime vertical; and being in the plane of the eye, this being supposed to be perpendicular over the pole of the meridian, will be projected into a right line ZN, as will be shown in our treatise on Projection and Perspective. The same will also be the case with the equator EQ, the horizon HR, and the great circle POP′, supposed to be drawn perpendicular to the meridian; the common intersection of all these circles being in the point O, the pole of the meridian.

Draw the small circles, or parallels of declinations, w H, $m\,t$, $a\,e$, Rv, $y\,x$, which will represent the circles described by any of the heavenly bodies; and as the great circle POP′ bisects the equator, it will bisect all the small circles parallel to it, consequently $m\,t$, $a\,e$, are bisected in r and c; and we shall have $a\,c = c\,e$, and $m\,t = r\,t$, each equal to a quadrant, or 90°.

Now if we conceive the figure referred to as exhibiting the eastern hemisphere, the several arcs QE, $a\,e$, $t\,m$, &c. will represent the paths of bodies placed at those distances from the pole, as they ascend from the meridian under the horizon to the meridian above; and the points b, O, s, will be the places where they rise, or begin to appear above the horizon; and t E e, the points where they attain their greatest or meridian altitude; as $a\,e$, QE, $m\,t$, are bisected in c, O, r; $e\,b$ must be greater than $b\,a$; QO, equal to OE, and $t\,s$ less than $s\,m$.

Whence it follows, that a body on the same side of the equator as the spectator, will be longer above the horizon than below it; when the body is in the equator it will be as long below as above the horizon; and when it is on the contrary side of the equator to the spectator, it will be longer below the equator than above it; for the arc $e\,b$ is greater than $b\,a$, EO=OQ, and $t\,s$ less than $s\,m$, and the motion with which these arcs are described are uniform.

The bodies describing $a\,e$, $m\,t$, rise at b and s; and as O is the east point of the horizon, and H and R the north and south points; a body on the same side of the equator as the spectator rises between the east and the north; and a body on the contrary side, between the east and the south, the spectator being supposed in north latitude; and a body in the equator rises in the east at O.

Circles of perpetual apparition.

59. When bodies come to d and r, they are in the prime vertical, or in the east; hence, a body on the same side of the equator as the spectator comes to the east after it has risen; a body in the equator rises in the east, and one on the contrary side of the equator has passed the east before it rises. The body which describes the circle Rv, or any one nearer to P, never sets; and such circles are called circles of *perpetual apparition*, and the stars which describe them *circumpolar stars*. The body which describes the circle w H, just become visible at H, and then instantly descends below the horizon; but those bodies which describe the circles nearer to P′ are never visible.

Such is the apparent diurnal motion of the heavenly bodies, when the spectator is situated any where between the equator and either poles; and this is called an *oblique* sphere; because all bodies rise and set obliquely to the horizon.

In the above deductions we have supposed the figure to represent the eastern hemisphere, and the bodies to ascend through their respective arcs; but it may be equally supposed to denote the western hemisphere, only in this case, these arcs will represent the paths of the body as they descend from their greatest altitude above the horizon to their meridian below the horizon. And hence, it is obvious, that supposing a body not to change its declination, it will be at equal altitudes at equal times before and after it has attained its meridian altitude.

Right sphere

60. In the preceding article we have supposed the spectator to be in north latitude, or, which is the same, the zenith of the spectator to be between the equator and north pole; and it is obvious, that what we have said would apply equally to a spectator similarly situated in south latitude; but when he is situated either in the equator or in one of the poles,

the considerations become less complicated; in the former case we call it a *right sphere*, and in the latter a parallel sphere. If the spectator be at the equator, then E coincides with Z, and Q with N, as in (fig. 22); consequently also PP' coincides with HR; and the declination circles ea, tm, which are always parallel to the equator, are in this case perpendicular to the horizon; and as these circles are always bisected by PP', they must now be bisected by the horizon HR; hence in this position of the spectator all the heavenly bodies, which change not their declination, will be an equal time above and below the horizon, and will rise perpendicularly to it; whence the denomination of the right sphere.

Fig. 22.

Parallel sphere.

Fig. 23.

On the other hand, if the spectator be situated at the pole, then the sphere will be as represented in fig. 23, that is, we shall have P coincide with Z, and EQ, with HR, or the equator will coincide with the horizon; and all the parallels of declination, ae, mt, described by the heavenly bodies, will be therefore parallel to the horizon; any body, therefore, which is above the horizon, and which changes not its declination, will remain constantly above the horizon, and at the same altitude; and those which are below the horizon will continue constantly below. Consequently, a spectator at the pole would never see the heavenly bodies rise and set, but would observe them to describe circles in the heavens parallel to the horizon; whence the denomination *parallel sphere*.

Of the change of declination.

61. We have had two or three times occasion to use the words, " those heavenly bodies which change not their declination;" it may be here proper to explain to the reader, that by this we mean the fixed stars only, these being the only celestial body that are not subject to a change of declination; and some of these even are liable to such a change, it is however too inconsiderable to be attended to in this place. But the sun, moon, and planets are constantly changing their declination, in consequence of the proper motion of the earth and themselves; let us, therefore, now bestow a few words in explanation of these cases, particularly as regards the sun.

Apparent motion of the sun.

We have already stated in our introduction (art.27) that by attentively observing the stars which set and rise with the sun during the course of the year, that he appears to have described a great circle of the celestial sphere, forming with the ecliptic an angle of about $23\frac{1}{2}$°, or more exactly 23° 28′. This circle is called the ecliptic, and is denoted by the line LC, into which it is projected in the three last figures; which circle cuts the equator, as we have seen, in two points called the equinoctional points. The sun, therefore, during one part of the year, is on one side of the equator, and in the other, on the contrary side; and by this means his rising and setting is subject to all that variety which we have noticed in the stars in the two hemispheres, and hence the cause of the different lengths of the days at different times of the year, the succession of seasons, and the several phenomena attending them, as we have already endeavoured to explain in a popular manner in the preceding introduction.

Change of seasons.

Let us now endeavour to illustrate the same a little more particularly by referring again to our fig. 21. Here since in the course of the year, the sun appears to describe the circle of which LC is the projection, it is obvious that he will be sometimes to the north of the equator, as in q, at others to the south, as in p, and at others in the equator, as at O. In the former case, that is, when he is in q, P denoting the north pole, it is obvious, as we have already remarked respecting any body describing the declination circle ae, that he will rise between the north and east, attain to the prime vertical, after he has risen, and will be longer above the horizon than below it, which is the case in our latitudes, from about the 21st of March to the 22d of September, these being nearly the times when the sun crosses the equator. On those two days he rises in the east, and is an equal time above and below the horizon to every part of the globe except the two poles; and the days and nights being then equal, these points are called the *equinoctial* points, and the sun itself is said to be in the *equinoxes*. The former of these is called the *vernal*, and the latter the *autumnal* equinox. When the sun is on the south side of the equator, as at p, then the same remarks apply as we have already made with respect to any body describing the declination circle, mt, that is, he will rise between the south and the east, and will be longer below than above the horizon, and our days will be shorter than our nights, as is the case from the autumnal to the vernal equinox, that is, from about the 22d of September to the 21st of March.

When the spectator is in the equator, then the sphere being right (see fig. 23), the sun will be always an equal time above and below the horizon; and when the sun is also in the equator, he will rise east and describe a great circle corresponding with the prime vertical, and will be vertical over the head of the spectator in the middle of his course: at other times he will rise between the north and the east or the south and the east, according as his declination is north or south. There is, therefore, even in these regions, a change of seasons; but as the sun will dart his vertical beams upon every point of the equator, twice in the course of one revolution, the inhabitants may be said to have two summers and two winters in the course of a year. When the spectator is at the pole, the sphere will be parallel (see fig. 23), and the sun from the vernal to the autumnal equinox, will be constantly visible to the north pole, and perpetually hidden below the horizon during the other half of the year; and the contrary for the south pole. That is, in the former, he will be visible from the time he passes from O to L, and from L to O, and be invisible while he is describing the other half of the ecliptic. At each of the poles, therefore, the days and nights are each half a year in length. It must not, however, be understood here, that the length of the days and nights are each exactly equal to half a year, for we shall see hereafter, that the sun is not so long on the northern as on the southern side of the equator, the cause of which will likewise be illustrated in a subsequent chapter. At present, it will be sufficient to observe, that such is known to be the case from observation.

Zones described.

62. We have seen in our definitions that the earth is considered as divided into five zones, by referring to the earth the two polar circles, and the two tropics. Now from what has been above stated, it is obvious, that to an observer in either hemisphere, in the latitude of 23° 28′, his zenith will coincide with the sun's

place at noon, on that day when it has attained its greatest north declination, if the observer be in north latitude; or south, if he be situated in south latitude; consequently, in either situation, on one day in a year the sun will be vertical to the inhabitants of either tropic, and of course to all places situated between them, except those on the equator, who, as we have seen, have the sun vertical twice.

Beyond the tropics, either to north or south, the sun is never vertical, the zenith of all such places being farther from the equator than the extreme declination of the sun or the obliquity of the ecliptic.

Those who inhabit the two polar circles will have one day and one night of twenty-four hours, or there will be one day in each of those circles when the sun will not set, another on which he will not rise, as will be immediately obvious by referring to the preceding figures.

From these circles to the poles themselves the sun will be for a greater or less time above and below the horizon, till in the actual poles, the nights and days will be half a year each, as we have already stated.

Synopsis of spherical trigonometry.

3. *Synopsis of spherical trigonometry.*

62. As in the course of the following articles, we shall have frequent occasion to refer to the several cases of spherical trigonometry, we conceive that it will be very convenient for the reader to have a general synoptic table of all the principal results and theorems belonging to this doctrine, the investigations of which will be given in our Treatise on Trigonometry, Part I. We collect them here merely for the convenience of reference, and have chosen those only which are most general in their application. They are sufficient for the solution of any spherical problem, although, in certain cases, they may not offer the most expeditious mode of solution; we shall not therefore uniformly adopt the formulæ given in the table, but when a more expeditious form presents itself, we shall avail ourselves of it; in the greater number of cases, however, our solutions will be deduced from the tabula formulæ.

TABLE I. *For the solution of all the cases of right angled spherical triangles.*

Given.	Required.	Value of the Terms required.	Cases in which the terms required are less than 90°.
I. Hypothenuse and one leg.	Angle opposite the given leg.	Its sin $= \frac{\text{sin given leg}}{\text{sin hypoth.}}$	If the given leg be less than 90°.
	Angle adjacent to the given leg.	Its cos $= \frac{\text{tan given leg}}{\text{tan hypoth}}$	If the things given be of the same affection*.
	Other leg.	Its cos $= \frac{\text{cos hypoth.}}{\text{cos given leg}}$	Idem.
II. One leg and its opposite angle.	Hypothenuse.	Its sin $= \frac{\text{sin given leg}}{\text{sin given ang.}}$	Ambiguous.
	Other leg.	Its sin $= \frac{\text{tan given leg}}{\text{tan given ang.}}$	Idem.
	Other angle.	Its sin $= \frac{\text{cos given ang.}}{\text{cos given leg}}$	Idem.
III. One leg, and the adjacent angle.	Hypothenuse.	Its tan $= \frac{\text{tan given leg}}{\text{cos given ang.}}$	If the things given be of like affection.
	Other angle.	Its cos = cos giv. leg × sin giv. ang.	If the given leg be less than 90°.
	Other leg.	Its tan = sin giv. leg tan giv. ang.	If the given angle be less than 90°.
IV. Hypothenuse and one angle.	Adjacent leg.	Its tan = tan hyp. × cos giv. ang.	If the things given be of like affection.
	Leg opposite to the given angle.	Its sin = sin hyp. × sin giv. ang.	If the given angle be acute.
	Other angle.	Its tan $= \frac{\text{cot giv. angle}}{\text{cos hypothen}}$	If the things given be of like affection.
V. The two legs.	Hypothenuse.	Its cos = rectan. cos given legs	If the given legs be of like affection.
	Either of the angles.	Its tan $= \frac{\text{tan opposite leg}}{\text{sin adjacent leg}}$	If the opposite leg be less than 90°.
VI. The two angles.	Hypothenuse.	Its cos = rect. cot given angles	If the angles be of like affection.
	Either of the legs.	Its cos $= \frac{\text{cos opposite angles}}{\text{sin adjacent angle}}$	If the opposite be acute.

* Angles or sides are of the same affection when they are both greater or both less than a right angle; and in the third column of our Table we have stated when the result is of the same affection with the things given, and when it is ambiguous.

In working by logarithms the reader must observe, that when the resulting logarithm is the log. of a quotient, 10 must be added to the index; and when it is the log. of a product, 10 must be subtracted from the index. Thus, when the two angles are given,

log. cos hypothenuse = log. cos one angle + log. cos other angle − 10.

log. cos either leg = log. cos opp. angle − log. sin adjac. angle + 10.

Quadrantal triangles.

In a quadrantal triangle if the quadrantal side be called radius, the supplement to the angle opposite to that side be called hypothenuse, the other sides be called angles, and their opposite angles be called legs; then the solution of all the cases will be as in the above table for right angled spherical triangles.

NAPIER'S *analogies for right angled spherical triangles.*

Napier's circular parts are, the complements of the two angles that are not right angles, the complement of the hypothenuse and the other two sides; that is, denoting the sides by a, b, c, and the angles by A, B, C, A being the right angle, the parts are $90°-a$, $90°-$B, $90°-$C, and b, c; of these, any one may be the middle part, and the two parts next adjacent, one to either hand (not including the right angle) the adjacent parts; and the other two the opposite parts; then the analogies are

rad × sin middle part = rectangle of the tangents of adjacent parts.

rad × sin middle part = rectangle of the cosines of the opposite parts.

And by making the transformations above explained, these will also apply to quadrantal spherical triangles.

TABLE II. *For the solution of oblique angled spherical triangles.*

Given.	Required.	Values of the quantities required.	
I. Two angles and a side opposite to one of them.	Side opposite to other angle.	By common analogy	Sines of angles, are as sines of opp. sides.
	Third side.	Let fall a perpendicular upon the side contained between the given angles.	Tan 1 seg. of this side = cos adj. angle × tan given side. Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{tan ang. adj. given side}}{\text{tan ang. opp. given side}}$.
	Third angle.	Let fall a per. as before	Cot 1 seg. of this ang. = cos giv. side × tan adj. ang. Sin 2. seg. × $\frac{\text{sin 1 seg.} \times \text{cos ang. opp. given side}}{\text{cos ang. adj. given side}}$.
II. Two sides and an angle opposite to one of them.	The angle opposite to the other side.	By the common analogy.	Sines of sides are as sines of their opposite angles.
	Angle included between the given sides.	Let fall a perpendicular from the included angle.	Cot 1 seg. ang. req. = tan giv. ang. × cos adj. side. Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{tan giv. side adj. giv. ang.}}{\text{tan side opp. given angle}}$. Tan 1 seg. side req. = cos giv. ang. × tan adj. side.
	Third side.	Let fall a perpendicular as before.	Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{cos side opp. giv. ang.}}{\text{cos side adj. given angle}}$.
III. Two sides and the included angle.	An angle opposite to one of the given sides.	Let fall a perpendicular from the third angle.	Tan 1 seg. of div. side = cos giv. ang. × tan side opp. ang. sought. Tan ang. sought = $\frac{\text{tan giv. ang.} \times \text{sin 1 seg.}}{\text{sin 2 seg. of div. side}}$. Tan 1 seg. of div. side = cos giv. ang. × tan other given side.
	Third side.	Let fall a perpendicular on one of the given sides.	Cos side sought = $\frac{\text{cos side not div.} \times \text{cos 2 seg.}}{\text{cos 1 seg. of side divided}}$.
IV. A side and the two adjacent angles.	A side opposite to one of the given angles.	Let fall a perpendicular on the third side.	Cot 1 seg. of div. ang. = cos giv. side × tan ang. opp. side sought. Tan side sought = $\frac{\text{tan giv. side} \times \text{cos 1 seg. div. ang.}}{\text{cos 2 seg. of divided angle}}$.
	Third angle.	Let fall a perpendicular from one of the given angles.	Cot 1 seg. div. ang. = cos given side × tan other giv. angle. Cos angle sought = $\frac{\text{cos ang. not div.} \times \text{sin 2 seg.}}{\text{sin 1 seg. div. angle}}$.

TABLE II.—*Continued.*

Given.	Required.	Value of the quantities required.
V. The three sides.	An angle by the sine or cosine of its half.	Let a, b, c, be the three sides; A, B, C, the angles; b and c, including the angles sought, and $s = a + b + c$. Then, $\sin\frac{1}{2}A = \sqrt{\frac{\sin.\,(\frac{1}{2}s - b)\,\sin\,(\frac{1}{2}s - c)}{\sin b\,\sin c.}}$ $\cos\frac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}s.\,\sin\,(\frac{1}{2}s - a)}{\sin b.\,\sin c.}}$
VI. The three angles.	A side by the sine or cosine of its half.	Let S be the sum of the angles A, B, C, and B and C be adjacent to a, the side required; then, $\sin\frac{1}{2}A = \sqrt{-\frac{\cos\frac{1}{2}S\,\cos\,(\frac{1}{2}S - A)}{\sin B.\,\sin C.}}$ $\cos\frac{1}{2}A = \sqrt{-\frac{\sin\,(\frac{1}{2}S - B)\,\sin\,(\frac{1}{2}S - A)}{\sin B.\,\sin C.}}$

TABLE III.

For the solution of all the cases of oblique angled spherical triangles, by the analogies of Napier.

Given.	Required.	Value of the terms required.
I. Two angles and the sides opposite to one of them.	Side opposite to the other given angle	By the common analogy; viz. The sines of the angles are as the sines of the opposite sides
	Third side.	tan of its half $\begin{cases} = \dfrac{\tan\frac{1}{2}\text{ dif. giv. sides} \times \sin\frac{1}{2}\text{ sum opp. ang.}}{\sin\frac{1}{2}\text{ dif. of those angles.}} \\ = \dfrac{\tan\frac{1}{2}\text{ sum giv. sides} \times \cos\frac{1}{2}\text{ sum opp. ang.}}{\cos\frac{1}{2}\text{ dif. those angles.}} \end{cases}$
	Third angle.	By the common analogy.
II. Two sides and an opposite angle.	Angle opposite to the other known side.	By the common analogy.
	Third angle.	cot of its half $\begin{cases} = \dfrac{\tan\frac{1}{2}\text{ dif. other two ang.} \times \sin\frac{1}{2}\text{ sum giv. sides}}{\sin\frac{1}{2}\text{ dif. those sides.}} \\ = \dfrac{\tan\frac{1}{2}\text{ sum other two ang.} \times \cos\frac{1}{2}\text{ sum giv. sides}}{\cos\frac{1}{2}\text{ dif. those sides.}} \end{cases}$
	Third side.	By common analogy.
III. Two sides and the included angle.	The two other angles.	$\tan\frac{1}{2}\text{ dif.} = \dfrac{\cot\frac{1}{2}\text{ giv. angle} \times \sin\frac{1}{2}\text{ dif. given sides}}{\sin\frac{1}{2}\text{ sum of those sides.}}$ $\tan\frac{1}{2}\text{ sum.} = \dfrac{\cot\frac{1}{2}\text{ giv. angle} \times \cos\frac{1}{2}\text{ dif. given sides}}{\cos\frac{1}{2}\text{ sum of those sides.}}$
	Third side.	By the common analogy.
IV. Two angles and the included side.	The other two sides.	$\tan\frac{1}{2}\text{ dif.} = \dfrac{\tan\frac{1}{2}\text{ giv. side} \times \sin\frac{1}{2}\text{ dif. giv. angle}}{\sin\frac{1}{2}\text{ sum of those angles}}$ $\tan\frac{1}{2}\text{ sum} = \dfrac{\tan\frac{1}{2}\text{ giv. side} \times \cos\frac{1}{2}\text{ dif. giv. angle}}{\cos\frac{1}{2}\text{ sum of those angles.}}$
	Third angle.	By common analogy.
V. The three sides.	Either of the angles.	Let fall a perpendicular on the side adjacent to the angle sought; then, $\left\{\begin{matrix}\tan\frac{1}{2}\text{ sum or }\frac{1}{2}\text{ dif. of the seg-} \\ \text{ments of the base}\end{matrix}\right\} = \dfrac{\tan\frac{1}{2}\text{ sum} \times \tan\frac{1}{2}\text{ dif. of the sides}}{\tan\frac{1}{2}\text{ base.}}$ Cos angle sought = tan adj. seg. × cot adj. sides.
VI. The three angles.	Either of the sides.	These will be determined, by finding the corresponding angle, by the last case, of a triangle, which has all its parts supplemental to those of the triangles, whose three sides are given.

4. *Problems relative to the determination of the position of the heavenly bodies.*

Rising and setting of the heavenly bodies.

63. It is obvious from what we have stated relative to the position of the different circles of the sphere, that one of the most important data in the solution of astronomical problems is the latitude of the place of the observer, from which the zenith Z in the preceding figures is determined. This may be found as follow.

Problem I.

To find the latitude of any place on the earth's surface.

To determine the latitude of the place.

1. Observe the altitude of the pole above the horizon of any place, and that altitude will be equal to the latitude.

The latitude of any place on the earth is measured by the arc EZ, that is, by the arc subtended between the equator and zenith. But EZ + ZP = 90° and ZP + PR = 90°; whence

$$EZ + ZP = ZP + PR$$

$$\text{Consequently } EZ = PR$$

That is, the latitude of any place is equal to the elevation of the pole above the horizon of that place.

By observations on the circumpolar stars.

The elevation of the pole above the horizon, may be practically determined by observing the greatest and least altitude of any of the circumpolar stars, and taking half the sum of the two altitudes; the proper corrections being made for refraction, parallax, &c. according to the principles explained in a subsequent chapter.

For let xy, fig. 21, represent the circle of declination described by any circumpolar star, then Rx will be its greatest meridian altitude, and Ry its least, and it is obvious that

$$R\dot{P} = \tfrac{1}{2}(Ry + Rx)$$

By observations on the sun.

2. The latitude may also be found by observing the altitudes of the sun when he has attained his greatest north and greatest south declination. Half the sum will be the elevation of the equator above the horizon, and the complement of that angle the latitude of the place of the observer.

Referring to the same figure, let $e\,a$ be the declination circle described by the sun when he has the greatest north declination, then eH will be his greatest altitude on that day; let $s\,t$ in like manner be the declination circle described on the day when he has the greatest south declination; then He, will be its meridian altitude on that day; and since Ee = Es, it is obvious that

$$HE = \tfrac{1}{2}(He + Hs)$$

$$\text{And } 90° - HE = EZ \text{ the latitude.}$$

64. It is obvious, also, from what is stated above, and referring to our definition of the *obliquity* of the ecliptic, that this angle is measured by half the arc $s\,e$, that

$$\tfrac{1}{2}(He - Hs) = \text{the obliquity of the ecliptic}$$

Problem II.

To find the time of the rising, setting, &c. of the heavenly bodies.

To find the time of rising, &c

65. Let the proposed body be the sun, and let us suppose that its declination remains constant during its passage from one meridian to the other, and that a clock is adjusted to go 24 hours during this one apparent revolution of the sun, and moreover, that the clock shows 12 exactly, when the sun is on the meridian; *to find the time of its rising, and its azimuth at that time; the latitude of the place and the declination of the sun being given.*

Fig. 24.

Referring to fig. 24, and comparing it with what has been stated with refcrence to fig. 21 (art. 59,) it appears that the sun rises when it comes to b; that it is twelve o'clock when the sun is upon the meridian at e, and that the whole circle $a\,e\,a$ is described in 24 hours; that is uniformly at the rate of $\frac{360}{24}°$ = 15° per hour; to find the time of rising, therefore, we have only to compute the angle ZPb, and to convert it into time at the rate of 15° to an hour in time; and to find its azimuth from the north we must compute the angle RPb, for which computations we have the following data:

$bZ = 90°$, Ee = declination; $eP = Pb$ co-declination
EZ = latitude; ZP = co-latitude

Hence in the triangle ZPb, we have the three sides given and one of them Zb = 90°, to find the angle ZPb. This case may therefore be solved by our fifth form in the preceding Table II. for oblique angled triangles, but one of the sides being 90°, it will be more readily solved by means of Napier's analogy; viz

$$\text{rad} : \cot bP :: \cot ZP : \cos ZPb = \text{hour angle}$$
$$\text{or rad} : \tan.\text{dec} :: \tan.\text{lat} : \cos ZPb = \text{hour angle}$$

Exam. 1. Let us, for example, suppose the latitude of the place to be 52° 13′ north; the declination 23° 28′, to find the time of sun's rising.

By the above analogy,

rad	10·0000000
tan 23° 28′..	9·6376106
tan 52° 13′..	10·1105786
Cos 124° 2′	9·7481892

(taking the supplement of the tabular angle 55° 58′, the angle being obviously greater than 90°.)

According to either solution, therefore, we have the hour angle = 124° 2′, which converted into time gives

15° : 124° 2′ : : 1h. : 8h. : 19½′ time from noon

consequcntly,

h.	m.	″.	
12	0	0	
8	19½	0	
3	40½	0	the timc of rising

That is in the latitude 52° 13′, on the longest day, or when the sun has 23° 28′ north declination, he will rise at 3h. 40½m.

To find the azimuth from the north, we have in the same triangle the same data to find the angle PZb, which is the measure of the azimuth sought. This may therefore be determined by means of our form 5, oblique spherical triangles; but more concisely by the following analogy:—

$$\sin ZP : \text{rad.} :: \cos bP : \sin PZb = \text{azimuth.}$$

Hence ar. com. cos 52° 13′........	0·2127683
rad.	10·0000000
sin 23 28	9·6001181
cos 49° 32′ azim...........	9·8128864

EXAM. 2. Let it be required to determine the time of sun rising at the same place, lat. 52° 13′ on the 25th of February, 1818.

The declination on this day by nautical almanack is 9° 11½′ south. Consequently, referring to the same fig. 24, b′ = 99° 11½′. Hence, then, in the triangle ZPb′, we have as before, ZP = co. lat. = 37° 47′, P b′ codec. 99° 11½′, Z b′ = 90° Whence, as in the former case,

rad. .... 10·0000000
tan dec. 9° 11½′........ 9·2090197
tan 52° 13′ .... 10·1105786

cos 77° 57′ hour angle.. 9·3195983

This converted into time gives 5h. 12m. from noon; which, taken from 12 hours, gives 6h. 48m. for the time sought.

PROBLEM III.

To find the sun's altitude at 6 o'clock, azimuth, &c.

Altitude at 6 o'clock. Fig. 25.

66. Here, since PP′ (fig. 25) bisects *e a*, it is clear that the sun will be at *c*, at 6 o'clock; and we have therefore, in this case, the hour angle ZP*c* = 90° given, and the two sides ZP, P*c*, to find Z*c*, the co-altitude; that is, two sides and the contained angle are given to find the third side. Whence, by our fifth form for right-angled spherical triangles, we have

$$\cos Zc = \frac{\cos ZP \times \cos Pc}{\text{rad.}}$$

Or, which is the same, (adopting the data of the first of the preceding examples,)

rad. .... 10·0000000
sin lat. 52° 13′ ...... 9·8978103
sin dec. 23° 28′ ...... 9·6001181

sin. alt. 18° 21′ ...... 9·4979284

Time of easting.

67. To find at what time in the day the sun will be east and west; that is, in the prime vertical, and its altitude at that time; taking the latitude of the place 52° 12′ 35″, and the sun's declination 23° 28′.

Here, taking ZP to denote the co-latitude = 37° 47′ 25″ P *b* the co-declination; and the angle *b* ZP being 90°, we have the two sides ZP, P *b*, of the right-angled spherical triangle *b* ZP, and the angle *b* ZP, a right angle, to find the hour angle ZP *b*, and the co-altitude Z *b*.

By preceding Table 1, form 1,

$$\cos Zb = \frac{\cos Pb}{\cos ZP} = \frac{\text{sin dec.}}{\text{sin lat.}} = \text{sin alt.}$$

$$\cos ZPb = \frac{\tan ZP}{\tan Pb} = \frac{\text{cos lat.}}{\text{cos dec.}} = \text{hour angle.}$$

sin dec. 23° 23′ = 9·6001181
sin lat. 52° 12′ 35″ = 9·8977695

sin alt. = 30° 15′ 31″ 9·7023486

Again,

cos lat. 52° 12′ 35″ = 9.8995301
cos dec. 23 28 = 10·3623894

cos hour angle = 70° 19′ 44″.... 9·5371407

Which, converted into time, gives 4h. 41′ 19″ from apparent noon.

Azimuth and hour.

68. Given the latitude of the place, the sun's declination; and altitude to find the hour and azimuth.

Here, referring to the same figure, we have the co-latitude ZP, the co-altitude Z *b*, and the co-declination P *b* to find the hour angle ZP *b*, and the azimuth PZ *b*. That is, in a spherical triangle, we have three sides to find the angles.

Whence, calling the sum of the three sides *s*, we have, by form 5, of our Table II.,

$$\tfrac{1}{2}\cos ZPb = \sqrt{\left\{\frac{\sin\frac{1}{2}s\ \sin\left(\frac{1}{2}s - Zb\right)}{\sin ZP\ .\ \sin Pb}\right\}}$$

Let the latitude be 34° 55′, the declination 22° 22′ 57″ N, the altitude 36° 59′ 39″. Then ZP = 55° 5′, Z *b* = 53° 0′ 21″, P *b* = 67° 37′ 3″. Hence the operation,

P *b* =	67°	37′	3″	sin arith. comp.	0·0340191
ZP =	55	5	0	sin arith. comp.	0·0861939
Z *b* =	53	0	21		
sum	175	42	24		
½ *s*	87	51	12	sin =	9·9996942
Z *b*	53	0	21	sin =	9·7569320
½ *s*—Z*b* =	34	50	51		2)19·8768392

½ cos ZP *b* = 29° 47′ 44″ = 9·9384196

Hence, ZP *b* = 59° 35′ 28″, which, reduced to time, gives 3h. 58′ 22″ the time from apparent noon. In the same manner

$$\tfrac{1}{2}\cos PZb = \sqrt{\left\{\frac{\sin\frac{1}{2}s\ .\ \sin\left(\frac{1}{2}s - Pb\right)}{\sin Zb\ .\ \sin ZP}\right\}}$$

The numerical solution of which, we leave as an exercise for the reader.

Meridian line.

69. By means of the azimuth determined as above, it will not be difficult to draw a meridian line, by simply determining its position, so as to make the requisite angle with the direction of the sun at that time. This, however, of course supposes that the altitude of the sun has been properly corrected for refraction and parallax, which being subjects we have not yet touched upon, we merely indicate the principle of the determination, and shall leave our further discussion on it to a subsequent chapter.

Error in altitude.

70. In the preceding examples relative to those questions in which the altitude is supposed to be determined from observation, it is obvious that we not only suppose the requisite corrections to have been applied, but also that the observation is made without any appreciable error; but as such error is easily made, particularly if the instrument be not of the most perfect construction, it may not be amiss to ascertain what effect any such error will produce in the computed time, and the azimuth the sun ought to have, so that an error in altitude shall produce the least error in time; we propose therefore the following problem.

PROBLEM IV.

Given the error in altitude to find the error in time.

Error in time. Fig. 26.

71. Let EQ, fig. 26, represent the equator, P the pole, *a e* the parallel of declination, in which the sun, or

any other heavenly body is on the day of observation; and let r be real, and s the apparent place of the body, or $r\,s$ the error in altitude. Draw $m\,s$ parallel to the horizon; from Z, the zenith, draw $Z\,m$, $Z\,r$; and from P the pole, the meridians $P\,m\,p$, $P\,r\,q$, to pass through $m\,r$, and to cut the equator in the points $p\,q$, then m and r will be the real and apparent places of the body, on the parallel of declination; and the arc $m\,r$ on that circle, or the arc $p\,q$ on the equator, will measure the angle $m\,P\,r$, the corresponding error in time. Now the triangle $m\,s\,r$ being of course exceedingly small, it may be regarded as a rectilinear right-angled triangle, right angled at s;
and hence we have

$$s\,r : r\,m :: \sin s\,m\,r : \text{rad.}$$

by spherics $$m\,r : p\,q :: \cos q\,r : \text{rad.}$$

Multiplying these together, rejecting the like factors, we have

$$s\,r : p\,q :: \sin s\,m\,r \cos q\,r : \text{rad.}^2$$

Whence

$$p\,q = s\,r \times \frac{\text{rad.}^2}{\sin s\,m\,r \cdot \cos q\,r}$$

But $Z\,r\,P = s\,m\,r$, $s\,r\,m$ being the complement of each, and $\sin Z\,r\,P = (\sin s\,m\,r) : \sin PZ :: \sin r\,ZP : \sin Pr$
Consequently,

$$\sin s\,m\,r \,.\, \sin P\,r = \sin s\,m\,r \,.\, \cos q\,r = \sin PZ \,.\, \sin r\,ZP$$

Whence, substituting for the denominator in the above expression, its equivalent in the last, we have,

$$p\,q = r\,s \times \frac{\text{rad.}^2}{\sin PZ \cdot \sin r\,ZP} = \frac{r\,s}{\cos \text{ lat.} \sin r\,ZP}$$

taking radius = 1.

Hence, since all the quantities in this expression are supposed to be given, except $\sin r\,ZP$, it is obvious, that $p\,q$ will *cæteris paribus* be the least, when $\sin r\,ZP$ is the greatest, or when the azimuth is the greatest; that is, when the body, whose altitude is observed, is in the prime vertical: it is best, therefore, to deduce the time from an observation, when the body is in or near that circle.

As an example in numbers, suppose the latitude to be 52° 12′ 35″; the declination 15° 24′ 25″; and the altitude as corrected, 40°; required the error in time, supposing an error of 1″ in the altitude.
This gives

$$1' \times \frac{1}{\cos \text{ lat. } \sin \text{ azim.}} = 1'.882$$

which answers to 7.528 seconds in time.

Again, supposing the azimuth to be 46° 22′, the latitude 52° 12′, and the error in altitude 1′, we have

$$1' \times \frac{1}{.612 \times .690} = 2'.334 \text{ of a degree} = 9''.336 \text{ in}$$

time.

Time of the sun ascending through an arc equal to its diameter

72. By means of this solution, we may ascertain the time in which the sun will pass either the horizontal or vertical wire of a telescope: we have seen, for instance, that the time during which the sun will ascend through any small altitude $= r\,s$, or the arc that he will describe when referred to the equator during that time, is

$$p\,q = \frac{r\,s}{\cos \text{ lat. } \sin r\ ZP}$$

in which, substituting d'' the apparent diameter of the sun in seconds for $r\,s$, this becomes

$$\frac{d''}{\cos \text{ lat. } \sin .\, \text{azimuth}}$$

consequently,

$$\frac{d''}{15 \,.\, \cos \text{ lat. } \sin \,.\, \text{azim.}} = \text{time in seconds.}$$

Time of its passing a vertical wire

73. To find the time in which the sun will pass over the vertical wire, we may take $m\,r$ to denote the apparent diameter of the sun; then, as the seconds in $m\,r$, considered as a small circle, must be increased in proportion as the radius is diminished; (because, when the arc is given, the angle is inversely as the radius,) we have
$\sin P\,r$, or $\cos .\, \text{dec.} : \text{rad} :: d''$ the seconds in $m\,r$. considered as a great circle, to the seconds in the same considered as a small circle, which are the seconds in $p\,q$; whence

$$p\,q = \frac{\text{rad.}\, d''}{\cos \text{ dec.}} = d''.\ \sec.\ \text{dec.}$$

and, consequently,

$$\text{the time} = \frac{d'' \times \sec.\ \text{dec.}}{15''}$$

Hence $p\,q = d''$ sec. dec. expresses the sun's diameter in right ascension. If therefore we assume his apparent diameter at 32′ = 1920″, and its declination 20°, its right ascension is

$$1920'' \times \sec.\ 20° = 34'\ 2''\ .\ 88$$

which, divided by 15, will give the same in time.

In the Nautical Almanack a column is given for indicating the time in which the semi-diameter of the sun passes over a vertical wire, by means of which a single observation on either limb, may be referred to the centre.

Problem V.

To find the time when the apparent diurnal motion of a fixed star is perpendicular to the horizon.

74. Let $x\,y$ (fig. 26) be the parallel of declination described by the star; draw the vertical $Z\,h$, touching it at o; then, when the star arrives at o, its apparent motion will be perpendicular to the horizon; the two circles having in the point o, a common tangent. And $Z\,o\,P$ is a right angle, we have

$$\text{rad} : \tan o\,P :: \cos PZ : \cos ZP\,o$$

that is, $$\text{rad} : \cos \text{dec.} :: \tan \text{lat.} : \cos ZP\,o$$

which, converted into time, gives the time from the star's being on the meridian; the latter therefore being supposed known, the former is readily determined.

Problem VI.

Given the right ascension and declination of a heavenly body, and the obliquity of the ecliptic, to find its latitude and longitude.

To determine the latitude and longitude. Fig. 27.

75. Let s (fig. 27) be the body, VC the ecliptic, VQ the equator, the angle QVC denoting the given angle of the obliquity of the ecliptic; let s V be joined by the arc of a great circle s V, and let fall the perpendiculars $s\,p$, $s\,r$; then it is clear, from our definitions, that V p will be the right ascension of the body s, $s\,p$

its declination, Vr its longitude, and sr its latitude: we have therefore given the Vp, sp, and the angle pVr to find Vr, and rs.

Now, first in the right-angled spherical triangle Vps, we have, by form 5, Table I.

$$\tan sVp = \frac{\tan sp}{\sin Vp}$$

whence the angle sVp becomes known, and consequently also sVr, because

$$sVr = sVp + pVr$$

Again,

$$\cos Vs = \cos Vp \,.\, \cos ps$$

whence Vs also becomes known.

Therefore, in the right-angled triangle rVs, we have the hypothenuse Vs, and the angle rVs to find the two sides Vr, rs:

which, by our form 4, Table I. are determined as follows:

$$\tan Vr = \tan Vs \times \cos sVr = \tan \text{long.}$$
$$\sin sr = \sin Vs \times \sin sVr = \sin \text{lat.}$$

In like manner, the right ascension and declination of any body may be found, when its latitude and longitude are given; but generally the problem is to determine the latter from the former, these being first ascertained from actual observation: it is thus the tables of the latitudes and longitudes have been computed.

But as both the ecliptic and the equator are subject to a change in their positions, the right ascension, declination, latitude, and longitude of all the fixed stars are constantly varying, and therefore those tables formed for any particular epoch, will not answer correctly after a certain time; the annual variations, however, being computed, their right ascensions, &c. may be determined for any proposed time, as will be hereafter explained.

5. *Of the crepusculum, or twilight.*

Twilight.

76. The crepusculum, or twilight, is that faint light which is perceived before the rising of the sun, and after its setting. It is occasioned by the terrestrial atmosphere, refracting the rays of the sun, and reflecting them amongst its particles.

The depression of the sun below the horizon, at the beginning of the morning and at the end of the evening twilight, has been variously stated at different seasons, and by different authors; for example,

Alhazen observed it to be	19°
Tycho	17°
Rothman	24°
Stevinus	18°
Cassini	15°
Riccioli, at the equinox	16°
summer solstice	21°
winter solstice	17¼°

but we more commonly now in our latitudes assume it 18°; the same both for morning and evening and for all seasons of the year.

Problem I.

Given the latitude of the place and the sun's declination, to find the time when twilight begins.

Time when it commences. Fig. 28.

77. Here S denoting the place of the sun when twilight begins, we have in the triangle ZPS, (fig. 28) ZS = 90° + 18° = 108°, the co-latitude ZP, and the co-declination PS, to find the hour angle ZPS from apparent noon.

Having thus the three sides, the angle required may be found by our 5th form, Table II.; that is, assuming ZP + ZS + PS = s

$$\tfrac{1}{2} \cos ZPS = \sqrt{\frac{\sin \frac{1}{2} s \,.\, \sin (\frac{1}{2} s - ZS)}{\sin ZP \sin PS}}$$

Whence the angle ZPS may be determined, and consequently the time from apparent noon.

Problem II.

Given the latitude of the place to find the time and duration of the shortest twilight.

Time of shortest twilight.

78. Let P fig. 28, denote the pole, Z the zenith of the observer, S the sun, 18° below the horizon ZS = 90° + 18° = 108°, or more generally, = 90° + $2a$, where we suppose the twilight to begin when the sun is $2a$ degrees below the horizon; or that the observer whose zenith is Z, will then see the commencement of the morning twilight.

Now in consequence of the diurnal rotation, the declination circle PS, turning about the axis will bring the sun from S to S′, in the horizon, or which is the same, the zenith Z will approach nearer to S, by describing about P the little circle $ZmbQ$, such that the distances Zm, Pm, Pb, PQ are all equal.

When, therefore, the zenith shall have descended from Z to any point m, such that $mS = 90°$, the sun will appear at 90° from the zenith, the day will commence and the twilight terminate. And the arc Zm of the small circle will be the measure of the angle ZPm, and consequently, of the duration of the twilight.

In order to determine this angle, draw the arc ZBm of a great circle, and to its middle B the perpendicular arc PB, then will

$$\sin \tfrac{1}{2}\, ZPm = \sin ZPB = \frac{\sin \frac{1}{2}\, Zm}{\sin PZ} = \frac{\sin \frac{1}{2}\, Zm}{\cos \text{lat.}},$$

now in the spherical triangle ZmS, we have

$$Sm + mZ \;7\; ZS, \text{ or}$$
$$90° + mZ \;7\; 90° + 2a$$

consequently, $mZ \;7\; 2a$ and $\frac{1}{2}\, mZ \;7\; a$

Let $\frac{1}{2}\, mZ = a + x$

then,

$$\sin \tfrac{1}{2}\, ZPm = \frac{\sin (a + x)}{\cos \text{lat.}} = \frac{\sin a \cos x + \sin x \cos a}{\cos \text{lat.}}$$
$$= \frac{\sin a + \cos a \sin x - 2 \sin a \sin^2 \frac{1}{2} x}{\cos \text{lat.}}$$

because, $1 - \cos x = 2 \sin^2 \frac{1}{2} x$ (See Trigonometry)

Now the last expression is equivalent to

$$\sin \tfrac{1}{2}\, ZPm = \frac{\sin a}{\cos \text{lat.}} + \frac{2 \sin \frac{1}{2} x\, (\cos a + \frac{1}{2} x)}{\cos \text{lat.}}$$

And here $\frac{1}{2} x$ is essentially positive, and a and $\frac{1}{2} x$ small angles, such that $a + \frac{1}{2} x \angle 90°$. Whence the latter part of the expression will be positive, and we shall have

$$\sin \tfrac{1}{2}\, ZPm \;7\; \frac{\sin a}{\cos \text{lat.}}$$

It is further evident that the twilight will be longer as x is greater, and shorter as x is less, and that it will be the shortest possible when x is zero, because in that case, the second member of the second side of the equation will vanish.

But this is what occurs when the triangle ZmS is reduced to the arc ZS, that is, when the point m falls on b, or when the distance PS is such, that the part Zb, of the vertical ZS, lying within the small circle ZmQ is equal to $2a$, and the exterior part bS $= 90°$. It is also manifest, that if PS increases, the opposite angle PZS will in like manner increase, and that on the contrary that angle will diminish as PS diminishes. In these variations, the point m will approach to or recede from Z, the intercepted part bZ will vary between the limits 0 and ZQ=2 PZ. Thus the intercepted part may have all values from 0 to $2\ (90° - \text{lat.}) = 180° - 2\ \text{lat.}$ and consequently may have the value $2\ a$: hence in the case where $Zb = r\,a$, and $bS = 90°$, the shortest twilight will obtain; and the semi duration in degrees will be found by means of the equation

$$\sin \text{ZPB} = \frac{\sin a}{\cos \text{lat.}} = \sin a \sec \text{lat.}$$

Whence the duration in time is readily determined.

Fig. 29.

79. Other theorems are deducible from the same construction and investigations; for on the arc $Zb = 2a$, and with the complement of ZP of the latitude constitute the isosceles triangle ZPb, (fig. 29) and let fall the perpendicular Pm; then ZPb will be the angle which measures the duration. Prolong Zb, till bS $= 90°$, and draw the arc PS which will be the sun's co-declination for the day of the shortest twilight.

Sun's azimuth.

Now,

$$\cos m\text{P} = \frac{\cos \text{PZ}}{\cos \text{Z}m} = \frac{\cos \text{PS}}{\cos m\text{S}}$$

Wherefore,

$$\cos \text{Z}m : \cos m\text{S} :: \cos \text{PZ} : \cos \text{PS}$$

or, $\cos a : \cos (90° + a) : \sin \text{lat.} : \sin \text{dec.}$

or, $\cos a : -\sin a :: \sin \text{lat.} : \sin \text{dec.}$

whence, $\sin \text{dec.} = \dfrac{-\sin a}{\cos a} \sin \text{lat.} = -\tan a. \sin \text{lat}.$

Again, in the right angled triangle ZPm, we have

$$\tan \text{Z}m = \cos \text{Z} \tan \text{PZ}$$

and $\cos \text{Z} = \tan \text{Z}m \cot \text{PZ} = \tan a. \tan \text{lat.}$

Now the angle Z or PZm is the sun's azimuth at the commencement of the twilight, and

$$\text{PZ}b = \text{P}b\text{Z} = 180° - \text{P}b\text{S}$$

But PbS is the sun's azimuth at the instant, when his centre is upon the rational horizon; wherefore *the sun's azimuth at the beginning and ending of the twilight are supplements to each other on the day of the shortest twilight,*

Hence, since $\cos \text{PZS} = \tan a. \tan \text{lat.}$

we have $\cos \text{P}b\text{S} = -\tan a. \tan \text{lat.}$

Hour angle.

80. In like manner ZbS and bPS are the hour angles, at the beginning and end of the twilight; let the former be denoted by P′ and the latter by P; then

$$\text{ZPS} - b\text{PS} = \text{P}' - \text{P} = \text{ZP}b$$

which is the angle that measures the duration of twilight. Hence we have from what has been done above

$$\sin \tfrac{1}{2} (\text{P}' - \text{P}) = \frac{\sin a}{\sin \text{lat.}}$$

Also,

$$m\text{PS} = b\text{PS} + m\text{P}b = \text{P} + \tfrac{1}{2} (\text{P}' - \text{P}) = \tfrac{1}{2} (\text{P}' + \text{P})$$

and

$$\sin m\text{PS} = \frac{\sin \text{S}m}{\sin \text{PS}} = \frac{\sin (90° + a)}{\cos \text{dec.}} = \frac{\cos a}{\cos \text{dec.}}$$

that is, $\sin \tfrac{1}{2} (\text{P}' + \text{P}) = \dfrac{\cos a}{\cos \text{dec.}}$

Whence from the two equations,

$$\sin \tfrac{1}{2} (\text{P}' - \text{P}) = \frac{\cos a}{\cos \text{dec.}}$$

$$\sin \tfrac{1}{2} (\text{P}' + \text{P}) = \frac{\sin a}{\cos \text{lat.}}$$

the hour angles P′ and P are easily determined, the declination itself being given by the equation

$$\sin \text{dec} = -\tan a. \sin \text{lat.}$$

We have also,

$$\cos \text{PSZ} = \frac{\sin \text{lat.}}{\cos \text{dec.}}$$

Lastly, let ST $= 2a =$ Zb; and draw PT, then ZT $= 90°$, and T is a point in the horizon for the moment when the zenith was in Z. Whence the triangle PZT gives

$$\cos \text{PT} = \cos \text{Z} \sin \text{PZ} \sin \text{ZT} + \cos \text{PZ} \cos \text{ZT}$$

$$= \cos \text{Z} \sin \text{PZ} = \tan a. \tan \text{lat.} \cos \text{lat.}$$

$$= \tan a. \sin \text{lat.} = \sin \text{dec.}$$

Therefore, $90° - \text{PT} = \text{dec.}$, or $\text{PT} = 90° - \text{dec.}$

But $\text{PS} = 90° + \text{dec.}$

therefore, $\text{PT} + \text{PS} = 180°$

which is another remarkable property of the shortest twilight.

81. In all the above deductions, the latitude has been left general, or indeterminate, as has also the quantity $2\ a$, which denotes the number of degrees that the sun is below the horizon when the twilight begins. The latitude may therefore be introduced for any given place, as may also $2\,a$; but commonly, we assume $2a = 18°$, that is, the twilight is supposed to commence when the sun is 18° below the horizon.

Example.

EXAMPLE. Required the day on which the twilight is the shortest at Woolwich, the latitude of which is 51° 28′ 40″, in the year 1820, with its duration, and the time of its beginning and end.

First, for the declination we have

$$\sin \text{dec.} = -\tan a \sin \text{lat.}$$

Now $\log \tan a = 9° \quad = 9{\cdot}1997125$

$\log \sin \text{lat. or } 51°\ 28'\tfrac{2}{3} = 9{\cdot}8934104$

$\log \sin \text{dec.} = -7°\ 7'\ 5'' \quad 9{\cdot}0931229$

The declination, therefore, is 7° 7′ 5″ south, which answers to March 2 and October 11.

Again, for the duration, we have

$$\sin \tfrac{1}{2} (\text{P}' - \text{P}) = \frac{\sin a}{\cos \text{lat.}}$$

from $\log \sin 9° = 9{\cdot}1943324$

take $\log \cos 51°\ 28'\tfrac{2}{3} = 9{\cdot}7943613$

$\log \sin 14°\ 32'\ 41'' \quad 9{\cdot}3999711$

Whence $\text{P}' - \text{P} = 29°\ 5'\ 22''$

Also $\sin \tfrac{1}{2} (\text{P}' + \text{P}) = \dfrac{\cos a}{\cos \text{dec.}}$

Hence from $\cos 9° = 9.9946199$

take $\cos 7°\ 7'\ 5'' = 9.9966399$

$\sin 95°\ 31'\ 19'' \quad 9.9979800$

$$\begin{aligned} \text{whence } P' + P &= 191^\circ\, 2'\, 38'' \\ P'' - P &= 29\ \ 5\ \ 22 \\ \hline 2)220\ \ 8 & \end{aligned}$$

half sum = ... 110 — 4 = angle P′
half dif. = ... 80 58 — 38 = angle P

	h	m	s
The former in time answers to	7	20	$16\frac{1}{4}$
and the latter to	5	23	$54\frac{3}{4}$

The former showing the time when the evening twilight ends, and the latter the time of the sun's setting, or the time of its beginning; consequently their difference, 1h. 56m. $21\frac{1}{2}$s., is the duration.

The above numbers, taken respectively from 12, will leave the time of the beginning and end of the morning twilight.

Distinction between mean and apparent time.

82. In all the preceding investigations, we have considered only apparent time; that is, we have supposed it to be 12 o'clock when the sun is on the meridian, and that it is exactly 24 hours in passing from one meridian to the same again; but if a clock be adjusted to go thus for one day, that is, if it show exactly 24 hours between the time of the sun being twice successively in the same place, it will not continue to show 12 o'clock every day when the sun comes to the meridian, because the intervals of time from the sun's leaving a meridian to his return to it again, are not always equal. This difference between the sun and a well regulated clock is called the *equation of time*, which will be treated of in a subsequent chapter; at present, we shall not enter farther upon the subject, it is sufficient to apprize the reader that the time as determined in all the preceding problems, is what is called *apparent time*, or the time shown by the sun, and not *mean or true time*, which is that shown by a well regulated clock.

Corrections for parallax, &c.

83. Beside this correction for the time, there are also other corrections which must be introduced into the data of the preceding examples, in order to render the results perfectly conformable with observation. Thus we have all along supposed the body to rise as soon as it is found in the rational horizon; but all bodies in the heavens, when in or near the horizon, are elevated 33′ by refraction above their true places; this, therefore, would make them appear when they are actually 33′ below the horizon, or when they are 90° 33′ from the zenith; and all the way from the horizon to the zenith refraction has the effect of elevating the apparent places of the heavenly bodies, but in a less degree as the altitude is greater; till it vanishes in the zenith. The altitudes, therefore, as we have given them, are supposed to have been subjected to these corrections, the method of making which will be explained hereafter. This is one of the principal corrections for the fixed stars; but for the sun or any other of the bodies of our system, a different correction becomes necessary, these being all depressed below their true places by the effect of parallax, as will also be explained in a subsequent chapter; that is, we have confounded the sensible and rational horizon, which is admissible as far as relates to the fixed stars, in consequence of their immense distance; but the angle subtended by them or by the earth's radius, at any of the bodies of our system, is a sensible quantity that must not be neglected in any computations relative to such bodies. The parallax has therefore a tendency to increase the apparent zenith distance of any body in our system, while the refraction tends to diminish it; therefore the actual zenith distance of a body when it first becomes visible to a spectator on the earth, is equal to 90° — hor. parallax + refraction.

84. What has been hitherto done, has been merely to indicate the nature of the calculations after certain observations have been made and corrected; we now propose to explain the instruments used for making these observations, the method of employing them for these purposes, and the nature and quantity of the corrections that are requisite for reducing the results thus obtained, so as to render them proper for the purposes of astronomical computation.

§ IV. *Description and use of the most indispensable astronomical instruments.*

Astronomical instruments.

85. We shall not attempt in this place a general description of astronomical instruments, this will be given under a distinct head in another part of this work; our purpose here is only to describe those whose use is indispensable in the pursuit of this science, and with the principle of whose construction and operation, it is essential that the student be well informed, if he wish to proceed upon any other data than those furnished by the report of other observers.

1. *Of the astronomical telescope.*

Astronomical telescope.

86. We have already in our treatise on Optics described and illustrated the principles of this instrument, we have therefore in this place merely to speak of its application to astronomical observations.

Fig. 30.

Let, then, AB (fig. 30) represent the diameter of any heavenly body, as the sun, moon, or planet; *m* its middle point, will send out a luminous pencil of rays, which by means of the refraction of the lenses will be united in the focus of the telescope. The same will be the case with all the points of the disc, and an image of the object GH will be formed, as we have already explained in our treatise on Optics; H being the image of the point B, and G of the point A; that is, the image will be reversed with respect to its actual position. The motion of the object is also reversed, so that if it moves from left to right in the heavens, it will move from right to left in the telescope; but the effect being the same for all bodies, there will result from it no practical inconvenience; it is sufficient that we are apprized of it.

Let us farther observe, that the angle H*e*G = ACB, and if the focal distance *e*F is to the breadth HG in such a ratio that

$$\frac{\frac{1}{2}\mathrm{HG}}{e\mathrm{F}} = \tan \tfrac{1}{2}\,\mathrm{ACB}$$

the object will be shown entire in the telescope; but if the focal distance be *ef* 7 CF, the image *hg*, will be larger than the field of the telescope, and consequently, we shall not then see it entire. On the contrary, if the focal distance be CQ, less than CF, the image *h*′Q*g*′ will not fill the telescope.

Generally, the field of the telescope is found by the formula

$$\tan \tfrac{1}{2}\,\text{field} = \frac{\tfrac{1}{2}\ \text{diam. of the tube}}{\text{focal length}}$$

$$\tan \tfrac{1}{2}\,\text{field} = \frac{\tfrac{1}{2}\ \text{diam. of the tube}}{\text{radius of curvature of the object glass}}$$

Or in case of a diaphragm being placed in the tube, as is commonly practised to prevent the reflection of the oblique rays, then

$$\tan \tfrac{1}{2}\ \text{field} = \frac{\tfrac{1}{2}\ \text{diam. diaphragm}}{\text{focal length}}$$

These results, which are immediately deducible from what has been done in the preceding treatise, are stated here merely for the sake of avoiding frequent references to those articles; with the same view we may state the following relative to the magnifying power of such an instrument as we are here speaking of, viz.

$$\text{magnifying power} = \frac{\text{focal distance obj. glass}}{\text{focal distance eye glass}}$$

By which is to be understood, that the angle under which we view the image, is equal to that under which we should see the object if it were brought so many times nearer as is indicated by the above fraction.

Having said thus much with regard to the telescope more commonly employed in astronomical observations, let us offer a few remarks relative to the apparatus attached to it, for rendering those observations more precise than they could be obtained with this instrument in the simple state in which it has been described above.

Use of the cross wires
Fig. 31.

87. *Of the reticule or cross wires.*—Let ABDF (fig.31) represent a section of the telescopic tube or of the diaphragm with which it is furnished. On the edges of this ring or circle, are attached with two screws a metallic thread or fine wire DF; this is in a transit instrument called the horizontal wire. Perpendicular to this thread are placed five others; the centre one AB bisecting DF in C, and the other four at equal distances, two on each side of AB.

A star or any other heavenly body passes the field of view of a telescope in different times according to the diameter of the instrument, and the polar distance of the body; therefore, if the telescope were unprovided with some such an apparatus as here described, it would be difficult or even impossible to say precisely the moment when it was in the axis or in the centre of the instrument, and the accuracy of modern science requires this determination to the utmost accuracy: let us see what precision is attainable by means of the cross wires above indicated.

The point C or G being supposed either in the centre of DF, or in the vertical line bisecting DF, at the moment a star passes it, if the diameter of the thread be equal to that of the star, it will be entirely hidden by it, and that moment will be the time of its passage; but more commonly the thread is sufficiently fine, that at the instant of passage, it will bisect the star, an equal portion of the latter being observable on each side of the former; and thus the time of passage might be found to within one-fourth or one-fifth of a second. But it is obvious that we may observe the same with respect to all the other four vertical wires, which being at equal distances, the times of passing each in succession from the first will form an arithmetical progression; and by taking the mean of the five, we shall have the time of the star passing the centre wire still more exactly, and by this means, we may generally depend upon our observation to within one-tenth of a second.

Our figure and the description of it, applies to the case of an instrument fixed in the plane of the meridian, in which case the motion of a heavenly body will be apparently horizontal. In any other case, the star ascends or descends obliquely, and then it is necessary to give to the wire DF, a similar inclination, so that the motion of the star may be parallel to it, as in the line ES, as shown in fig. 32, the proper apparatus being supplied for this purpose. The only difficulty of observation is when the night is very dark, and when we are unable to see the threads except for the moment when the star is bisected by them; which being almost instantaneous, we are not sufficiently prepared for noting the time. In order to obviate this difficulty, the interior of the tube is illuminated by the following apparatus.

Fig. 32.

Method of enlightening the tube.

88. In the side of the tube of the telescope, and commonly in the axis on which it turns, is made a small hole, directly opposite to which is placed, in the tube, a small mirror, inclined to the axis or sides of the telescope, at an angle of 45°. The light of a small lamp falls on the mirror, and forming with it an angle of 45°, it is reflected at the same angle, and therefore passes in a line parallel to the axis of the instrument, and thus renders the wires sufficiently visible. If the star on which the observation is made be of the 9th or 10th magnitude, we must however be careful to modify the intensity of the illumination, as otherwise, the artificial light will render the natural light of the star imperceptible, and it will be in danger of passing unobserved.

Observations on the sun or moon.

89. At present we have spoken only of the transit of a fixed star, if it be the sun or moon that we are observing, we must ascertain the time when their respective centres pass the axis of the instrument. For this purpose we note very accurately, the instant when the eastern or western limb comes in contact with each of the five wires in succession, and the sum of the times divided by 5, will be the instant when that limb passed the centre wire. These five observations being made, the other limb of the sun or moon will be just about leaving the first wire, we therefore, in like manner, note these other five instants, the mean of which will give the time when the last limb passed the centre wire, and the mean of the two will be the time of the transit of the centre.

90. Having said thus much with reference to the telescope and the apparatus with which it is supplied, it remains for us to describe the instruments to which it is attached, and the nature of their adjustment; for the accuracy of observations indicated above with reference to the object traversing the axis of the telescope would be to little purpose, unless we could be equally precise in the determination of the direction of that axis, both as regards the horizontal and vertical position of the instrument.

By far the greater number of astronomical observations are made in the plane of the meridian; but some are made at different azimuths, and different instruments are best employed for these purposes; at the same time some are so constructed as to answer in both cases; at present we shall confine our explanation to only one of the most perfect of each sort, but in a future part we shall enter upon the subject more at length, and endeavour to illustrate the advantages and defects of different constructions.

Of astronomical quadrants.

91. These may be either portable or fixed; in the former case they are commonly mounted on a tripod, and may be used for taking altitudes in any azimuth, or be made to follow the body observed in its apparent path; but in the latter case, the instrument is fixed in the plane of the meridian against a substantial wall, and is hence denominated a *mural quadrant.*

2. *Portable astronomical quadrant.*

Portable astronomical quadrant.

92. The great variety of forms under which this instrument has appeared, the numerous methods proposed by different artists for adjusting it to its verticality, &c. would be sufficient of themselves, were we to enter at length upon the subject, to form a considerable volume; we shall, therefore, select one of those esteemed the most perfect, and limit ourselves to the description of that only; which will be amply sufficient for our present purpose.

Fig. 33.

Fig. 33 represents a portable quadrant, constructed by Ramsden for the observatory of Christ's College, Cambridge. The tripod on which it is mounted has screws of adjustment to set the stem, on which the horizontal motion is performed, perpendicular, which is proved to be so in all directions when the plumb-line bisects both the superior and inferior dots during the whole revolution in a horizontal circle. The visible stem is a brass tube, and through it ascends a solid steel vertical axis, which fitting closely at the upper and lower extremity, has not the least shake, and preserves the position once given to it, so long as the feet screws are unmolested. The telescope is of the achromatic construction, and has the usual apparatus for a slow motion. The telescope lies on a bar that carries the counterpoise, and in which is the centre of its motion. It has a system of wires in the focus of the eye glass, which are adjustable by screws both upwards and sideways, as well as in a circular direction, so that the adjustments for collimation, and for zero in the altitude of the circle, may be thereby effected. The point of suspension of the plumb-line is also adjustable by a proper screw apparatus. At the top of the vertical tube or stem, is a small horizontal circle with a clamping apparatus for slow horizontal motion, by means of which the whole quadrant with its attached telescope turns gradually round in azimuth. When the observation is made in or near the zenith, the plumb-line of this instrument falls in the way of the telescope, and is obliged to be removed. This, however, is supplied by the addition of a spirit level, suspended from an adjustable horizontal brass rod, under the uppermost radical bar of the quadrant, which level not only supplies the place of the plumb-line when taken off, but at all times serves as a check on its adjustment, and when furnished with a graduated scale, may very well be made its substitute. We shall not here attempt to describe the nature and division of the vernier scale, it will be sufficient to observe, that the angles may be ascertained to the 10ths of seconds. We proceed next to the adjustments of the instrument.

To adjust the axis of the pedestal vertical.

Adjustment of the instrument to verticality.

93. This adjustment may be performed either by the plumb-line or by the level. When the plumb-line is used, turn the quadrant in azimuth till its plane, or which is the same thing, till the telescope lies parallel to a line joining any two of its three feet, and turn one of the two screws of the feet till the wire bisects the lower dot, and with the proper screw bring the upper dot to the same wire; then reverse the telescope by turning 180° in azimuth, and if both dots are again bisected, the axis is vertical in the direction that the telescope has pointed; in the next place, turn the telescope the space of a quadrant till it points in the same direction as the third foot of the tripod, and make the wire bisect the lower dot by the screw of this foot, and it will be found to bisect the upper dot also, if the first adjustment was properly made, but if not, repeat the operation till both dots are bisected in all the reversed situations of the telescope, and then the axis will be vertical in every direction.

In making this adjustment by the level alone, the process must be thus; first, the level must be made parallel to the rod which it hangs on, and secondly, this rod must be put perfectly horizontal, and the level will be horizontal also, with the bubble in the middle. In order to make the level parallel to the rod, place it parallel to a line joining two of the feet screws, and bring the bubble to the middle by one of the feet screws in question; then take off and reverse the position of the level, and if the bubble is found in the middle now, the parallelism is perfect; if not, one half of the error must be rectified by the same foot screw, and the other half by the adjusting screws at the end of the rod, by releasing one and screwing up the other. A repetition or two of this process will make the bubble stand in the middle in both of the reversed situations. In the next place, with the level thus parallel to the rod of suspension, turn the quadrant round its axis an entire semicircle as nearly as can be estimated, and if the bubble will now rest in the middle, the rod is level, and being at right angles with the axis of the quadrant's motion, proves that this axis is vertical in every direction; but if the bubble be found to run to one end of the tube, bring it one half way back by the adjusting screws of the rod, releasing one and fixing the other, as the case may require, and the other half by the proper foot screw. A repetition of this process will soon settle the bubble in the middle during a whole revolution in azimuth, and then the adjustment of the axis is perfect, as well as of the rod and level.

Adjustment for collimation.

94. The second adjustment is that by which the line of collimation of the telescope, is made parallel to the horizontal line that passes to the centre of the quadrant to zero on the limb, or quadrantal arc, at the same time that zero on the vernier coincides with zero on the limb. This important adjustment may be made in several ways, some of which are tedious and otherwise objectionable; but we shall confine ourselves to two, which apply one to the vertical, and the other to the horizontal line of the quadrant, which two methods, when duly effected, will not only check each other, but detect the error of

the total arc, if there be any at the same time; which is an acquisition of the utmost importance. First then to adjust by the vertical line, let the axis of the quadrant be made truly perpendicular in all directions by the adjustment we have already described, and fix on a star within a few degrees of the zenith when exactly on the meridian, and measure its altitude by the cross wire in the field of view in the usual way, and note down the result; if these readings prove to be at equal distances from the point 90°, one on the quadrant arc, and the other on the arc of excess beyond 90°, the horizontal wire is truly placed in the eye piece, but if not, half of the difference of the readings must be corrected by the proper screw for raising or depressing the said wire. This may be done by directing the telescope to a distant mark till the cross wire bisects it, then by moving the screw of slow motion of the vernier the half quantity required, and by bringing back the cross wire thus displaced to its original mark again. This operation repeated will place the cross wire in such situation, that zero on the vernier will be in its proper place with respect to the point 90°; or the half difference thus ascertained may remain, without altering the cross wire, as an error of adjustment to be constantly applied with the sign + or —, as the case may be, in all subsequent observations. Again, to adjust by the horizontal line passing through the zero of the quadrantal arc, it will be necessary to have a second telescope turning on pivots in adjustable *y*'s attached to the back of the quadrant, on the same level with the said horizontal line of the quadrant. This telescope may be called the adjusting telescope, and may be also used to watch a distant mark, before and after an altitude is taken, in order to detect any deviation in the position of the vertical axis that may happen during the operation of measuring. Let the adjusting telescope bisect a fine distant mark with its cross wire, and turn the tube of the telescope round one half way on its pivots, as it lies in a horizontal position, and if the wire now bisects the same mark it is truly fixed, if not, look out for a new mark a little higher or lower, as the case may require, and make it cut that in the reversed positions of the cross wire, by means of the proper screw for the purpose; now this adjusting telescope will be adjusted for collimation. In the next place, put zero on the vernier to zero on the limb, and direct the telescope of observation to the distant mark, by which the adjusting telescope had its wire adjusted, and let this mark be bisected by both telescopes, the level and plumb-line at the same time showing that the vertical axis is perpendicular; now turn the quadrant half round its azimuth, and reverse the adjusting telescope so as to view the same distant mark again, and if it be found to bisect it as before, the horizontal line of the quadrant is right, and all the quadrantal arc without error, supposing the telescope of observation to have its adjustment for collimation as fixed by the point 90°, above described; but if this adjustment of the point zero on the limb be first made, half the apparent error must be rectified by the screw at the eye piece by means of reversed positions and marks; and then afterwards the adjustment by a new star near the zenith will detect the error of the whole arc. If, however, no error in the total arc exists, then the adjustment for collimation may be made either from the horizontal or from the vertical measurement, as may be most convenient; one of which is more practicable by day, and the other by night. When this delicate and most essential adjustment is finally settled, the object glass of the telescope should not be disturbed, and therefore it would be adviseable to have its interior surface well cleaned previously. It was taken for granted that the cross wire was perfectly horizontal during the time the preceding adjustment was made, or, which is the same thing, that the parallel wires were perpendicular to the horizon. This is proved in a simple manner thus; direct the telescope to a fine small distant mark, or make the adjustment for vision, if necessary, then if one of the vertical wires will continue to bisect the said mark through the whole field of view while the telescope is elevated or depressed, the wires are right, but if not, they must be made so by the proper screws for that purpose, near the focus of the eye-glas. This preparation ought to precede the last adjustment, and when once made, seldom requires altering, except in case of accidental injury. It has also been assumed in the preceding adjustment, that the maker of the instrument placed the plane of the quadrant parallel to the axis of its motion, and also its line of collimation of the telescope parallel to the said plane. The former may be known to be true thus: if, when the plumb-line is adjusted at its centre of suspension, just to escape touching the limb (which should always be the case) the motion of the quadrant in azimuth will not alter it in this respect, the plane is truly fixed; but if not, the screws that fix the quadrant to its axis must be resorted to for its alteration, which is best done by the maker. When there is no plumb-line, a small spirit level fixed at right angles to the plane of the quadrant will answer the same purpose; for the resting of the bubble during the quadrant's revolution in azimuth, will be a proof that the plane to which it is at right angles is vertical. With respect to the parallel position of the telescope, as this is guided by the vernier sliding on the limb, it is the business of the maker to adjust it properly, which he will best do by a comparison with a good transit instrument of the passages of a high and of a low star in each of the two instruments, but a small deviation of the telescope with respect to parallelism, though to be avoided, if practicable, will not sensibly affect the measurement of altitudes, which is the sole business of this instrument. If, however, this deviation be considerable, the eye end of the telescope must be set nearer to or farther from the limb, as the case may require, by the maker himself. We have been the more minute in describing these adjustments, not only because they are indispensably necessary in making good observations, but because they will apply, one or other of them, by means of the plumb-line or of the spirit level, to all other astronomical quadrants that have a motion in azimuth.

3. *Of the transit instrument.*

Transit instrument.

95. Transit instruments, as they are now constructed, may be considered either as fixed or portable; the former of which was the original construction, and is still that commonly made use of in permanent observatories, for the purpose of determining in conjunction with a good astronomical clock, the right ascensions of the heavenly bodies; but the latter

may be used in any place for ascertaining the rate of a clock or chronometer, and when nicely brought into the meridian, for determining also the right ascension with considerable accuracy. It is one of the latter only that we proposed to describe in this place.

Fig. 34. The best construction of a portable transit instrument, is that represented in our fig. 34, where are shown all the parts that are necessary to be described under this article. This instrument is one of the numerous inventions of Troughton, and by no means one of the least useful.

The telescope of this transit is 20 inches in length, and magnifies from 20 to 35 times, according to the eye pieces that are employed; two of which are usually of the prismatic or diagonal kind, to be used at considerable altitudes: the aperture is 1¾ inches, and the power is sufficient to enable us to see the pole star by clear day light. The base of the instrument is a thick ring or rim of brass, which receives three equidistant screws for feet, besides the four screws that fix the two vertical frames to the base, and which constitute the supports of the axis; one of these is shown complete in the first of our two figures. These supports are kept perpendicular by the interior bracing bars, of which two are seen in the second figure; they are attached by thumb screws at both ends to the base and upright frames respectively. The circular figure of the base is not only firm, but preserves its shape in all degrees of temperature; and when the parts are detached, by loosening the thumb screws, the whole pack into a box, which is of convenient size for carriage; the diameter of the circular base, and the consequent length of the axis, is a foot within, and the height of the supports thirteen inches.

The graduated circle being of six inches diameter, admits of reading by each of the two opposite verniers to minutes, which is sufficient for finding the meridian altitude of any celestial body of which the declination is known, when the latitude is given; or for determining the latitude when unknown, to the accuracy of the nearest minute.

If the circle were made a little larger, and three verniers substituted for the two, a longer level might be used, and the readings made accurate to 20″ or 30″; but as the intention of the inventor was not to make it an altitude instrument, he has limited himself only to such conditions as were requisite for constituting an useful transit instrument in a portable form. The level of this transit is wholly detached, is equal in length to the axis itself; and is intended to be placed over the same, by resting upon it, having notches on its end pieces, which become tangents to the cylindrical parts of the axis or pivot; so that the reversion is performed without any inconvenience: but it is necessary to remove the level when the altitudes are great, in order to avoid its being displaced and broken by any alteration in the elevation of the telescope. There are usually three studs of brass included, with the darkening glasses, lantern, and other appendages; two of which studs have conical holes to receive the points of the screws or feet of the circular base; and for this purpose all the studs must be made fast to the slab or pillar which supports the instrument, by plaster of Paris or putty, inserted into as many holes in the plane of the marble or stone, care being taken that the line which joins the two conical points be in the direction of the meridian, or so nearly so, that the adjusting screws of one of the Y's will bring it into that situation.

Mr. Jones, of Charing-cross, has made several 30 and 42 inch transit instruments of the portable sort, supported by oblong frames of cast iron, which look neat, and answer the purpose very well. These have all the advantages of the instrument last described, and at the same time of course have greater powers in the telescope, and are cheaper in proportion to their size. He has also made some with telescopes of only 20 inches, for the sake of being more portable.

Of the adjustments.

1. *Of the level.*

When the level hangs on, or is made fast to the axis, put the telescope in its place, and observe to which end of the level the bubble runs, which will always be the more elevated end; bring it back to the middle by the Y screw for vertical motion, or by the foot screw under the end of the axis, and then invert the axis end for end; then, if the bubble is again found in the middle, the level is already parallel to the axis; but if not, adjust one half of the error by the adjusting screw of the level, and the other half by the Y screw, or that at the foot of the support, and let the operation of reversing and adjusting by halves be repeated until the bubble will remain stationary in either position of the axis, in which case the level will be right. When the detached level is used, that notch must be made a little deeper where the bubble is, by scraping it with a penknife, instead of using an adjusting screw, with which it is not commonly provided. And when the notches which rest on the pivots are once made right, they will seldom require a second rectification.

2. *To place the axis of the telescope horizontal.*

If the spirit level be made use of, the same operation which we have just described, will put the axis level, at the same time that it brings the level parallel to the axis; for unless both these conditions be fulfilled, the adjustment of the level will be deranged by reversion; and when this is not the case, it is a proof that both the level and the axis are truly horizontal. Hence, when the level is previously adjusted, it will be sufficient to bring the bubble to the middle of the level, by the Y screw, or the foot screw alone, as the construction may require.

It is not necessary for our present purpose to describe the adjustments of the larger instruments, but we may just observe, that when in these the plumb line is employed, it is applied to a frame suspended by the pivots of the axis, that will reverse in position according to Ramsden's method, or hanging on the tube of the telescope parallel to the line of collimation; in either case a dot is bisected by the plumb-line near the point of suspension, and another near the lower end of the line, in both the reversed positions of the axis, when the adjustment is truly made by the proper screws as above directed.

3. *To adjust the telescope.*

That is, to place an eye-glass and object-glass at such a distance from each other, that their respective foci may coincide: after which, the wires are to be brought into their common focus. To effect this, some telescopes have the eye-glass and cell, which carries the wires, moveable, while the object-glass is fixed: others have the wires fixed, and the two glasses moveable. In the former case, by pushing in or drawing out the eye piece, adjust the telescope so that the sun or a planet appears perfectly distinct through it; then move the wires nearer to, or farther from, the eye-glass, as may be required, until they also appear perfectly distinct, and the telescope will be adjusted ready for use. In the latter construction, push in, or draw out, the eye piece, till the wires appear perfectly distinct; then alter the object-glass until the sun, or a planet, appears perfectly distinct also, and the telescope will be adjusted ready for use.

As it is of importance to have the telescope adjusted very exactly in this respect, the following method of trying whether it be so or not, may be practised.

The telescope being adjusted to distinct vision for distant objects, when a fixed star is on the meridian, bring the horizontal wire to bisect it very exactly, and the star will run along the wire through the whole extent of the field of the telescope. While the star is thus running along the wire, move your eye a little upward or downward; and if the wires be not exactly in the common focus of the two glasses, the star will appear to quit the wire when the eye is moved. If this be the case, the wires or glasses must be altered until the star will not quit the wire by the motion of the eye; the objects appearing perfectly distinct at the same time.

To bring a transit instrument into the plane of the meridian.

Take the altitude of the sun, noting the times by the watch or clock, and thence find the apparent time, the latitude of the place, and the sun's declination being known. The difference between this time and the mean of the times shown by the watch when the observations were made, will be what the watch is too fast or too slow, for apparent time.

If the watch is too fast, add the difference to 12 hours: but if it be too slow, subtract it from 12 hours, and you will have the time by the watch when the sun will be on the meridian, as near as the going of the watch can be depended upon. Take the time which the sun's semidiameter is in passing the meridian from the Nautical Almanack, and add it to, and subtract it from the time by the watch, when the sun will be on the meridian, and you will have the times when the sun's eastern and western limbs will be on the meridian. A few minutes before the time when the western limb will be on the meridian, let your assistant count the seconds as they pass, by the watch; but instead of calling the 60th second, let him name the minute the watch is then at. While he is doing this, you must bring the sun into the telescope by elevating it to the proper altitude; and turning the whole instrument round on the screw pin U. Having by this means brought the middle wire apparently to the eastward of what appears to be the eastern limb of the sun, (because the sun will appear to move that way in the telescope) tighten the screw U by turning the nut; and when the sun's limb arrives at the middle wire, keep it on it by turning the screw *g*, at the rate the sun moves, till your assistant calls the second by the watch at which you had computed the western limb of the sun would be on the meridian; and the instrument will be nearly in the meridian. Let your assistant count on till the watch arrives at the second, when, according to your calculation, the eastern limb of the sun should be on the meridian; and, if it is not exactly on it, you will have another opportunity of rectifying the instrument by turning the screw *g*.

Having thus brought the instrument into, or very near the meridian, its real situation with respect to the meridian may be verified several ways; of which we shall point out two. If the latitude of the place be considerable, that is, 30 degrees or upward, there are a variety of stars in both hemispheres sufficiently bright, which never set: and consequently, they may be observed with the instrument both above and below the pole.

Let the transits of such a star over the meridian be observed above and below the pole; and it is manifest, that if the time of the first transit above the pole be subtracted from the time of the second transit above the pole (adding 24 hours if necessary), the remainder will be the time by the watch, in which the earth (or the star apparently) makes one diurnal revolution. It is also evident, that if the two intervals between the time of the transit below the pole be equal, the instrument must be exactly in the meridian. If the interval between the first transit above the pole, and the transit below the pole be greater than the interval between the transit below the pole and the second transit above it, the object end of the telescope, when directed toward the elevated pole, lies to the east of the true meridian; but if the latter interval be greatest, the object end of the telescope, when directed towards the elevated pole, lies west of the true meridian.

To correct the error, bring the instrument into the meridian; add 24 hours to the time of the latter transit above the pole, subtract the time of the former from it, and take half the remainder. Take the difference between this and the interval between the transits above and below the pole, and take half this difference. Then, as the time by the watch of an entire revolution is to 24 hours, so is this half difference to the half difference in sideral time. Add to the logarithm of this half difference, the logarithmic tangent of the star's polar distance; and the logarithmic secant of the latitude of the place, the sum, rejecting 20 from the index, will be the logarithm of the number of seconds in time, which expresses the angle made by the instrument and the meridian.

Consider what part this angle makes of the interval, between the wires which are in the focus of the telescope; and turn the instrument on its axis till the telescope points at the horizon. Look out for some tolerably distant object which is cut by one of the wires; and by turning the screw *g*, remove the wire to the east or west of this object (as may be required), such a part of the space between that wire and the next to it, as the angular error which the instrument makes of that interval. You must then proceed to examine the position of the instrument again, either by the same, or some other circumpolar star, and to correct it, if it requires correction, until you get it exactly into the plane of the meridian; and when you have, a mark must be set up in the meridian at as great a distance from the instrument as may be convenient, or as it can be seen distinctly; and the telescope must be carefully adjusted to this mark before every observation. See also vol. 1 of the Transactions of the Astronomical Society.

4. *To adjust a clock or watch by observation.*

96. In the preceding article we have supposed the watch to be already regulated, and have shown how from this the transit may be adjusted; we shall now suppose the instrument correctly fixed, and show the method of adjusting the clock or watch.

Regulation of the clock.

The instrument then being thus adjusted, observe the hour, minute, second, &c. when any particular star traverses over the centre of the field of the telescope, by taking the mean of the times when it passes each wire as described (art. 87). Observe the same the next day, and for several days in succession, as also with different stars; and if the clock be properly adjusted to sideral time, each star ought to transit the meridian at the same instant every day that it is observed. If it marks different times on the different days, it will be easy to determine from the mean of these observations, how much it gains or loses per day; and if this error be considerable, the pendulum must be lengthened or shortened by the proper adjusting screw according as it gains or loses; but if the error be not more than a fraction of a second, or even a second or two, and uniform for every equal interval of time, the pendulum may be allowed to remain, and the proper correction applied whenever any observation is made.

We here suppose the clock to be adjusted to sideral time, or to register exactly 24 hours from one transit of any fixed star to another; if the clock be adjusted to mean solar time, like those employed for common purposes, it ought to show only 23h. 56′ 41″.1, which is the length of a sideral day in mean solar time. But for all the purposes of an observatory, sideral time is to be preferred. The clock may also be regulated by the transit of the sun; but before we can employ it for this purpose, it will be necessary to enter at some length into an explanation of what is termed the equation of time. We shall therefore only further observe, that the best way of observing the transit of any heavenly body, is to watch it first into the telescope; then the clock being supposed close at hand, note the hour, minute, and second of its entrance; and with your eye then applied to the telescope, count the beats of the pendulum till the body passes the first wire, and note down the exact time; between the time of its passing the first and second wire observe again the time by the clock, and proceed again as before, and so on with all the wires; and the mean of the several results will be the true time of its passing the centre wire as already explained. (art. 87.)

Estimation of right ascension in time.

97. The right ascension of any heavenly body is, as we have already observed, the arc of the equator intercepted between the first point of aries, or that point where the equator is cut by the ecliptic, and the point where a secondary to the equator, passing through the body, meets the latter circle. But as the motion of the earth, or the apparent motion of the heavens, is uniform, we may also denote any arc of right ascension, by the interval which elapses between the time when the first point of Aries passes the meridian of any place, and that of the transit of the proposed body. The right ascension may be therefore estimated in measure or time; but the latter is the most common: in the Nautical Almanack, for example, we always find the sun's right ascension noted in sideral hours, minutes, seconds, &c. The right ascension of any heavenly body is then readily obtained by means of our clock and transit instrument; for the former being set to 0 hours at the moment when the first point of Aries passes the meridian of any place; and being supposed correctly adjusted to sideral time by means of the preceding observations, the right ascension of any body will be shown in time by the clock, and this, when requisite, may be immediately reduced to angular measure by saying, as 24 hours : to the time shown by the clock :: 15° : to the measure sought; or the reduction may be made by means of a table computed for the purpose. If the clock has any rate, that is, if it does not show correct sideral time, the proper correction must be applied, as indicated above; and if the clock be not so adjusted as to show 0h. 0m. 0s. when the first point of Aries passes the meridian, then it is obvious that the difference of the two times will be the right ascension sought.

Certain requisite corrections.

It is proper, however, to make here one important remark, which is, that what we have called above the first point of Aries, is not a fixed point, but that it changes its place by a slow retrograde motion from year to year, called *the precession of the equinoxes*; therefore, the right ascension of the stars is also variable, and stand in need of constant corrections; we shall not, however, enter upon this subject at present, our purpose here was only to show how the right ascension of a heavenly body might be determined, that of any particular body being given.

Most of the principal stars have had their right ascensions ascertained with the utmost precision, particularly 36 of them, by Dr. Maskelyne, as well as their annual variations; these, therefore, are now commonly employed by most astronomers for the purpose of regulating their clocks, and then by means of the clock, the right ascension of the sun, moon, and planets, at any time, is ascertained; as well as that of any fixed star, (which is supposed to be not correctly established,) may also be determined according to the principles above explained.

Corrections for observed altitudes

98. We have seen in our illustration of the circles of the sphere, that the altitude of any body when on the meridian will be sufficient for determining its declination; for the altitude of the equator or point E in our fig. 22, is equal to the co-latitude; and the difference therefore, between the meridian altitude and the co-latitude of the place of observation is equal to the declination, which in our latitude will be north or south, according as the former of those quantities is greater or less than the latter; and having the right ascension and declination, the latitude and longitude of the body may be computed as explained in art. 75; or, on the contrary, the latitude and longitude being given, the right ascension and declination may be computed, the obliquity of the ecliptic being supposed given. But as we have before observed, the altitude of a body, as determined from observation, stands in need of certain corrections for parallax, refraction, &c. which we have not yet investigated, consequently, the student cannot yet proceed to apply

such observations to actual astronomical determinations.

§ V. *Of sideral and mean solar days, years, &c.*

99. We have seen that the earth performs its revolution on its axis with a motion perfectly uniform; and that the interval between the return of any fixed star to the same meridian, is what is called a sideral day; observing that by the same meridian is here to be understood, the star appearing at the same meridian altitude; for the circumpolar stars, as we have seen, may be observed twice on the same meridian in the course of 24 hours, once in their inferior and once in their superior passage; but they are only once at the same meridian altitude. This interval, then, is the length of the sideral day; but it remains for us now to explain what is to be understood by a mean solar day, or that day which is employed in the common concerns of life.

1. *Of the mean solar day.*

Mean solar day.

100. It will be seen in the following articles, that the interval time between the sun's leaving the first point of Aries to its return again to the same, which is what is denominated a solar year, is performed in about 365¼ solar days; or the sun will have appeared in that interval 365 times on the meridian, and will besides have performed nearly one-fourth of his 366th revolution: hence, if all the solar days were equal, that is, if the sun returned to the meridian of an observer always after the same interval, the increase of his right ascension every day, or the additional angle which the earth (having performed a complete revolution) would have to move through to bring the sun again upon the meridian of the observer, would be equal to $\frac{360^\circ}{365\frac{1}{4}} = 59' \, 8''{\cdot}2$; if, therefore, to the sideral day we add the time which the earth employs in describing 59′ 8″·2, we shall have the length of the mean solar day; that is, the sideral day is to the mean solar day as 360° : 360° 59′ 8″·2.

Consequently, if we call the mean solar day 24 hours, according to common reckoning of time, we shall have

$$360^\circ \, 59' \, 8'' : 360^\circ :: 24\text{h.} : 23\text{h.} \, 56\text{m.} \, 4{\cdot}1''$$

which is the length of a sideral day in mean solar hours; and by reversing the first two terms of the proportion, we shall have the mean solar day expressed in sideral hours.

2. *Sideral year.*

Sideral year.

101. It appears from what is stated above, and we have before made the same remark in a general way, that the sun has a continual motion in the heavens from east to west; and that after a certain period, he will again obtain with respect to the same star, the same relative situation; so that if he were in conjunction with it in the first instance, he will return again to conjunction after this interval, which is therefore called a *sideral year*.

Determined by observation.

In order to determine the duration of this period from observation, take on any day the difference between the sun's right ascension and that of the star, and when the sun returns to the same part of the heavens the next year, compare its right ascension with that of the same star for two days, one when their difference of right ascension is less, and the other when greater than the difference before observed; then it will be obvious that at some instant between these two observations, the right ascension will be exactly the same as it was when first observed.

Now in order to find the precise instant when this happens, let us suppose D to be the difference in right ascension as observed on the two consecutive days; and at the difference between the differences of the sun's and stars' right ascension on the first of these two days, and on the day when the observation was made the day before; and t be the exact time between the intervals of the two transits of the sun over the meridian on the two days; then assuming the motion of the sun to be equal in right ascension during this interval, we shall have

$$\text{D} : d :: t : \frac{d\,t}{\text{D}}$$

the time from the passage of the sun over the meridian on the first day, to the instant when it had the same right ascension, compared with the star, which it had the year before; and the interval between these two times when the difference of right ascension was the same, is the length of the sideral year.

Or if instead of supposing the second observation to have been repeated on the second year, there is an interval of several years between the two observations, and the observed interval of time be divided by the number of years, the length of the year will be had more exactly, any error in the observations being thus rendered less important by being divided into the great number of parts. The best time for these observations is, when the sun is in or near one of the equinoxes or one of the solstices, his motion in right ascension being then exactly or very nearly uniform.

As an example of this kind, we may state the following:—

By MM. Picard and La Caille.

102. April 1, 1669, at 0h. 3m. 47s. of mean solar time, M. Picard observed the difference between the longitude of the sun and the star Procyon to be 3s. 8° 59′ 36″. And M. La Caille found the difference of longitude between the sun and star to be the same, on April 2, 1745, at 11h. 10′ 45″. The sun, therefore, made 76 complete revolutions with regard to the same fixed star in 76 years 1 day 11h. 6m. 58s.; or in 27,759 days 11h. 6m. 58s.; we have, therefore, by dividing this interval by 76, 365 days 6h 8′ 47″ for the length of the sideral year; more recent observations, however, give 365 days 6h. 9m. 11·5s. for this interval.

3. *Of the tropical year.*

The length of the tropical year.

103. The length of the tropical year is the interval between the sun leaving either equinoctial point to its return again to the same; which it does in a less time than it passes from any fixed star to the same again, the latter we have seen is what is termed the sideral year; and the former being that on which the change of seasons depend is called the *solar* or *tropical* year. To determine the length of this year, we may proceed as follows:

Determined from observation.

Observe the meridian altitude of the sun on the day nearest the equinox, and the next year take its meridian altitude again on two successive days, on the one when its altitude is greater, and on the other when it is less, than in the first observation; then it is obvious, that at some intermediate time between the two last observations, the sun must have had the same altitude or declination as in the first instance. Now, to find the precise instant, let D be the difference of altitude in the two last observations, t the interval between them, which may be here taken as 24 hours; also d the difference between the altitude as observed on the first of the two latter days, and that taken the year before; then, assuming the declination to be uniform, as it actually is at this time, say, as $D : d :: 24 \text{ hours} : \frac{24\,d}{D}$ the time from the first of the two latter observations, to the instant when the declination was the same as in the preceding year. This time, therefore, being added to the number of days between the two first observations, will give the true length of the solar year.

Here again, as in last case, if the two observations be repeated after an interval of several years, we may look for a result more nearly approximating to the truth.

By the Cassinis.

104. The two following observations were made by Cassini and his son, after an interval of 44 years.

March 20, 1672, Meridian alt. sun's upper limb .... 41° 43′ 0″
March 20, 1722 ditto ...... 41° 27′ 10″
March 21 ditto ...... 41° 51′ 0″

$$\begin{array}{rrrcrrr} 41 & 51 & 0 & & 41 & 43 & 0 \\ 41 & 27 & 10 & & 41 & 27 & 10 \\ \hline & 23 & 50 & : & 15 & 50 & :: 24 : 15\text{h. } 56\text{m. } 39\text{s.} \end{array}$$

Whence on the 20th of March, at 15h. 56m. 39s., the sun's declination was the same as on the 20th of March, 1672, at noon.

Now the interval between the two first of these observations was 44 years, 34 of these were common years of 365 days each, and 10 of 366 days each, making in all an interval of 16,070 days, and therefore the interval between the two periods when the sun's declination was the same was, 16,070 days 15h. 56m. 39s. and this interval embraces 44 tropical years, whence $\frac{16{,}070\text{d. } 15\text{h. } 56\text{m. } 39\text{s.}}{44}$ 365d. 5h. 49′ 0″ 53‴ the length of the tropical year, as determined from these results; more recent observations give for the true length of the tropical year, 365 days 5 hours 48′ 48″.

4. *Precession of the equinoxes.*

Precession of the equinoxes.

105. It appears from what is stated above, that the sun returns to the equinoxes every year, before it returns again to the same fixed star, or to the same point in the heavens; the equinoctial points must, therefore, have a retrograde motion with respect to that of the earth, the cause of which it is not for us at present to explain; we shall, however, hereafter, see that it is due to a regular mechanical effect, viz. the attraction of the sun and moon upon the earth in consequence of its spheroidal figure. The effect of this is, that the longitude of the stars, which are always estimated from the intersection of the equator and ecliptic, or from the equinoctial point, or first point of Aries, must constantly increase, and by comparing the longitude of the same stars at different times, the mean motion of the equinoctial points, or the precession of the equinoxes may be determined.

We have observations of this kind from the time of Timocharis and Hipparchus; but we may be allowed to entertain considerable doubt as to accuracy; it will be sufficient to observe, that from a comparison of the best observation, the secular precession, or that which takes place in 100 years, amounts to 1° 23′ 45″ or to 50·34″ annually.

5. *Anomalistic year.*

Anomalistic year.

106. At a certain time of the year, the sun's diameter, if measured instrumentally, would be found to be the least; at which time it is obvious he would be the most distant from the earth or in his apogee, that is, the earth will then be at one extremity of the transverse axis of its orbit, and at that extremity which is furthest from the sun. Now, if at the end of a certain interval, a year from the first observation, a second were made, and the sun were found in precisely the same relative situation with regard to certain fixed stars, when his diameter was least, then it would be obvious that the sun had always the least apparent diameter after the completion of a sideral year; but the astronomical fact is not so; the sun does not return to a point in the heavens where his diameter is least in a sideral year; but in an interval a little exceeding it and this interval is what astronomers have called the *anomalistic year*, the apogee has therefore a progressive motion, as we have seen the equinoxes have a precession; the quantity of the former, like that of the latter, being found from observation. According to the most recent determination, the increase in longitude of the sun's apogee in 100 years, is 1° 42′ 28″, or 1′ 2·2″ annually; but since the precession, which is a regressive motion, is 50·34″ annually, the annual sideral progression is 62·2″ − 50·34″ = 11·86″. Now, the time of describing 11·86″ added to the length of the sideral year, will compose an anomalistic year; and since the sun near its apogee moves in longitude about 58′ in 24 hours; the time will be about 4m. 50s.; hence the length of the anomalistic year, is equal to

365d. 6h. 9m. 11·4s. + 4m. 50s. = 365d. 6h. 14m. 1·4s.

6. *Obliquity of the ecliptic.*

Obliquity of the ecliptic.

107. The angle contained between the plane of the equator and the ecliptic is what is denominated the obliquity of the ecliptic; which is shown from repeated observations to be variable, like the other quantities we have been just examining. We have already, in the preceding section, indicated the method of determining the measure of this angle by means of the greatest and least meridian altitudes of the sun; it will therefore be sufficient in this place to show the result of a long succession of such observations by different astronomers, which are as follows:—

Determined by different astronomers.

	°	′	″
Eratosthenes, 230 years B.C.	23	51	20
Hipparchus, 140 years B.C.	23	51	20
Ptolemy, A.D. 140	23	51	10
Pappus, A.D. 390	23	30	0
Albatenius, in 880	23	35	40
Arzachel, in 1070	23	34	0
Prophatius, in 1300	23	32	0
Regiomontanus, in 1460	23	30	0

	°	′	″
Waltherus, in 1490	23	29	47
Copernicus, in 1500	23	28	24
Tycho, in 1587	23	29	30
Cassini (the father), in 1656	23	29	2
Cassini (the son), in 1672	23	28	54
Flamstead, in 1690	23	28	48
De la Caille, in 1750	23	28	19
Dr. Bradley, in 1750	23	28	18
Mayer, in 1750	23	28	18
Dr. Maskelyne, in 1769	23	28	8,5
M. de Lalande, in 1768	23	28	0
M. Pond, Ast. Roy. 1816	23	27	50

The observations of Albatenius, an Arabian, are here corrected for refraction. Those of Waltherus, M. De la Caille computed. The obliquity by Tycho is put down as correctly computed from his observations; also the obliquity, as determined by Flamstead, is corrected for the nutation of the earth's axis; these corrections M. de Lalande applied. It is manifest from the above observations, that the obliquity of the ecliptic continually decreases; and the irregularity which here appears in the diminution, we may ascribe to the inaccuracy of the ancient observations, as we know that they are subject to greater errors than the irregularity of this variation. If we compare the first and last observations, they give a diminution of 70″ in 100 years. If we compare the observation of Lalande with that of Tycho, it gives 45″. The same compared with that of Flamstead gives 50″. If we compare that of Dr. Maskelyne with Dr. Bradley's and Mayer's, it gives 50″. The comparisons of Dr. Maskelyne's determination, with that of M. de Lalande, which he took as the mean of several results, gives 50″, as determined from the most accurate observations. This result agrees very well with that deduced from theory; but the observation of Mr. Pond, as compared with those of Bradley, gives 66″ for the variation in the obliquity in 100 years, or 0·40″ annually.

§ VI. *Of the corrections for refraction, parallax, &c.*

1. *Of refraction.*

Refraction.

108. When a ray of light passes out of a vacuum into any medium, or out of a medium into any other of a greater density, it is found to deviate from its regular course, towards a perpendicular to the surface of the medium into which it enters. (See Optics, p. 427) Hence light passing out of a vacuum into the atmosphere, will, where it enters, be bent or deflected towards a radius drawn to the earth's centre, the extreme surface of the atmosphere being supposed spherical and concentric with the centre of the earth; and as in approaching the earth's surface the density of the atmosphere continually increases, the rays of light are constantly entering into a denser medium, and therefore the course of the rays will continually deviate from a right line, and describe a curve; whence, at the surface of the earth, the rays of light enter the eye of a spectator, in a different direction from that in which they would have entered it, if there had been no atmosphere. Consequently, the apparent place of a body from which the light comes must be different from the true place, as shown in fig. 35.

Fig. 35.

Observed by the ancients.

109. Although we here state the fact of the refraction of the atmosphere as a necessary consequence of established laws in Optics, it must not be understood to have been introduced into astronomy as such; for it was observed by the ancients long before they were able to trace its cause to optical principles. Nothing is indeed more easy to be detected in astronomical observation; for by taking the greatest and least altitudes of the circumpolar stars, it will be seen that their apparent north polar distances will be different accordingly as it is taken at the time of their superior or inferior passage; and this variation is observed to be very nearly constant for the same place and the same star, and for all stars that have the same declination; but to vary according to a certain law, in stars that pass at different altitudes; it is also determinable from observation, that refraction does not alter the azimuth of bodies, and the same may also be demonstrated on physical principles, as we have shown in our treatise on Optics. Refraction, therefore, has a tendency only to increase the apparent altitude of a heavenly body, its entire effect being produced in a vertical circle, and its effect is less and less from the horizon to the zenith, where it vanishes.

Tycho endeavoured to estimate its quantity.

Both Ptolemy and Alhazen were acquainted with this irregularity, and attributed it to its true cause, but neither of them undertook to determine the quantity of it. Tycho perceiving that the altitude of the equator, as deduced from the two solstices, was not the exact complement of the height of the pole, endeavoured to determine the quantity of refraction due to each zenith distance; he made the horizontal refraction 34″, and supposed it to become insensible at 45° of altitude; in the former, he was not far from the truth, but the latter conclusion was wholly erroneous; it is, in fact, to Dominic Cassini that we are indebted for the first regular hypothesis on the subject of astronomical refraction; but as his solution leads to an expression in which the refraction is made proportional to the tangent of the zenith distance, diminished by a quantity which itself depends upon the refraction sought; we shall not insist upon it in this place, but proceed to illustrate the formulæ given by Bradley.

Bradley's formulæ.

Bradley's formulæ.

110. Various tables of refraction more or less correct had been already formed, when the above celebrated astronomer commenced his observations for the purpose of deducing more exact formulæ than were at that time in existence, and of these tables he availed himself in settling the law of this important astronomical correction.

By means of numerous observations on Polaris and other circumpolar stars, Bradley deduced the apparent zenith distance P of the pole. By observations also on the sun at the equinoxes, when this body had the same zenith distance but opposite right ascensions, he deduced the height of the equator. In this instance, as in the former, it was only the apparent altitude that was obtained on account of the refraction, and therefore greater than the true height, and consequently, the apparent zenith distance was less than the true; whence the sum of the two zenith distances of the poles and the equator (which if true ought to be = 90°) will be less than 90°, by the sum of the two refractions due respectively to the zenith distances

P and Q. Conceive, for instance, the two refractions to be p and q, then,

$$P + Q = 90° - (p + q)$$

and $$p + q = 90° - (P + Q)$$

consequently, the sum of the two refractions $p + q$ is given, but the object is to determine them separately; but, by referring to the best tables extant on this subject, it was found, that the difference $q - p$ was about equal to 2 seconds, and hence the two equations

$$p + q = 90° - (P + Q)$$
$$p - q = 2''$$

gave

$$q = 45°\ 0'\ 1'' - \tfrac{1}{2}(P + Q)$$
$$p = 44°\ 59'\ 59'' - \tfrac{1}{2}(P + Q)$$

Now, according to Bradley's observation, he found

$$P = 38°\ 30'\ 35''$$
$$Q = 51°\ 27'\ 28''$$

whence, $p = 57''{\cdot}5$ and $q = 59''{\cdot}5$

but this being only an approximation, in order to obtain a more correct result; the author separated the sum $p + q = 1'\ 57''$ into two parts $p' + q'$ which should be to one another as the tangents of the observed zenith distances; whence he obtained

$$p' = (p' + q') \times \frac{\tan 38°\ 31'\ 32''{\cdot}5}{\tan 38°\ 31'\ 32''{\cdot}5 + \tan 51°\ 28'\ 27\tfrac{1}{2}''}$$

$$q' = (p' + q') \times \frac{\tan 51°\ 28'\ 27\tfrac{1}{2}''}{\tan 38°\ 31'\ 32\tfrac{1}{2}'' + \tan 51°\ 28'\ 27\tfrac{1}{2}''}$$

and by means of the new operation, he found

$p' = 45''{\cdot}5$ at the co-altitude $38°\ 31'\ 20\tfrac{1}{2}''$
$q' = 1'\ 11''{\cdot}5$ ditto $51°\ 28'\ 39\tfrac{1}{2}''$

111. We have stated above that Bradley assumed the refraction to vary as the tangent of the zenith distance; the principle on which this assumption is founded, may be illustrated as follows:—

Fig. 36. Let CAn, fig. 36. be the angle of incidence, CAm the angle of refraction; and, consequently, mAn the quantity of refraction; let CT be the tangent of the arc Cm, mr its sine, nw the sine of Cm, and draw rm parallel to vw; then as the refraction of the arc is very small, we may consider mrn as a rectilinear triangle; and hence by similar triangles,

$$Av : Am :: rn : mn = \frac{Am \times rn}{Av}$$

but Am is constant, and as the ratio of mv to mw is also constant by the laws of refraction; their difference rn must vary as mv; hence mn varies as $\frac{mv}{Av}$; but $CT = \frac{Am \times mv}{Av}$; which varies as $\frac{mv}{Av}$ because Am is constant: consequently, the refraction mn varies as CT, the tangent of the apparent zenith distance of the star; for the angle of refraction CAm, is the angle between the refracted ray and the perpendicular to the surface of the medium, which perpendicular is directed to the zenith; therefore, while the refraction is very small, so that rmn may be considered as rectilinear, this rule may be considered as furnishing a good approximation.

Formation of tables. 112. Assuming, therefore, the preceding quantities (art. 110) as the true or actual refractions, at the altitude of the pole and equator; and adopting the above analogy with reference to the tangents of the zenith distances, Bradley deduced the refractions for other altitudes exceeding that of the equator, and for less ones or greater zenith distances, he employed the circumpolar stars. That is, supposing xy, (fig. 37) to be the true places of a circumpolar star at its least and greatest altitude, then Zx, by correcting the observed distance is known, consequently ZP is known, and Px is so likewise. Again, the apparent zenith distance of the star at y, is observed, and subtracting from it ZP, the apparent distance from P is known, which is less than the true distance Py, by a certain quantity which is determinable because Px = Py, and the former has been determined, and this difference is obviously the correction due to the zenith distance Zy.

Fig. 37.

As an example, it was found by observation on Cassiopea at its greatest altitude, that the apparent zenith distance was

	13°	48′	12½″
Correct for refraction	0	0	14
True zenith dist.	13	48	26½
Zenith dist. of pole	38	31	20½
North polar dist.	24	42	54

Again, by observation, the star being at its least altitude above the horizon, the zenith distance was found to be

	63°	13′	21″·8
Subtract	38	31	20·5
App. N. P. dist.	24	42	1·3
True ditto	24	42	54
Refraction at 63° 13′ 21″·8 Z. D.	0	0	52·7

By means of similar observations, Bradley determined the refractions for other altitudes; and after tabulating the results and a due examination of them, he found that the law of the refraction instead of being simply proportional to the tangent of the zenith distance, was of the form

$$r = \frac{m}{n} \tan (z - 3\,r)$$

And deducing from observation the values of m and n under different temperatures and different barometrical pressures, he obtained the formula

$$\text{refrac.} = \frac{a}{29{\cdot}6} \times \tan (Z - 3\,r) \times 57'' \times \frac{400}{350 + h}$$

In which a is the altitude of the barometer in inches
Z the zenith distance,
$r = 57'' \times \tan Z$,
p = height of Fahrenheit's thermometer,
29·6 = mean height of barometer.

Groombridge's formulæ. 113. This formula is found to apply with considerable accuracy for all altitudes greater than 10°, but for less altitudes it is very erroneous; and different formulæ and tables have accordingly been computed by more recent astronomers which approach nearer to the truth: of these the results published by Mr. Groombridge in the *Phil. Tran. for* 1814, are perhaps the most valuable; his formula, under the medium temperature and pressure, is as follows:

$$\text{refrac.} = \tan (z - 3{\cdot}6342956\ r) \times 58''{\cdot}132967$$

which answers for all altitudes above 3°; for less altitudes a farther correction becomes necessary; viz.

for every minute below 3° of altitude; or for every minute more than 87° of zenith distance, the result found as above must be reduced ·00462. By means of these formulæ, Mr. Groombridge has computed a very extensive table of refractions, with the requisite tables of corrections for the different states of barometrical pressure, as well as for thermometrical temperatures, both for the outside and inside of the observatory. We cannot allow ourselves to transcribe this table in the extended form given to it by its author, but the following abridgment of it will, it is presumed, be found highly acceptable to our readers. It will be understood that the correction for the barometer and thermometer will be the sum of the two factors in Table II. and Table III. multiplied into the mean refraction, and the product added or subtracted therefrom, according as the sum of the factors is plus or minus.

Table of mean refractions, computed from the preceding formula.

Zen. Dist	Refrac.	Zen. Dist.	Refrac.	Zen. Dist.	Refrac.	Zen. Dist.	Refrac.	Zen. Dist.	Refrac.
° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″
0 0	0 0·00	22 0	0 23·46	44 0	0 56·03	66 0	2 9·77	87 8	14 46·37
0 30	0 0·51	22 30	0 24·05	44 30	0 57·01	66 30	2 12·85	87 16	15 14·40
1 0	0 1·01	23 0	0 24·65	45 0	0 58·01	67 0	2 16·05	87 24	15 44·00
1 30	0 1·52	23 30	0 25·25	45 30	0 59·03	67 30	2 19·38	87 32	16 15·27
2 0	0 2·03	24 0	0 25·85	46 0	1 0·07	68 0	2 22·85	87 40	16 48·35
2 30	0 2·53	24 30	0 26·46	46 30	1 1·13	68 30	2 26·47	87 48	17 23·37
3 0	0 3·04	25 0	0 27·07	47 0	1 2·20	69 0	2 30·25	87 56	18 0·46
3 30	0 3·55	25 30	0 27·69	47 30	1 3·30	69 30	2 34·21	88 0	18 19·83
4 0	0 4·06	26 0	0 28·32	48 0	1 4·41	70 0	2 38·34	88 6	18 49·98
4 30	0 4·57	26 30	0 28·95	48 30	1 5·55	70 30	2 42·68	88 12	19 21·51
5 0	0 5·08	27 0	0 29·58	49 0	1 6·72	71 0	2 47·23	88 18	19 54·47
5 30	0 5·59	27 30	0 29·80	49 30	1 7·90	71 30	2 52·01	88 24	20 28·89
6 0	0 6·10	28 0	0 30·87	50 0	1 9·11	72 0	2 57·03	88 30	21 5·08
6 30	0 6·62	28 30	0 31·52	50 30	1 10·34	72 30	3 2·33	88 36	21 42·88
7 0	0 7·13	29 0	0 32·18	51 0	1 11·60	73 0	3 7·92	88 42	22 22·47
7 30	0 7·64	29 30	0 32·85	51 30	1 12·89	73 30	3 13·82	88 48	23 3·95
8 0	0 8·16	30 0	0 33·52	52 0	1 14·21	74 0	3 20·07	88 54	23 47·40
8 30	0 8·68	30 30	0 34·20	52 30	1 15·55	74 30	3 26·69	89 0	24 32·94
9 0	0 9·20	31 0	0 34·88	53 0	1 16·93	75 0	3 33·73	89 4	25 4·51
9 30	0 9·72	31 30	0 35·57	53 30	1 18·33	75 30	3 41·22	89 8	25 37·09
10 0	0 10·24	32 0	0 36·27	54 0	1 19·78	76 0	3 49·21	89 12	26 10·71
10 30	0 10·76	32 30	0 36·98	54 30	1 21·25	76 30	3 57·75	89 16	26 45·40
11 0	0 11·29	33 0	0 37·70	55 0	1 22·77	77 0	4 6·89	89 20	27 21·20
11 30	0 11·81	33 30	0 38·42	55 30	1 24·31	77 30	4 16·72	89 24	27 58·14
12 0	0 12·34	34 0	0 39·15	56 0	1 25·91	78 0	4 27·30	89 28	28 36·26
12 30	0 12·87	34 30	0 39·89	56 30	1 27·54	78 30	4 37·72	89 32	29 15·60
13 0	0 13·40	35 0	0 40·64	57 0	1 29·21	79 0	4 51·09	89 36	29 56·19
13 30	0 13·94	35 30	0 41·40	57 30	1 30·93	79 30	5 4·53	89 40	30 38·07
14 0	0 14·48	36 0	0 42·17	58 0	1 32·69	80 0	5 19·18	89 44	31 21·28
14 30	0 15·02	36 30	0 42·95	58 30	1 34·51	80 30	5 35·21	89 48	32 5·85
15 0	0 15·56	37 0	0 43·74	59 0	1 36·38	81 0	5 52·83	89 52	32 51·82
15 30	0 16·10	37 30	0 44·53	59 30	1 38·30	81 30	6 12·26	89 56	33 39·84
16 0	0 16·65	38 0	0 45·34	60 0	1 40·28	82 0	6 33·79	90 0	34 28·13
16 30	0 17·20	38 30	0 46·16	60 30	1 42·32	82 30	6 57·78	90 2	34 53·15
17 0	0 17·75	39 0	0 47·00	61 0	1 44·42	83 0	7 24·63	90 4	35 18·55
17 30	0 18·31	39 30	0 47·84	61 30	1 46·59	83 30	7 54·87	90 6	35 44·34
18 0	0 18·86	40 0	0 48·69	62 0	1 48·83	84 0	8 29·13	90 8	36 10·52
18 30	0 19·43	40 30	0 49·56	62 30	1 51·14	84 30	9 8·18	90 10	36 37·10
19 0	0 19·99	41 0	0 50·44	63 0	1 53·53	85 0	9 53·03	90 12	37 4·08
19 30	0 20·56	41 30	0 51·34	63 30	1 56·00	85 30	10 44·88	90 14	37 31·47
20 0	0 21·13	42 0	0 52·25	64 0	1 58·56	86 0	11 45·57	90 16	37 59·28
20 30	0 21·71	42 30	0 53·17	64 30	2 1·21	86 24	12 41·02	90 18	38 27·50
21 0	0 22·29	43 0	0 54·11	65 0	2 3·96	87 48	13 44·62		
21 30	0 22·87	43 30	0 55·06	65 30	2 6·81	87 0	14 19·80		

Correction to preceding table of refraction.

BAROMETER.				FAHRENHEIT'S THERMOMETER.					
Inches.	Correc. −	Inches.	Correc. +	Degree.	Within. +	Without. +	Degree.	Within. −	Without.
28·60	·0350	29·60	·0000	24·0	·0575	·0420	49·0	·0000	·0080
62	·0342	62	·0007	24·5	·0563	·0410	49·5	·0011	·0090
64	·0335	64	·0014	25·0	·0552	·0400	50·0	·0022	·0100
66	·0328	66	·0020	25·5	·0540	·0390	50·5	·0033	·0110
68	·0321	68	·0027	26·0	·0529	·0380	51·0	·0044	·0120
28·70	·0314	29·70	·0034	26·5	·0517	·0370	51·5	·0055	·0130
72	·0306	72	·0041	27·0	·0506	·0360	52·0	·0066	·0140
74	·0299	74	·0047	27·5	·0494	·0350	52·5	·0077	·0150
76	·0292	76	·0054	28·0	·0483	·0340	53·0	·0088	·0160
78	·0285	78	·0061	28·5	·0471	·0330	53·5	·0099	·0170
28·80	·0278	29·80	·0068	29·0	·0460	·0320	54·0	·0110	·0180
82	·0271	82	·0074	29·5	·0448	·0310	54·5	·0121	·0190
84	·0264	84	·0081	30·0	·0437	·0300	55·0	·0132	·0200
86	·0256	86	·0088	30·5	·0425	·0290	55·5	·0143	·0210
88	·0249	88	·0095	31·0	·0414	·0280	56·0	·0154	·0220
28·90	·0242	29·90	·0101	31·5	·0402	·0270	56·5	·0165	·0230
92	·0235	92	·0108	32·0	·0391	·0260	57·0	·0176	·0240
94	·0228	94	·0115	32·5	·0379	·0250	57·5	·0187	·0250
96	·0221	96	·0122	33·0	·0368	·0240	58·0	·0198	·0260
98	·0214	98	·0128	33·5	·0356	·0230	58·5	·0209	·0270
29·00	·0207	30·00	·0135	34·0	·0345	·0220	59·0	·0220	·0280
02	·0200	02	·0142	34·5	·0333	·0210	59·5	·0231	·0290
04	·0193	04	·0149	35·0	·0322	·0200	60·0	·0242	·0300
06	·0186	06	·0155	35·5	·0310	·0190	60·5	·0253	·0310
08	·0179	08	·0162	36·0	·0299	·0180	61·0	·0264	·0320
29·10	·0172	30·10	·0169	36·5	·0287	·0170	61·5	·0275	·0330
12	·0165	12	·0176	37·0	·0276	·0160	62·0	·0286	·0340
14	·0158	14	·0182	37·5	·0264	·0150	62·5	·0297	·0350
16	·0151	16	·0189	38·0	·0253	·0140	63·0	·0308	·0360
18	·0144	18	·0196	38·5	·0241	·0130	63·5	·0319	·0370
29·20	·0137	30·20	·0203	39·0	·0230	·0120	64·0	·0330	·0380
22	·0130	22	·0210	39·5	·0218	·0110	64·5	·0341	·0390
24	·0123	24	·0216	40·0	·0207	·0100	65·0	·0352	·0400
26	·0116	26	·0223	40·5	·0195	·0090	65·5	·0363	·0410
28	·0109	28	·0230	41·0	·0184	·0080	66·0	·0374	·0420
29·30	·0102	30·30	·0237	41·5	·0172	·0070	66·5	·0385	·0430
32	·0096	32	·0243	42·0	·0161	·0060	67·0	·0396	·0440
34	·0089	34	·0250	42·5	·0149	·0050	67·5	·0407	·0450
36	·0082	36	·0257	43·0	·0138	·0040	68·0	·0418	·0460
38	·0075	38	·0264	43·5	·0126	·0030	68·5	·0429	·0470
29·40	·0068	30·40	·0270	44·0	·0115	·0020	69·0	·0440	·0480
42	·0061	42	·0277	44·5	·0103	·0010	69·5	·0451	·0490
44	·0054	44	·0284	45·0	·0092	—	70·0	·0462	·0500
46	·0048	46	·0291	45·5	·0080	·0010	70·5	·0473	·0510
48	·0041	48	·0297	46·0	·0069	·0020	71·0	·0484	·0520
29·50	·0034	30·50	·0304	46·5	·0057	·0030	71·5	·0495	·0530
52	·0027	52	·0311	47·0	·0046	·0040	72·0	·0506	·0540
54	·0020	54	·0318	47·5	·0034	·0050	72·5	·0517	·0550
56	·0014	56	·0324	48·0	·0023	·0060	73·0	·0528	·0560
58	·0007	53	·0331	48·5	·0011	·0070	73·5	·0539	·0570

Other effects of refraction.

114. What we have hitherto stated, relates principally to corrections requisite to be made in astronomical observations in consequence of the effect of refraction in elevating all the heavenly bodies, but the same principles will also explain some other astronomical phenomena, as for instance, the morning and evening twilight, the oval appearance of the sun in the horizon, the horizontal moon, &c.

Cause of the twilight

115. With respect to the twilight, it arises both from the refraction and reflection of the sun's rays by the atmosphere. It is probable, that the reflection arises principally from the exhalations of different kinds with which the lower beds of the atmosphere are charged; for the twilight lasts till the sun is farther below the horizon in the evening than it is in the morning when it begins, and it is longer in summer than in winter. Now in the former case, the heat of the day has raised the vapours and exhalations; and in the latter, they will be more elevated from the heat of the season; therefore, supposing the reflection to be made by them, the twilight ought to be longer in the evening than in the morning, and longer in the summer than in the winter.

Commonly, it is assumed, that the twilight begins when the sun is 18° below the horizon; but in our investigations relative to the time of shortest twilight (art. 78), we have left the solution general for any number of degrees, and which may therefore be supplied at pleasure.

Oval figure of the sun and moon in the horizon.

116. Another effect of refraction, as we have above observed, is that of giving the sun and moon an oval appearance in consequence of the refraction of the lower limb being greater than the upper; whereby the vertical diameter is diminished. For assuming the diameter of the sun to be 32′, and the lower limb to touch the horizon, then the mean refraction at that limb is 33′; but the altitude of the upper limb being 32′, its refraction is only 28′ 6″, the difference of which is 4′ 54″, the quantity by which the vertical diameter appears shorter than that parallel to the horizon. This, however, will only happen when the body is in or very near the horizon, for when the altitude is any considerable quantity, the refraction of both limbs being then very nearly the same, the apparent disc will not differ sensibly from a circle.

2. *Of parallaxes.*

Paralla

117. Parallax is a term used by astronomers to denote an arc of the heavens intercepted between the true and apparent place of a star, or other heavenly body, or its place as viewed from the centre and the surface of the earth.

Fig. 37.

Let *s* (fig. 37) represent a star, C the centre of the earth, Z the zenith of a spectator, then the observed zenith distance of *s* is the angle Z A *s*. but its actual zenith distance, as viewed from the centre C, is ZC*s*, and the difference between the angles ZC*s* and ZA*s* is is A*s*C, which is called the angle of parallax.

Now we know from the principles of trigonometry that

$$\sin \mathrm{C}s\mathrm{A} = \sin \mathrm{CA}s \times \frac{\mathrm{CA}}{\mathrm{C}s} = \sin \mathrm{ZA}s \times \frac{\mathrm{CA}}{\mathrm{C}s}$$

whence if CA, C*s* remain the same, the sine of C*s*A, that is, the sine of the parallax varies as the sine of the star's zenith distance. Consequently, the parallax must be greater, the greater the zenith distance; it must therefore be greatest when the body is seen in the horizon, or when the zenith distance is 90°. Let p represent the parallax at any zenith distance Z, P the greatest or horizontal parallax, then we shall have

$$\sin p = \frac{\mathrm{CA}}{\mathrm{C}s} \times \sin \mathrm{Z}, \text{ and } \sin \mathrm{P} = \frac{\mathrm{CA}}{\mathrm{C}s} \times \sin 90^\circ = \frac{\mathrm{CA}}{\mathrm{C}s}$$

Consequently, $\sin p = \sin \mathrm{P} \sin \mathrm{Z}$

If therefore the parallax be known for any one zenith distance, it may be determined for any other; and moreover, if CA the earth's radius, and C*s* the distance of the body, were given, the parallax P would become known; and conversely, if the parallax were given, the distance C*s* might also be determined, the radius of the earth being supposed known from geodetic operations. It is, in fact, from knowing the parallax of one or more of the heavenly bodies, that their distances have been determined.

The correction for parallax is one of the most important in practical astronomy, and accordingly, various methods have been proposed by different astronomers for determining it; but of these we shall only specify one or two of the most obvious.

Parallax of the moon.

118. First, to find the parallax of the moon. Take the meridian altitudes of the moon when it has its greatest north and south latitudes, and correct them for refraction, as explained in the preceding chapter. Then, if there were no parallax, or if the parallax were the same at both altitudes, the difference of the altitudes thus corrected would be equal to the sum of the latitudes, and consequently, what those quantities want of equality will be equal to the difference of the parallax. We have, therefore, by means of these observations, the difference of the parallaxes at these altitudes, and it is required to find them separately. For this purpose, let us denote the two zenith distances, by Z, z, the parallaxes by P, p: then from what has been stated above we have

$$\sin \mathrm{Z} : \sin z :: \mathrm{P} : p$$

or $$\sin \mathrm{Z} - \sin z : \sin. z :: \mathrm{P} - p : p$$

whence $$p = \frac{\sin z\,(\mathrm{P} - p)}{\sin \mathrm{Z} - \sin z}$$

the parallax at the greatest altitude.

We here suppose the moon to be at the same distance from the earth at both observations; when this is not the case, one of the observations must be reduced to what it would have been had the distance been the same.

Determined by observations. Fig. 38

119. Let a body P (fig. 38) be observed from two places, A, B, in the same meridian, then the whole angle APB is the sum of the two parallaxes at those two places. Now we have seen in article 117, that the

$$\text{parallax APC or } p = \mathrm{P} \times \sin \mathrm{PAL}$$
$$\text{parallax PBC or } p' = \mathrm{P} \times \sin \mathrm{PBM}$$

Hence $\mathrm{APB} = p + p' = \mathrm{P} \times (\sin \mathrm{PAL} + \sin \mathrm{PBM})$

Consequently

$$\text{hor. parallax (P)} = \frac{p + p'}{\sin \mathrm{PAL} + \sin \mathrm{PBM}}$$

In order to illustrate this by actual observation, we may state the following example:

October 5, 1751, M. De la Caille, at the Cape of Good Hope, observed Mars to be $1'\,25''\cdot 8$ below the parallel of λ in Aquarius, and at 25° distance from the zenith. On the same day, at Stockholm, Mars was observed to be 1′ 57″·7 below the parallel of the same star, and at 68° 14′ zenith distance. Hence

$$\begin{array}{ll} \text{from} & 1' \; 57''{\cdot}7 \\ \text{take} & 1 \;\; 25{\cdot}8 \\ \hline \end{array}$$

$$\text{angle APB} = p + p' = 0 \;\; 31{\cdot}9$$

$$\text{Whence} \;\; \text{P} = \frac{31{\cdot}9''}{\sin 25^\circ + \sin 68^\circ\, 14'} = 23{\cdot}6$$

The horizontal parallax of Mars being thus determined, we may hence find that for the sun; for we have seen that, generally,

$$\text{P} = \frac{\text{CA}}{\text{C}s} \qquad \text{(see fig. 36)}$$

Consequently, since CA, the radius of the earth, is constant, the parallax will vary reciprocally as the distance of the body; knowing, therefore, the proportional distance of the sun and Mars from the earth at the time of observation; the parallax of the former may be determined when that of the latter is given. This method, however, of determining the solar parallax is not sufficiently accurate for the purposes of modern astronomy.

120. Method of determining the parallax in right ascension and declination.

Parallax in right ascension and declination. Fig. 39.

Let EQ (fig. 39) be the equator, P its pole, Z the zenith, *v* the true place of the body, and *r* the apparent place, as depressed by parallax in the vertical circle *Zk*, and draw the secondaries P*va*, P*rb*, then *ab* is the parallax in right ascension, and *rs* in declination.

$$\text{Now} \qquad vr : vs :: \text{rad} : \sin vrs \text{ or } \text{Z}v\text{P}$$

$$\text{and} \qquad vs : ab :: \cos va : \text{rad}$$

Hence by multiplication, and rejecting the like factors,

$$vr : ab : \cos va : \sin \text{Z}v\text{P}$$

$$\text{therefore} \quad ab = \frac{vr \sin \text{Z}v\text{P}}{\cos va}$$

$$\text{but} \qquad vr = \text{hor. par. (P)} \times v\text{Z}$$

$$\text{and} \qquad \sin v\text{Z} : \sin \text{ZP} :: \sin \text{ZP}v : \sin \text{Z}v\text{P}$$

$$\text{hence} \qquad \sin \text{Z}v\text{P} = \frac{\sin \text{ZP} \times \sin \text{ZP}v}{\sin v\text{Z}}$$

whence by substitution

$$ab = \frac{\text{P} \times \sin \text{ZP} \times \sin \text{ZP}v}{\cos va}$$

Hence it follows, that for the same star, where the *hor. par.* or (P) is given, the parallax in right ascension varies as the sine of the hour angle.

Also,

$$\textit{hor. par.} = \frac{ab \cos va}{\sin \text{ZP} \times \sin \text{ZP}v}$$

In the eastern hemisphere, the apparent place *b* lies on the equator to the east of *a* its true place, and therefore the right ascension is diminished by parallax; but in the western hemisphere *b* lies to the west of *a*, and therefore the right ascension is increased. Hence, if the right ascension be taken before and after the meridian, the whole change of parallax in right ascension between the two observations, is the sum (*s*) of the two parts before and after the meridian, we have therefore

$$s = \frac{vr}{\cos va} \times \text{S}$$

S denoting the sum of the sines of the two hour angles, and

$$\textit{hor. par.} \text{ (P)} = \frac{s \cos va}{\sin \text{ZP} \times \text{S}}$$

On the meridian there is no parallax in right ascension.

Illustration of this method.

In order to apply this rule, observe the right ascension of the planet when it passes the meridian, compared with that of a fixed star, at which time there is no parallax in right ascension; about six hours after, take the difference of their right ascensions again, and observe how much the difference (*d*) between the apparent right ascension of the planet and fixed star has changed in that time. Next observe the right ascension of the planet for three or four days when it passes the meridian in order to get its true motion in right ascension. Then if its motion in right ascension in the above interval of time, between the taking of the right ascensions of the fixed star and planet on and off the meridian, be equal to *d*, the planet has no parallax in right ascension; but if it be not equal to *d*, the difference is the parallax in right ascension, and hence, on the above principles, the horizontal parallax will be known. Or one observation may be made before the planet comes to the meridian and another after, by which a greater difference will be obtained.

By example.

121. In order to illustrate this method by an example, let the following be taken:

On August 15, 1719, Mars was very near a star of the 5th magnitude in the eastern shoulder in Aquarius: and at 9h. 18m. in the evening, Mars followed the star in 10′ 17″; and on the 16th, at 4h. 21m. in the morning, it followed it in 10′ 1″, therefore in that interval, the apparent right ascension of Mars had increased 16″ in time.

But according to observations made in the meridian for several days after, it appeared that Mars approached the star only 14″ in that time, from its proper motion; therefore 2″ in time or 30″ in motion was the effect of parallax in the interval of the observations.

Now the declination of Mars was 15°

$$\begin{array}{ll} \text{the co-latitude} & 41^\circ \;\; 10' \\ \text{the two hour angles} & \begin{cases} 49^\circ \;\; 15' \\ 56^\circ \;\; 39' \end{cases} \end{array}$$

Consequently, the horizontal parallax

$$\text{P} = \frac{30'' \times \cos 15^\circ}{\sin 41^\circ\, 10' \, (\sin 49^\circ\, 15' + \sin 56^\circ\, 39')} = 27\tfrac{1}{2}$$

At the time of these observations, the distance of the earth from Mars was to its distance from the sun as 37 : 100; whence the sun's horizontal parallax is found to be 10·17″.

Effect of parallax in latitude and longitude

122. Besides the effect of parallax in right ascension and declination, it is manifest that the latitude and longitude of the moon and planets must also be effected by it, and as the determination of this, in respect to the moon, is, in many cases, particularly in solar eclipses, of great importance, we shall proceed to show how it may be computed, supposing the latitude of the place, the time, and consequently the sun's right ascension, the moon's true latitude and longitude, and her horizontal parallax to be given.

Fig. 40.

Let HZR (fig. 40) be the meridian, ♈EQ the equator, *p* its pole; ♈C the ecliptic, P its pole; ♈ the first point of Aries, HQR the horizon, Z the zenith, ZL a secondary to the horizon, passing through the true place *r*, and apparent place *t*, of the moon; draw P*t*, P*r*, which produce to *s*, drawing the small circle *ts* parallel to *ov*, and *rs* is the parallax in latitude, and *ov* the parallax in longitude. Draw the great circles ♈P, PZAB, P*p d e*, and ZW perpendicular to *pe*; then as ♈P = 90°, ♈*p* = 90°, ♈ must be the pole of P*p d e*, and therefore *d* ♈ = 90°; consequently, *d* is one of the solstitial points; viz. either ♋ or ♑, draw also Z*x* perpendicular to P*r*, and join Z♈, *p*♈. Now ♈E, or the angle ♈*p*E, or Z*p*♈ is the right ascension of the mid heaven which is known; PZ = AB, (because A Z the complement to both) the altitude of the highest point A of the ecliptic above the horizon, called the nonagesimal degree, and ♈A, or the angle ♈PA, is its longitude.

Now in the right angled triangle Z*p*W, we have Z*p* the co-latitude, and the angle Z*p*W, the difference between the right ascension of the mid heaven ♈*p*E and ♈*e*, to find *p*W; hence PW = *p*W ± *p*P, where the upper sign is to be taken when the right ascension of the mid heaven is less than 180°, and the under when greater.

Again, in the triangle WZ*p*, WZP, we have sin W*p* : sin WP :: cot W*p*Z : cot WPZ, or tan AP♈ and as we know ♈ *o*, or ♈P*o*, the true longitude of the moon, we know AP*o*, or ZP*x*: also cos WPZ, or sin ♈PZ : rad :: WP : tan ZP.

Hence in the triangle ZP*r*, we know ZP, P*r*, and the angle P, from which the angles Z*r*P, or *trs*, and Z*r* may be found; for in the right angled triangle ZP*x* we know ZP, and the angle P to find P*x*, therefore we know *rx*, and hence, as the sines of the segments of the base of any triangle are inversely as the tangents of the angles at the base adjacent to which they lie, we may find the angle Z*rx*, with which and *rx*, we may find Z*r*, the *true* zenith distance, to which, as if it were the *apparent* zenith distance find the parallax by (art. 117) and add it to the true zenith distance, and we shall have very nearly the apparent zenith distance, corresponding to which find the parallax *rt*; then in the right angled triangle *rst*, which may be considered as plane, we know *rt* and the angle *r* to find *rs* the parallax in *latitude*; find *ts*, which multiplied by the secant of *tv*, the apparent latitude gives the arc *ov* the parallax in longitude.

EXAMPLE.

Example. 123. On January 1, 1771, at 9 hours apparent time, in latitude 53° north, the moon's true longitude was 3s. 18° 27′ 35″, and latitude 4° 5′ 30″ S; also its horizontal parallax was 61′ 9″, to find its parallax in latitude and longitude.

The sun's right ascension was by the tables 282° 22′ 2″, and its distance from the meridian 135°, also the right ascension ♈E of the mid heaven was 57° 22′ 2″, hence the whole operation for the solution of the triangles may stand thus:

Triangle Z*p*W

° ′ ″
Z*p*W = 32 37 58 ...... cos 9·9253864
Z*p* = 37 0 0 ...... tan 9·8871144

*p*W = 32 23 57 ...... tan 9·8025008
P*p* = 23 28 0

PW = 55 51 57

Triangle W*p*Z

*p*W = 32 23 57........AC sin 0·2709855
PW = 55 51 57 ....sin 9·9178865
Z*p*W = 32 37 58 ....cot 10·1935941

AP♈ = 67 29 8 ....tan 10·3824661
*o*P♈ = 108 27 35

*o*PA = 40 58 27

Triangle WPZ

APZ = 67 29 8 ....sin 9·9655700
WP = 55 51 57 ....tan + 10 − 20·1688210

ZP = 57 56 36 ....tan 10·2032510

Triangle ZP*x*

ZP = 57 56 36 ....tan 10·2032555
ZP*x* = 40 58 27 ....cos 9·8779500

P*x* = 50 19 33 ....tan 10·0812055

P*r* = 94 5 30

Triangles ZP*x*, Z*rx*

rx = 43 45 57........AC sin 0.1600743
P*x* = 50 19 33 ....sin 9·8863144
ZP*x* = 40 58 27 ....tan 9·9387676

Z*rx* = 44 1 16 ....tan 9·9851563
Z*rx* = 44 1 16 ....cos + 10 19·8567795
rx = 43 45 57 ....tan 9·9812846

Z*r* = 53 6 10 ....cot 9·8754949

Z*r* = 53 6 10 ....sin 9·9029362
hor. par. = 61′ 9″ = 3669″....log 3·5645477

rt, uncorrected 2934″ = 48′ 54″ log 3·4674839

App. Z. Dist. Z*t* 53° 55′ 4″ nearly sin 9·9075042
hor. par 3669″ ....log 3·5645477

par. *rt* corrected = 2965″ = 49′ 25″ log 3·4720519

Triangles trs

par. *rt* corr. 2965 = 49′ 25″ ....log 3·4720519
trs = 44° 1′ 16″ ....cos 9·8567795

rs par. in lat. = 2132″ 35′ 32″ ..log 3·3288314

rt corrected = 2965″ ....log 3·4720519
trs = 44° 1′ 16................sin 9·8419369

ts = 2061″ = 34′ 21″ ....log 3·3139888
true lat. *ro* = 4° 5′ 30″
app. lat. *tv* = *ro* − *rs* = 4° 41′ 2″ sec 10·0014528

ov par in long. = 2067″ = 34′ 27 log 3·3154416

The value of *tv* is equal to *ro* ∓ *rs*, according as the moon has north or south latitude.

Remarks. 124. The above operation supposes the moon's horizontal parallax to have been determined. According to the tables of Mayer, the greatest parallax of the moon, or when she is in her perigee and in opposition, is 61′ 32″; the least parallax (or when she is in her apogee and in conjunction) is 53′ 52″ in the latitude of Paris: now as the parallax varies inversely as the distance, the parallax at the mean distance of the moon is 57′ 24″ viz. an harmonical mean between the two former.

But Delambre recalculated the parallax from the same observations from which Mayer calculated it, and found a slight disagreement in the two results. He made the equatorial parallax 57′ 11″·4; Lalande made it 57′ 5″ at the equator, 56′ 53″·2 at the pole, and 57′ 1″ for the mean radius of the earth; upon a supposition that the ratio of the equatorial and polar axes was as 300 : 299.

Assuming then the mean parallax to be 57′ 1″, we have, referring to fig. 36,

AC : mean radius r :: D, dist. of moon : sin 57′ 1″

hence D = 60·3 rad = 60·3 × 3964 = 239029 miles the mean distance of the moon from the earth.—*Vince's Astronomy.*

The preceding methods by which the parallaxes of the moon and of Mars have been determined are not sufficiently exact for us to employ them in determining that of the sun. And since this is in astronomy a most important element, and requires the most exact determination, it has, as we have before remarked, engaged the most anxious attention of philosophers, and no one has rendered in this respect a more essential service to the science than Dr. Maskelyne, our late worthy astronomer royal; but the method which he employed, and which was first pointed out by the celebrated astronomer Dr. Halley, cannot be with propriety illustrated in this place, because it supposes the planetary motions to be determined to the utmost accuracy: in a subsequent chapter we shall, however, enter at some length upon the explanation of this method, at present it will be sufficient to state that it depends upon the transit of either of the two inferior planets, but particularly that of Venus over the sun's disc; and that it has been thus found to be 8″·75 according to Maskelyne, but 8″·81 according to Laplace; whereas we have seen (art. 121) that as deduced from observations on Mars, it was found to be 10″·17.

From what has been stated it appears, that the parallaxes of the planets answer two important purposes, for we hence may determine their actual distances from the earth, and moreover, without a knowledge of the quantity of this important datum, we should be unable to correct our observations, and much uncertainty would consequently attend all our deductions. The parallax of the sun being very small, its mean horizontal parallax may be considered as constant, viz. 8″·75, and consequently, the parallax for any altitude is readily determined by means of the formula

$$p = P \times \text{sin zen. dist., or}$$
$$p = 8''{\cdot}75 \times \text{sin zen. dist.}$$

But for the moon, the parallax being considerable, we ought, in delicate observations, to compute it for her actual distance, and accordingly, in the *Nautical Almanack*, we find the lunar parallax, as well as her semi diameter, stated for 12 o'clock, both at noon and midnight, for every day in the year.

125. Let us now propose an example to show the method of introducing the corrections for parallax and refraction, agreeably to what has been taught in the preceding articles.

Observation.

	°	′	″	
Alt. ⊙ upper limb....	62	44	11	
Lower limb....	62	15	41	
2)	124	59	52	
App. alt. ⊙ centre ..	62	29	56	
App. zen. dist.	27	30	4	sin = 4617
+ Refraction........	0	0	29	
	27	30	33	
(8″¾ × ·4617) parallax	0	0	4	
Corr. alt. ⊙'s centre..	27	30	29	
Latitude of the place..	48	50	14	
⊙'s declination......	21	29	45	

3. *Of the correction for aberration.*

Aberration.

126. In our historical chapter, we have given some account of this important astronomical discovery; it remains for us in this place to enter a little more at length into an illustration of the principles, and to describe the method of computing and applying the requisite corrections.

The situation of any object in the heavens is determined by the position of the axis of the telescope, attached to the instrument with which we measure; for such a position is given to the telescope, that the rays of light from the object may descend down the axis, and in that situation, the index shows the angular distance required; if, therefore, the observer were in a state of absolute rest, while he was making his observation, the direction of his telescope would coincide with that of the object, as it would also, although he were in motion, if light was instantaneously transmitted from the luminous body to the eye. But if, as is actually the case, the motion of light is progressive, while the observer is also carried forward in space, then, except in the particular instance in which both motions take place in the same line, a different direction must necessarily be given to the axis of the telescope; and consequently, the place measured in the heavens will be different from the true place.

Illustrated.
Fig. 41.

127. This may be illustrated in a general manner as follows. Let S′ (fig. 41) be a fixed star, VF the direction of the earth's motion, S′F the direction of a particle of light, entering the axis ac of the telescope, at a, and moving through aF, while the earth moves from c to F; and if the telescope be kept parallel to itself, the light will descend in the axis. For let the axis nm, Fw, continue parallel to ac; and if each motion be considered as uniform, that of the spectator occasioned by the earth's rotation being disregarded on account of its being too small to produce any sensible effect, the spaces described in the same time will preserve the same proportion; but cF and Fa being described in the same time, and as we have

$$c\text{F} : \text{F}a :: cn : av$$

it follows, that cn and av will be described in the same time; therefore, when the telescope is in the situation nm, the particle of light will be at v in the telescope; and the case being the same at every moment of its descent, the place measured by the telescope at F is s', and the angle $S'F's'$, is the aberration or the difference between the true place of the star and the place measured by the instrument.

Hence it appears, that if we take

$$\text{FS} : \text{F}t :: \text{vel. of light} : \text{vel. of the earth},$$

join St and complete the parallelogram Ft Ss, the aberration will be equal to FSt; S will be the true place of the star, and s the place measured by the instrument, and this latter is the same with the apparent place of the object as it would be seen by the naked eye. This will appear as follows.

The place shown by the telescope the same as shown to the naked eye.

128. If a ray of light fall upon the eye in motion, its relative motion with respect to the eye, will be the same as if equal motions were impressed in the same direction upon each, at the instant of contact; for equal motions in the same direction impressed upon two bodies will not effect their relative motions, and therefore the effect one upon the other will not be altered. Let then VF be a tangent to the earth's orbit at F, and therefore represent the direction of the earth's motion at that instant, and S′ a star: join S′F and produce it to G; take

$$\text{FG} : \text{F}n :: \text{vel. of light} : \text{vel. of the earth},$$

and complete the parallelogram FGHn; join also FH: then since FG, and Fn represent the motions of light and of the earth, we shall have on the principle of the composition of motion explained (art. 24) Mechanics, FH, for the corresponding resulting motion; that is, the object will appear in the direction of the diagonal FH, and GFH or its equal S′Fs' or FSt will be the aberration; consequently, the apparent place to the naked eye is the same as that determined by the telescope.

Ratio of vel. of light to the vel. of earth.

129. Now we have by trigonometry,

$$\sin \text{FS}t : \sin \text{F}t\text{S} :: \text{F}t : \text{FS} :: \text{vel. of earth} : \text{the vel. of light}$$ whence

$$\sin \text{FS}t = \sin \text{F}t\text{S} \times \frac{\text{F}t}{\text{F}s}$$

$$\text{or} \quad \sin \text{of aberration} = \sin \text{F}t\text{S} \times \frac{\text{vel. of earth}}{\text{vel. of light}}$$

therefore considering the ratio of the velocity of light and of the earth as constant, the sine of aberration or the aberration itself, will vary as the sine of FtS, and is therefore greatest when that angle is a right angle, and it will be zero when the angle FtS vanishes, that is, when the motions of the earth and of light are made in the same right line.

By observations it has been determined, that the greatest effect of aberration is 20″, and as this corresponds to the case of FtS = 90°, or sin FtS = 1 we have

$$\sin 20'' = 1 \times \frac{\text{vel. of earth}}{\text{vel. of light}}$$

$$\text{vel. of earth} : \text{vel. of light as } \sin 20'' : \text{rad.} :: 1 : 10314$$

Compared with other observations.

130. This result is obtained from observation, and is independent of any deductions drawn from the actual velocity of the luminous rays; it only shows that if light moved with the velocity indicated, that such phenomena ought to have place, and that it may therefore be employed as an illustration of them. But the actual velocity of light has been otherwise determined; let us see, therefore, how nearly the results in the two cases agree with each other.

For this purpose, call the radius of the earth's orbit r, and we shall have $3{\cdot}1416 \times 2r$ for the circumference which is described by the earth in $365\frac{1}{4}$ days, or 31557600 seconds; consequently, $\dfrac{3{\cdot}1416 \times 2r}{31557600}$ will be the velocity per second, and by the above determination,

$$\frac{10314 \times 3{\cdot}1416 \times 2r}{31557600} = \text{the velocity of light per second.}$$

$$\text{Hence as } \frac{10314 \times 3{\cdot}1416 \times 2r}{31557600} : 2r : 1 \text{ sec.}$$

$$\frac{31557600}{10314 \times 31416} = 97''4 \text{ or } 16\tfrac{1}{4} \text{ minutes, the time}$$

light employs in traversing a space equal to the diameter of the earth's orbit, which agrees very nearly with the results previously deduced by astronomers from observations on the eclipses of Jupiter's satellites.

Considered as a proof of the earth's revolution.

The coincidence of these deductions, founded upon observations wholly independent of each other is generally considered as furnishing one of the most satisfactory proofs of the truths of the Copernican or modern system of astronomy.

Direction of aberration.

131. The aberration S′s' lies from the true place of a star in a direction parallel to that of the earth's motion, and towards the same part; and its effect is therefore produced at one time wholly in declination, at another wholly in right ascension, and at others, it will effect both these quantities, but in a greater or less degree, according to circumstances.

Fig. 42.

In order to illustrate this, let E, E′, E″, E‴, (fig. 42) represent four positions of the earth when the sun is in the signs ♈, ♑, ♎, and ♋, and let tT, t'T′, t''T″, &c. be tangents to the earth's orbit at those points; s, s' the same star seen on the meridian of any place. Now in the position E, corresponding to the vernal equinox, the sun is in the equator; therefore, if Pmp be a meridian passing through the poles of the equator and ecliptic, Pmp will coincide with the solstitial colure, and a line drawn from S to E will be perpendicular to the plane of the meridian Pmpn: therefore, if the plane of the ecliptic EE′E″, be conceived to lie in the plane of the paper, that of Pmpn must be perpendicular to it; and SE will be also perpendicular to the tangent tT, which tangent, therefore, is in the plane of the meridian Pmpn.

Now the earth being supposed to rotate on its axis in the direction nEm, and SE being perpendicular to the plane mPpn, the position as shown in the figure corresponds to that of 6 o'clock in the evening; and supposing the star s, to be on the meridian at this instant, it is obvious that it will be situated in a plane corresponding with the line of the earth's motion; the aberration will therefore produce its effect wholly in that plane, and will tend to elevate the star s, from s to s', that is, it will increase its declination, or diminish its zenith distance; and it is obvious that a directly contrary effect will be produced in the opposite position E″ corresponding to the autumnal equinox; for the direction of light being then in the line SE″, while that of the earth is from E″ towards t'', and in the plane of sE″t'' the aberration will be produced wholly in the same plane, and will tend to depress the star; that is, to diminish its declination; or to increase its zenith distance.

In the second position E′ at the summer solstice, the star will come with the sun upon the meridian P′m′p′, that is, it will be upon the meridian at 12 o'clock at noon. In this position T′E′t′ will be perpendicular to the plane of the meridian; therefore, since the aberration takes place in a plane passing through E′s, and T′E′t′ it will take place in a plane perpendicular to the plane of the meridian P′m′p′, and towards E′t′. that is, to the right or west of the meridian; consequently, the declination of the star, which is estimated in such plane, will not be at all affected, but the right ascension only, which will be lessened; and in the opposite position E‴ corresponding to the winter solstice, the effect will still be the same, except that as the motion of the earth is directly opposite to that at E′, the effect of the aberration will be to increase the right ascension of the star, instead of diminishing it as in the former instance.

Hence in the particular instances referred to it appears,

1. That a star which comes to the meridian at 6 o'clock in the evening, on the 20th March, will be affected by aberration only in its declination which will be increased.

2. That a star which comes to the meridian at 12 o'clock at noon, will be affected only in right ascension, which will be diminished.

3. The star which passes the meridian at 6 o'clock in the morning, September 23d, will like that which passes on March 20 have its declination only affected, which in this case will be diminished.

4. The star which comes upon the meridian at 12 o'clock at night, Dec. 23, will experience the effect of aberration only in right ascension, which will be increased.

Directly the reverse of all this will happen with respect to a star whose right ascension is twelve hours less than that we have supposed; for then the declination will be diminished in the first case and increased in the third; and the right ascension increased in the second and diminished in the fourth, as is obvious from what is above stated.

The star γ Draconis on which Bradley first made his observations, corresponds very nearly in position with the supposition in the first case above.

This particular instance has been selected for the purpose of showing, that in particular cases the effect of aberration may be wholly in right ascension or declination; but generally both these quantities will require correction, and consequently also the latitude and longitude, and therefore we shall now proceed to show how these effects may be separated from each other and computed; in which we shall follow the method laid down by professor Vince in his Treatise on Astronomy.

To estimate the effect of aberration in latitude and longitude.

Aberration in latitude and longitude.
Fig. 43.

132. Let ABCD (fig. 43) be the earth's orbit supposed to be a circle, with the sun in the centre at x; let Px be a line drawn from x perpendicular to ABCD, P representing the pole of the ecliptic: let S be the true place of the star, and conceive $apcq$ to be the circle of aberration parallel to the ecliptic, and $abcd$ the ellipse into which it is projected; let γT represent an arc of the ecliptic, and draw to it a secondary PSG, which will coincide with the minor axis bd, into which the diameter pq is projected. Draw GCxA, and it is parallel to pq; and BxD perpendicular to AC, must be parallel to the major axis ac; then when the earth is at A, the star is in conjunction, and in opposition when at C. Now as the place of the star in the circle of aberration is always 90° before the earth in its orbit, when the earth is at A, B, C, D, the apparent place of the star in the circle will be at a, p, c, q, or in the ellipse at a, b, c, d: hence to find the place of the star in the circle when the earth is at any point E, take the angle $p\mathrm{S}s = \mathrm{E}x\mathrm{B}$, and s will be the corresponding place of the star in the circle. To find its projected place in the ellipse, draw sv perpendicular to Sc, and vt perpendicular to the same; in the plane of the ellipse join st, and it will be perpendicular to vt, because the projection of the circle into the ellipse is in lines perpendicular to the latter. Draw the secondary P$vt$K, which will, as to sense, coincide with vt; unless the star be very near the pole of the ecliptic; and the rules here given will be sufficiently accurate except in that case. Now as cvS is parallel to the ecliptic, S and v must have the same latitude, hence vt is the aberration in latitude; and as G is the true and K the apparent place of the star in the ecliptic, GK is the aberration in longitude.

In order to find these quantities, we may observe, that the angle $s\mathrm{S}c$ or CxE is the angle of the earth's distance from syzygies, and as the angle svt = complement of stars latitude, vst will be the latitude itself. Now putting Sa, or Ss = r = 20″ the greatest effect of aberration; we have in the right angled triangle Ssv.

$$\mathrm{rad} = 1 : \sin s\mathrm{S}c :: r : vs = r \sin s\mathrm{S}c$$

and in the triangle vts

$$1 : \sin vst :: vs : tv = r \sin s\mathrm{S}c \sin vst$$

Therefore, $r \sin s\mathrm{S}c \sin vst$ = *aberration in latitude*

Again, in the triangle Ssv, we have

$$1 : \cos s\mathrm{S}c : r : vs = r \cos s\mathrm{S}c$$

But $\cos vst : 1 :: \mathrm{S}v : \mathrm{GK}$

or $\cos vst : 1 :: r \cos s\mathrm{S}c : \dfrac{r \cos s\mathrm{S}c = \mathrm{GK}}{\cos vst}$

the *aberration in longitude*.

133. When the earth is in syzygies, $\sin s\mathrm{S}c = o$, therefore, there is then no aberration in latitude, and as $\cos s\mathrm{S}c$ is then greatest, the aberration in longitude is at its maximum, as we have already explained in a particular case with reference to declination and right ascension.

If the earth be at A, or the star in conjunction, the apparent place of the star is at a, and reduced to the ecliptic at H, therefore GH is the aberration which diminishes the longitude of the star, the order of the signs being γ, G, T; but when the earth is at C, or the star in opposition, the apparent place c reduced to the ecliptic is at F, and the aberration GF increases the longitude, hence the longitude is the greatest when the star is in opposition, and least when in conjunction; corresponding with what has been already shown in a particular case (art. 131) with respect to the right ascension.

When the earth is in quadratures at D or B, then $\cos s\mathrm{S}c = o$, and $\sin s\mathrm{S}c$ is greatest; there is, therefore, then no aberration in longitude, while that in latitude is the greatest. When the earth is at D, the apparent place of the star is at d, and the latitude is increased, but when at B, the apparent place being at b, the latitude is diminished. Hence the latitude is least in quadrature before opposition, and greatest in quadrature after. From the mean of a great number of observations, Dr. Bradley determined the value of the greatest aberration to be 20″ as we have already stated (art. 129.)

Deductions.

134. It follows from what has been shown above,

1. That the greatest *aberration in latitude* is equal to 20″ multiplied by the sine of the star's latitude, and that the aberration in latitude *for any time* is equal to 20″ multiplied by the star's latitude and by the sine of the elongation found for the same time; and that it is subtractive before opposition and additive after it.

2. The *greatest aberration in longitude* is equal to 20″ divided by the cosine of the latitude; and the aberration for *any time* equal to 20″ multiplied by the cosine of the elongation, and divided by the cosine of the latitude; it will be subtractive in the first and last quadrant of the argument, or of the difference between the longitude of the sun and star, and additive in the second and third.

Examples.

1. Find the greatest aberration of γ *Ursa Minoris*, whose latitude is 75° 13′.

Here sin 75° 13′ is ·9669
Mult. by 20″

19″·34 the *greatest aberration in lat.*

And $\frac{20''}{\cos\ 75^\circ\ 13'} = \frac{20''}{\cdot 2551} = 78''\cdot 4$ the *greatest aberration in longitude.*

2. Required the aberration in latitude and longitude of the same star when the earth is 30° from syzygies.

$\sin 30^\circ = m = \cdot 5$; hence

$20'' \times (\sin 75^\circ\ 13') + \cdot 5 = 9''\cdot 67$ aberration in latitude; and since $\cos 30^\circ = \cdot 866$

$\frac{20'' + \cdot 866}{\cdot 2551} = 67''89$ the aberration in longitude.

In the case of the sun, we have always $\sin sSc = o$ and $\cos = 1$, also $\cos \text{lat.} = 1$; and, consequently, $20'' \times \sin sSc = o$, or there is no aberration in latitude, and the aberration in longitude is constant and equal to 20″. This quantity 20″ of aberration of the sun answers to its mean motion in 8′ 7″½; and is therefore the time in which light moves from the sun to the earth, at its mean distance, agreeably to what we have already stated (art. 130.) Hence the sun always appears 20″ behind its true place.

The following table is intended to expedite the calculation in the preceding cases.

The argument for the longitude is *long sun — long star*. The argument for the latitude is *long sun — long star — 3 signs.*

Degree.	0 VI. — +	I VII. — +	II. VIII. — +	Degree.
0	20″·00	17″·32	10″· 0	30
1	20·00	17·14	9·70	29
2	19·99	16·96	9·39	28
3	19·97	16·77	9· 8	27
4	19·95	16·58	8·77	26
5	19·92	16·58	8·45	25
6	19·89	16·18	8·13	24
7	19·85	15·97	7·81	23
8	19·81	15·76	7·49	22
9	19·75	15·54	7·17	21
10	19·70	15·32	6·84	20
11	19·63	15· 9	6·51	19
12	19·56	14·86	6·18	18
13	19·49	14·63	5·85	17
14	19·41	14·39	5·51	16
15	19·32	14·14	5·18	15
16	19·23	13·89	4·84	14
17	19·13	13·64	4·50	13
18	19· 2	13·38	4·16	12
19	18·91	13·12	3·81	11
20	18·80	12·86	3·47	10
21	18·67	12·59	3·12	9
22	18·54	12·21	2·78	8
23	18·41	12· 4	2·44	7
24	18·27	11·76	2· 9	6
25	18·13	11·47	1·74	5
26	17·98	11·18·	1·40	4
27	17·82	10·89	1·50	3
28	17·66	10·60	0·70	2
29	17·49	10·30	0·35	1
30	17·32	10· 0	0· 0	0

135. For the aberration *in longitude*, multiply the corresponding quantities in the table, by the secant of the star's latitude.

Application of the table.

For the aberration *in latitude* multiply the quantities taken from the table by the sine of the star's latitude.

Example.

1. Let the longitude of the sun be 7s 5° 18′, the longitude of the star 5s. 11° 14′, and its latitude 31° 10′; to find its aberration in latitude and longitude.

long. ☉ 7s. 5° 18′
long. ⚹ 5 18 14′

1 17 4 correspond in tab. to 13″·62
see 31° 10′........ 1″·69

aberration in long. — 15″·92 product

For the latitude.

s ° ′
long.☉—long. ⚹ = 1 17 4
— 3 signs 3 0 0

10 17 4 cor. t a .—14″·65
sin 31° 10 0·5175

aberration in latitude — 7·58 prod.

To find the aberration in declination and right ascension.

136. Let AEL (fig. 44) represent the equator, *p* its pole, ACL the ecliptic, P its pole, S the true place of a star, *s* the apparent place in the ellipse; draw the great circles PS*a*, P*sb*, *p*S*w*, *p*S*v*, and S*t*, S*v* per-

Aberration in right ascension and declination.

pendicular to pv, Pb. Now vs, and vS, being found as in the preceding articles, we shall have,

Fig. 44.

$$\frac{vS}{vs} = \tan Ssv$$

which angle hence becomes known.

Again in the triangle Psp, the three sides being given, compute the angle of position Psp, and hence find

$$Ssp = Ssv \mp Psp$$

Then again,

$$\cos Ssv : \cos Ssp :: sv : st = \frac{\cos Ssp.\,sv}{\cos Ssv} =$$

aberration in declination.

and $\sin Ssv : \sin Ssp :: Sv : St = \dfrac{\sin Ssp.Sv}{\sin Ssv}$

Consequently,

$$vw = \frac{St}{\cos \text{dec.}} = \frac{\sin Ssp.\ Sv}{\sin Ssv.\cos \text{dec.}} = \text{ the } aberration$$

in right ascension.

137. Notwithstanding the process of solution is rendered very obvious by the preceding investigation, it still involves considerable numerical computation, to facilitate which, different tables and formulæ have been contrived, of which the following, due to Delambre, is deserving of preference.

General TABLES *for the aberration of the fixed stars.*

Note.—A = right ascension, D the declination of the star, and S = the longitude of the sun.

TABLE I. Arg. A – S.				
Signs.	0 VI.	I.—VII.	II. VIII.	S.
Deg.	— +	— +	— +	Deg.
0	″19·17	″16·60	″9·59	30
1	19·17	16·43	9·30	29
2	19·16	16·26	9·00	28
3	19·15	16·08	8·70	27
4	19·13	15·89	8·40	26
5	19·10	15·71	8·10	25
6	19·07	15·51	7·80	24
7	19·53	15·31	7·49	23
8	18·99	15·11	7·19	22
9	18·94	14·90	6·87	21
10	18·88	14·69	6·56	20
11	18·82	14·47	6·24	19
12	18·75	14·25	5·93	18
13	18·68	14·02	5·61	17
14	18·60	13·79	5·28	16
15	18·52	13·56	4·96	15
16	18·43	13·32	4·64	14
17	18·33	13·08	4·31	13
18	18·23	12·83	3·99	12
19	18·13	12·56	3·66	11
20	18·02	12·32	3·33	10
21	17·90	12·07	3·00	9
22	17·78	11·80	2·67	8
23	17·65	11·54	2·34	7
24	17·52	11·27	2·00	6
25	17·38	11·00	1·67	5
26	17·23	10·72	1·34	4
27	17·08	10·44	1·00	3
28	16·93	10·16	0·67	2
29	16·77	9·87	0·33	1
30	16·60	9·59	0·00	0
Deg.	— +	— +	— +	Deg.
Signs.	XI. V.	X. IV.	IX. III.	S.

TABLE II. Arg. A + S.				
S.	0 VI.	I. VII.	II. VIII.	S.
Deg.	+ —	+ —	+ —	Deg.
0	″0·83	″0·72	″0·41	30
1	0·83	0·71	0·40	29
2	0·82	0·70	0·39	28
3	0·82	0·69	0·38	27
4	0·82	0·68	0·37	26
5	0·82	0·67	0·35	25
6	0·82	0·67	0·33	24
7	0·82	0·66	0·32	23
8	0·82	0·65	0·30	22
9	0·82	0·64	0·29	21
10	0·82	0·63	0·28	20
11	0·82	0·62	0·27	19
12	0·82	0·61	0·25	18
13	0·81	0·61	0·24	17
14	0·81	0·60	0·23	16
15	0·80	0·58	0·22	15
16	0·80	0·57	0·20	14
17	0·80	0·56	0·19	13
18	0·79	0·55	0·17	12
19	0·78	0·54	0·15	11
20	0·78	0·53	0·14	10
21	0·77	0·52	0·12	9
22	0·76	0·51	0·11	8
23	0·76	0·50	0·10	7
24	0·75	0·49	0·09	6
25	0·75	0·47	0·07	5
26	0·75	0·46	0·06	4
27	0·74	0·45	0·05	3
28	0·73	0·44	0·03	2
29	0·72	0·43	0·02	1
30	0·72	0·41	0·00	0
Deg.	+ —	+ —	+ —	Deg.
S.	XI. V.	X. IV.	IX. III.	S.

TABLE III. Arg. S + D & S – D.				
S.	0 VI.	I. VII.	II. VIII.	S.
Deg.	— +	— +	— +	Deg.
0	″3·98	″3·45	″1·99	30
1	3·98	3·42	1·93	29
2	3 98	3·38	1·87	28
3	3·98	3·34	1·81	27
4	3·97	3·30	1·75	26
5	3·97	3·26	1·68	25
6	3·96	3·22	1·62	24
7	3·95	3·18	1·56	23
8	3·94	3·14	1·49	22
9	3·93	3·10	1·43	21
10	3·92	3·05	1·36	20
11	3·91	3·10	1·30	19
12	3·90	2·97	1·23	18
13	3·89	2·92	1·17	17
14	3·87	2·87	1·10	16
15	3·85	2·82	1·03	15
16	3·83	2·77	0·97	14
17	3·81	2·72	0·90	13
18	3·79	2·67	0·83	12
19	3·77	2·62	0·66	11
20	3·74	2·56	0·69	10
21	3·72	2·51	0·63	9
22	3·70	2·46	0·56	8
23	3·67	2·40	0·49	7
24	3·64	2·34	0·42	6
25	3·61	2·28	0·35	5
26	3·58	2·23	0·28	4
27	3·55	2·17	0·22	3
28	3·52	2·11	0·14	2
29	3·49	2·05	0·07	1
20	3·45	1·99	0·00	0
Deg.	— +	— +	— +	
S.	XI. V.	X. IV.	IX. III.	

Use of the Tables

For the aberration in right ascension.

Use of the tables.

138. Enter Table I, with the argument A — S, and Table II. with A + S. Then the sum of the two corresponding numbers, multiplied by the sec. of D will be the aberration in *right ascension.*

For the aberration in declination.

Enter Table I. with the argument A — S + 3 signs, and Table II. with A + S + 3 signs, and the sum of the two corresponding numbers multiplied by the sine of D, will be the *first part* of the aberration in declination.

Enter Table III., with the arguments S + D and S — D, by which will be found the other two parts of the aberration in declination, and the sum of the three will give the whole of the *aberration in declination.*

If the declination of the star be south, add 6 signs to the arguments S + D and S — D.

Example.

Illustrated by an example.

Required the aberration of *a* Aquila, Feb. 15, 1819, at 12 o'clock in the evening.

Here by the tables, A = 9s 25° 29′ D = 8° 24′ 25″
S = 10 26 2

A — S =	10s 29° 27′	Tab. I. . .	—16″·5
A + S =	8 21 31	Tab. II. .	— 0·15
			— 16·6
Sec. dec. 8° 24′ 25″ =			1·0108
Aberration in *right ascen.*			16·66 Product
A — S + 3 signs	1s 29° 27′	Tab. I.	—16″·68
A + S + 3 signs	11 21 31	Tab. III.	+ 0.82
			— 15·86
Sin dec. 8° 24′ 25″			·1461
S =	10 26° 2′ 9″		— 2·317
S + D =	11 4° 26′ 34″	Tab. III.	— 3·595
S — D =	10 17 37 44	Tab. III.	— 2·945
Aberration in declination			— 8·857

If the declination had been south, the two latter arguments would have been S + D + 6 signs, and S — D + 6 signs, as stated above.

4. *Of nutation.*

Nutation.

139. We have in the preceding articles endeavoured to illustrate the principles of the most important astronomical corrections, and have shown the method of computation; there however still remains for explanation the doctrine of nutation, but as the complete developement of the principles upon which this theory rests, involves considerations of a physical nature which we have not hitherto examined, we must in this place content ourselves with a very general view of the subject, leaving the more minute particulars for our treatise on physical astronomy. By nutation is to be understood a kind of trepidation or tremulous motion of the earth's axis, whereby its inclination to the plane of the ecliptic is not always the same; but vibrates within certain limits, never, however, exceeding a few seconds; the period of variation is also limited to a certain number of revolutions of the earth in its orbit. This inequality in the terrestrial motion was, like that described in the foregoing chapter, discovered by Dr. Bradley, to whom we owe likewise a just explanation of the cause of it, and a near approximation of its effects.

Physical cause of.

140. We have observed above, that it is impossible in this place to give more than a very general explanation of this doctrine; it will be perhaps sufficient to observe here, that the first cause of nutation is due to the mutual gravitation of matter, and to the laws which it is known to observe; viz. that it is directly as the mass, and inversely as the square of the distance. If the orbit of the earth were a circle, and the terrestrial globe a perfect sphere, the attraction of the sun would have no other effect than to keep it in its orbit, and would cause no irregularity in the position of its axis; but neither of these conditions takes place, the earth is not a perfect sphere, nor is its orbit a circle: when the position of the earth is such, that the plane of its equator passes through the centre of the sun, the attractive power of the latter body will still have no other tendency than that of drawing the earth towards it, and the parallelism of its axis will not be disturbed: this happens in the equinoxes. But as the earth recedes from these points, the sun deviates so much the more from the plane of the equator, and the latter, in consequence of its protuberance is more powerfully attracted than the rest of the globe, which causes some alteration in its position, that is, in the inclination of its axis to the plane of the ecliptic; and at that part of the orbit, which is described between the vernal and autumnal equinox, is less than that passed over between the latter and the former; it follows that the irregularity caused by the sun, during its passage through the northern signs, is not entirely compensated by that which takes place during the other part of the revolution; and consequently, that the parallelism of the terrestrial axis, and its inclination to the ecliptic will be a little changed.

Effect of the moon.

141. Again, the same effect which the sun produces upon the earth by its attraction, or at least an analogous effect, is also produced by the moon, which is more powerful in proportion as it is more distant from the equator; and therefore when its nodes concur with the equinoctial points, the power which causes the irregularity in the position of the terrestrial axis acts with the greatest force; and the revolution of the moon's nodes being performed in about eighteen years, the nodes will twice in this period concur with the equinoctial points; and consequently, twice in the same period, or once every nine years, the earth's axis will be more influenced than at any other time; and during this interval, the pole of the earth will describe an ellipse in the heavens, whose transverse axis is 19″·2, and conjugate axis 15″, which correspond with the ratio between the cosine of the obliquity and the cosine of twice the obliquity of the ecliptic; or to the ratio of cos 23° 28′ and cos 46° 56′.

Computed. Fig. 45.

142. Let π (fig. 45) be the pole of the ecliptic, and P the mean place of the pole of the equator, ADC a circle whose radius is equal to the semi transverse axis of the ellipse C*d* A described by the pole as above stated, A the true pole of the equator when the ascending node of the moon's orbit is at ♈, and let A be supposed to move contrary to the order of the signs. Take

APO equal to the distance of the ascending node from ♈; then a line O*i*, draw from O perpendicular to PA, will cut the ellipse in a point *p*, which is the true place of the pole, the true place having been at A, when the node of the moon was at ♈.

Conceive now a perpendicular to be let fall from P upon *s′p*, produced if necessary, and meeting *s′p* in *u*, then the effect of nutation in *north polar distance* is $\pm pu = \pm Ps' - ps'$,

and in *right ascension*, $ts = Pu \dfrac{\sin s s'}{\sin s' p}$

These expressions may be rendered more commodious for computation by means of certain trigonometrical transformations, which reduce them to

pu = P*p* sin (⚹'s RA + AP*p*) for *north polar dist.*
ts = P*p* cos (⚹'s RA + AP*p*) tan dec. for RA

And these again to

1″·1 sin (⚹'s RA + long. ☽'s ☊)
+ 8″·5 sin (⚹'s RA — long ☽'s ☊) in N.P.D.

and in right ascension

± 8″·5 tan dec. cos (long ☽'s ☊ — ⚹'s RA)
+ 1″·1 tan dec. cos (long. ☽'s ☊ + ⚹'s RA)
+ 17″·2 sin long ☽'s ☊

Examples.

143. In the first of these formulæ by substituting 180 + ⚹'s RA, for ⚹'s RA, the expression will answer to the case of nutation in N. P. D. of a star with an opposite right ascension to the former. The reader will find a complete investigation of these theorems in Woodhouse's Treatise of Astronomy, chap. 16, from which we extract the two following examples for illustrating their application.

Exam. 1. Required the quantity of nutation in N. P. D. of γ Draconis, on May 10, 1802.

long. ☽'s ☊ .... 11s 17° 43′
⚹'s R A.................. 8 28 0

long. ☽'s ☊ + ⚹'s R A = 8 15 43
long. ☽'s ☊ — ⚹'s R A — 2 19 43
sin 8s 15° 43′ = — ·96909
Mult. by 1·1

— 1″·06
sin 2s 19° 43′ = ·98394
Mult. by 8·5

8″·36

Hence — 1″·06 — 8″·36 = — 9″·42 nutation in north polar distance.

Exam. 2. Required the nutation in R A of *a* Serpentis, on April 30, 1810.

⚹'s dec................ 0s 7′ 4′ tan = ·12397
long ☽'s ☊ .... 6 13 31 sin = ·23373
⚹'s R A — long ☽'s ☊ 1 10 17 cos = ·76473
⚹'s R A + long ☽'s ☊ 2 7 7 cos = ·38886

Hence

17″·2 × ·23373 — (8″·5 × ·76473 + 1″·1 × ·38886) × ·12397 = 3″·16, or in time = 0s·2 *nearly* the nutation in right ascension.

Tables for nutation.

Tables of nutation.

144. As the above corrections, like those for aberration, involve considerable numerical computations, recourse is commonly had to tables in order to facilitate the labour as much as possible. The arguments of such tables must necessarily involve the right ascension of the star and the longitude of the moon's ascending node.

By means of the two first of the following tables, the moon's longitude may be found for any time, with which as an argument enter the general table of nutation; take out the corresponding numbers marked B, *c*, and log *b*. Also denote the right ascension of the star by A and its declination by D.

Find the natural number answering to

± log *b* + log cos (☊ ± B — A) + log tan D

and call it *d*, then

d ± *c* = nutation *in right ascension.*

Find also the natural number to

± log *b* + log sin (☊ ± B — A) and it will be the nutation in *declination.*

Examples

145 This rule will be rendered obvious by the following example.

Required the nutation in right ascension and declination of *a* Cygni, on Dec. 17, 1807.

By inspection in Tables I. and II. it appears that ☊ = 7s 29° 19′; and by the table of fixed stars, the right ascension of *a* Cygni at the proposed time was 10s 8° 43′ and declination 44s 35′ 58″. Hence by Table III. we have

B = — 7° 53′, log *b* = — 0·8976, *c* = + 14·23

s ° ′
Now ☊ = 7 29 19
B = — 0 7 53

☊ — B = 7 21 26
A = 10 8 43

(☊ — B — A) = 9 12 43
log *b* = — 0·8976
log cos (9° 12′ 43″) = 9·3426
log 44° 35′ 58″ = 9·9939
=

d = log — 1·71 0·2341
c = 14·23

d + *c* = 12·52 = nutation in *right ascension.*
log *b* = 0·8976
log sin (9s 12° 43′) = 9·9892

log 7″·705 0·8868
Therefore 7″·705 = nutation in *declination.*

Epochs of the mean longitude of the moon's ascending node.

Years.	S.	D.	M.	Years.	S.	D.	M.	Years.	S.	D.	M.	Years	S.	D.	M.
1752	8	1	37	1785	10	23	20	1817	2	4	27	1849	5	15	31
1753	7	12	17	1786	10	4	1	1818	1	15	7	1850	4	26	11
1754	6	22	57	1787	9	14	41	1819	0	25	48	1851	4	6	51
1755	6	3	37	1788	8	25	18	1820	0	6	25	1852	3	17	28
1756	5	14	14	1789	8	5	58	1821	11	17	5	1853	2	28	8
1757	4	24	55	1790	7	16	39	1822	10	27	45	1854	2	8	48
1758	4	5	35	1791	6	27	19	1823	10	8	26	1855	1	19	29
1759	3	16	15	1792	6	7	56	1824	9	19	3	1856	1	0	6
1760	2	26	52	1793	5	18	36	1825	8	29	43	1857	0	10	46
1761	2	7	33	1794	4	29	17	1826	8	10	23	1858	11	21	26
1762	1	18	13	1795	4	9	57	1827	7	21	4	1859	11	2	7
1763	0	28	53	1796	3	20	34	1828	7	1	41	1860	10	12	44
1764	0	9	30	1797	3	1	14	1829	6	12	21	1861	9	23	24
1765	11	20	11	1798	2	11	55	1830	6	23	1	1862	9	4	4
1766	11	0	51	1799	1	22	35	1831	5	3	42	1863	8	14	45
1767	10	11	31	1800	1	3	16	1832	4	14	19	1864	7	25	22
1768	9	22	8	1801	0	13	55	1833	3	24	59	1865	7	6	2
1769	9	2	49	1802	11	24	36	1834	3	5	39	1866	6	16	42
1770	8	13	29	1803	11	5	16	1835	2	16	20	1867	5	27	23
1771	7	24	9	1804	10	15	53	1836	1	26	57	1868	5	8	0
1772	7	4	46	1805	9	26	33	1837	1	7	37	1869	4	18	40
1773	6	15	27	1806	9	7	14	1838	0	18	17	1870	3	29	20
1774	5	26	7	1807	8	17	54	1839	11	28	58	1871	3	10	1
1775	5	6	47	1808	7	28	31	1840	11	9	35	1872	2	20	38
1776	4	17	24	1809	7	9	11	1841	10	20	15	1873	2	1	18
1777	3	28	5	1810	6	19	52	1842	10	0	55	1874	1	11	58
1778	3	8	45	1811	6	0	32	1843	9	11	35	1875	0	22	39
1779	2	19	25	1812	5	11	9	1844	8	22	13	1876	0	3	16
1780	2	0	2	1813	4	21	49	1845	8	2	53	1877	11	13	56
1781	1	10	42	1814	4	2	30	1846	7	13	33	1878	10	24	36
1782	0	21	23	1815	3	13	10	1847	6	24	13	1879	10	5	17
1783	0	2	3	1816	2	23	47	1848	6	4	51	1880	9	15	54
1784	11	12	40												

Retrograde motion of the moon's nodes to every day in the year, to be subtracted from the longitude of the epoch.

Days.	Jan.		Feb.		Mar.		Apr.		May.		June.		July.		Aug.		Sept.		Oct.		Nov.		Dec.	
	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.	D.	M.
1	0	3	1	42	3	11	4	49	6	24	8	3	9	38	11	17	12	55	14	31	16	9	17	44
2	0	6	1	45	3	14	4	52	6	28	8	6	9	41	11	20	12	58	14	34	16	12	17	48
3	0	10	1	48	3	17	4	55	6	31	8	9	9	45	11	23	13	2	14	37	16	15	17	51
4	0	13	1	51	3	20	4	59	6	34	8	12	9	48	11	26	13	5	14	40	16	19	17	54
5	0	16	1	54	3	23	5	2	6	37	8	16	9	51	11	29	13	8	14	43	16	22	17	57
6	0	19	1	58	3	27	5	5	6	40	8	19	9	54	11	33	13	11	14	46	16	25	18	0
7	0	22	2	1	3	30	5	8	6	44	8	22	9	57	11	36	13	14	14	50	16	28	18	3
8	0	25	2	4	3	33	5	11	6	47	8	25	10	1	11	39	13	17	14	53	16	31	18	7
9	0	29	2	7	3	36	5	15	6	50	8	28	10	4	11	42	13	21	14	56	16	34	18	10
10	0	32	2	10	3	39	5	18	6	53	8	32	10	7	11	45	13	24	14	59	16	38	18	13
11	0	35	2	13	3	42	5	21	6	56	8	35	10	10	11	49	13	27	15	2	16	41	18	16
12	0	38	2	17	3	46	5	24	6	59	8	38	10	13	11	52	13	30	15	6	16	44	18	19
13	0	41	2	20	3	49	5	27	7	3	8	41	10	16	11	55	13	33	15	9	16	47	18	23
14	0	44	2	23	3	52	5	30	7	6	8	44	10	20	11	58	13	37	15	12	16	50	18	26
15	0	48	2	26	3	55	5	34	7	9	8	47	10	23	12	1	13	40	15	15	16	54	18	29
16	0	51	2	29	3	58	5	37	7	12	8	51	10	26	12	4	13	43	15	18	16	57	18	32
17	0	54	2	33	4	1	5	40	7	15	8	54	10	29	12	8	13	46	15	21	17	0	18	35
18	0	57	2	36	4	5	5	43	7	18	8	57	10	32	12	11	13	49	15	25	17	3	18	38
19	1	0	2	39	4	8	5	46	7	22	9	0	10	36	12	14	13	52	15	28	17	6	18	42
20	1	4	2	42	4	11	5	50	7	25	9	3	10	39	12	17	13	56	15	31	17	9	18	45
21	1	7	2	45	4	14	5	53	7	28	9	7	10	42	12	10	13	59	15	34	17	13	18	48
22	1	10	2	48	4	17	5	56	7	31	9	10	10	45	12	23	14	2	15	37	17	16	18	51
23	1	13	2	52	4	21	5	59	7	34	9	13	10	48	12	27	14	5	15	40	17	19	18	54
24	1	16	2	55	4	24	6	2	7	38	9	16	10	51	12	30	14	8	15	44	17	22	18	57
25	1	19	2	58	4	27	6	5	7	41	9	19	10	55	12	33	14	11	15	47	17	25	19	1
26	1	23	3	1	4	30	6	9	7	44	9	22	10	58	12	36	14	15	15	50	17	28	19	4
27	1	26	3	4	4	33	6	12	7	47	9	26	11	1	12	39	14	18	15	53	17	32	19	7
28	1	29	3	7	4	36	6	15	7	50	9	29	11	4	12	43	14	21	15	56	17	35	19	10
29	1	32			4	40	6	18	7	53	9	32	11	7	12	46	14	24	16	0	17	38	19	13
30	1	35			4	43	6	21	7	57	9	35	11	10	12	49	14	27	16	3	17	41	19	17
31	1	38			4	46			8	0			11	14	12	52			16	6			19	20

In the months Jan. and Feb. leap year, take out for the day preceding the given day.

TABLE *of nutation*

(148) Argument the mean longitude of the node.

Degrees.	0s.		VI	Deg.	Deg.	I.		VII	Deg.	Deg.	II.		VIII.	Deg.
	log.b−	B −	C − +			log.b−	B −	C − +			log.b−	B −	C − +	
0	0·9944	0° 0′	0″·00	30	0	0·9588	6° 45′	8″·27	30	0	0·8960	7° 48	14″·35	30
1	0·9944	0 15	0·29	29	1	0·9571	6 54	8·52	29	1	0·8939	7 40	14·47	29
2	0·9943	0 31	0·58	28	2	0·9554	7 3	8·77	28	2	0·8917	7 32	14·61	28
3	0·9942	0 46	0·87	27	3	0·9536	7 12	9·01	27	3	0·8896	7 23	14·74	27
4	0·9940	1 1	1·15	26	4	0·9518	7 20	9·25	26	4	0·8875	7 14	14·87	26
5	0·9937	1 16	1·44	25	5	0·9500	7 28	9·49	25	5	0·8854	7 4	14·99	25
6	0·9834	1 32	1·75	24	6	0·9481	7 36	9·72	24	6	0·8834	6 53	15·11	24
7	0·9830	1 47	2·02	23	7	0·9462	7 43	9·96	23	7	0·8814	6 42	15·23	23
8	0·9825	2 2	2·03	22	8	0·9442	7 49	10·19	22	8	0·8795	6 29	15·34	22
9	0·9821	2 17	2·59	21	9	0·9422	7 55	10·41	21	9	0·8776	6 17	15·45	21
10	0·9815	2 31	2·87	20	10	0·9402	8 1	10·63	20	10	0·8758	6 3	15·55	20
11	0·9809	2 46	3·16	19	11	0·9382	8 6	10·85	19	11	0·8740	5 49	15·64	19
12	0·9802	3 1	3·44	18	12	0·9361	8 10	11·07	18	12	0·8723	5 35	15·73	18
13	0·9795	3 15	3·72	17	13	0·9340	8 14	11·28	17	13	0·8707	5 20	15·82	17
14	0·9787	3 29	4·00	16	14	0·9318	8 17	11·49	16	14	0·8692	5 4	15·90	16
15	0·9779	3 45	4·28	15	15	0·9297	8 20	11·70	15	15	0·8677	4 48	15·98	15
16	0·9770	3 57	4·56	14	16	0·9275	8 23	11·90	14	16	0·8663	4 31	16·05	14
17	0·9760	4 11	4·84	13	17	0·9253	8 24	12·10	13	17	0·8659	4 14	16·12	13
18	0·9750	4 24	5·11	12	18	0·9231	8 25	12·30	12	18	0·8637	3 56	16·18	12
19	0·9739	4 37	5·39	11	19	0·9208	8 25	12·49	11	19	0·8625	3 38	16·24	11
20	0·9728	4 50	5·66	10	20	0·9186	8 25	12·67	10	20	0·8615	3 20	16·29	10
21	0·9716	5 3	5·93	9	21	0·9163	8 24	12·86	9	21	0·8605	3 1	16·34	9
22	0·9704	5 16	6·20	8	22	0·9140	8 23	13·04	8	22	0·8596	2 41	16·38	8
23	0·9691	5 28	6·46	7	23	0·9118	8 21	13·21	7	23	0·8588	2 22	16·42	7
24	0·9678	5 40	6·73	6	24	0·9095	8 18	13·38	6	24	0·8582	2 2	16·45	6
25	0·9664	5 51	6·99	5	25	0·9072	8 15	13·55	5	25	0·8576	1 42	16·48	5
26	0·9650	6 3	7·25	4	26	0·9050	8 11	13·72	4	26	0·8571	1 22	16·50	4
27	0·9635	6 14	7·51	3	27	0·9027	8 6	13·83	3	27	0·8568	1 2	16·52	3
28	0·9620	6 24	7·77	2	28	0·9005	8 1	14·03	2	28	0·8565	0 41	16·53	2
29	0·9604	6 35	8·02	1	29	0·8983	7 55	14·18	1	29	0·8563	0 21	16·54	1
30	0·9588	6 45	8·27	0	30	0·8960	7 48	14·33	0	30	0·8563	0 0	16·54	0
	log b	+ B	− + C			log. b	+ B	− + C			log. b	+ B	− + C	
	V.		XI.			IV.		X.			III.		IX.	

§ VII. *Of the motion of the planets in their orbits.*

1. *Kepler's laws.*

Kepler's laws.

146. We have already observed in our introduction that each of the planets revolve about the sun in elliptic orbits, having that body in one of the foci, but we have offered at present no proof that such is actually the case. We propose, however, now to examine the subject with more precision, and to explain the principles upon which such a deduction has been made.

It is to Kepler that astronomers are indebted for the first discovery of the laws of the planetary motions which he deduced from observation only, and these principally upon the planet Mars. It would carry us too far to undertake a minute description of the labours of this celebrated astronomer, before he arrived at that important deduction which has immortalized his name; but a brief statement of the principles upon which he proceeded will be expected, and they may be stated as follow. Let S (fig. 46) be the sun, M the planet Mars, D and E two places in the earth's orbit, AQDE. Now first, when the earth was at D, Kepler observed the difference in the longitude between the sun and Mars, or the angle MDS, and in like manner, when the earth was at E, he observed the angle MES. Then, the places D, E of the earth being supposed known, the distance DS, ES, and the angle DSE were easily determined; and therefore, in the triangle DSE, the two sides SD, SE and the angle DSE are given to find DE, and the other two angles SDE, SED; hence the angles MDE, MED are determined; again, in the triangle MDE, we know DE and the angles MDE, MED, to find MD; and lastly, in the triangle MDS, MD, DS, and the angle MDS are given to find MS, the distance of Mars from the sun.

Fig. 46.

Kepler found also the angle MSD, the difference of the heliocentric longitude of Mars and the earth. In this manner, having previously determined the line of the apsides, independent of the form of the orbit of the planet, by observations when Mars was in its aphelion and perihelion, he determined the former distance from the sun to be 166,780 and the latter 138,500 in units, of which the mean distance of the earth from the sun was 100,000; hence the mean distance of Mars was 152,640, and the eccentricity of its orbit ·14140. He afterwards determined in like manner three other distances, and found them to be 147,750, 163,100, and 166,255; he then calculated what these distances ought to be, supposing Mars actually described a circle about the sun, and found them to be 148,539, 163,883, 166,605; they were therefore in error 789, 783, and 350. Being thus unable to reconcile his observations with the circular hypothesis, he was under the necessity of introducing an orbit of an oval form, and as of all ovals the ellipse was the most simple, he adopted this curve, and placing the sun in one of its foci, he found it to correspond with all his observations, as well on Mars as on all the other planets of the system; and continuing his labours with the most persevering industry, he established this and the other two laws which still bear his name; and which are commonly given in the following order, viz.

1. The planets move in plane curves, and their *radii vectores* describe about the sun, as a centre, areas which are proportional to the times.

2. The orbits or trajectores of the planets, are ellipses having the sun in one of their foci.

3. The squares of the times of the revolutions of the planets about the sun are to each other, as the cubes of the major axes of their orbits.

These laws, as we have observed, were deduced wholly from observation, independent of any hypothesis whatever; but we have shown in our Section XII. of Dynamics, that the same laws are the necessary consequence of the mechanical principles established in that treatise, in case of one body revolving about another, by which it is attracted with a force that varies inversely as the square of the distance; whence therefore we may justly conclude, that the sun attracts the earth and all the other planets of our system with a force observing the above law, and that each of the primary planets acts upon its secondaries in the same manner.

2. *Kepler's problem.*

To find the place of a planet in its orbit.

Kepler's problem.

147. Since by the first of the preceding laws the planets describe about the sun areas which are proportional to the times, or which is the same, equal areas in equal times, the sun occupying one of the foci of the ellipse, it follows that they must move with unequal angular velocities, and the question which we here propose to resolve is, *given the periodic time of a planet as deduced from observation with the time when it passed through its aphelion or perihelion, to determine its place at any other time.* This is commonly called Kepler's problem

We have examined this question on physical principles in that section of our treatise of Dynamics above referred to; we propose, however, here to take a different view of it, and to give the solution on other principles. Let AGB (fig. 47) be the ellipse described by the planet about the sun at E in one of its foci, AB the major, GC the semi minor axis, A the aphelion, B the perihelion, P the place of the planet, AVBE, a circle, and C its centre. Draw DPN perpendicular to AB, join PE DE and DC, on which produced let fall the perpendicular TE.

Fig. 47.

Now conceive a body to move uniformly in a circle, in such a manner as to perform its complete revolution in the same time as the planet performs an elliptical revolution; then this velocity is what is called the *mean* angular velocity of the planet; conceive, also, that while the body revolves from A to any point M, the planet has moved with its variable motion from A to P; then the angle ACM is what is called the *mean anomaly:* and the angle AEP the *true anomaly;* and the difference between these two angles is denominated the *equation of the planet's centre.*

Let p denote the periodic time in the ellipse or circle, t the time in describing AM, or AP; then as the bodies in the ellipse and circle describe equal areas in equal times about E and C respectively, we have

$$\text{area AMC} : \text{area of the circle} :: t : p$$
$$\text{area of ellipse} : \text{area AEP} :: p : t$$

therefore,

$$\text{area AMC} : \text{area AEP} :: \text{area AED} : \text{area AEP}$$

hence AMC = AED; take away the area ACD, which is common to both, and the area MCD = EDC, but MCD = $\frac{1}{2}$ DM × CD; and EDNC = $\frac{1}{2}$ ET × CD, therefore ET = DM.

Now if t be given, the arc AM will be given; for as the body in the circle moves uniformly,

$$p : t :: 360^\circ : AD$$

Thus the mean anomaly may be found for any given period, the time of the planets passing its aphelion being supposed known; and therefore, if we can find ET or DM, we shall know the angle DCA called the *eccentric* anomaly; and hence, also, we shall be able to find the angle AEP the true anomaly. With a view to determine the eccentric anomaly, Woodhouse gives the following easy approximation.

Approximate determination of eccentric anomaly.

148. (1.) Draw MO parallel to DC; find half the sum of the angles at the base of the triangle ECM; which may be done by means of the formula

$$\tan\tfrac{1}{2}\{CEM - CME\} = \tan\tfrac{1}{2}\{CEM + CME\} \times \frac{1-e}{1+e}$$

(2.) Find CEM by the addition of $\frac{1}{2}$ (CEM − CME,) and $\frac{1}{2}$ (CEM + CME), and use this angle as an approximate value of the eccentric anomaly DCA; from which, however, it really differs by the angle EMO.

Use this approximate value of ∠ DCA = ∠ ECT, in computing ET, which is equal to the arc DM; for since $t = \frac{P}{\text{area} \odot} \times DEA$, and the body being supposed to revolve in the circle ADM,

$$t = \frac{P}{\text{area} \odot} \times ACM.$$

therefore area DEA = area ACM.

Or the area DEC + area ACD = area DCM + area ACD, consequently, area DEC = area DCM.

Hence expressing their values,

$$\frac{ET \times DC}{2} = \frac{DM \times DC}{2}, \text{ or } ET = DM$$

Having then computed ET = DM, find the sine of the resulting arc DM, which sine = OT; the difference then of the arc and sine (ET − OT) gives EO.

(3.) Use EO in computing the angle EMO, the real difference between the eccentric anomaly DCA, and the angle MEC: add the computed angle EMO to MEC in order to obtain ∠ DCA: which result, however, is not the exact value of DCA, since EMO has been computed only approximatively; that is, by a process which commenced by assuming ∠ MEC, for the value of ∠ DCA.

(4.) For the purpose of finding the eccentric anomaly, the above is the entire description of the process, which, if greater accuracy be required, must be repeated; that is, from the last found value of ∠ DCA = ∠ ECT, ET, EO, and ∠ EMO must be again computed.

(5.) A sufficiently correct value of the eccentric anomaly being found, investigate the true anomaly as follows.

True anomaly determined.

149. Let v be the other focus, and make AC = 1; then

$$EP^2 - Pv^2 = vE^2 + 2\,vE \times vN =$$
$$(vE + 2\,vN) \times vE = (2\,Cv + 2\,vN) \times 2EC =$$
$$2\,CN \times 2\,EC$$

Hence $(EP + Pv) : 2\,CN :: 2\,EC : (EP - Pv)$, or

$$2 : 2\,CN :: 2\,EC : \{EP - (2 - EP)\}$$

$$1 : CN :: EC : EP - 1$$

Therefore, $EP = 1 + \overline{CE \times CN} = 1 + CE \times \cos ACD$

Now by trigonometry,

$$\frac{1 - \cos AEP}{1 + \cos AEP} = \tan^2\tfrac{1}{2}\,AEP$$

And $EP : \text{rad.} :: EN : \cos AEP$

or $1 + EC \cos ACD : 1 :: CE + CN : \cos AEP$

whence $\cos AEP = \dfrac{CE + \cos ACD}{1 + CE \cos ACD}$

Hence $\tan^2\tfrac{1}{2}\,AEP = \dfrac{1 - \cos AEP}{1 + \cos AEP} =$

$$\frac{1 + CE \cos ACD - CE - \cos ACD}{1 + CE \cos ACD + CE + \cos ACD}$$
$$= \frac{1 - CE + \cos ACD\,(CE - 1)}{1 + CE + \cos ACD\,(CE + 1)}$$
$$= \frac{EB - \cos ACD \times EB}{EA + \cos ACD \times EA}$$
$$= \frac{1 - \cos ACD}{1 + \cos ACD} \times \frac{EB}{EA} = \tan\tfrac{1}{2}ACD \times \frac{EB}{EA}$$

Hence $\sqrt{EA} : \sqrt{EB} :: \tan\tfrac{1}{2}\,ACD : \tan\tfrac{1}{2}\,AEP$ consequently, we get AEP the *true anomaly*.

This may be more concisely expressed as follows,

$$\tan\tfrac{1}{2}\,v = \sqrt{\frac{1-e}{1+e}} \times \tan\tfrac{1}{2}\,u$$

Where v is the true anomaly, u the eccentric anomaly, and e the eccentricity of the orbit.

150. The following examples will illustrate the above approximation.

EXAMPLES.

Examples.

1. The eccentricity of the earth's orbit being assumed at ·01691, and the mean anomaly 30°, it is required to find the eccentric and true anomalies.

(1) log. tan 15 .... 9·4280525
log $(1-e)$ or log ·98309 .... −1·9925933
arith. com. $(1+e)$ or of 1·01691 .... −1·9927218

log tan $\frac{1}{2}$ (CEM−CME) = 14° 31′ 22″ 9·4133676
(2.) $\frac{1}{2}$ (CEM − CME) = 14° 31′ 22″
$\frac{1}{2}$ (CEM + CME) = 15 0 0

CEM = 29 31 22 1st approx. of DCA

(3) log sin 29° 31′ 22″ .... 9·6926438
log ·01691 .... −2·2281436
+ log (arc = rad) .... 5·3144251

log DM in seconds = 1718·7 ...... 3·2352125

DM = 28′ 38″·7, and its sine expressed as seconds differs from the arc DM, by less than half a second.

(4.) The operation prescribed under this number (art. 149) is unnecessary in this case, the correction for the angle MEC being so small, that the first eccentric value may be considered as correct.

(5.) log tan $\frac{1}{2}$ u, or log tan 14° 45′ 41″ .... } = 9·4207651
$\frac{1}{2}$ log $(1-e)$ or of .... ·98309 = − 1·9962966
A.C. $\frac{1}{2}$ log $(1+e)$ or of 1·01691 = 1·9963609

log tan $\frac{1}{2}$ v = 14° 31′ 28″ .... 9·4134226
Mult. 2

29 2 56 true anomaly

Mean anomaly = 30° 0′ 0″
true anomaly = 29 2 56

equation of centre 0 57 4

Examples.

2. As a second example, take the following from Woodhouse's Astronomy, p. 197. The eccentricity of the orbit, that of Pallas, for example, being 0·259, the mean anomaly 45°. To find the eccentric and true anomalies,

(1) log. tan 22° 30′............9·6172243
log. tan ·741............−1·8698182
arith. com. 1·259..........9·8999743

log. tan $\frac{1}{2}$ (CEM − CME) = 13° 42′ 3″·3 } 9·3870168

(2) $\frac{1}{2}$ (CEM−CME)=13° 42′ 3′·3
$\frac{1}{2}$ (CEM+CME)=22 30

CEM=36 12 3·3 1st approx. ∠ CDA
CME= 8 47 56·7

(3) log. sin 36° 12′ 3″·3........9·7713071
log. ·259 ....−1·4132998
log. (arc = rad.)..........5·3144251

log. MD in sec. = 31552·4 − 4·4990320

therefore DM = 31552″·4 = 8° 45′ 52″.4
log. sin..............9·1829067
log. (arc. − rad.) ....5·3144251

log. 31429 ....4·4973318
therefore, since DM = 31552·4
and sin DM = 31429

EO = 123·4

(4) log. ·259............−1·4132998
log. sin 45° ....9·8494850

9·2627848
log. sin 8° 47′ 56″·7....9·1845968

·0781880
log. (arc. = rad.)......5·3144251

5·3926131

log. rad. ....10·0000000
log. 123·4............2·0913152

12·0913152
log. (arc=rad.)+log. EM − 5·3926131

log. sin EMO ....6·6987021
therefore EMO = 1′ 43″·1
Hence, since CDA = 36° 12′ 3″·3
and EMO = 0 1 43 ·1

corrected val. CDA or *eccentric anom.* } = 36 13 46 ·4
log. tan 18° 6′ 53″·2 ....9·5147282
$\frac{1}{2}$ log. (1 − ·259)..........−1·9349091
A.C $\frac{1}{2}$ log. (1 + ·259) ......−1·9499871

log. tan $\frac{1}{2}v$ = 14° 5′ 19″ ....9·3996244
and v = 28° 10′ 38″ the *true anomaly.*

To find the distance of a planet, the anomaly being given.

Distance determined.

151. The eccentricity of an orbit, and either the true or eccentric anomaly being given, the corresponding distance of the planet from the sun may be readily determined, by means of what is denominated the polar equation of the ellipse: viz. calling a the semi-transverse axis, e the eccentricity, v the true anomaly, and ρ the distance sought; we shall have, (See art. 58 DYNAMICS)

$$\rho = \frac{a\ (1 - e^2)}{1 - e \cos v}$$

and hence, again,

$$\rho = a\ (1 + e \cos u)$$

where u is the eccentric anomaly.

One example will be a sufficient illustration of the latter formula; and we will take the conditions proposed in the last example to determine the corresponding distance of that planet from the sun.

We there found the eccentric anomaly, or the value of u to be 36° 13′ 46″·4, and the eccentricity was assumed ·259. Hence, we have

log. cos u = 36° 13′ 36″·4 ....9·9066881
log. ϵ = ·259 ....−1·4132998

log. ·20892 ....−1·3199879

Hence 1 + ·20892 = 1·20892 = ρ the distance sought.

We might determine precisely the same with formula 1, using the true anomaly, v, but the operation is obviously more laborious.

Of the greatest equation of the centre.

Greatest equation of the centre.

152. We have seen (art. 147) that the equation of the centre is the difference between the mean and the true anomalies; the greatest equation of the centre then is obviously as its name imports, the greatest difference between the mean and true anomalies; and this must necessarily happen at that time when the planet is moving with its mean angular velocity. For if we conceive a body to move uniformly in a circle about E as a centre, with the mean angular velocity of the planet, so that, departing from BA, the line of the apsides, it shall arrive at it again in the same time as the planet: then it is plain, that at the commencement of the motion, P moving, we will suppose, with its least angular velocity, describes about E a less angle than the fictitious body does in a given time, a day for example: the second day it also describes a less angle than the supposititious body, although a greater one than on the preceding; and so on for the third, fourth, &c days. The angular distances of the two bodies from the aphelion will therefore differ more and more from each other, till, by the still increasing angular velocity of P, it at length attains to and surpasses the uniform angular velocity of the fictitious body; after which the difference between the two angles will decrease, till at length they coincide again in the perihelion. Consequently, at that time, when the angular velocities of the two bodies are equal, the difference of the two angles, or the angle of the centre, will be the greatest.

Radius vector.

153. The point in the elliptic orbit at which this takes place is easily determined; for it must be that where the circle in which the body moves with the mean angular velocity, intersects the elliptic orbit. Let x be the radius of that circle; then, since the

circle and ellipse are described in equal times, by Kepler's laws the areas must be equal; hence, denoting 3·14159, and by π, we shall have

$$\pi x^2 = \pi \times \text{semi axis major} \times \text{semi axis minor}$$

whence

$$x = \sqrt{(\text{semi axis major} \times \text{semi axis minor})}$$

The semi axis minor $= \sqrt{(1 - e^2)}$, the semi transverse being unity; therefore

$$x = (1 - e^2)^{\frac{1}{4}} \text{ or}$$

$$x = a(1 - e^2)^{\frac{1}{4}}$$

when the semi transverse $= a$; this value of x therefore corresponds to that particular value of ρ in either of the equations

$$\rho = \frac{a\,(1 - e^2)}{1 - e \cos a}, \text{ or}$$

$$\rho = a\,(+ e \cos u)$$

That is, we shall have, in the case of the greatest equation of the centre,

$$(1 - e)^{\frac{1}{4}} = \frac{(1 - e^2)}{1 - e \cos v}$$

$$(1 - e)^{\frac{1}{4}} = (1 + e \cos u)$$

And since by these equations the values of v and u become known, we may also hence find the mean anomaly, $n\,t$, viz.

$$n\,t = u + e \cos u$$

and hence, again, the greatest equation of the centre; or

$$gr.\,eq = \pm\, v - n\,t$$

Hence, it appears, that from the preceding value of ρ, or of the radius vector at the time of the greatest equation, the true and eccentric anomalies may be computed; as also the mean anomaly, and the greatest equation of the centre. We shall not stop, however, to illustrate these formulæ by example; because, it is generally a subject rather of observation than of calculation. It will therefore be better to explain the method of determining the greatest equation by observation.

The greatest equation determined by observation.

For this purpose we must observe the longitude of the body, when its angular velocity is equal to its mean angular velocity, which may be done by taking the difference of the longitude in two successive days; and, hence, by comparing it with the mean motion, the greatest equation will be obtained.

Thus Lacaille found on Oct. 7, 1751, the sun's angular velocity to be equal to the mean angular velocity; and again, on March 28th, 1752; at which times the longitudes were as follows:

	S	°	′	″
1751 Oct. 7, ⊙'s longitude......	6	13	37	13·7
1752 March 28, ⊙'s longitude....	0	8	9	25·5
Diff. of two long.	5	24	22	11·8
Mean motion proportional to the above interval	5	20	31	43·2
Difference................		3	50	28·6

Now this difference is obviously equal to double the greatest equation of the centre; consequently, that difference was at that time 1° 55′ 14″·3: more recent observations give it 1° 55′ 33″

Eccentricity determined from the greatest equation.

154. It appears from the investigation preceding the last example, that the eccentricity being given, we may thence compute the greatest equation of the centre we may also, conversely, when the greatest equation of the centre is given, find the eccentricity, and it has been sometimes employed for this purpose; with which view Lambert gives the following series; taking E to denote the greatest equation, and assuming

$$\frac{\text{E}}{57^{\circ}{\cdot}2957795} = k$$

we have for the eccentricity,

$$e = \frac{k}{2} - \frac{11\,k^3}{3{\cdot}2^8} - \frac{587\,k^5}{3{\cdot}5{\cdot}2^{16}} - \&c.$$

In the case of the earth's orbit k being a small fraction, all its powers beyond the first may be omitted; and we shall have simply

$$e = \frac{k}{2}$$

If we assume E $= 1^{\circ}\ 55'\ 36''{\cdot}5$ for the greatest equation in the orbit of the earth, as given for 1750, we shall have

$$e = \frac{k}{2} = \frac{1^{\circ}\ 55'\ 36''{\cdot}5}{2 \times 57^{\circ}{\cdot}2957795} \doteq {\cdot}016814$$

for the eccentricity.

If we assume E $= 1^{\circ}\ 55'\ 26''{\cdot}8$, as given for 1800, we find

$$e = \frac{k}{2} = \frac{1^{\circ}\ 55'\ 26'{\cdot}8}{2 \times 57^{\circ}{\cdot}2957795} = {\cdot}016791$$

If the greatest equation of the centre is much larger than that of the earth; as, for example, that of Saturn, which is 6° 26′ 42″, two at least of the terms of the preceding series must be employed. In the case in question, we have

$$k = \frac{6^{\circ}\ 26'\ 42''}{57{\cdot}2957795} = {\cdot}112486$$

$$\frac{k}{2} = {\cdot}056243$$

$$-\frac{11k^3}{3{\cdot}2^8} = {\cdot}000031$$

whence $\quad {\cdot}056212 =$ the eccentricity of Saturn's orbit.

3. *Kepler's problem applied to the determination of the motion of the aphelion.*

Progression of the aphelion or apogee.

155. We have seen that the placês of the aphelion and perihelion of a planet are subject to a certain annual variation, which it is important to determine, and we have different methods of arriving at this determination; some of them, however, rather abstract and purely theoretical; and others, which may be submitted to actual practice.

Determined by sun's apparent diameter.

Of the first, we may mention that which has reference to the sun's apparent diameter; which is, of course, greatest when the earth is in its perihelion, and least when in its aphelion; consequently, if our instruments were sufficiently accurate to determine when these conditions obtained, the two points in question would be known; but it is impossible to determine these with sufficient accuracy, even by the most delicate observations.

Another method.

156. Another method, and more practicable than the above, is, by computing, day by day, from the observed right ascensions and declinations, the sun's

longitude; and thus to ascertain the several daily increments in longitudes; and when these are the least, the sun will be in apogee, the sun's angular velocity at that point being the least. Then, by repeating the same again after any considerable interval, we should know by the difference of the longitudes of the apogee during this interval, the annual motion of the apogee in longitude. In this manner, by a comparison of an observation made by Waltherus, in 1496, with one by Lacaille, in 1749, it appears that the progressive motion in 235 years was 4° 41′ 3″, making the mean annual progression 1′ 6″: but more recent observations make 1′ 2″, or rather 61″.9.

Determined by Kepler's law.

157. The most accurate method however of determining the progression of the apogee rests upon a very simple principle, which, as given by Woodhouse in his Astronomy, p. 207, is as follows:—

Fig. 48.

Let SEr (fig. 48) be a right line, and draw TEt, making with the axis major AB of the ellipse, an angle TEA = SEA. Now, the time through rBtS, is less than the time through the remaining arc SATr; for the equal and similar areas SEt, TEr, are described in equal times; but the area rEt, is less than SET; therefore, by the laws of Kepler, it is described in less time; consequently, the areas rEt and SEt, or the area SE$rt$S is described in less time than the areas SET, TEr, which make up the area SErTS; that is the body moves through the arc rBtS in less time than through STrB; and this property belongs to every line through E, except the line AEB, the major axis, or the line of the apsides, which joins the aphelion and perihelion of the orbit.

Consequently, if on comparing two observations of the sun at S and r, that is, when the difference of the longitudes is 180°, it is found that the time elapsed is not half a year, we may be certain that the sun has not been observed in his perigee and apogee; but if, on the contrary, the interval is exactly, or very nearly, half a year, we may be equally certain that the sun has been observed either exactly, or very nearly, in the line of the apsides.

In practice, however, it is not probable that the distance of time should correspond exactly to half a year; it is indeed impossible if the observations are made in the plane of the meridian; it remains therefore to be explained how, from what may be called an approximate observation, the exact time of, or situation, of the apogee or perigee is to be found. This is as follows:—the time from r to S is equal to the time from r to B, *plus* the time from B to A, minus the time from S to A: therefore

$$\begin{array}{l}\text{the time from B to A} - \text{time from } r \text{ to S} = \ldots \ (a)\\ \qquad\text{time from S to A} - \text{time from } r \text{ to B}\end{array}$$

Now the first difference is known, being the difference between half an anomalistic year, (art. 106) and the observed interval of observation; and of the second difference, the second term may be expressed by means of the first: thus, let the first term = t; then, by Kepler's law,

$$\text{time from } r \text{ to BT} = t \times \frac{\text{area } r\text{EB}}{\text{area SEA}}$$

which, since r and S are near the apsides, becomes

$$T = t \times \frac{r\text{B} \times \text{EB}}{\text{SA} \times \text{EA}}, \text{ or}$$

$$T = t \times \frac{r\text{B}}{\text{EB}} \times \frac{\text{EA}}{\text{SA}} \times \frac{\text{EB}^2}{\text{EA}^2}$$

Or since $\frac{r\text{B}}{\text{EB}} = \angle\, r\text{EB} = \angle\, \text{SBA} = \frac{\text{SA}}{\text{EA}}$, we have

$$T = t \times \frac{\text{EB}^2}{\text{EA}^2}, \text{ or}$$

$$T = t \times \frac{\text{angular vel. at A}}{\text{angular vel. at B}}$$

Now the angular velocities at A and B, or the increment of the sun's longitude at the apogee and perigee being known from observation, and the time from r to B, being expressed in terms of those velocities and of t, the quantity t is the only unknown quantity in the equation (a) and may therefore be determined from it. The time t being thus known, we may ascertain the exact time when the sun S is at the apogee A, and his longitude computed for that time will be the longitude of the apogee.

Example.

158. The following example is from Lacaille's Astronomy, p. 205.

1743, Dec. 30, 0h. 3m. 7s ☉'s longit. 9s 8° 29′ 12″·5
1744, June 30, 0h. 3m. 0s ☉'s longit. 3s 8° 51′ 15″

diff. of longitude 6 0 21 49

Therefore at the second observation June 30, the sun was past S. In order to find when he was exactly at that point, that is, when the difference of longitude was exactly 6s (or supposing his perigee to have been progressive through 31″), when the difference of longitude was 6s 0° 0′ 31″, we must find the time of describing the difference of 21′ 49″ and 31″. Now since on the 30th of June, the sun's daily motion in longitude was 57′ 12″, we have

57′ 12″ : 24h. :: 21′ 18″ : 8h. 56m. 13s.

Hence	from	June 30......	0h.	3m.	0s.
	take		8	56	13
	gives	June 29	15	6	47

for the time when the difference of longitudes of the sun at r, and near S was 6s 0° 0′ 31″.

The interval between this last time and the first, viz. Dec. 30, 0h. 3m. 7s., is 182d. 15h. 3m. 40s nearly the time from r to S, but this time is less than half an anomalistic year, which is 182d 15h. 7m. 1s. (see art. 106) Now

$$t - \text{time from } r \text{ to B} = 3\text{m. } 21\text{s.}$$

and the time from r to B = $t \times \frac{57'\ 12''}{61'\ 12''}$

whence substituting

$$t = \frac{4'}{61'\ 12''} = 3\text{m. } 21\text{s.}$$

we find t = 47m. 54s.
and adding this to the time last found, viz. June 29, 15h. 6m. 47s. when the sun was at S, and we shall have June 29, 15h. 54m. 41s. for the time when the sun was in apogee, viz. 8h. 8m. 19s. prior to the second observation.

Now ☉'s longitude June 30, 0h. 3m. = 3s. 8° 51′ 1″·5, and its motion in longitude at this time being 57′ 12″ daily, we have

24h. : 8h. 8m. 19s. :: 57′ 12″ : 19′ 21″

Whence from 3s 8° 51′ 1″·5
take 19 21
leaves 3 8 31 40·5

the longitude of the apogee, viz. 8° 31′ 40″·5 past the summer solstice.

Results.

159. At the time of Hipparchus, 140 B. C., the sun was in apogee in 5° 30′ of Gemini; consequently, from that time to 1744, a space of 1884 years, it advanced 33°, which divided by 1884 gives

$$\frac{33^\circ}{1884} = 1'\,3'' \text{ for the annual progression.}$$

More recent observations state the longitude of the perigee to be as follows, viz.

1750 ....9s 8° 37′ 28″
1800 ....9 9 29 3
1810 ....9 9 39 22

The first difference, divided by 50, gives

$$\frac{51'\,35''}{50} = 61'\,9''$$

which is now generally assumed for the annual motion of the perigee, and consequently, for the apogee.

The progressive motion of the apogee or perigee being determined, the position of either, or of the axis of the solar ellipse for any other time may be found by a simple proportion. Thus, if the time were required, when the line of the apsides was perpendicular to the line of the equinoxes, that is, the time when the longitude of the perigee was 9s. or 270°, we should find

$$\frac{8^\circ\,37'\,28''}{61{\cdot}9} = \text{about 500 years}$$

and consequently, the year sought 1250 of the Christian era.

The epoch when the axis coincided with the line of the apsides will be found in like manner by dividing 3s 8° 37′ 28″ by 61·9, which answers to about 5720 years, or about 4000 years before the birth of Christ, at which time chronologists have fixed the creation of the world.

Remarks.

160. It may not be amiss to observe, that there is one apparent objection to the latter method of determining the longitude of the apogee; viz. that the length of the anomalistic year, which depends upon the progression in question is supposed to be already known to some degree of accuracy; this, however, as we have frequently had occasion to remark, is a defect in most astronomical determinations; for in this more than in any other science, we are obliged to proceed step by step, from approximation to approximation, till we arrive at the conclusion sought, which is, however, still only a more accurate approximation than the former. In the present instance, if we suppose only the tropical revolution to be known, we must employ it as we have done the anomalistic period, and thus obtain a first value, and then using this again, we should soon arrive at the conclusion as above given.

6. *Of the equation of time.*

Equation of time.

161. We have already (art. 82) had occasion to explain the difference between solar and mean time. We have seen that the sun returns from one meridian to the same again after unequal intervals, at different seasons of the year, at one time preceding and at another subsequent to the noon or 12 o'clock, as shown by a well regulated chronometer properly adjusted to mean time, and this difference is what is denominated the equation of time. It remains for us now to enter a little more minutely into this doctrine, and to show how the difference in question may be computed.

First, then, we may observe, that if we imagine two meridians to be drawn through the two extremities of the arc of the ecliptic described by the sun in one day, the arc of the equator which they intercept will measure the difference between the solar and the sideral day; for when the complete revolution of the celestial sphere is terminated, this small arc of the equator must traverse the meridian before the centre of the sun can arrive there; and as from two distinct causes, which are explained below, this arc is not always of the same length, the solar days must necessarily, as we have already stated, be also unequal.

In the first place, the motion of the sun being in the ecliptic, it is obvious, that if he described in this circle, equal arcs every day, these arcs when referred to the equator would be unequal, and consequently, the solar days also; but we have seen that the sun's motion in the ecliptic is not uniform, being greatest at the time he is in or near his perigee, and least in apogee; this, therefore, is another cause of the unequal lengths of the arcs of the equator intercepted between the two meridians supposed to pass through the extremities of the arc of the ecliptic described by the sun in a day. Thus the inequality in the solar days arises from two distinct causes, the obliquity of the ecliptic and the unequal motion of the earth in its orbit.

Mean solar day defined.

162. The mean solar day to which we have referred is marked by no obvious phenomenon, but the noon of which may be conceived to be shown by a fictitious sun or star moving in the equator with the mean motion of the real sun in right ascension, viz. with such an uniform motion, that supposing both to depart from the equinox at the same instant, they may both arrive at it again together, after one complete revolution, and as the length of the tropical year, or the time in which the sun passes through 360° in right ascension is 365d. 5h. 48m. 48s., we shall have

$$\frac{360^\circ}{365\text{d. } 5\text{h. } 48\text{m. } 48\text{s.}} = 59'\,8''{\cdot}3$$

for its mean daily motion in right ascension, and therefore for the mean motion of the fictitious sun in the equator; and hence the length of the mean day will be the time during which any point on the earth revolves through 360° 59′ 8″·3, and it is this interval which answers to 24 hours by a clock well regulated to mean time, and the difference between the noon, or 12 o'clock, shown by such a time piece, and that of the real noon when the sun is on the meridian, is the equation of time, which is sometimes, as we have before stated, additive, at others subtractive, and four times in the year zero.

Equation of time illustrated.

163. In order to a more complete illustration of this phenomenon, let us imagine two suns S″ and S‴ to set out at the same time from the equinox, both moving with an uniform motion, the one S″ in the ecliptic, the other S‴ in the equator, and let the motion of each be referred to the latter plane. Now on this supposition, S‴ will first advance upon the me-

ridian of S″ because in a spherical right angled triangle whose base is less than 90°, the hypothenuse will be less than the base; but afterwards, the latter S″ will approach to S‴, and they will arrive together at the solstice, the hypothenuse and base being then equal; after this, the meridian of S″ will advance upon that of S‴, the hypothenuse being now less than the base, till again at the second equinox they will once more coincide; the same circumstances will occur in the second half of the orbit, and thus the meridian of the two suns will coincide four times in the year.

The circumstances, however, as above stated, do not actually obtain, because the sun does not, as supposed above, move uniformly in the ecliptic, and therefore, the points in which the two meridians coincide will not be exactly those pointed out above; the general facts, however, will be the same, and only the place of coincidence will be changed.

Let us now take S′ to denote the real sun, and consider its motion with reference to two fictitious suns, supposed to set out from the autumnal equinox, and proceeding towards the perigee. Here the true sun S′ which describes the ecliptic being referred by its meridian to the plane of the equator, will be found behind the others, for it is preceded by S″ to the perigee. Till the moment of the solstice, S‴ precedes S″, and S″ precedes S′, their order is therefore S′, S″, S‴; at the solstice S″ and S‴ unite, and afterwards, the order is S′, S‴, S″; but at the perigee, S′ coincides with S″, and it afterwards precedes it, consequently, in this interval, the meridians S′ and S‴ must have coincided. The order then becomes S‴, S″, S′; thus between the winter solstice and the perigee, the true sun falls in with the fictitious equatorial sun, and the equation of course becomes zero. From the perigee to the vernal equinox, the like circumstances again occur, but in a contrary order, and the same again from the vernal equinox to the apogee, and lastly from the apogee to the autumnal equinox.

Hence it appears, that in consequence of the obliquity of the ecliptic, and the unequal motion of the sun, the equation of time becomes zero four times in a year, viz. about the 24th of December, the 15th of April, the 15th of June, and the 31st of August.

Such is a popular view of this phenomenon, let us now endeavour to explain the principles upon which the amount of it for any given time may be computed.

Dr. Maskelyne's rule for computing the equation of time. Fig. 49.

164. Let APLS (fig. 49) be the ecliptic, ALv the equator, A the first point of Aries, P the sun's apogee, S any other place of the sun. Draw Sv perpendicular to the equator, and take An = PA. Now when the sun departs from P, let the imaginary sun set out at n, with the mean motion of the sun in right ascension or longitude, at the rate of 59′ 8″·2 in a day, and when n passes the meridian, conceive the clock to be adjusted to 12. Take nm = Ps, and when the fictitious sun comes to m, the place of the true sun, if it moved uniformly, with its mean motion, would be at s, but at that time let S be the place of the sun. Now let the sun at S, and consequently at v, be upon the meridian, and then as m is the place of the imaginary sun, at that instant, $m\,v$ will be the equation of time.

The sun's mean place is at s, and as An = AP, and nm = PS, therefore Am = APs; consequently,

$$mv = \mathrm{A}v - \mathrm{A}m = \mathrm{A}v - \mathrm{AP}s$$

Let a be the mean equinox, and draw az perpendicular to AL; then

$$\mathrm{A}m = zm + \mathrm{A}z = \mathrm{A}a \times \cos a\mathrm{A}z + zm$$

and because cos of obliquity or cos 23°28′ $= \frac{11}{12}$ nearly this becomes

$$\frac{11}{12}\,\mathrm{A}a + zm$$

Hence

$$mv = \mathrm{A}v - zm - \frac{11}{12}\mathrm{A}a$$

but Av is the sun's true right ascension, zm is the mean right ascension, or mean longitude, and $\frac{11}{12}$ Aa (Az) is the equation of the equinoxes, in right ascension; *hence the equation of time is equal to the difference of the sun's true right ascension, and its mean longitude corrected by the equation of the equinoxes in right ascension.* When Am is less than Av, mean time precedes the apparent, and when greater, apparent time precedes the mean; for as the earth turns on its axis in the direction Av, or in the order of the right ascension, that body whose right ascension is the least, must come first upon the meridian. That is, when the sun's true right ascension is greater than its mean longitude corrected as above, we must add the equation of time to the apparent to get the mean time, and when it is less, we subtract. To convert mean time into apparent, we must subtract in the former case, and add in the latter.

This rule for computing the equation of time was first given by Dr. Maskelyne in *Phil. Trans.* 1764.

Remarks.

165. As a meridian of the earth when it leaves m returns to it again in 24 hours, it may be considered when it leaves that point as approaching a point at that time 360° from it, and at which it arrives in 24 hours; hence the relative velocity with which a meridian accedes to, or recedes from m, is at the rate 15° in an hour; therefore, when the meridian passes through v, the arc $v\,m$ reduced into time at the rate of 15° in an hour, gives the equation of time at that instant; hence the equation of time is computed for the instant of apparent noon. Now the time of apparent noon in mean solar time, for which we compute, can only be known by knowing the equation of time: to compute, therefore, the equation for any day, we must assume the equation the same as on that day four years preceding, from which it will differ but very little; and it will give the time of apparent noon sufficiently accurate for the purpose of computing the equation. If the equation be not known for the same day four years back, we may compute the equation for noon mean time, and we shall have apparent noon sufficiently accurate for most purposes.

Tables of equation of time.

166. Tables of the equation are computed and published in the Nautical and some other Almanacks for every day in each year; others are computed for every degree of longitude of the sun, and may be considered as perpetual when the secular variation is also given, such is the following, calculated for 1810.

TABLE *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Sign 0.	Diff.	Secular Variation. ±	Diff.	Sign I.	Diff.	Secular Variation. +	Diff.	Sign II.	Diff.	Secular Variation. +	Diff.
0	+6′ 59″·2	18″·8	2″·97	25	−1′ 31″·2	12″·5	4″·31	24	−3′ 43″·4	4″·5	10″·83	18
1	6 40·4	18·9	2·72	24	1 43·7	12·0	4·55	24	3 38·9	5·0	11·01	17
2	6 21·5	18·9	2·48	24	1 55·7	11·4	4·79	23	3 33·9	5·5	11·18	18
3	6 2·6	18·8	2·24	24	2 7·1	10·8	5·02	24	3 28·4	6·1	11·36	17
4	5 43·8	18·9	2·00	25	2 17·9	10·4	5·26	23	3 22·3	6·6	11·53	16
5	5 24·9	19·0	1·75	24	2 28·3	9·8	5·49	24	3 15·7	6·9	11·69	16
6	5 5·9	18·8	1·51	24	2 38·1	9·3	5·73	23	3 8·8	7·7	11·85	16
7	4 4·1	18·8	1·27	25	2 47·4	8·8	5·96	23	3 1·1	8·0	12·01	15
8	4 28·3	18·7	1·02	24	2 56·2	8·3	6·19	23	2 53·1	8·4	12·16	15
9	4 9·6	18·6	0·78	24	3 4·5	7·6	6·42	23	2 44·7	8·8	12·31	14
10	3 51·0	18·4	0·54	25	3 12·1	7·1	6·65	23	2 35·9	9·2	12·45	14
11	3 32·6	18·5	0·29	24	3 19·2	6·6	6·88	23	2 26·7	9·8	12·59	14
12	3 14·1	18·3	0·05	25	3 25·8	5·9	7·11	22	2 16·9	10·0	12·73	13
13	2 55·8	18·0	0·20	24	3 31·7	5·4	7·33	23	2 6·9	10·3	12·86	13
14	2 37·8	17·9	0·44	24	3 37·1	4·9	7·56	22	1 56·6	10·8	12·99	12
15	2 19·9	17·6	0·68	25	3 42·0	4·2	7·78	22	1 45·8	11·2	13·11	12
16	2 2·3	17·3	0·93	24	3 46·2	3·4	8·00	22	1 34·6	11·4	13·23	11
17	1 45·0	17·1	1·17	24	3 49·6	3·0	8·22	21	1 23·2	11·6	13·34	11
18	1 27·9	16·9	1·41	25	3 52·6	2·5	8·43	22	1 11·6	11·9	13·45	10
19	1 11·0	16·7	1·66	24	3 55·1	1·9	8·65	20	0 59·7	12·1	13·55	9
20	0 54·3	16·4	1·90	24	3 57·0	1·3	8·85	21	0 47·6	12·3	13·64	9
21	0 37·9	16·0	2·14	25	3 58·3	0·6	9·06	21	0 35·3	12·5	13·73	9
22	0 21·9	15·5	2·39	24	3 58·9	0·1	9·27	21	0 22·8	12·7	13·82	8
23	+0 6·4	15·1	2·63	24	3 59·0	0·5	9·48	20	−0 10·1	12·8	13·90	8
24	−0 8·7	14·7	2·87	24	3 58·5	1·1	9·68	20	+0 2·7	12·9	13·98	7
25	0 23·4	14·3	3·11	24	3 57·4	1·7	9·88	19	0 15·6	13·1	14·05	6
26	0 37·7	13·9	3·35	24	3 55·7	2·2	10·07	20	0 28·7	13·1	14·11	6
27	0 51·6	13·5	3·59	24	3 53·5	2·8	10·27	20	0 41·8	13·2	14·17	5
28	1 5·1	13·2	3·83	24	3 50·7	3·4	10·47	18	0 55·0	13·1	14·22	5
29	1 18·3	12·9	4·07	24	3 47·3	3·9	10·65	18	1 8·1	13·0	14·27	4
30	1 31·2		4·31		3 43·4		10·83		1 21·1		13·31	

TABLE *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Sign II.	Diff.	Secular Variation. +	Diff.	Signs IV.	Diff.	Secular Variation. +	Diff.	Signs V.	Diff.	Secular Variation. +	Diff.
0	+1′ 21″·1	13″·0	14″·31	4	+5′ 59″·3	2″·1	13″·14	11	+2′ 46″·2	15″·0	8″·78	17
1	1 34·1	13·1	14·35	3	6 1·4	1·6	13·03	12	2 31·2	15·5	8·61	17
2	1 47·2	13·1	14·38	2	6 3·0	1·2	12·91	12	2 15·7	15·9	8·44	17
3	2 0·3	13·0	14·40	2	6 4·2	0·6	12·79	12	1 59·8	16·4	8·27	17
4	2 13·3	12·8	14·42	2	6 4·8	0·1	12·67	12	1 43·4	16·8	8·10	17
5	2 26·1	12·6	14·44	1	6 4·7	0·7	12·55	13	1 26·6	17·2	7·93	17
6	2 38·7	12·4	14·45	0	6 4·0	1·4	12·42	13	1 9·4	17·6	7·76	17
7	2 51·1	12·2	14·45	0	6 2·6	2·0	12·29	13	0 51·8	18·0	7·59	17
8	3 3·3	12·0	14·45	1	6 0·6	2·5	12·16	13	0 33·8	18·4	7·42	17
9	3 15·3	11·7	14·44	1	5 58·1	3·3	12·03	14	+0 15·4	18·6	7·25	17
10	3 27·0	11·4	14·43	2	5 54·8	3·8	11·89	14	−0 3·2	18·9	7·08	17
11	3 38·4	11·1	14·41	2	5 51·0	4·4	11·75	14	0 22·1	19·2	6·91	17
12	3 49·5	10·8	14·39	3	5 46·6	5·0	11·61	14	0 41·3	19·5	6·74	18
13	4 0·3	10·5	14·36	4	5 41·6	5·7	11·47	15	1 0·8	19·8	7·56	17
14	4 10·8	10·1	14·32	4	5 35·9	6·4	11·32	15	1 20·6	20·0	7·39	18
15	4 20·9	9·7	14·28	4	5 29·5	6·9	11·17	15	1 40·6	20·2	7·21	17
16	4 30·6	9·4	14·24	5	5 22·6	7·5	11·02	15	2 0·8	20·5	6·04	17
17	4 40·0	9·0	14·19	6	5 15·1	8·1	10·87	15	2 21·3	20·6	5·87	18
18	4 49·0	8·5	14·13	6	5 7·0	8·7	10·72	15	2 41·9	20·8	5·69	17
19	4 57·5	8·1	14·07	6	4 58·3	9·3	10·57	16	3 2·7	20·9	5·52	18
20	5 5·6	7·6	14·01	7	4 49·0	9·8	10·41	16	3 23·6	21·1	5·34	18
21	5 13·2	7·4	13·97	7	4 39·2	10·4	10·25	16	3 44·7	21·1	5·16	18
22	5 20·6	6·9	13·87	8	4 28·8	10·9	10·09	16	4 5·8	21·2	4·98	17
23	5 27·5	6·2	13·79	8	4 17·9	11·5	9·93	16	4 27·0	21·3	4·81	18
24	5 33·7	5·7	13·71	9	4 6·4	12·1	9·77	17	4 48·3	21·4	4·63	18
25	5 39·4	5·1	13·62	9	3 54·3	12·7	9·60	16	5 9·7	21·3	4·45	18
26	5 44·5	4·5	13·53	9	3 41·6	13·0	9·44	16	5 31·0	21·3	4·27	18
27	5 49·0	3·9	13·44	10	3 28·6	13·6	9·28	16	5 52·3	21·4	4·09	18
28	5 52·9	3·4	13·34	10	3 15·0	14·2	9·12	17	6 13·7	21·3	3·91	18
29	5 56·3	3·0	13·24	10	3 0·8	14·6	8·95	17	6 35·0	21·3	3·73	19
30	5 59·3		13·14		2 46·2		8·78		6 36·3		3·54	

TABLE *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Signs VI.	Diff.	Secular Variation. ±	Diff.	Signs VII.	Diff.	Secular Variation. −	Diff.	Signs VIII.	Diff.	Secular Variation. −	Diff.
	′ ″	″	″		′ ″	″	″		′ ″	″	″	
0	− 6 56·3	21·2	3·54	18	− 15 17·8	8·8	2·56	23	− 13 54·9	16·0	9·64	24
1	7 17·5	21·0	3·36	19	15 26·6	8·1	2·79	23	13 38·9	16·8	9·88	23
2	7 38·5	20·8	3·17	18	15 34·7	7·4	3·02	23	13 22·1	17·5	10·11	22
3	7 59·3	20·7	2·99	19	15 42·1	6·7	3·25	23	13 4·6	18·2	10·33	22
4	8 20·0	20·6	2·80	19	15 48·8	6·0	3·48	23	12 46·4	19·1	10·55	22
5	8 40·0	20·4	2·61	19	15 54·8	5·2	3·71	23	12 27·3	19·9	10·77	21
6	9 1·0	20·1	2·42	19	16 0·0	4·4	3·94	24	12 7·4	20·6	10·98	21
7	9 21·1	19·9	2·23	19	16 4·4	3·7	4·18	23	12 46·8	21·3	11·19	21
8	9 41·0	19·7	2·04	19	16 8·1	2·9	4·41	24	11 25·5	22·0	11·40	21
9	10 0·7	19·4	1·85	19	16 11·0	2·0	4·64	24	11 3·5	22·7	11·61	20
10	10 20·1	19·2	1·66	20	16 13·0	1·2	4·89	24	10 40·8	23·4	11·81	20
11	10 39·3	18·8	1·46	20	16 14·2	0·5	5·13	24	10 17·4	24·0	12·01	19
12	10 58·1	18·4	1·26	20	16 14·7	0·5	5·37	24	9 53·4	24·5	12·20	19
13	11 16·5	18·0	1·06	20	16 14·2	1·3	5·61	24	9 28·9	25·2	12·39	19
14	11 34·5	17·7	0·86	20	16 12·9	2·1	5·81	24	9 3·7	25·8	12·58	19
15	11 52·2	17·3	0·66	20	16 10·8	3·0	6·09	24	8 37·9	26·4	12·77	18
16	12 9·5	16·8	0·46	21	16 7·8	3·8	6·33	24	8 11·5	26·9	12·95	17
17	12 26·3	16·4	0·25	21	16 4·0	4·7	6·57	24	7 44·6	27·3	13·12	17
18	12 42·7	16·0	0·04	20	15 59·3	5·6	6·81	24	7 17·5	27·6	13·29	16
19	12 58·7	15·5	0·16	21	15 53·7	6·4	7·05	24	6 49·7	28·1	13·45	16
20	13 14·2	14·9	0·37	21	15 47·3	7·3	7·29	24	6 21·6	28·5	13·61	15
21	13 29·1	14·4	0·58	22	15 40·0	8·3	7·53	24	5 53·1	28·8	13·76	15
22	13 43·5	13·9	0·80	21	15 31·7	9·1	7·77	24	5 24·3	29·2	13·91	14
23	13 57·4	13·4	1·00	22	15 22·6	10·0	8·01	24	4 55·1	29·5	14·05	14
24	14 10·8	12·8	1·23	22	15 12·6	10·9	8·25	24	4 25·6	29·7	14·19	13
25	14 23·6	12·1	1·45	22	15 1·7	11·7	8·49	23	3 55·9	30·0	14·32	12
26	14 35·7	11·5	1·67	22	14 50·2	12·5	8·72	24	3 25·9	30·1	14·44	12
27	14 47·2	10·8	1·89	22	14 37·5	13·4	8·96	23	2 55·8	30·3	14·56	11
28	14 58·0	10·2	2·11	22	14 24·1	14·2	9·19	23	2 25·5	30·5	14·67	11
29	15 8·2	9·6	2·33	23	14 9·9	15·0	9·42	22	1 55·0	30·3	14·78	10
30	15 17·8		2·56		13 54·9		9·64		1 24·7		14·88	

Table *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Signs IX.	Diff.	Secular Variation. —	Diff.	Signs X.	Diff.	Secular Variation. —	Diff.	Signs XI.	Diff.	Secular Variation. —	Diff.
	′ ″	″	″		′ ″	″	″		′ ″	″	″	
0	− 1 24·7	30·5	14·88	10	+11 36·3	17·0	14·89	10	+14 6·9	7·0	9·96	22
1	0 54·2	30·5	14·98	9	11 53·3	16·2	14·79	11	13 59·9	7·8	9·74	22
2	− 0 23·7	30·5	15·07	8	12 9·5	15·3	14·68	11	13 52·1	8·5	9·52	22
3	+ 0 6·8	30·4	15·15	8	12 24·8	14·5	14·57	11	13 43·6	9·1	9·30	22
4	0 37·2	30·3	15·23	7	12 39·3	13·8	14·46	12	13 34·5	9·6	9·08	22
5	1 7·5	30·1	15·30	6	12 53·1	13·0	14·34	13	13 24·9	10·2	8·86	23
6	1 37·6	29·9	15·36	5	13 6·1	12·1	14·21	13	13 14·7	10·7	8·63	22
7	2 7·5	29·7	15·41	5	13 18·2	11·2	14·08	14	13 4·0	11·6	8·41	23
8	2 37·2	29·5	15·46	5	13 29·4	10·4	13·94	14	12 52·4	12·0	8·18	22
9	3 6·5	29·3	15·51	4	13 39·8	9·5	13·80	15	12 40·4	12·5	7·96	23
10	3 36·0	28·9	15·55	3	13 49·3	8·7	13·65	15	12 27·9	13·0	7·73	24
11	4 4·9	28·6	15·58	2	13 58·0	7·9	13·50	15	12 14·9	13·4	7·49	23
12	4 33·5	28·3	15·60	1	14 5·9	7·0	13·35	16	12 1·5	14·0	7·26	23
13	5 1·8	27·8	15·61	1	14 12·9	6·1	13·19	16	11 47·5	14·4	7·03	24
14	5 29·6	27·3	15·62	0	14 19·0	5·2	13·03	17	11 33·1	14·8	6·79	24
15	5 56·9	26·8	15·62	0	14 24·2	4·4	12·86	17	11 18·3	15·1	6·55	23
16	6 23·7	26·3	15·62	1	14 28·6	3·7	12·69	17	11 3·2	15·6	6·32	23
17	6 50·0	25·9	15·61	2	14 32·3	2·8	12·52	18	10 47·6	16·0	6·09	24
18	7 15·9	25·3	15·59	2	14 35·1	2·0	12·34	18	10 31·6	16·3	5·85	24
19	7 41·0	24·6	15·57	3	14 37·1	1·1	12·16	19	10 15·3	16·7	5·61	24
20	8 5·8	24·1	15·54	4	14 38·2	0·3	11·97	19	9 58·6	17·0	5·37	24
21	8 29·9	23·4	15·50	5	14 38·5	0·6	11·78	19	9 41·6	17·2	5·13	23
22	8 53·3	22·8	15·45	4	14 37·9	1·3	11·59	19	9 24·4	17·4	4·90	24
23	9 16·1	22·1	15·41	6	14 36·6	2·0	11·40	20	9 7·0	17·7	4·66	24
24	9 38·2	21·5	15·35	6	14 34·6	2·7	11·20	20	8 49·3	17·9	4·42	24
25	9 59·7	20·9	15·29	7	14 37·9	3·6	11·00	21	8 31·4	18·1	4·18	24
26	10 20·6	20·2	15·22	8	14 28·3	4·4	10·79	20	8 13·3	18·3	3·94	24
27	10 40·8	19·3	15·14	8	14 23·9	5·0	10·59	21	7 55·0	18·5	3·74	25
28	11 0·1	18·5	15·06	8	14 18·9	5·6	10·38	21	7 36·5	18·6	3·45	24
29	11 18·6	17·7	14·98	9	14 13·3	6·4	10·17	21	7 17·9	18·7	3·21	24
30	+11 36·3		14·89		+14 6·9		9·96		+ 6 59·2		2·97	

§ VIII. *Of the phenomena and theory of the planetary motions.*

1. *Phases of the planets, &c.*

Phenomena of the planets.

167. Having in the preceding sections treated of the theory of the earth's motion about the sun, as well as examined some of the circumstances attending the planetary orbits generally, it remains for us now, first, to explain some of the more obvious phenomena which these bodies exhibit; and then to illustrate the principles upon which all the circumstances of their motion are computed. The hypothesis of the motion of the earth in its orbit, combined with its rotation on its axis, have been found sufficient to illustrate all the observed phenomena of the sun and stars; let us now see how the phenomena of the planets can be reconciled with the same hypothesis, when we add to it the motions of those bodies in their respective orbits. The circumstances to which we here allude, are the phases of the inferior, and the stationary positions and retrograde and direct motions of the whole; to which we have already slightly referred in our introductory chapter.

Of the phases of the inferior planets.

Phases of Venus.

168. As the same remarks will obviously apply to both our inferior planets, we shall select Venus for the subject of our explanation, her phases being far more conspicuous than those of Mercury.

Venus is seldom seen but for a certain time before sun rising, or after he is set; though sometimes her lustre is sufficient to render her visible while that body is yet above the horizon. When seen in the east, she is called the *morning* star; and when in the west, the *evening* star.

The motion of this planet sometimes direct, at others retrograde.

In the latter case, if observed for several successive nights, she will be found to vary her distance from the sun; first increasing it to about 45°, after which she approaches the sun again, and is finally lost in the more refulgent rays of this great luminary. If now we watch in the east for a few days, we shall perceive Venus rising before the sun, increasing her distance from him as before, till, after having attained her maximum in this direction, she again approaches him, and is again lost in the lustre of his beams.

These appearances, which continually recur, shows decidedly that Venus does not revolve about the earth; for, if this were the case, she must sometimes be seen in opposition to the sun; but it does not prove with the same force that her revolution is made about the sun; this, however, is otherwise rendered incontrovertible.

Venus sometimes a crescent, full, &c.

To the naked sight the disc of this planet, like that of all the others, appears circular, and always nearly of the same magnitude; but the telescope and its micrometer, both prove this to be a deception or optical illusion. For when thus viewed as an evening star, at her greatest elongation from the sun, she assumes the form of a crescent, like the moon in its first quarter, the horns being opposite to the sun. As she approaches the sun the horns diminish; and after her reappearance in the morning, we see her again assume the crescent form, but her horns pointing now in a contrary direction. Moreover, by means of the micrometer, it is found that her apparent diameter is greatest when she is least illuminated; that is, when she is ceasing to be an evening star; and least when most illuminated, or when ceasing to be a morning star; the ratio of her diameter being in these two cases as 166 to 26; that is, 2′ 46″ and 26″

Deductions from observations.

From these facts, deduced from observation, it appears, *first*, that Venus never attains an angle of elongation that far exceeds 45°; *secondly*, that she is at various distances from the earth; and other observations, not referred to above, have shown that she sometimes passes like a black spot over the solar disc: *thirdly*, it appears, that Venus, when an evening star, and separating from the sun, moves from west to east, or according to the order of the signs; but when she returns, her motion is necessarily in a contrary direction, or contrary to the order of signs; and the same phenomena have place in the opposite direction, after this planet first becomes a morning star.

These appearances are what it is requisite for us to show to be the necessary consequences of Venus revolving in an orbit interior to that of the earth, and of her being an opaque body enlightened by the sun.

The above consistent with the received hypothesis. Fig. 50.

169. First, that the phases of Venus correspond very correctly with such an hypothesis, will appear by referring to (fig. 50), in which S represents the sun, KVN the orbit of Venus, and EE′ part of the orbit of the earth. Let W, N, V, be three different positions of Venus in her orbit with respect to the earth at E; then, since that hemisphere of Venus which is perpendicular to the sun is the enlightened hemisphere, and that perpendicular to the earth is the visible one to the inhabitants of this planet, it is clear that all the phenomena above described, ought to obtain, as represented in the figure; viz. Venus will appear as a crescent at W, half illuminated at N, NE being a tangent to her orbit, gibbous at V, and full at K.

The same appearances precisely obtain with respect to Mercury; but being small, at a greater distance from the earth, and much nearer to the sun, they are not so easily observed as in the case of Venus.

It is also equally obvious, without our entering into a minute detail of the particulars, that, according to the supposition we have made, Venus ought to appear to move from K towards M according to the order of the signs; and from M to N contrary to that order; while about the points M and N her motion will be imperceptible, and she will appear stationary.

As we had here a view only to the general illustration, we have not considered the motion of the earth, but in any computation relative to these phenomena, it is obvious that this circumstance must be introduced, as we shall see in a subsequent article.

Stations, &c. of the superior planets.

170. None of the superior planets, with the exception of Mars, offer any phases such as those we have been explaining above; their distance from the earth being such as to render these phenomena imperceptible; but Mars not being so far distant, does sometimes exhibit a gibbous appearance, but is never horned, as we have shown to be the case with Venus and Mercury. We have seen, that neither Venus nor Mercury are ever seen in opposition to the sun, from which we have concluded, that they do not either revolve about the earth as a centre, nor include the orbit of the earth in their revolution about the sun; but Mars, and all the other superior planets, being found sometimes in opposition, and at others in conjunction, we are equally certain, that they do either revolve about the

earth, or enclose its orbit in their's. If they revolved about the earth as a centre, then their motions ought to be always direct; whereas, it is found from observation, that they are somctimes direct, at others stationary, and others retrograde; which is strictly what ought to take place, supposing them to revolve about the sun in circular or elliptic orbits superior to that of the earth. Their phases also, although not variable, are consistent with the same hypothesis; namely, that they revolve about the sun; that they are dark opaque bodies, which shine only by the borrowed light of the great central luminary.

Problem I.

To find the position of a planet when stationary.

Problem. 171. To solve this problem with the utmost accuracy, the distances and relative motions of the earth and planets ought to be first ascertained, but the principle of computation may be explained independent of these data, and the subject having an intimate relation with the preceding parts of this chapter, we shall make no apology for introducing the question here, although we may not be able to give the most correct numerical results.

Fig. 51. Let S (fig. 51) be the sun, E the earth, P the cotemporary position of a superior planet, and MN the sphere of the fixed stars, to which all motions are referred. Let EE′, PQ, be two indefinitely small arcs described in the same time; and let EP, E′Q produced, meet at L; then it is manifest, that while the earth moves from E to E′, the planet will appear stationary at L; and, in consequence of the immense distance of the fixed stars EPL, E′QL may be considered as parallel. Draw the other lines as in the figure: and then since EP, E′Q are considered as parallel, we shall have

$$\angle \mathrm{QE'S} - \angle \mathrm{PES} = \angle \mathrm{P}w\mathrm{S} - \angle \mathrm{PES} = \mathrm{ESE'}$$

$$\text{and} \quad \angle \mathrm{SP}\,w - \angle \mathrm{SQE'} = \angle \mathrm{S}\,v\,\mathrm{E'} - \mathrm{SQE'} = \mathrm{PSQ}$$

Hence the cotemporary variations of the angles at E and P, are as the angles ESE′, PSQ; or, since the angular velocities are inversely as the periodic times, or as the $\frac{2}{3}$d power of the distances, we shall find those angles to each other as $\mathrm{SP}^{\frac{3}{2}} : \mathrm{SE}^{\frac{3}{2}}$, or denoting the ratio of the distances by $a : 1$, they will be as $a^{\frac{3}{2}} : 1$.

Now the sines of the angles of a triangle being in the ratio of their opposite sides, we shall have

$$\sin \mathrm{E} : \sin \mathrm{P} : \mathrm{SP} : \mathrm{SE} : a : 1$$

and the variation of angles, whose sines are constant being as their tangents; we shall have also,

$$\sin \mathrm{E} : \sin \mathrm{P} : a : 1$$

$$\frac{\sin \mathrm{E}}{\cos \mathrm{E}} (= \tan \mathrm{E}) : \frac{\sin \mathrm{P}}{\cos \mathrm{P}} (= \tan \mathrm{P}) :: a^{\frac{3}{2}} : 1$$

$$\text{or} \quad \frac{\sin \mathrm{E}}{\sqrt{(1-\sin^2 \mathrm{E})}} : \frac{\sin \mathrm{P}}{\sqrt{(1-\sin^2 \mathrm{P})}} : a^{\frac{3}{2}} : 1$$

$$\text{whence} \quad \sin^2 \mathrm{E} = \frac{a^3 - a^2}{a^3 - 1} = \frac{a^2}{a^2 + a + 1}$$

$$\text{and} \quad \sin \mathrm{E} = \frac{a}{\sqrt{(a^2 + a + 1)}}$$

the planet's elongation from the sun when stationary.

By taking P to represent the earth, and E an inferior planet, the same results will apply to the case of Venus or Mercury.

Problem II.

To find the time when a planet is stationary; the time of opposition, if a superior planet, or of its inferior conjunction, if an inferior one, being given.

172. Let m and n be the daily motions of the earth and planet; and v, the angle PSE when the planet is stationary, then $m - n$, or $n - m$ is the daily variation of the angle at the sun between the earth and planet, according as it is a superior or inferior planet; hence,

$$\left.\begin{matrix} m - n \\ \text{or } n - m \end{matrix}\right\} : v :: \overset{\text{day}}{1} : \frac{v}{m \backsim n}$$

the time from opposition or conjunction to the stationary points both before and after.

Hence the planet must be stationary every *synodic* revolution.

173. It is to be observed, that the above solutions depend upon the supposition of the orbits being circular, which we know is not precisely the case; but as the subject is altogether one rather of curiosity than utility, we have not thought it necessary to enter upon it at greater length. It will be sufficient to observe, that in the case of elliptic orbits, the stationary positions will depend upon the position of the apsides, and the places of the bodies at the time. A near approximation however may be obtained by computing, on the supposition of circular orbits, the time when the planet would be stationary, and finding for several days about that time the geocentric place of the planet, so as to obtain two results, such, that on one day the motion of the planet be direct, and on the other retrograde, within which interval it must have been stationary; and the point of time when this happens may be determined by interpolation. Remarks.

174. It is obvious from what has now been stated, that a superior planet will be retrograde in opposition, and an inferior in its inferior conjunction; and that the latter would appear to retrograde whether the earth were in motion or at rest, but that the apparent retrogradation of a superior planet depends wholly upon the motion of the earth; this will, however, perhaps be rendered more evident by referring to fig. 52, in which we shall suppose E to represent the earth; P a superior planet, in opposition; then, as the velocities are as the inverse square roots of the radii of the orbits, the superior planet moves slowest; hence, if EF, PQ, be two indefinitely small cotemporary arcs, PQ is less than EF; and, on account of the immense distance of the sphere of the fixed stars, FQ must cut EP in some point x between P and m; consequently, the planet appears retrograde from m to n; whereas, if the earth were at rest, then the motion of P must necessarily appear throughout its whole circle to move in the same direction. Superior planets retrograde in opposition. Fig. 52.

175. Again, if P be the earth, and E an inferior planet in its inferior conjunction, it will appear retrograde from v to w; which being due to the more rapid motion of the planet, it is clear that the same phenomena would occur, whether P were at rest or moving with a velocity less than that of the planet. The periods and positions of retrogradations would however be obviously changed in the former case.

PROBLEM III.

To delineate the phase of a planet in any given position.

Delineation of the phases.

176. The principle of this construction will appear from what has been stated in the preceding part of this section, where we have observed that the enlightened hemisphere will be that whose plane or base is perpendicular to a line drawn from the sun, and the hemisphere of vision that whose plane is perpendicular to a line drawn from the earth, each being directed to the centre of the planet. If, therefore, in fig. 53, we take S to denote the sun, E the earth, V any planet, aVb will be the plane of the illuminated hemisphere, or the plane of illumination, and cVd perpendicular to EV the plane of vision; draw av, perpendicularly to cd then ca is the breadth of the illuminated part which is projected into cv, the versed sine of cVa, or SVZ, SVc being the complement to both. Now the circle terminating, the illuminated part of the planet being seen obliquely, appears to be an ellipse, therefore if $cmdn$ represent the projected hemisphere of the planet next the earth, mn, cd, being two diameters perpendicular to each other, and we take $cv =$ the versed sine of SVZ. and describe the ellipse mvn, then mcn vm will represent the visible enlightened part, as it appears at the earth; and from the property of the ellipse this varies as cv. Hence

Fig. 53.

The visible enlightened part
Is to the whole disc
As the versed sine of SVZ
Is to the diameter.

From this construction it appears, that the more remote a planet is from the sun with respect to the earth, the nearer will the illuminated part as seen from the earth approach to a circle; in Mars a gibbous form may sometimes be perceived, but in the more distant bodies the angle SVZ never differs enough from two right angles to give them the same appearance, that is, they must necessarily always appear full.

2. *On the reduction of observations made on the planets from the earth, to the corresponding observations that would be made from the sun.*

Determination of the heliocentric longitude of a planet.

177. The earth being in constant motion in its orbit, and being thus very differently situated with regard to the other bodies of our system, as well in respect to distance as position, it is actually necessary for the purpose of comparing different observations with each other, to refer them all to one common point of observation; and it is obvious that no point can be better suited for this purpose than the sun, the common centre of all their motions. The theory of the earth's motion being determined, the dimensions of its orbit may be taken as the base of the geometric operations necessary for the reduction in question, of which we have had occasion to speak in describing the laws of Kepler; we shall now illustrate the same a little more at length.

Fig. 54.

Let E (fig. 54) be the earth's place in its orbit, S the sun, and P any planet, P′ its projected place in the ecliptic, SEP′ the observed angle, measuring the difference between the longitude of the sun and that of the planet seen from the earth. After an entire revolution of the planet, it will return to the same point P; let then the earth be at E′, and the angle P′E′S be observed. Then by the theory of the earth's motion, the angle ESE′, and the lines SE, E′S are known, therefore the angles SEE′, SE′E, and the line EE′ may be also determined.

Again, in the triangle EME′, there is known EE′ and the angles P′EE′, P′E′E, therefore the sides P′E, P′E′ are likewise known or may be found. Lastly, in the triangle P′SE, the sides P′E, ES and the included angle P′ES are given to find SP′, the sun's distance from the planet at the point P′ of its orbit, as also the angle ESP′, whence the difference between γSE, the longitude of the earth, and γSP′, the longitude of the planet, become also determined.

Heliocentric latitude

178. To find the latitude seen from the sun, the following proportion may be used,

As the sine of the angle P′ES : sine of ESP′ : : the tangent of the latitude of the planet, observed from the point E, to the tangent of its heliocentric latitude, that is, to its latitude as observed from the sun. To find the true distance of the planet from the sun, that is, the line SP, say, as the cosine of the planet's heliocentric latitude is to radius, so is the SP′ to the distance sought.

We may otherwise determine the longitude, latitude, and distance by means of the triangle SEP. By either of these means the distance of a planet, its longitude and latitude may be found for various points of its orbit, but as will be seen in the following problem, three such points being found, the orbit itself may be described.

PROBLEM.

To describe an ellipse through three given points, that is, naving given three distances, and the three corresponding heliocentric longitudes, to find the orbit.

To determine the orbit from three longitudes and distances.

179. In the preceding part of this section, we have shown the method by which the distance and heliocentric longitude of a planet may be determined from observation, the dimensions of the earth's orbit being supposed known; but the distances thus determined apply only to those particular points of the orbit in which the planet is at the times of observation, and we are still at a loss to determine the nature and elements of its entire orbit; the present problem then is intended to obviate this difficulty, and to show that, from three points being given, the orbit itself may be ascertained. This may be effected either geometrically or analytically, but we prefer the latter.

Fig. 55.

Let SP, SP′, SP″, fig. 55, be three given distances, and γSP, γSP′, γSP″, three given longitudes, and AB the line of the apsides of the planet's orbit. Let the angles PSP′, PSP″, which are known, be represented by m, n, and the angle ASP which is unknown by x; let the three distances SP, SP′ SP″ be denoted by ρ ρ' ρ''; the semi transverse Ac by c, and the eccentricity by e; then the three unknown quantities which are to be determined are, a, e, and the angle x, and the three equations from which their values are to be deduced are,

$$\rho = \frac{a(1-e^2)}{1+e\cos x} \qquad (1)$$

$$\rho = \frac{a\,(1-e^2)}{1 + e \cos\,(m+x)} \qquad (2)$$

$$\rho'' = \frac{a\,(1-e^2)}{1 + e \cos\,(n+x)} \qquad (3)$$

Sec art. 58. Dynamics.

From these three we deduce the two following, involving only e and x; viz.

$$\rho\,(1 + e \cos x) = \rho' \left\{1 + e \cos\,(m+x)\right\}$$
$$\rho\,(1 + e \cos x) = \rho'' \left\{1 + e \cos\,(n+x)\right\}$$

or which are still the same,

$$e = \frac{\rho'' - \rho}{\rho \cos x - \rho'' \cos\,(n+x)} \qquad (4)$$

$$e = \frac{\rho' - \rho}{\rho \cos x - \rho' \cos\,(m+x)} \qquad (5)$$

Let $\rho'' - \rho = r'$ and $\rho' - \rho = r'$; then

$r' \rho \cos x - r' \rho'' \cos\,(n+x) = r'' \rho \cos x - r'' \rho' \cos\,(m+x)$

or

$\cos x\,(r' \rho - r'' \rho) = r' \rho'' \cos\,(n+x) - r'' \rho' \cos\,(m+x)$

substituting for $\left\}\begin{array}{l}\cos\,(n+x) = \cos n \cos x - \sin n \sin x \\ \cos\,(m+x) = \cos m \cos x - \sin m \sin x\end{array}\right.$

and dividing by $\cos x$, we have,

$r'\rho - r''\rho = r'\rho''\,(\cos n - \sin n \tan x) - r''\rho'\,(\cos m - \sin m \tan x)$

Hence $(r''\rho' \sin m - r'\rho'' \sin m) \tan x = r'\rho - r''\rho - (r'\rho'' \cos n - r''\rho' \cos m)$

Therefore, finally,

$$\tan x = \frac{r'\rho - r''\rho - (r'\rho'' \cos n - r''\rho' \cos m)}{r''\rho' \sin m - r'\rho'' \sin m}$$

The angle x being thus determined, the three angular distances from the perihelion become known; also x being given, the eccentricity e may be computed from either equation 4 or 5, and hence again a, the semi transverse axis from equation 1, 2, or 3.

Cases in which the above solution will not apply.

180. We have thus the means of determining very accurately all the preceding elements of a planet's orbit, but it will be perceived, that it rests upon one important condition, namely, that the planet has been observed from the same point of its orbit after one, or some number of complete revolutions; for it is on this condition that the heliocentric longitudes are determined. The problem, therefore, will still not directly apply to any newly discovered body, as for example, Uranus or the Georgian planet, whose period of revolution being about 80 years, we must have waited till this period had passed by, before its heliocentric longitudes could have been computed on the principles explained in (art.179). But as such a delay is wholly incompatible with the anxiety and impatience of astronomers, recourse has been had to a method of trial and error, and by assuming a distance in the first instance, and comparing the results as determined according to Kepler's laws, with those deduced from observations, and making such successive corrections as the case requires, astronomers have found the method of approximating to the elements of a planet's orbit, from three geocentric longitudes only, the time between these observations being of course also given. It was by these means that the elements of the orbit of Uranus were determined by Lalande to a very considerable degree of accuracy, within one year of the first discovery of that planet by Sir W. Herschel.

Of the methods of determining the elements of a planetary orbit.

Elements of a planet's orbit.

181. We have in the preceding articles explained the methods of determining certain of the elements of a planet's orbit; but there are others equally necessary to be known, in order to complete the theory of the planetary motions.

The quantities known under the denomination of elements, are the seven following.

1. The longitude of the ascending node.
2. The inclination of the orbit to the plane of the ecliptic.
3. The mean motion of the planet about the sun.
4. The mean distance from the sun.
5. The eccentricity of the orbit.
6. The longitude of the aphelion.
7. The epoch of the planet being in its aphelion

1. *Determination of the place of the nodes.*

Longitude of the node.

182. The *nodes* of a planet's orbit are the two points where the orbit intersects the plane of the ecliptic, and the line joining those points is called the line of the nodes. That node at which the planet passes from the southern to the northern side of the ecliptic is called the *ascending* node, and is marked ☊, the other is called the *descending* node, and is marked ☋.

Since the node is the intersection of the orbit with the ecliptic, and the latitude of a planet being always estimated from that plane, it is obvious that if by a comparison of observations made on a planet when near the node, and when its latitude is decreasing, we compute according to the principles explained in several preceding articles the exact time when its latitude was zero, that time will be the exact time of the node. Whence again, as explained in the last section, find the corresponding heliocentric longitude of the planet, and the result will be the longitude of the planet when in the ecliptic or the *longitude of the node.*

Second method.

Second method. Fig. 56.

183. Let N, (fig.56,) be the place of the node, nMm a portion of the ecliptic, aNb, a portion of the planet's orbit, am bn, two heliocentric latitudes, reduced from their geocentric latitudes; then by the analogies of Napier.

$$\text{rad.} \sin Nm = \cot N \tan am \qquad (1)$$
$$\text{rad.} \sin Nn = \cot N \tan bn \qquad (2)$$

Whence

$$\frac{\sin Nm}{\sin Nn} \text{ or } \frac{\sin\,(mn - Nn)}{\sin Nn} = \frac{\tan am}{\tan bn}$$

and by reducing

$$\sin nm \times \cot Nn - \cos mn = \frac{\tan am}{\tan bn}$$

Whence $\tan Nn = \dfrac{\sin nm \tan bn}{\cos mn \tan bn + \tan an}$

where bn, am, the two reduced latitudes are known; also nm, the difference of the longitudes in the interval of the observations; and hence Nn, or the difference of the longitudes n and N is also known; and the longitude of *n* being given, that of N, the node, is determined.

2. *To find the inclination of the orbit.*

Inclination of the orbit.

184. This is immediately deducible from the preceding investigation; for as N*n* is determined, and *bn* known, we have from equation (2)

$$\cot\ N = r \times \frac{\sin\ Nn}{\tan\ bn} \text{ or } \tan N = r \times \frac{\tan\ bn}{\sin\ Nn}$$

3. *To determine the periodic time of a planet.*

Periodic times

185. This may be effected in different ways; viz. we may determine by means of (art. 179 et seq.) the semi axis of its orbit, and hence compute its periodic time by Kepler's laws; or we may observe the planet in its node, and again, when it returns hither; and the interval elapsed will be the period nearly but not exactly, by reason of the retrogradation of the nodes and some other minor circumstances.

Another method, and the most accurate of any, is to observe the planet, if a superior one when in opposition, and if an inferior when in conjunction, and to observe the same again after one or some number of complete revolutions, whence the time may be found very correctly, and the tables of its motion corrected. The time of opposition, it will be perceived, is selected for this determination, in consequence of the heliocentric and geocentric longitudes being then the same; whereby that reduction necessary in all other cases is avoided.

Periodic times by opposition.

186. Therefore when the planet is approaching towards opposition, observe the mean time that it passes the meridian, its right ascension and meridian altitude, whence by means of the corrections already explained, find its declination on those days; and, hence again, compute its latitude and longitude; make similar observations on the sun, and determine its longitude in like manner. By comparing these results with each other, it will be seen that the difference of longitude from being more than 180°, has become less than 180°, consequently, the opposition must have taken place in that interval. that is, between the two observations that give these opposite results.

In order now to find the exact instant, compute the sun's longitude at the moment when the planet was on the meridian, for that observation that gives the difference in longitude too little; and let 180° — *d* denote the difference of longitude between the sun and planet at this instant; let *r* represent the sun's daily increase in longitude at this time, and *r′* the daily motion in that of the planet; then since when a planet is in opposition its motion is retrograde, the sun and planet will approach each other at the rate of *r*+*r′*, per day; and we shall have

$$r + r' : d : 1 \text{ day } \frac{d}{r+r'}$$

the time prior to the latter of the two observations, when the difference of longitude was exactly 180°; which time, therefore, subtracted from the time of the said observation will be the time of opposition required.

If the above be, as we have supposed, repeated for several days, and the mean of the several results be taken, the time will be had more accurately, and still more so if the observations be repeated after a number of complete revolutions; as by this means the errors will be so much the more divided, and rendered of less consequence. If the time of opposition, as determined above, be compared with the time given in the tables, the difference will be the error of the tables, and may be employed as a means of correcting them.

Illustrated by example.

187. Mr. Vince in his Astronomy, gives the following example of this kind of determination. On October 14, 1763, M. Lalande observed the difference between the right ascension of β Aries and Saturn, (which latter passed the meridian at 12h. 17m. 17s. apparent time) to be 8° 5′ 7″, the star passing first. Now the apparent right ascension of the star at that time was 25° 24′ 33″·6, hence the apparent right ascension of Saturn was 1s. 3° 29′ 40′·6 at 12h. 17m. 17s. apparent time, or 12h. 1′ 37″ mean time. On the same day, he found from observation of the meridian altitude of Saturn, that its declination was 10° 35′ 20″ N. Hence the longitude was found to be 1s. 4° 50′ 56″, and latitude 2° 43′ 25″ S. At the same time the sun's longitude was found by calculation to be 7s. 1° 19′ 22″, which subtracted from 1s. 4° 50′ 56″, gives 6s. 3° 31′ 34″; hence Saturn was 3° 31′ 34″ beyond opposition, but being retrograde must afterwards come into opposition. Now, from the observations made on several days at that time, Saturn's longitude was found to decrease 4′ 50″ in 24 hours, and by computation, the sun's longitude increased 59′ 59″, in the same time, therefore *r* + *r′*, or the sum of the two, was 64′ 49″: hence

64′ 49″ : 3° 31′ 24″ : 24 : 78h. 20m. 20s.

which added to Oct. 24, 12h. 1m. 37″ gives 27d. 18h. 21′ 57″ for the time of opposition.

To find Saturn's longitude at this time, say

24h. : 78h. 20m. 20s. 4′ 50″ : 15′ 47″

which latter is the retrograde motion of Saturn in 78h. 20m. 20s., and this subtracted from 1s 4° 50′ 56″ leaves 1s. 4° 35′ 9″, the longitude of Saturn at the time of opposition; and by a similar process, the longitude of the sun for the same time was found to be 7s. 4° 35′ 9″.

In a manner nearly analogous, we may find the latitude of Saturn at the instant of opposition; viz. by observing the daily variation in latitude. In the present case between the time of observation and opposition, the latitude had increased 6″, and consequently, the latitude was 2° 43′ 31″ S at the time of opposition.

188. The most ancient observation of this kind on record is one by Hipparchus; from which it appears that

228 B.C. Mar. 2, 1h. Saturn's longitude in opposition was .... } 98° 23′ 0″
lat. N 2° 50′
1714, A.D. Feb. 26, 8h. 15m. his longitude in opposition was ...... } 97° 56′ 46″
lat. N 2° 3′

Differen. of longitude in 1943y. 105d. 7h. 15m. = 26′ 14″

Now we find the time of describing this 26′ 14″ to be 13d. 14h. Adding this to the former interval, we have 1943y. 118d. 21h. 15m. for the interval between these two oppositions of Saturn, and the period of this planet is known to be about 29½ years, this time must therefore contain 66 complete periods of revolution, and hence by division we have

$$\frac{1943\text{y. } 118\text{d. } 21\text{h. } 15\text{m.}}{66} = 29\text{y. } 162\text{d. } 4\text{h. } 27\text{m.}$$

for the exact time of one revolution.

Sideral revolution of Saturn.

189. It will be observed, that the longitudes in both these cases is reckoned from the first point of Aries, and in consequence of the annual precession of this point of 50″·1, the entire precession due to the above interval of 29y. 162d. 4h. 27m. will be 24′ 35″. The period then determined for Saturn belongs to his tropical revolution, and is shorter than his sideral by the time that he employs in describing 24′ 35″, which is about 12d. 7h.; hence his sideral revolution will be 29y. 174d. 11h. 27m. His mean annual motion is hence found to be 12° 13′ 35″ 14‴ and his daily motion 2′·0097.

Exactly the same process may be observed with all the superior planets.

Method for the inferior planets.

190. With respect to the inferior planets, we may proceed in the same manner when circumstances will admit of our making the requisite observations, but it is obvious that this will only happen when the elongation of the planet from the sun, near conjunction, is sufficient to render the former visible. The most accurate method of observing the time of an inferior conjunction of either Mercury or Venus is from observations made on them in their transits over the sun's disc.

3. *To find the semi axis major or mean distance.*

Of the mean distance.

191. Let d denote the mean distance of the earth, and t the time of its sideral revolution, D the required mean distance of the planet, and T its period of revolution, as determined in the last article. Then by Kepler's law

$$t^2 : T^2 :: d^3 : D^3 = d^3 \times \frac{T^2}{t^2}$$

whence $D = d \times \left(\frac{T}{t}\right)^{\frac{2}{3}}$ the distance sought.

4. *To determine the eccentricity of the orbit.*

Eccentricity.

192. This may be done by means of our general equation.

$$\rho = \frac{a\,(1 - e^2)}{1 + e \cos x}$$

the angle x, and radius vector ρ being determined as in arts. 179 and 180, and a as in the preceding article; but more conveniently from the equation (4) art. 179. viz.

$$e = \frac{\rho'' - \rho}{\rho \cos x - \rho'' \cos (n + x)}$$

in which all the quantities on the second side of the equation are supposed to have been determined.

5. *Of the aphelion of a planet's orbit.*

Longitude of aphelion.

193. This is deducible immediately from article 179, where we have.

$$\tan x = \frac{r' \rho - r'' \rho - (r' \rho'' \cos m - r'' \rho' \cos n}{r'' \rho' \sin m - r' \rho'' \sin m}$$

for the quantity x is the angular distance of the aphelion from the observed place of the planet, the longitude of which is known.

Second method.

194. Another method of determining the same element, is by means of the greatest equation of the centre, for example, let the planet be observed when near A (fig. 48); if it should be exactly there, it will be known by the mean angle proportional to the time from M to A, minus the difference of the observed longitudes at M and A, being equal to the greatest equation of the centre. But suppose it to be less by some quantity e, then the planet is at some point S; make a second observation, and now let the difference of the true and mean motions be greater than the greatest equation, by another quantity e'; then the planet is at T, past the aphelion. Now the longitudes of S and T are known; consequently the difference, SET is known, and the intervals between the two observations; then, the points S and T being supposed very near to A, we shall have

$$e + e' : \angle \text{SET} :: e : \angle \text{SEA}$$

Add, therefore, the ∠ SEA to ∠ SEM, and ∠ MEA or the angular distance of M from the aphelion will be determined.

6. *To determine the epoch at which a planet is in the aphelion.*

Epoch determined.

195. This may be readily deduced from the preceding observations; for since the interval of time t, between the two observations at S and T are known, and the angle SEA has been determined, we have

$$e + e' : e :: t : \frac{e\,t}{e + e'}$$

the time from S to A: which time thus determined added to the time of observation at S, gives the time at which the planet was in its aphelion.

Table of the elements of the planets.

196. We shall here conclude this division of our treatise, by giving the following table of the elements of the planets, according to the most recent determination of Laplace, reduced where necessary to English measures.

Table of the elements of the planetary orbits.

Sideral periods of the planets.

	Days
Mercury	87·96258
Venus	224·700824
The Earth	364·256384
Mars	686·979619
Vesta	1335·205
Juno	1590·998
Ceres	1681·539
Pallas	1681·709
Jupiter	4332·596308
Saturn	10758·969840
Uranus, or the Georgian Planet	30688·712687

Mean distances, or semi axes major of the orbits.

	Days
Mercury	0·387098
Venus	0·723332
The Earth*	1·000000
Mars	1·523694
Vesta	2·373000

---

* The distance of the earth from the sun is computed to be 93,726,900 English miles.

Juno	2·667163
Ceres	2·767406
Pallas	2·767592
Jupiter	5·202791
Saturn	9·538770
Uranus, or the Georgian Planet	19·18305

Ratio of the eccentricities to the semi axes major at the beginning of 1801, *with the secular variation of the ratio. The sign — indicates a diminution.*

	Ratio of eccentricity.	Secular variation.
Mercury	0·205514	0·000003867
Venus	0·006853	0·000062711
The Earth	0·016853	0·000041632
Mars	0·093134	0·000090176
Juno	0·254944	not ascertained
Vesta	0·093220	
Ceres	0·078349	
Pallas	0·245384	
Jupiter	0·048178	0·000159350
Saturn	0·056168	0·000312402
Uranus	0·046670	0·000025072

Mean longitudes at the beginning of 1801, *reckoned from the mean equinox at the epoch of the mean noon, Jan.* 1*st*, 1801, *Greenwich.*

	°	′	″
Mercury	166	0	48·2
Venus	11	33	16·1
The Earth	100	39	10·0
Mars	64	22	57·5
Vesta	267	31	49·0
Juno	290	37	16·0
Ceres	264	51	34·0
Pallas	252	43	32·0
Jupiter	112	15	7·0
Saturn	135	21	32·0
Uranus	177	47	38·0

Mean longitudes of the perihelia, for the same epoch as above, with the sideral and secular variations.

	Mean lon. per.			Sec. var.	
	°	′	″	′	″
Mercury	74	21	46	9	43·5
Venus	128	37	08	4	28
The Earth	99	30	5	19	39
Mars	332	24	24	26	22
Vesta	249	43	0	not known	
Juno	53	18	41		
Ceres	146	39	39		
Pallas	121	14	1		
Jupiter	11	8	35	11	4
Saturn	89	8	58	32	17
Uranus	167	21	42	4	0

Inclination of the orbits to the ecliptic at the beginning of 1801, *with the secular variations.*

	Inclination.			Sec. var
	°	′	″	″
Mercury	7	0	1	19·8
Venus	3	23	32	4·5
The Earth	0	0	0	
Mars	1	51	3·6	1·5
Vesta	7	8	46	not known
Juno	13	3	28	
Ceres	10	37	34	
Pallas	34	37	7·6	
Jupiter	1	18	51	23
Saturn	2	29	34·8	15·5
Uranns	0	46	26	3·7

Longitudes of the ascending nodes on the ecliptic at the beginning of 1801, *with the sideral and secular motions.*

	Lon. of ascen. node.			Sec. var.
	°	′	″	″
Mercury	45	57	31	13·2
Venus	74	52	38·6	31 10
The Earth	0	0	0	
Mars	48	14	38	38 48
Vesta	103	0	6	not known
Juno	171	6	37	
Ceres	80	55	2	
Pallas	172	32	35	
Jupiter	98	25	34	26 17
Saturn	111	55	46	37 54
Uranus	72	51	14	59 57

§ IX. *On the computation of eclipses.*

Computation of eclipses.

197. We have already given a popular illustration of the phenomena of eclipses; and have shown that they arise from the interposition of the earth between the moon and sun, or of the moon between the earth and sun. If the lunar orbit were in the plane of the ecliptic, we must necessarily have an eclipse at every conjunction and opposition of the sun and moon, and these phenomena would therefore, in this case, be governed by the phases of the moon; but the duration of the obscuration, and the other circumstances attending the eclipse, would still depend upon the relative distances and apparent magnitudes of these bodies with respect to each other.

When the moon is at her mean distance, the apparent diameters of the earth and sun as seen from the moon's centre are 1° 55′ 8″ and 31′ 59″·08. But the angles subtended by the sun and moon at the extremity of the earth's shadow in a lunar eclipse, are the same, about 64′; consequently, the moon must fall within that shadow at a very considerable distance from its vertex; and by computation it will be seen, that the length of the shadow is actually about equal to four times the distance of the moon from the the earth. This has reference, however, only to the mean distance, but the eccentricity of the lunar orbit being only ·0548553, the ratio between the length of the shadow and the distance of the moon is never materially changed. For example, the greatest distance of the moon from the earth never exceeds 64 terrestrial radii; and the length of the shadow when the moon is in perigee is .... 212·896 rad.
mean distance .... 216·531
in apogee .... 220·231

Consequently, as we have observed above, if the lunar orbit, were wholly in the plane of the ecliptic, we must necessarily have an eclipse of the moon at every opposition.

The case, however, is somewhat different in a solar eclipse; for here the shadow of the moon in no instance passes far beyond the earth's centre, and in some it does not even reach the surface; but still the eclipse must, on the supposition we have made, actually take place with every conjunction of the sun and moon: this phenomenon depending, not upon the shadow of the moon, but upon the actual interposition of that body between the sun and earth.

The moon, however, revolves in an orbit which inclines to the plane of the ecliptic at an angle of 5° 9′; and, consequently, she may be in opposition, that is, the difference of longitude of the sun and moon may be 180°, and yet the latter body be either above or below the terrestrial shadow, in which case no eclipse will take place; and the same of course applies to the moon being in conjunction, or having the same longitude as the sun. It is not necessary, however, that the moon be actually in her node at the time of opposition, but only that she be within certain limits of that position; for the shadow of the earth subtending a larger angle at its vertex than the moon does, an eclipse may happen although the moon be not in her node at the time of opposition; and the same again, except that the limits are more narrow, at the time of conjunction. Let us, therefore, first examine what are called the *ecliptic limits*, that is to say, the least degree of proximity of the moon that will render an eclipse possible.

Fig. 57. Ecliptic limits.

198. For this purpose, let SO (fig. 57) denote the radius of the sun, TE that of the earth, and L the centre of the moon at the moment of opposition: the three centres being in the same right line. Let OENC represent a solar ray, being a tangent to the sun at O, and to the earth at E: SO and TE will be perpendicular to the radius OC; and CS will be the axis of the conical shadow projected by the earth, and which, therefore, will be dark between the earth and the vertex C.

Let LuK be a portion of the lunar orbit; then its tangent LN will represent the semi diameter of a section of the cone, drawn perpendicularly to the axis in the region of the moon, and this section will be a circle; u will be the point where the edge of the lunar disc enters into the conical shadow; the centre of the moon being in K, or Ku being supposed equal to a semi diameter of the moon at that time, as seen from the earth at T. Let now TE, the terrestrial radius = 1, then

$$\text{TS} = \text{dist. of sun from the earth} = \frac{1}{\sin p}$$

$$\text{TL} = \text{dist. of moon from the earth} = \frac{1}{\sin \text{P}}$$

$$\text{TC} = \text{axis of conical shadow} = \frac{1}{\sin \text{C}}$$

where p denotes the parallax of the sun, P that of the moon, and C the angle of the cone, or C = TCE.

$$\text{Now LC} = \text{TC} - \text{TL} = \frac{1}{\sin \text{C}} - \frac{1}{\sin \text{P}}, \text{ or}$$

$$\text{LC} = \frac{\sin \text{P} - \sin \text{C}}{\sin \text{C} \sin \text{P}} = \frac{2 \sin \tfrac{1}{2} (\text{P}-\text{C}) \cos \tfrac{1}{2} (\text{P}+\text{C})}{\sin \text{C} \sin \text{P}}$$

Let the apparent semi diameter of the sun, as seen from the earth be called δ, then

$$\text{SO} = \text{TS} \sin \delta = \frac{\sin \delta}{\sin p}$$

$$\text{and TC} = \frac{\text{SC TE}}{\text{SO}} = \frac{\text{SC}}{\text{SO}} = \frac{\text{ST} + \text{TC}}{\text{SO}}$$

Or, substituting for these quantities their respective values, we have

$$\frac{1}{\sin \text{C}} = \frac{\frac{1}{\sin p} + \frac{1}{\sin \text{C}}}{\left(\frac{\sin \delta}{\sin p}\right)} = \frac{1 + \frac{\sin p}{\sin \text{C}}}{\sin \delta}$$

Whence,

$$\frac{\sin \delta}{\sin \text{C}} = 1 + \frac{\sin p}{\sin \text{C}} = \frac{\sin \text{C} + \sin p}{\sin \text{C}}$$

Therefore,

or,

$$\sin \delta = \sin \text{C} + \sin p$$
$$\sin \text{C} = \sin \delta - \sin p$$

Now, $\sin \text{C} = 2 \sin \tfrac{1}{2} \text{C} \cos \tfrac{1}{2} \text{C}$

and $\sin \delta - \sin p = 2 \sin \tfrac{1}{2} (\delta - p) \cos \tfrac{1}{2} (\delta + p)$

Wherefore,

$$2 \sin \tfrac{1}{2} \text{C} \cos \tfrac{1}{2} \text{C} = 2 \sin \tfrac{1}{2} (\delta - p) \cos \tfrac{1}{2} (\delta + p)$$

In this expression, if $(\delta - p)$ and $(\delta + p)$ were equal to each other, we should have, accurately,

$$2 \sin \tfrac{1}{2} \text{C} = 2 \sin \tfrac{1}{2} (\delta - p)$$

or

$$\text{C} = \delta - p$$

which may, in fact, be taken as an approximate equation; for since δ never surpasses 16′ 18″, nor p 7 9″, we have the greatest value of $\tfrac{1}{2} (\delta + p) = 8' \, 14''$ and of $\tfrac{1}{2} (\delta - p) = 8' \, 5''$; consequently, the cos $\tfrac{1}{2}$ $\tfrac{1}{2} (\delta - p) = \cos \tfrac{1}{2} (\delta + p)$ *very nearly*. Hence the angle subtended by the side and axis of the conical shadow, is equal to the sun's apparent semi diameter minus the sun's parallax, or,

$$\text{C} = \delta - p$$

Now CTE = 90° — C = 90° + p — δ = bTE = bE
 2 bE = 180° + 2p — 2 δ = E b F = 180° — 2 (δ — p)

E b F is the obscure part of the earth's surface, which is less than a hemisphere by a zone whose altitude is C, or $\delta - p$; and the enlightened part is for the like reason greater than a hemisphere by the same quantity.

The angle LTu = TuE — C = P — C = P — δ + p = P + p — δ,

Hence the sum of the two parallaxes, minus the sun's semidiameter, will be the angular distance of the moon from the axis of the conical shadow at the moment when the edge of the lunar disc enters into the shadow. To this angle, adding the moon's semidiameter (d), we shall have the angular distance of the axis of the cone from the moon's centre for the same instant. The two expressions therefore are

$$\text{P} + p - \delta, \text{ and}$$
$$\text{P} + p - \delta + d,$$

and they are those generally adopted; although, as we have seen, they are in fact only approximative, but very closely so.

199. If we conceive the moon to be placed between the earth and the sun in L′, L′n' being a part of its orbit, n' will be the point at which the moon enters into the luminous cone, and we shall have

$$\text{L}'\text{T}n' = \text{T}n'\text{E} + \text{C} = \text{P} + \delta - p = \text{P} - p + \delta.$$

Whence the difference of the parallaxes, plus the semidiameter of the sun, will be the angular distance of the centre of the moon from the axis of the cone, at the moment when the centre of the moon enters into the luminous cone; and to this expression, adding the semidiameter (d) of the moon, we shall have the angular distance corresponding to the moment when the edge of the lunar disc enters the cone.

The formulæ therefore for solar eclipses are

$$\mathrm{P} - p + \delta$$
$$\mathrm{P} - p + \delta + d.$$

We have thus the general expressions for the limits beyond which no eclipse of either luminary can take place.

Limits of total eclipses.

200. Within these limits an eclipse will happen, but they must be still further restricted in the case of the sun, in order that the eclipse may be total. With respect to the moon, the eclipse is necessarily total, provided the latitude of this luminary be such that it enters completely within the shadow. But this condition only is not sufficient in the case of a solar eclipse; for although the moon may be completely interposed between the earth and sun, yet its shadow may not arrive at the earth; and in this case the eclipse will not be total, but annular or partial.

Draw KVO, a tangent to the moon and sun; then, if KL′ be equal to TL′, the eclipse is necessarily total wherever it is central; and the duration of it will be greater or less according to the situation of the observer on the terrestrial surface. The same will also happen if KL′ be greater than TL′; but the obscuration will continue longer. If KL′ be less than L′T, but not less than L′a, the eclipse will still be total to those observers who are nearer the moon than the vertex of the cone, and total for an instant only to those who are precisely at the same distance as the vertex: to others the eclipse will be partial or annular. If KL′ be less than a L′, then the eclipse cannot be total to any observer on the terrestrial surface.

Now it may be shown, that

$$\mathrm{TL}' - \mathrm{KL}' = \frac{\dfrac{1}{\sin \mathrm{P}} - \dfrac{\sin d}{\sin \mathrm{P} \sin \delta}}{1 - \dfrac{\sin \delta \sin \mathrm{P}}{\sin \delta \sin \mathrm{P}}}$$

Hence assuming $d = 16' \; 28''$, $\delta = 16'$, $\mathrm{P} = 60' \; 20''$ $p - 8''{\cdot}6$; we have, expressing the above in numbers,

$$\mathrm{TL}' - \mathrm{KL}' = \frac{\dfrac{1}{\sin \mathrm{P}}\left(1 - \dfrac{\sin d}{\sin \delta}\right)}{1 - {\cdot}002445107}$$
$$= \frac{1}{\sin \mathrm{P}}\,(0{\cdot}00245110)\left(1 - \frac{\sin d}{\sin \delta}\right)$$
$$= \frac{1}{\sin \mathrm{P}} - \frac{1{\cdot}2736100}{\sin \delta} + \frac{0{\cdot}0024511}{\sin \mathrm{P}}$$

Consequently, when this expression is equal to zero, we shall have

$$\sin \mathrm{P} = 3{\cdot}663792 \sin \delta$$

and when it is equal to unity, that is, to T a,

$$\sin \mathrm{P}' = \frac{1{\cdot}002508336}{1 + \dfrac{0{\cdot}279996}{\sin \delta}}$$

and it is only between these limits, in the value of sin P, that the eclipse can be total.

201. By means of these two expressions, we may compute the several values of P corresponding to the different apparent diameters of the sun that must have place to produce a total or annular eclipse. For example, in the following table, the semidiameter of the sun being such as stated in the first column, the lunar parallax (P) must not be less than that stated in the second column, in order that the eclipse be necessarily total: if the parallax be less than in the second column, but greater than in the third, it may be total to certain observers; but if the parallax be less than that given in the third column, it can be only annular or partial.

Limits of total and annular eclipses.

δ		P		P′		P−P′
′	″	′	″	′	″	″
15	45	57	42	56	45	57
15	50	58	01	57	3	58
15	55	58	19	57	21	58
16	00	58	37	57	39	58
16	05	58	36	57	56	60
16	10	59	14	58	14	60
16	15	59	32	58	32	60
16	20	59	51	58	50	61

1. *Ecilpses of the moon.*

Eclipses of the moon.

202. Having in the preceding articles explained the principles by which the ecliptic limits may be determined, we proceed now to the computation; beginning with those of the moon.

Fig. 58.

Let NZ (fig. 58) be the ecliptic, O the point of it opposite the sun, or the nadir of the sun; the position of which will always be known by adding 180° to the sun's longitude. This point O will therefore be in the axis of the cone, and be the centre of the circular section cut by a plane perpendicular to the axis. Let
$\mathrm{OE} = \mathrm{OG} = \mathrm{P} + p - d$, be the semidiameter of the shadow.

Conceive NAV to represent the orbit of the moon inclined about 5° 9′ to the ecliptic; then OA will be the latitude of the moon at the moment of conjunction. We shall suppose the point O immoveable during the time of the eclipse, although in fact it advances along the ecliptic about $2'\tfrac{1}{2}$ in an hour, but the moon's motion is about $32'\tfrac{1}{2}$ in an hour; we may therefore consider the point O constant, and conceive the moon to advance at the rate of about 30′, this being called its relative velocity or motion in the ecliptic.

Fig. 59.

While the moon advances from N to A (fig. 59), her latitude, which is nothing at N, will be OA when she has arrived at A, or is in conjunction.

Let m M denote the motion of the moon in longitude, and $m\lambda$ her motion in latitude, and m S the motion of the sun in longitude.

Then $\dfrac{m\lambda}{m\mathrm{M}}$ = tan of inclination of the true orbit = tan N;

$\dfrac{m\lambda}{m\mathrm{M} - m\mathrm{S}}$ = tan of inclination of relative orbit = tan I;

for while the moon passes from N to A by a combination of the motions $m\mathrm{M}$, $m\lambda$, the nadir of the sun will have passed from S to O; so that the moon and the nadir of the sun are found at the same instant in the same circle of latitude AO.

Now in the right angled triangle ONA, we have

$$\tan \mathrm{N} = \frac{\mathrm{OA}}{\mathrm{NO}}$$

but by placing the nadir immoveable in O, it is necessary to suppose that the moon left the point I, such that $\mathrm{NI} = \mathrm{SO}$; whence IO will be the relative motion, which we have denoted by $m\mathrm{M} - m\mathrm{S}$: the motion in latitude will be actually OA; and the triangle OAI gives

$$\mathrm{an\ I} = \frac{\mathrm{AO}}{\mathrm{IO}} = \frac{m\lambda}{m\mathrm{M} - m\mathrm{S}}$$

as stated above.

The motions being supposed uniform during the eclipse, if we assume $m\mathrm{M}$, $m\lambda$, $m\mathrm{S}$, to be the hourly motions, and n any fraction of an hour; then $nm\mathrm{M}$, $nm\lambda$, $nm\mathrm{S}$, will be the actual motions in the time n, and we shall still have

$$\frac{nm\lambda}{nm\mathrm{M} - nm\mathrm{S}} = \frac{m\lambda}{m\mathrm{M} - m\mathrm{S}} = \tan \mathrm{I}.$$

Hence the relative motion of the moon to the sun, will always have the same inclination to the ecliptic; and if we suppose the nadir of the sun immoveable, the relative orbit of the moon will be IA, of which the inclination is I

These suppositions are made for the sake of simplifying the calculation; and they are so nearly true for the interval we have employed, that they produce no sensible error in the calculation: but if more accuracy should be thought desirable in any case, we may employ the common methods, for calculating the distances of the centres; or we may conceive the period of duration of the eclipse divided into a certain number of equal parts, for each of which the angle I of inclination may be computed. All the motions to which we have referred are the true motions as seen from the centre of the earth; and no notice is taken of the parallax; for in order to know whether the moon actually enters into the shadow, and loses the light which she receives from the sun, it is not necessary to know the point in the heavens to which the moon is referred, but the point where she actually is, before we can ascertain whether she is in the light or in the shadow.

Calculation. 203. These preliminaries being established, and supposing the time of opposition to have been determined by tables, or otherwise to within half an hour of the truth, we commence the calculation of the eclipse as follows:

Calculate first,

$$\tan \mathrm{I} = \frac{m\lambda}{m\mathrm{M} - m\mathrm{S}}$$

$$\text{make } ♂ = 180 + \mathrm{S}$$

then

$$(m\mathrm{M} - m\mathrm{S}) : 1 \text{ hour} = (3600'') : ♂ - \mathrm{M} : t = \frac{(♂ - \mathrm{M})\, 3600''}{m\mathrm{M} - m\mathrm{S}}$$

So that T being the time given by the first calculation, the time of opposition will be

$$\mathrm{T} + t = \mathrm{T} + \frac{(♂ - \mathrm{M})\, 3600''}{m\mathrm{M} - m\mathrm{S}}$$

In which t will be negative if $\mathrm{M} > ♂$.

Again,

$$3600'' : m\lambda :: t : \frac{tm\lambda}{3600''}$$

Hence the latitude at the opposition will be

$$\lambda + \frac{tm\lambda}{3600''} = \lambda + \left(\frac{♂ - \mathrm{M}}{m\mathrm{M} - m\mathrm{S}}\right) m\lambda = l,$$

where λ is supposed to be the latitude by computation.

Now $l \cos \mathrm{I}$, will be the shortest distance; for, letting fall the perpendicular $m\mathrm{O}$, fig. 58, on the relative orbit NA, the point m will be the nearest to the centre O of the shadow; $\mathrm{O}m$ will be the shortest distance from the centre of the moon to the point O, and

$$\mathrm{O}m = \mathrm{OA} \cos m\mathrm{OA} = l \cos \mathrm{I}$$

and it is evident that $m\mathrm{OA} = 90^\circ - m\mathrm{ON} = 90^\circ - 1$, also, $m\mathrm{A}$ will be the path of the moon in her orbit between the time of the shortest distance in m, and the time of the conjunction in A, or

$$m\mathrm{A} = \mathrm{OA} \sin m\mathrm{OA} = l \sin \mathrm{I} = \mathrm{O}m \tan \mathrm{I}.$$

Let fall the perpendicular mm' on the ecliptic, and we shall have

$$m'\mathrm{O} = m\mathrm{O} \cos m\mathrm{O}m' = l \cos \mathrm{I} \sin m\mathrm{OA} = l \sin \mathrm{I} \cos \mathrm{I}$$

$$(m\mathrm{M} - m\mathrm{S}) : 3600'' :: m'\mathrm{O} : \text{time in } m'\mathrm{O} = \frac{l \sin \mathrm{I} \cos \mathrm{I}\, 3600''}{m\mathrm{M} - m\mathrm{S}};$$

Or,

$$\text{time in } m\ \mathrm{O} = \frac{l \sin \mathrm{I}\, 3600''}{\dfrac{m\mathrm{M} - m\mathrm{S}}{\cos \mathrm{I}}} = \frac{m\mathrm{A} \cdot 3600''}{\dfrac{m\mathrm{M} - m\mathrm{S}}{\cos \mathrm{I}}}$$

Thus, in order to convert the arc $m'\mathrm{A}$ into time. that is to say, to determine the time which the moon employs to describe any arc of its relative orbit, it is sufficient to multiply this arc by the constant quantity

$$\frac{3600''}{\dfrac{m\mathrm{M} - m\mathrm{S}}{\cos \mathrm{I}}} = \frac{3600'' \cos \mathrm{I}}{m\mathrm{M} - m\mathrm{S}}$$

and reciprocally, to know the arc of the orbit when the time is given, we have only to divide by the above quantity, or multiply by its reciprocal,

$$\frac{m\mathrm{M} - m\mathrm{S}}{3600'' \cos \mathrm{I}}$$

This is denominated the horary motion in the relative orbit.

204. From the centre O, (fig. 58) and with the radius $\mathrm{P} + p - \delta = \mathrm{OE}$, describe the circle ERG, which will be the circle of the shadow, or the section of the dark cone. With the radius $\mathrm{OL} = \mathrm{OV} = \mathrm{P} + p - \delta + d$ mark the points L and V on the relative orbit; and from the points L and V, on the relative orbit, and with the radius $\mathrm{LE} = \mathrm{LV} = d$, describe two circles, which will represent the disc of the moon; and which two circles will be tangents in E and in G to the circle of the shadow. The points L and V will therefore be those of the beginning and end of the eclipse: the right angled triangles $m\mathrm{OL}$, $m\mathrm{OV}$, will have two sides equal each to each; and hence it follows, that

$$mL = mV = \sqrt{}(OL^2 - Om^2) = \sqrt{}(OV^2 - Om^2)$$
$$= \sqrt{}\{(OL - Om)(OL + Om)\}$$
$$= \sqrt{}\{(P + p - \delta + d - l\cos I)(P + p - \delta + d + l\cos I)\}$$

whence the value of mL or mV becomes known.

Duration of the eclipse.

205. It follows from this, that the point m is equally distant from the beginning in L, and from the ending in V; it is therefore the middle of the eclipse, and the time at m is called the time of the middle of the eclipse; the time in Lm is the half duration; wherefore the

$$\text{half duration} = \frac{mL \cdot 3600'' \cdot \cos I}{mM - mS}$$

middle of the eclipse $-$ $\frac{1}{2}$ duration $=$ time of commencement,

middle of the eclipse $+$ $\frac{1}{2}$ duration $=$ time of the end. Hence we may make

$$\sin OLm = \sin u = \frac{Om}{OL} = \frac{l\cos I}{P + p - \delta + d}$$

and

$$mL = OL\cos u = (P + p - \delta + d)\cos u.$$

Quantity of the eclipse.

As the moon advances from L towards m, its distance from the centre O will diminish, because we have always for any point L′,

$$L'O^2 = L'm^2 + Om^2;$$

Now the distance cannot diminish without a part of the disc entering into the shadow, and being eclipsed; and the part eclipsed will be

$LO - L'O = (P + p - \delta + d) -$ the actual dist. of centres.

It is obvious, supposing the centre to be in L′, that the semidiameter drawn from L′ towards O will be in the shadow; and that the part RL of another semidiameter will be also in the shadow. The eclipsed part therefore will be

$$d + RL' = d + OR' - L'O = d + P + p - \delta - L'O$$
$$= P + p - \delta + d - L'O.$$

The least value of $L'm$ is Om, and therefore the greatest quantity of the eclipse is found by substituting for $L'm$, Om, or its value $l\cos I$, in which case it becomes,

$$P + p - \delta + d - l\cos I.$$

If the part eclipsed be equal to the lunar disc $= 2d$, we shall have,

$$2d = P + p - \delta + d - L'O$$

or $L'O = P + p - \delta + d - 2d = (P + p) - (\delta + d)$

$=$ sum of the parallaxes $-$ $\frac{1}{2}$ sum of diameters.

Fig. 60. Projection of an eclipse.

206. Take $L'O = (P + p) - (\delta + d)$, (fig. 60,) and describe about the two points L′, L′ two circles touching the circle of the shadow OR in the points R, R′, and we shall have

$$mI = \sqrt{}\{(OL' - Om)(OL' + Om)\}$$
$$= \sqrt{}\{(P + p - \delta - d\,l\cos I)(P + p - \delta + d + l\cos I)\}$$

and the half duration of the total eclipse will be

$$\tfrac{1}{2}\text{ duration} = \frac{mL'\,3600\cos I}{mM - mS}$$

In order to make these calculations of the half duration, the radius of the shadow is commonly assumed equal to $\frac{61}{60}(P + p - d)$; because it has been found that the observed duration always exceeds the computed; a circumstance which is attributed to the atmosphere of the earth, viz. its intercepting the light of the sun, and thereby producing an effect equivalent to an increase of $\frac{1}{60}$ in the diameter of the earth.

It is customary to express the quantity of the eclipse in digits, or $\frac{1}{12}$ths of the lunar diameter, under which form our equation (art. 205.) becomes

$$\text{eclipse} = \frac{12}{2d}(P + p - \delta + d - l\cos I)$$

which, when $l = o$, or when the eclipse happens in the node, becomes.

$$\text{eclipse} = (P + p) + (d - \delta) = (P + d) + (p - \delta)$$
$$= \frac{60{\cdot}20}{16{\cdot}28}d + d - (\delta - p) = \frac{76{\cdot}48}{16{\cdot}28}d - (\delta - p)$$

which exceeds the quantity $2d$ very considerably. Whence we learn that a lunar eclipse may be more than total; that is, the number of digits may exceed twelve.

207. Again let us suppose

$$2d = \text{eclipse} = P + p + d - \delta - l\cos I$$

and it will follow, that

$$l\cos I = P + p - (\delta + d), \text{ and}$$
$$l = \frac{P + p - (d + \delta)}{\cos I}.$$

In this case the eclipse will be total, but only for an instant; if l be less than this quantity, the eclipse will be total with a greater or less duration: but if l be greater, the eclipse can only be partial.

The least value of $P + p$ is 53′, and in this case $d + \delta = 31'$ at the greatest: hence

$$\frac{P + p - (d - \delta)}{\cos I} = \frac{53' - 31'}{\cos I} = \frac{22'}{\cos I}$$ at the minimum;

consequently, if the latitude do not exceed 22′ we may conclude that the eclipse is necessarily total.

When we have measured the quantity of the eclipse

$$E = P + p - \delta + d - l\cos I,$$

we shall have

$$P = E - p + \delta - d + l\cos I$$

and we thus know the parallax of the moon, provided we know that of the sun, the semidiameters and the latitude; but practically this method, though employed by the ancients is by no means conclusive, on account of the difficulty in measuring the eclipse in consequence of the penumbra.

The figure of the shadow is always circular, which demonstrates the spherical figure of the earth; but as the arc which terminates the shadow is never more than a small part of the circumference, it is impossible from it to decide whether it be accurately a circle or an ellipse of small eccentricity.

Of the longitude of places.

208. As the penumbra, which encompasses the pure shadow, renders the beginning and end of the eclipse difficult to observe with certainty, astronomers prefer noting carefully the moment when the edge of the shadow arrives and it passes the divers spots which are always observed on the disc of the moon. Thus the same eclipse may present many different observations, which we may compare with others made in a different place, in order to ascertain the difference of longitude between the meridians; for these phenomena happen precisely at the same instant of time to all observers; and the time shown at these different places

will furnish of course the means of determining the difference of longitude between them.

The ancients, who had only the eclipses of the moon for finding the longitude, might easily be deceived as much as 2°, or even more, in their results; for two degrees in measure answers only to eight minutes in time; and we may readily imagine, that with the means they possessed, two observers might very easily be in error each four minutes in opposite directions, and thus cause a difference in the results to the extent we have stated, and even considerably beyond it.

Graphical construction of an eclipse.

Graphical construction. Fig. 61.

209. Let AX (fig. 61) be a line representing the ecliptic, and divided into six equal parts by six perpendiculars, which will be so many parallels of latitude. Take each interval, AA′, A′A″, &c. for the hourly relative motion in the ecliptic, and divide each of these intervals into 60 parts, which will be minutes in time; place the corresponding numbers at every five minutes, and mark with zero each intersection with a circle of latitude. This figure, once constructed and divided, will serve for all kinds of eclipses and occultations.

Let A‴ be the place of the moon for the instant given by calculation; then, on the first interval, take

$$\text{AB} = m\text{M} - m\text{S} = \left\{\begin{array}{l}\text{relative motion of moon in}\\ \text{the ecliptic.}\end{array}\right.$$

If, for example, the relative horary motion be 32′, we must make AB = 32′, and the perpendicular BD = 60′; and the lines BD, AB, will have to each other the ratio of the times to the angular motions. Through the points A and D draw the indefinite right line ADH. This being done, take A *c* equal to the latitude for the time shown by the calculation; and the perpendicular *c d* will be the latitude in parts of the figure: apply *c d* from A‴ to *a*‴, and *a*‴ will be the place of the moon at the time given by the calculation.

Take A *a* equal to the latitude an hour before the time by calculation, and apply the perpendicular *a b*, from A″ to *a*″, and the latter will be a second point in the relative orbit; draw the indefinite line *a*″ *a*‴, which will be the relative orbit.

Take A*e* equal to the difference of longitude between the moon and the sun at the time by calculation, and *ef* will be the distance of the conjunction in the figure; apply then *ef* from A‴ to O, and O will be the place of the conjunction, and the centre of the shadow. If the conjunction is past the time by calculation, OA‴ must be applied backwards on the ecliptic; but otherwise it must be taken forward towards X; and the divisions for the time will show how many minutes the point O precedes or follows the time by calculation: and we shall thus have the instant of opposition.

Draw the perpendicular O *m*, which will be the shortest distance: let fall the perpendicular *m n*, and the point *n* will show how much the middle exceeds the time by calculation, or how much it follows the time from A″; that is to say, the time in A‴ diminished by an hour.

Take $\text{AG}' = \text{P} + p + d - \delta$; make $\text{OC} = \text{G}'\text{H}'$, and with the radius OC mark the points C and F from the beginning and the end: and the perpendiculars, C *c*, F *f*, will give the instant of these two phases.

Again, take $\text{AG}'' = \text{P} + p - d - \delta$, and make $\text{OI} = \text{G}''\text{H}''$; and with this as a radius, mark the points I and E, which will be those of immersion and emersion; viz. from the commencement and end of the total eclipse. The perpendiculars I *i* and E *e* will give these two instants.

Take $\text{AG} = \text{P} + p - \delta$, and with the radius $\text{OG} = \text{GH}$ describe from the point O the circle GM, which will be the section of the shadow.

With the radius CG describe circles about the points C, I, *m*, E, F, and we shall have all the phases of the eclipse.

Produce O*m* to M; $(d + m\text{M})$ will be the quantity of the eclipse. If O *m* is less than $\text{P} + p - \delta - d$, the eclipse will be total; if O *m* = OM, the eclipse will be six digits; if O *m* be greater than $\text{P} + p - \delta$, the eclipse will be still less; and if $\text{O}\,m = \text{P} + p - \delta + d = \text{OC}$, the eclipse will not take place, being in this case only a simple contact.

We have stated, that the figure above described, when once constructed, will serve for all eclipses; it is therefore advisable, in order to preserve it, not to draw actually the several perpendiculars C *c*, I *i*, *mn*, &c., but to employ a square for simply getting the points of intersection, which is all that is required. See Delambre's *Abrégé d'Astronomie.*

2. *Eclipses of the sun.*

Eclipses of the sun.

210. At the instant when the centre of the moon enters into the luminous cone which extends from the sun to the earth, the angular distance of the moon from the axis of the cone is $\text{P} - p + \delta$; at the instant when the edge of the moon enters into the cone, the distance is $\text{P} - p + \delta + d$. But the edge of the moon cannot penetrate into the cone without obstructing the light of a part of the sun from an observer at E (fig. 57,) who will see the edge of the solar disc in the horizon, his zenith being in the prolongation of the radius TE; but an observer at E′ will see the centre of the sun in the line E′*r*S, and the inferior limb in the line E *o* O′, and consequently at a certain distance from the edge of the moon which enters at *n*′; the eclipse therefore will not have commenced to the observer at E′.

Let us conceive the plane L′*r o* to be a tangent to the orbit V′*n*′L′ of the moon. This may be called the *plane of projection*; *o r* will be the projection of the semidiameter of the sun; the point *r* the projection of the centre, and *o* that of the edge. The observers placed in different parts of the enlightened hemisphere, of which the section is E *a* F, will all see the centre of the sun answering to different points on the plane of projection.

Fig. 62.

The section of the cone by this plane will be circular, the radius of the circle seen from the centre of the earth, subtends an angle equal to $\text{P} - p + \delta$. Let therefore O (fig. 62) be the centre of the sun, considered as immoveable, and with the radius $\text{OE} = \text{P} - p + \delta$ describe the circle ED, which will represent the section of the cone. Let OA be the latitude of the moon in conjunction, NLAV the relative orbit; and the perpendicular O *m* will be the shortest distance of the centres. Let $\text{OL} = \text{P} - p + \delta + d$, and mark the points L, V, which will indicate the first and the last instant of the eclipse; the point E will be that at which the first edge of the moon enters into the cone; and the point S that at which the other edge leaves it; therefore from the points L and V, and with the radius LE = VS = d, describe two circles, which will represent the lunar disc; the points E and S will be the points of contact of these two circles with the circle of the section of the cone EDS.

We shall thus have, as in the eclipses of the moon,

$$\tan \mathrm{I} = \frac{m\,\lambda}{m\,\mathrm{M} - m\,\mathrm{S}}\,;\quad \mathrm{O}\,m = l \cos \mathrm{I};\ \mathrm{A}\,m = l \sin \mathrm{I};$$

$$\text{time in A}\,m = \frac{l \sin \mathrm{I} \cos \mathrm{I}.\ 3600''}{m\,\mathrm{M} - m\,\mathrm{S}}$$

$$m\,\mathrm{L} = \sqrt{\{(\mathrm{OL} + \mathrm{OM})\ (\mathrm{OL} - \mathrm{O}\,m)\}}$$

$$\tfrac{1}{2}\ \text{duration} = \frac{m\,\mathrm{L} \cdot 3600'' \cdot \cos \mathrm{I}}{m\,\mathrm{M} - m\,\mathrm{S}}$$

211. Instead of the trigonometrical calculation, we may employ the graphic construction, (art. 209) and we shall thus have all the general circumstances of the eclipse; viz. we shall have the instant when the shadow of the moon begins to abstract the light of the sun from certain points on the earth, and that, when the eclipse is entirely terminated; but the most important and the most difficult to determine, is the precise places on the earth where the eclipse will be visible; and to find rules for calculations of the eclipse for a given place; for these circumstances vary with the position of the observer relative to the moon and sun, which changes every instant by the diurnal motion.

In all these calculations it is necessary to take into account the parallax, which is different for every place on the earth, and of which the effect changes every instant; in short, it is involved in so much intricacy, notwithstanding the ingenious application which has been made of the plane of projection, that M. Delambre has preferred employing the direct solution, and in which we propose to follow him in the remaining part of this article.

Effect of parallax illustrated. Fig. 63.

212. Let T (fig. 63) represent the centre of the earth, O a point on its surface, of which Z is the zenith. Let S be the sun, whose zenith distance will be ZTS for the centre of the earth, and ZOS to the observer at O; let the moon be in the same vertical at the distance ZOL; then the observer will see the centre of the moon on the centre of the sun; and if we suppose the apparent diameters of both bodies to be the same, as they are nearly, the edges of the two discs will coincide, and the sun will be totally eclipsed to the observer at O, while to an observer supposed at the centre, T, the entire disc, will be luminous, as it will consequently to all points on the surface comprised between the radii a T, b T, and to observers situated towards D and E, the distance of the two bodies will be still greater; we see therefore how very considerable is the effect due to the parallax.

Fig. 64.

Let HR (fig. 64) be the horizon, and HMPR the meridian of any place P; PS the circle of declination of the sun S; PL, that of the moon L; La, the semi-diameter of the lunar disc; S b, that of the solar disc; at the centre of the earth, the distance of the edges of the moon and of the sun will be the arc $b\,a$, and therefore there will be no eclipse to an observer supposed at the centre; but if the arc $a\,b$ does not exceed a few minutes of a degree, the parallax, which has a tendency to diminish this distance, may bring the disc of the moon within the luminous cone, and produce an eclipse more or less considerable.

If it be desired to have the entire effect of the parallax, in the place of referring the moon to the zenith V, of the place P, and to the vertical VL, which is oblique to the distance of the centres SL, let us place it at Z on this distance SL, prolonged to M and Z; then the parallax will carry the edge a of the earth directly towards S. Make a Z = 90°, L a will descend towards S by all the horizontal parallax; $a\,b$ will be diminished by a quantity equal to it; and if therefore $a\,b$ is equal to this parallax, the distance of the edges of the sun and moon is reduced to zero, supposing no parallax of the sun, but this last is p, and therefore the actual diminution of the arc $a\,b$, will be P − p; consequently, when $a\,b$ = P − p, we shall have a simple contact at the place, whose zenith is Z, and which obviously happens when

$$\mathrm{SL} = \mathrm{P} - p + \delta + d$$

which is precisely the radius we have employed for determining the commencement and the end of the general eclipse. (art. 210.)

213. Let us conceive the arc ZP to be drawn, then PZ will be the meridian of the place, where the simple contact will be observed; PZ will be the distance from the pole, or the co-latitude of it; HMP is the meridian of the given place P, and consequently ZMP is the difference of the meridians. Now we know the time at P, for which we have calculated the true distance LS; and we have therefore the horary angle MPS. Hence, if we can find ZPS, we shall have ZPM = ZPS − MPS; but in the spherical triangle ZPS, we know

$$\mathrm{PS} = 90° - \text{dec. of sun} = 90° - \mathrm{D}$$
$$\mathrm{SZ} = \mathrm{S}\,b + 90° = 90° + \tfrac{1}{2}\,\text{diam.} \odot = 90° + \delta;$$

and we may calculate the angle PSL; we shall therefore have

$$\cos \mathrm{PZ} = \sin \text{lat.} = \cos \mathrm{S} \sin \mathrm{PS} \sin \mathrm{ZS} + \cos \mathrm{PS} \cos \mathrm{ZS}$$
$$= \cos \mathrm{S} \cos \mathrm{D} \cos \delta - \sin \mathrm{D} \sin \delta$$

$$\cot.\ \mathrm{ZPS} = \frac{\cot \mathrm{ZS} \sin \mathrm{PS}}{\sin \mathrm{S}} - \cos \mathrm{PS} \cot.\ \mathrm{S};$$

ZPS will therefore be the horary angle of the place, which is greater than the same angle corresponding to P, and the sun therefore more distant from the former place than from the latter: here we not only know the difference of the meridians, but we shall know also whether it be to the east or west, and the place required is therefore correctly known; that is,

$$\text{lon. sought} = \text{lon. P} + (\mathrm{ZPS} - \mathrm{MPS})$$
$$= \text{lon. P} - \text{horary ang. of P} + \mathrm{ZPS},$$

and this longitude will be west of P, supposing S to be in the east part of the heavens, and the hour at P to be in the morning.

If, on the contrary, we suppose the figure to represent the western part of the heavens, the time at P will be afternoon, and the east longitude of the place will be

$$= \text{lon. of P} + \mathrm{MPS}$$
$$= \text{lon. of P} + \mathrm{ZPS} - \mathrm{MPS}$$
$$= \text{lon. of P east} - \text{hor. ang. of P.}$$
$$+ \text{the hor. angle of the place.}$$

The point thus determined will be the first that has simple contact.

214. The instant after that at which the simple contact has place, the true distance LS will be diminished, and $a\,b$ will be less than P − p.

Let $ab = (P - p) \sin N$, or

$$\sin N = \frac{ab}{P - p} = \sin bZ$$

then N will be the distance from the zenith, which will give us the parallax in altitude $= ab$, sufficient for reducing the distance ab to zero. With

$$SZ = bZ + \delta, \text{ PS and angle S},$$

we shall have,

$$\cos PZ = \sin \text{lat.} = \cos S \sin PS \sin ZS + \cos PS \cos ZS;$$

$$\cot ZPS = \frac{\cot ZS \sin PS}{\sin S} - \cos PS \cot S,$$

the longitude will be known by the preceding formula, and ZS will be the true distance of the sun from the zenith.

There will be a time when $SL = P - p$: then supposing $SZ = 90°$, our formulæ will become

$$\cos PZ = \sin \text{lat.} = \cos S \sin PS$$
$$\cot ZPS = - \cos PS \cos S$$

this calculation will give the latitude and the longitude of the place, which will be first centrally eclipsed.

Afterwards the distance SL will be less than the parallax $P - p$; it will then be necessary to find bZ, SZ, and the quantities as before by the formulæ, (art. 214.) We shall thus have successively all the places which will be centrally eclipsed, both in the time at the given place, and in that of the places themselves.

As SL diminishes, we may, with different values of SZ, have different parallaxes; and, consequently, eclipses of different magnitudes, from the simple contact to the north of the sun to the central eclipse, and even beyond that in the south part of the sun; and sometimes merely a simple contact with the southern limb of the sun and the northern limb of the moon, providing we have

$$P - p = SL + \delta + d, \text{ or}$$
$$SL = P - p - \delta - d,$$

which is very possible.

Suppose even that the distance SL be less than $\delta + d$, so that the disc of the moon covers a part of that of the sun to the centre of the earth, it would still be possible to reduce this eclipse to simple contact, by finding a zenith Z on the prolongation of SL, which should give us a parallax.

$$(P - p) \sin bZ = ab, \text{ or}$$

$$\sin bZ = \frac{ab}{P - p}; \quad SZ = bZ - \delta.$$

Then, in the triangle PSZ, we shall have $PSZ = 180° - PSL$, SZ, and PS; whence we obtain PZ, and ZPS by the above formulæ; but in this case, the difference of longitude will not be the difference, but the sum of the horary angles.

For all these divers cases, there are always the same two trigonometrical formulæ, and the same formulæ for the parallax in altitude.

Again, when the distance ab shall be less than the horizontal parallax, and that we are obliged to have recourse to the parallax in altitude, to reduce the edges to contact, we may still obtain a simple contact by making the parallax act obliquely instead of directly in the arc LS.

Advantages of the above method.

215. This method presents many advantages: for example, let us suppose, that for the same hour at the place P, which may be considered to represent Paris, we shall have determined the place which will have a simple contact by the parallax in altitude; the central eclipse by another parallax of altitude, and the most southern eclipse by the horizontal parallax; all these places will be in the same vertical, and therefore necessarily in a great circle of the terrestrial sphere; and two of these points, viz. of contact, and that of the most southern eclipse, will be sufficient for finding all the intermediate places which will have an eclipse of intermediate magnitude; for two points being given on a globe, nothing is more easy than to join these points by an arc of a great circle, and such circles traced for every 10′ for the whole duration of the eclipse, will convey a sufficiently exact idea of all the places that will be eclipsed, or of the terrestrial zone traced by the shadow of the moon.

Oblique parallax illustrated. Fig. 65.

216. In order to illustrate the application and use of the oblique parallax, let

$$LS \angle (P - p) + (\delta + d), \text{ (fig. 65,)}$$

On this known side, form with the sides

$$SV = \delta + d \text{ and } LV = P - p,$$

the triangle SLV, which we may consider as rectilinear, in order to simplify a little the calculation; and let us compute

$$\sin^2 \tfrac{1}{2} SLV = \frac{\left(\frac{-SL+SV+LV}{2}\right)\left(\frac{SL+SV-LV}{2}\right)}{SL \cdot LV}$$

We shall then know SLV, which, subtracted from PLS, will give us PLV and $PLZ = 180° - PLV$, $SL = 90° - (P - p)$ and $PL = 90° -$ declination of the moon.

We may then calculate,

$$\cos PZ = \cos PLZ \sin ZL \sin PL + \cos ZL . \cos PL,$$

$$\cot ZPL = \frac{\cot ZL . \sin PL}{\sin ZLP} - \cos PL \cot ZLP$$

$$ZPS = ZPL + LPS = ZPL + (\text{Æ}\odot - \text{Æ}☾)$$

We have then the latitude and the longitude always by the same formula.

Let us now apply LV on the other side LV′, and the triangle LSV′ will be perfectly similar and equal to LSV; we have therefore already SLV′, and

$$PLV' = PLS + SLV' = PLZ = 180° - PLV';$$

and we may calculate PZ′, Z′PL, and ZPS, by the preceding formulæ, and we shall thus, for the same instant, have another point, where will be seen a simple contact in the horizon.

In order to abridge these calculations, we may form for the whole duration of the eclipse, for every ten minutes, a table of the sides PS, PL, SL, and of the angles LPS, PSL, and SLP.

Lastly, if we wish to know those places which have a simple contact, or an eclipse of a given number of digits for its greatest phase, the means are still equally easy.

Fig. 66.

For this, we may assume the centre V of the moon (fig. 66) on the line of the shortest distance mS, in such a manner, that $SV = \delta + d$, or $SV = \delta + d - n$ digits; then we shall know mV, we shall have mL by the relative orbit, and

$$\tan mLV = \frac{mV}{mL}, \quad LV = \frac{mL}{\cos mLV}$$

$$PLV = PLS - mLV; \quad PLZ = 180° - PLV$$

$$\sin ZV = \frac{LV}{P-p};\ ZL = ZV - LV$$

Then, in the triangle ZPL, we shall have ZL, PL, and the contained angle, to calculate PZ, ZPL, ZPS; and we have still the latitude and longitude by the same formulæ.

In making use of the oblique parallax, we may take for our data one of the angles, with the sides SL and SV, for example LSV, and hence find LV, ZV, and ZL; and for the rest, the calculation is the same as the above. When $SL \angle (P - p) + (\delta + d)$, we have two points V,V′ two equal and similar triangles SLV, SLV′; two zeniths, Z, Z′, which have a simple contact in the horizon: the triangles SLV, at first very oblique, have the angle at L very acute, the angles V or V′ very obtuse; L will open more and more, and V will close; the angle LSV will increase from 0 to 180°, and we shall then have, (fig. 65,)

$$P - p = SL + (\delta + d),\ \text{or}$$
$$SL = (P - p) - (\delta + d)$$

at this moment the points V and V′ are in one, we shall have only one zenith, one point of contact in the horizon; the curve formed by all these zeniths which give the contacts in the horizon, will finish as it begun, in one point only; it will form a sort of oval, of which we may have the breadth at each instant; for this breadth is the arc ZZ′, which separates the two zeniths, where the contact is simultaneous: now $ZL = 90 - LV = Z'L = 90° - LV'$; therefore

$$\sin \tfrac{1}{2} ZZ' = \sin ZL \sin SLV =$$
$$\sin \{90° - (P - p)\} \sin SLV =$$
$$\cos (P - p) \sin SLV.$$

The breadth ZZ′ can never become zero, except when sin SLV vanishes; therefore the two branches do not cross each other, and they touch each other only in the extreme points.

If the shortest distance $= (P - p) - (\delta + d)$, it will be at the middle of the general eclipse that the angle LSV becomes equal to 180°; and in the other half of the eclipse we shall see return in the inverse order, the same angles S, L, V, and the same triangles SLV, but not the same triangles LPZ, because the angles PLS, and the sides PL will have changed.

If the shortest distance $Sm \angle (P - p) - (\delta + d)$, it is before the middle of the eclipse that the angle LSV will be 180°, and the contacts at the horizon will cease, to commence again after the middle, when we shall have $SL = (P - p) - (\delta + d)$, by which interruption we see that the curve of the contacts will close, and form an oval of an unequal breadth.

While $SL \angle (P - p) - (\delta + d)$, the contacts at the horizon are impossible; and in order to have the simple contact, we must employ the parallax in altitude.

Fig. 67.

At the conjunction in right ascension, (fig. 67,) if $SL = (P - p) - (\delta + d)$, the zenith Z is on SP; so that $SZ = 90° - \delta$; but if SL 7 $(P - p) - (\delta + d)$, we shall have two zeniths Z and Z′, which will correspond to places having contacts at the horizon; $LZ = LZ' = 90° - (P - p)$: these arcs will be commonly greater than LP, and the distance ZZ′ of the zeniths will be found by the formula,

$$\sin \tfrac{1}{2} ZZ' = \cos (P - p) \sin SLV.$$

The difference of longitude is the sum of the two horary angles after the conjunction. The point V, which gave a zenith more westerly, will give one more easterly, the two branches will intersect each other in the interval, and the curve will resemble the figure of 8. We might extend these remarks to a greater length; but as the circumstances attending the different forms of the curve under particular data, are rather matters of curiosity than utility, we shall proceed to other inquiries.

Method of calculation.

217. In order to calculate by this method all the circumstances of an eclipse, we are supposed to have found very exactly by the tables of the sun and moon, and for a time not too distant from the middle of the eclipse, or the conjunction, the longitude of the sun, his horary motion, his parallax and semi-diameter; the longitude and latitude of the moon, its horary motions, the parallax, and its diurnal change, and semi-diameter.

From these first data we compute, hour by hour, the longitudes and the latitudes, from which we draw the right ascensions and declinations both of the moon and sun.

Example of the above method.

218. Let us take as an example the annular eclipse of 1764, the most remarkable which has been observed in Europe for many years, supposing that we have found the following quantities:

True time.	Ꝛ ☉	Declination ☉ = D	Ꝛ ☾	Declination ☾ = D′	P − p	d	Motion in Ꝛ = mot. Ꝛ	D′ — D
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″ ′ ″	° ′ ″	° ′ ″	′ ″
8	11 5 24	4 46 28	9 49 14	4 49 44	54 2 5	14 47 5	1 16 10	3 16
9	7 40	47 26	10 15 34	5 3 52	2 0	47 3	0 52 6	16 26
10	9 56	48 24	10 41 55	5 18 0	1 5	47 2	0 28 1	29 36
11	12 13	49 22	11 8 18	5 32 8	1 0	47 1	0 3 55	42 46
0	14 29	50 20	11 34 40	5 46 16	0 5	46 9	+0 20 11	55 56
1	16 46	51 18	12 1 4	6 0 24	54 0 0	14 46 8	+0 44 18	69 6

All these quantities increase with sufficient regularity, to enable us to extend the table to every ten minutes of time, by means of simple proportional parts. We may perform this operation on the declination of the sun and moon, for the differences of right ascension and parallax. And as to the diameters, we may suppose throughout $d = 14' 47''$, $\delta = 15' 57''$, and $\delta + d = \sigma = 30' 44''$. We may, moreover, suppose the relative motion in right ascension to be P = 24′ 6″ on the equator; it will then be (24′ 6″) cos D′ in the region of the moon, and the relative motion in declination = 13′ 10″.

The calculation of the relative orbit with regard to the equator, is a little less simple than when referred to the ecliptic, because the sun has a motion in declination as well as the moon. Astronomers, however, are satisfied with multiplying the relative motion in right ascension by the sine of the polar distance of the moon, in order to obtain the motion in the region of the moon, which motion will be (24′ 6″) cos 5° 25′ = 23′ 59″·5; the tangent of the inclination of the relative orbit will be

$$\tan \text{ I} = \frac{\text{mot. D}'}{\text{mot. Æ} \cos \text{D}} = \frac{13' \, 10''}{23' \, 59''\cdot 5} = 28° \, 45' \, 26''$$

and the shortest distance S m = 43′ 54″·6 cos I = 39′ 22″·3.

This method is sufficiently exact, taking, as we do, the declination of the moon corresponding to the middle of the interval.

We may determine the conjunction in right ascension by means of the following table A, where we see that at 11 hours the difference of right ascension was 3′ 55″, therefore

$$24''\cdot 6 : 1\text{h.} :: 3' \, 55'' : 9' \, 46'';$$

that is, the conjunction will be at 11h. 9′ 46″, and subtracting 47′ 20″·4, we shall have for the middle, 10h. 22′ 26″.

At 11 hours, the difference of declination is 42′ 46″, and it increases by 13′ 10″ per hour · hence

$$1\text{h.} : 13' \, 10'' :: 9' \, 46'' . \, 2' \, 8''\cdot 6,$$

and the declination in conjunction will be 44′ 54″·6, s in the preceding approximation.

219. In order to calculate the angles S and L, and the distance of the centres E = SL, we may employ the following formulæ:

$$\tan \tfrac{1}{2}(\text{L} - \text{S}) =$$

$$\frac{\sin \frac{1}{2} (\text{D}' - \text{D}) \cot \frac{1}{2}\text{P}}{\cos \frac{1}{2} (\text{D}' + \text{D})} = \frac{\frac{1}{2}(\text{D}' - \text{D})}{\frac{1}{2}\text{mot. Æ} \cos \frac{1}{2}(\text{D}' + \text{D})};$$

$$\tan \tfrac{1}{2}(\text{L} + \text{S}) = \frac{\cos \frac{1}{2}(\text{D}' - \text{D}) \cot \frac{1}{2}\text{P}}{\sin \frac{1}{2}(\text{D}' + \text{D})};$$

$$\text{L} = \tfrac{1}{2}(\text{L} + \text{S}) + \tfrac{1}{2}(\text{L} - \text{S})$$

$$\text{S} = \tfrac{1}{2}(\text{L} + \text{S}) - \tfrac{1}{2}(\text{L} - \text{S}),$$

$$\sin \text{E} = \frac{\cos \text{D} \sin \text{mot. Æ}}{\sin \text{L}} = \frac{\cos \text{D}' \sin \text{mot. Æ}}{\sin \text{S}}.$$

We may observe, that the angles at S are always diminishing till the conjunction, when they pass through zero, change their positions, and then increase to the end. The angle at L, on the contrary, increases till the conjunction, when it becomes 180°, and then diminishes to the end.

$$\left.\begin{array}{l} \text{P} - p = 54' \;\; 3'' \\ \delta - d = 30' \, 44'' \end{array}\right\} (\text{P} - p) + (\delta - d) = 84' \, 47''$$

the distance of the centres for the commencement. Now we see by the table A, that this difference happens a little before 7h. 40′, at which time, E = 83′ 56″, with a diminution of 4′ 3″ for 10 minutes; therefore

$$243'' : 10' \text{ or } 243'' : 600'' :: 51 : 126'' \text{ or } 2' \, 6'';$$

hence the commencement of the eclipse will take place at 7h. 37′ 54″.

The angle S′ changes 1° 23′ 20″ in 10 minutes; it therefore changes 25′ in 2′ 6″, it will consequently be 91° 6′ 33″ for 7h. 37′ 54″.

The declination of the sun changes 9″ every 10′, or 0″·9 in one minute; it will therefore be 4° 46′ 7″.

With these formulæ we find the horary angle H, viz.

$$\sin \text{H} = \cos \text{D} \cos \text{S} \sin \text{ZS} + \sin \text{D} \cos \text{ZS}$$

$$= \cos 4° \, 46' \, 7'' \cos 91° \, 6' \, 33'' = \sin \text{lat.} = 1° \, 6' \, 19'' \text{ S}$$

$$\cot. \text{ZPS} = \frac{\cos \text{D} \cot \text{ZS}'}{\sin \text{S}} - \sin \text{D} \cot. \text{S}$$

$$= -\sin 4° \, 46' \, 7'' \cot. 91° \, 6' \, 33'' = 89° \, 55' \, 17''$$

ang. at Paris (— 4h. 22′ 6″) =65 31 30

longitude west of Paris.............. 24 23 47
Hour at the place, or dist. of merid. .. 5h 59 41·8

By making ZS = 90° we have put the centre of the sun in the horizon; but we might, without sensibly lengthening the calculation, bring its upper limb in the horizon by making ZS = 90° 15′ 57″, and by making ZS = 90° 30′ 44″, we should bring the centre of the moon to the horizon.

There is no other calculation to make for this instant; this is the only place where the contact is seen, and it is seen in the horizon. This place being near the equator, we may employ the equatorial parallax, which is the greatest of any:

at 7h. 40′, the distance is only 83′ 56″
$\delta + d = \sigma =$ 30 44

The distance of the edges...... 53 12

It will be here sufficient, with a parallax of 53′ 12″ to work a contact in the vertical; the horizontal parallax produces an eclipse of 51″ in the northern limb of the sun.

220. In order to find the contact, we make

$$\sin \text{N} = \frac{53' \, 12''}{54 \cdot 3}, \; \text{ZS} = \text{N} + \delta;$$

and afterwards with D and S for 7h. 40′ and the formulæ,

$$\sin \text{H} = \cos \text{D} \cos \text{S} \sin \text{ZS} + \sin \text{D} \cos \text{ZS}$$

$$\cot \textit{ZPS} = \frac{\cos \text{D} \cot \text{ZS}}{\sin \text{S}} - \sin \text{D} \cot. \text{S}$$

we shall have the latitude and the horary angle, the hour and the longitude, as given in table B.

In this first calculation, the angle S may be acute or obtuse, according to the position of the moon with respect to the north pole, from which we reckon the angle PSL; but the rules for the signs of the trigonometrical lines are sufficient guides in this respect. At 13h. 10′. E 7 $(\text{P} - p) + \delta + d$, we ascertain by simple proportion when $\text{E} = \text{P} - + \delta + d$, as for the commencement, and by an operation, in all respects similar, we find the last point which will have a simple contact in the horizon.

221. For the following operation, that is, for forming table C, we have

$$\text{ZS} = 90° + \text{E} - (\text{P} - p) - d$$

thus, for 7h. 40m. 90° + E = 91° 23′ 56″
— (P — p) = —54 3
— d = —14 47

ZS = 90 15 06

D and S are found as in the preceding operation.

The quantity of the eclipse is

$$(P - p) - (\delta + d) - E = p' + \sigma - E$$

which will augment as E diminishes; but it can never surpass $2d = 29' \, 34''$, it is to this quantity that we must reduce it if it be too great. Thus at 9h.

$p' + \sigma = 84' \, 46''$, $E = 54' \, 27''$ $p' + \sigma - E = 30' \, 19''$.

This quantity is less than 2δ, and greater than $2d$, we must therefore reduce it to $29' \, 34''$. The moon has not then reached the southern limb of the sun, but it has passed the northern one, the eclipse is therefore annular.

At 9h. 10′ $p' + \sigma = 84' \, 46$, $E = 51' \, 25''$, $p' + \sigma - E = 33' \, 21''$,

which is greater than 2δ or $31' \, 54''$ by $1' \, 27''$; the edge of the moon is therefore beyond the sun, and the eclipse is $2d - 1' \, 27'' = 29' \, 34'' - 1' \, 27'' = 28' \, 7''$.

This found, by making ....$p' = 54' \;\; 2''$

$$\begin{array}{r} E = 51 \;\; 25 \\ \hline p' - E = \;\; 2 \;\; 37 \\ \sigma = 30 \;\; 44 \\ \hline \end{array}$$

$$E - (p' - \sigma) = E + \sigma - p' = \sigma \, (p - E) = 28 \;\; 7$$

This rule will serve till the eclipse surpasses $2d$, without being so great as 2δ, then we have to reduce it to $2d$, thus

at 11h. 40′......$E = 52' \, 56''$

$$\begin{array}{r} \sigma = 30 \;\; 44 \\ \hline E + \sigma = 83 \;\; 40 \\ p' = 54 \;\; 1 \\ \hline E + \sigma - p' = 29 \;\; 39 > 2d \end{array}$$

the eclipse will be.... $29 \;\; 34 = 2d$.

When the eclipse surpasses 2δ, we must have recourse to the formula

$$p' + \sigma - E = \sigma + (p' - E) = \sigma - (E - p'),$$

and the eclipse will again be northern, as in the commencement. The eclipse will be central for a point on the earth, when we have $E = P - p = 54' \, 2''$, which happens between the times 9h. and 9h. 10′. We may find this instant as we found the first contact. But from 9h. 10′ to 11h. 40′ inclusively, the distance being less than the horizontal parallax, we must use a parallax in altitude. We then search the distance from the zenith by the formula

$$\sin N = \frac{E}{P - p}; \;\; (N - p \sin N) = ZS;$$

but we may, without sensible error, make $ZS = N$; then we shall have

$$\sin H = \cos D \cos S \sin ZS + \sin D \cos ZS,$$

$$\cot. ZPS = \frac{\cos D}{\sin S}, \; \cot. ZS - \sin D \cot S,$$

where it is only necessary to pay attention to the signs. It is in this way that table D is formed, which shows all the points where the eclipse is central.

222. The places determined by the three first operations for the same instant are in the same vertical, and in the same great circle of the globe: therefore, in joining them by an arc of a great circle, we shall have on this arc all the places where the eclipse will be seen of divers magnitudes, from simple contact to where it is central, (when this happens,) and in all other cases to the *maximum* shown in table C.

223. After having made the calculations of the effects of the direct parallax, we must pass to the oblique parallax contained in tables E and F, where we find the contacts at the horizon.

We must at first calculate the angle $SLV = L'$ by the formula

$$\sin \tfrac{1}{2} L' = \frac{(p' + \sigma - E)\,(-p' + \sigma + E)}{2 \, p' \, E};$$

from which we draw

$$PLV = PLS - L' = (L - L'), \text{ and}$$
$$PLV' = PLS + L' = (L + L')$$

$PLZ = 180° - (L - L')$, $ZL = Z'L = 90° - (P - p)$,
$PLZ' = 180° - (L + L')$

$$\sin H = \cos PLZ \sin PL \sin ZL + \cos PL \cos ZL$$
$$= -\sin (L - L') \cos D' \cos (P - p) + \sin D' \sin (P - p)$$

$$\sin H' = -\sin (L + L') \cos D' \cos (P - p) + \sin D' \sin (P - p)$$

$$\cot. ZPL = -\frac{\cos D' \tan (P - p)}{\sin (L - L')} - \sin D' \cot (L - L')$$

$$\cot. Z'PL = -\frac{\cos D \tan (P - p)}{\sin (L + L')} - \sin D' \cot (L + L')$$

$ZPS = ZPL + LPS = ZPL + \text{mot. Æ}$,

$Z'PS = Z'PL + \text{mot. Æ}$.

These will be the horary angles of the two places; whence their longitudes may be deduced as above. These formulæ require no modification from the commencement to the conjunction in right ascension; but in order that the triangle LSV may be possible, it is necessary that

$E < (P - p) + (\delta + d)$ and $> (P - p) - (\delta + d)$.

If the shortest distance $S\,m = \text{mot. } D \cos I = (P - p) - (\delta + d)$ the points V and V′ are united in one on the arc LS, and there will be but one point of contact in the horizon, and this in the prolongation of $m\,S$.

If $S\,m < (P - p) - (\delta + d)$, the curve of contacts at the horizon of the first half will have no point common with the other half; and the two will be separated by an interval of greater or less extent.

When we have again $E = (P - p) - (\delta + d)$ the curve of contacts, recommences at one point. When $E > (P - p) - (\delta + d)$, there are two points, till $E = (P - p) + (\delta + d)$, then the curve is shut by a point only, which is the last of the eclipse.

For this second curve we have, as for the first, the same formulæ for L′, for PLV and PLV′, for PLZ and PLZ′, for H and H′, and for ZPL and Z′PL; but $ZPS = ZPL - \text{mot. Æ}$, and $Z'PS = Z'PL - \text{mot. Æ}$, because mot. Æ has a change of sign; by this means ZPS and Z′PS may be negative, if ZPL and Z′PL are very small: in this case, the contact at the horizon will be east, and it will be west when ZPS becomes positive: the same may be said of the triangle Z′PS.

The curve of contacts which embraces the pole will have a node. The two branches will cross each other, as we may perceive by putting the results of the calculation into table E and F; the former will contain the quantities given by the formula $(L - L')$, and the latter by $(L + L')$; but at 10h. 20′ the longitude and the latitude follow in more regular order, by putting the $(L + L')$ in table F, and the $(L - L')$

in table E. These considerations however are of but little consequence; for, by placing all these points on a globe, we see always the figure of the curve, and we may estimate, with sufficient accuracy, the place of the node, which would be difficult to determine by calculation.

In our example, the curve of contacts at the horizon (tab. E and F) is subject to no interruption, because $Sm > (P-p)-(\delta+d)$: all these points form a single curve, of which the branches intersect, but of which the inferior parts are at a considerable distance under the globe. The lower extremes may be joined by another curve, which gives the simple contact for the greatest phase.

In order to calculate it, on $Sm = \text{mot. D. cos. I} = -\epsilon$, let us take $SV = \sigma = \delta + d$, and draw LV, which will be the parallax; we shall then have

$$Lm = Sm \tan mLS = \epsilon \tan (S-I)$$

$$\tan mLV = \frac{mV}{mL} = \frac{\epsilon - \sigma}{\epsilon \tan (S-I)}$$

$$LV = \frac{mL}{\cos mLV}, \ \sin ZV = \frac{LV}{(P-p)}$$

$$ZL = ZV - (P-p)$$

$$\begin{aligned} PLV &= PLS - SLV = PLS - (SLm - mLV) \\ &= PLS + mLV - (90^\circ - SLm) = PLS + mLV \\ &\quad -90^\circ (S-I) \\ &= PLS + mLV + (S-I) - 90^\circ \\ PLZ &= 180^\circ - PLS - mLV + I - S + 90^\circ \\ &= 270^\circ + I - (L + S - mLV). \end{aligned}$$

Then we have for calculating the triangle PLZ the formula given in the preceding articles, and $ZPS = ZPL + \text{mot. AR}$.

This formula will serve for the commencement of the eclipse, by remarking that the angle I will change its sign, if the middle be after the conjunction.

After the middle, and before the conjunction,

$$mL = \epsilon \tan (S-I) = -\epsilon \tan (I-S)$$

because $I > S$, and mL changes position; but the formula will advertise us of the circumstance.

$$\tan mLV = \frac{\epsilon - \sigma}{\epsilon \tan (I-S)}$$

$$LV = \frac{mL}{\cos mLV}; \ \sin ZV = \frac{LV}{P-p}, \ ZL = ZV - LV$$

$$\begin{aligned} PLZ &= 180^\circ - PLV = 180^\circ - PLS - SLV \\ &= 180^\circ - L - (SLm - mLV) \\ &= 180^\circ + mLV - L - (90^\circ - mSL) \\ &= 180^\circ + mLV - L - 90^\circ + I - S \\ &= 90^\circ + mLV + I - (S + L) \\ &= 90^\circ + I + mLV - (S + L) \end{aligned}$$

After the conjunction,

$$mL = \epsilon \tan (I+S), \ \tan mLV = \frac{\epsilon - \sigma}{mL},$$

$$LV = \frac{mL}{\cos mLV}; \ \sin ZV = \frac{LV}{P-p}, \ ZL = ZV - LV$$

$$\begin{aligned} PLZ &= 180^\circ - PLV = 180^\circ - PLS + SLV, \\ &= 180^\circ - L + SLm - mLV, \\ &= 180^\circ - mLV - L + S + I, \\ &= 180^\circ + I - mLV - (L - S), \end{aligned}$$

where I changes signs, if the middle be after the conjunction.

$$ZPS = ZPL + \text{mot. AR}.$$

The two parts of this angle change their signs at the conjunction, and the place is to the west, provided ZPS be not negative, or greater than 180°.

224. In fig. 66, it is evident, that the moon advancing from L towards m, and the zenith towards PS, the two causes will concur in detaching the moon from the sun, and the contact will be that at the end. It is not certain that this has not been preceded by a small eclipse, but it must have been very small, for the motion of the moon is more rapid than that of the point V.

Fig. 67. In fig. 67, after the instant of calculation, the two causes will separate the moon from the sun for the same reason.

Fig. 68. In fig. 68, the motion of the moon will open the two edges, but the diurnal rotation may approximate them. When the declination of the moon is less than that of the sun, the eclipse is very little for our climates; and as the more southern parts of the globe are little inhabited, astronomers have hitherto paid but little attention to these kind of eclipses: they have generally been satisfied with calculating the line of centrality, which has always place when the shortest distance is less than the horizontal parallax; adding thereto the northern limit, and sometimes that of the contacts in the horizon, in all latitudes where this may be thought important: but for this it is commonly thought sufficient to make the calculation hour and hour.

When we have determined, by calculation, the principal curves, they may be placed on a map or chart, whose projection is such as to contain all the countries where the eclipse will be visible. The maps projected on the equator, and particularly those which are projected on a plane inclined to the equator at $23^\circ\frac{1}{2}$, or even to 45 or 50°, will also do very well.

If we have no chart already projected, it will be best to take the pole for the centre of projection, all the meridians being then right lines about the pole; and with any radius taken at pleasure, describe a circle to represent the equator; divide this into equal portions of 10° each, and at each division draw radii to represent so many meridians.

From the centre, and with radii equal to the tangents of the semi-polar distances, draw circles parallel to the equator; and on this chart trace the configurations of the principal continents and islands, and fix by their latitude and longitude some of the principal places in them; and then, by their latitudes and longitudes, the calculated points of the different curves. This practice, which was that of Lalande, is the most commodious, as it dispenses with tracing ellipses; every thing is done with a ruler and compass; and thus saves time, which would be otherwise lost without any corresponding recompense

General Eclipse.

Table A

Time.	D ☉	D′ ☾	Æ′ — Æ	S	L	E = LS Distance of centres.
h. m.	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
7 30	4 45 59	4 42 40	1 28 13	92 4 53	87 47 49	1 27 59
40	4 46 8	4 45 01	1 24 13	90 41 33	89 11 27	1 23 56
50	4 46 18	4 47 22	1 20 12	89 10 46	90 42 32	1 19 56
8 0	4 46 28	4 49 44	1 16 10	87 29 36	92 25 20	1 15 57
10	4 46 38	4 52 5	1 12 8	85 36 1	94 17 55	1 12 5
20	4 46 47	4 54 26	1 8 8	83 30 33	96 23 43	1 8 19
30	4 46 57	4 56 48	1 4 8	81 10 39	98 43 55	1 4 39
40	4 47 6	4 59 9	1 0 6	78 34 58	101 19 54	1 1 5
50	4 47 16	5 1 30	0 56 6	75 40 17	104 14 49	0 57 40
9 0	4 47 26	5 3 52	0 52 6	73 2 41	107 31 53	0 54 27
10	4 47 35	5 6 13	0 48 5	68 42 39	111 13 13	0 51 25
20	4 47 45	5 8 34	0 44 4	64 34 54	115 21 22	0 48 36
30	4 47 55	5 10 56	0 40 4	59 59 1	119 57 11	0 46 5
40	4 48 4	5 13 17	0 36 3	54 52 50	125 4 2	0 43 53
50	4 48 14	5 15 38	0 32 3	49 20 12	130 36 38	0 42 4
10 0	4 48 24	5 18 0	0 28 1	43 17 39	136 39 55	0 40 41
10	4 48 33	5 20 21	0 24 0	36 55 1	143 2 53	0 39 47
20	4 48 43	5 22 42	0 20 0	30 21 5	149 37 37	0 39 24
30	4 48 53	5 25 4	0 15 59	23 43 27	156 15 9	0 39 33
40	4 49 2	5 27 25	0 11 57	17 12 13	162 46 13	0 40 13
50	4 49 12	5 29 46	0 7 36	11 0 57	168 58 21	0 41 19
11 0	4 49 22	5 32 8	0 3 55	5 12 30	174 47 8	0 42 57
10	4 49 31	5 34 29	0 0 6	0 7 37	179 52 23	0 45 0
20	4 49 41	5 56 50	0 4 8	4 59 4	175 0 34	0 47 22
30	4 49 51	5 39 12	0 8 8	9 19 2	170 40 14	0 50 0
40	4 50 0	5 41 33	0 12 9	13 12 12	166 46 42	0 52 56
50	4 50 10	5 43 54	0 16 10	16 40 53	163 17 37	0 56 6
12 0	4 50 20	5 46 16	0 20 11	19 44 52	160 13 16	0 56 26
10	4 50 29	5 48 37	0 24 12	22 29 41	157 28 5	1 2 53
20	4 50 39	5 50 58	0 28 13	24 56 48	155 0 34	1 6 33
30	4 50 49	5 53 20	0 32 15	27 9 13	152 47 45	1 10 18
40	4 50 58	5 55 41	0 36 16	29 7 23	150 49 13	1 14 7
50	4 51 8	5 58 2	0 40 17	30 54 34	149 1 38	1 18 0
13 0	4 51 18	5 0 24	0 44 18	32 30 45	147 25 7	1 21 59
10	4 51 27	6 2 45	0 48 19	33 57 53	144 57 31	1 26 0

TABLE B.

Time at given place.	Longitude.	Latitude. north.	Time, place sought.
h. m.	° ′	° ′	h. ′
7 40	14 46W	0 10	— 5 19
7 50	3 11W	2 34	— 4 23
8 0	2 59E	4 42	— 3 48
8 10	6 40	6 26	— 3 23
8 20	10 44	7 58	— 2 57
8 30	13 33	9 19	— 2 36
8 40	15 54	10 22	— 2 16
8 50	17 59	11 26	— 1 58
9 0	20 11	11 58	— 1 39
9 10	21 11	12 25	— 1 25
9 20	22 9	12 44	— 1 11
9 30	22 52	12 50	— 0 58
9 40	24 13	12 45	— 0 47
9 50	23 6	12 36	— 0 38
10 0	22 33	12 29	— 0 30
10 10	21 34	12 30	— 0 24
10 20	20 23	12 46	— 0 19
10 30	18 37	13 25	— 0 16
10 40	16 53	14 35	— 0 12
10 50	15 16	15 55	— 0 9
11 0	13 46	17 51	— 0 5
11 10	12 32	20 10	+ 0 0
11 20	11 40	22 43	+ 0 7
11 30	11 10	25 27	+ 0 15
11 40	11 8	28 25	+ 0 25
11 50	11 36	31 32	+ 0 36
12 0	12 38	34 45	+ 0 51
12 10	14 19	38 2	+ 1 7
12 20	16 57	41 31	+ 1 28
12 30	20 59	44 51	+ 1 54
12 40	26 14	48 35	+ 2 25
12 50	34 35	52 7	+ 3 8
1 0	49 20	55 29	+ 4 17

TABLE C.

Time, given place.	Longitude.	Latitude.	Time, place sought.	Quantity of the eclipse.
h. m.	° ′	° ′	h. ′	° ′
7 40	25 11W	0 41 S	— 6 1	0 51N
7 50	27 23W	0 50N	— 6 0	4 51
8 0	29 55W	2 30	— 6 0	8 50
8 10	32 49W	4 23	— 6 1	12 41
8 20	35 33W	6 28	— 6 2	16 27
8 30	37 10W	8 48	— 6 3	20 7
8 40	40 50W	11 23	— 6 3	23 41
8 50	43 22W	14 16	— 6 4	27 6
9 0	46 16W	17 34	— 6 5	29 34
9 10	49 30W	22 14	— 6 6	28 7S
9 20	51 54W	25 21	— 6 8	25 18
9 30	54 50W	29 56	— 6 9	22 47
9 40	57 52W	35 1	— 6 11	20 35
9 50	61 2W	40 32	— 6 14	18 46
10 0	65 1W	46 33	— 6 20	17 23
10 10	68 25W	54 53	— 6 24	16 29
10 20	72 12W	59 23	— 6 29	16 6
10 30	77 8W	65 55	— 6 39	16 15
10 40	83 40W	72 17	— 6 55	16 55
10 50	93 48W	78 10	— 7 35	18 2
11 0	108 42W	83 12	— 8 15	19 40
11 10	169 35E	85 34	+11 54	21 53
11 20	141 47E	83 18	+ 8 47	24 5
11 30	123 7E	79 39	+ 7 22	26 43
11 40	113 44E	76 3	+ 7 15	29 34
11 50	107 30E	72 43	+ 6 59	28 39N
12 0	102 47E	69 43	+ 6 51	25 19
12 10	98 46E	67 2	+ 6 45	21 51
12 20	95 12E	64 38	+ 6 41	18 12
12 30	91 45E	62 8	+ 6 37	14 26
12 40	88 27E	60 30	+ 6 34	10 37
12 50	85 15E	58 46	+ 6 31	6 44
1 0	82 10E	57 12	+ 6 29	2 45

Central Eclipse.

Table D.

Time at given place.	Longitude.	Latitude. north.	Time at the place where the eclipse is central.
h. m.	° ′	° ′	h. ′
9 10	30 42W	21 42	— 4 53
9 20	23 36	24 55	— 4 14
9 30	19 14	27 58	— 3 47
9 40	15 47	30 58	— 3 23
9 50	13 2	33 55	— 3 2
10 0	10 15	36 57	— 2 41
10 10	7 47	40 2	— 2 21
10 20	5 22	43 12	— 2 1
10 30	2 49	46 29	— 1 41
10 40	0 1E	49 53	— 1 20
10 50	4 55	53 21	— 0 52
11 0	7 21	57 8	— 0 32
11 10	12 13	61 15	— 0 1
11 20	20 38	65 40	+ 0 42
11 30	34 2	70 24	+ 1 46
11 40	67 10	75 20	+ 4 9

The longitudes here are from the given place, Paris; and the negative time is the time during the morning, counted from the next noon. Table B gives the minimum, and table C the maximum in the same vertical. Table D shows the places where the eclipse is central. By the maximum, however, we must not understand the maximum absolutely, but the greatest immersion of the moon in the vertical of the sun. If the centre of the moon is above that of the sun, the eclipse is north; and it is south in the contrary case. The places situated on the same great circle have all an eclipse of greater or less magnitude, as they are nearer or more distant from the point where it is seen central; the distance between the first and second place is the greater axis or length of the figure of the shadow. The places marked in Tables E and F, by their distance, give the breadth of the shadow.

Contact in the horizon.

Table E.

Time, given place.	Longitude.	Latitude. north.	Time at place sought.
h. m.	° ′	° ′	h ′
7 40	24 57W	5 21	— 6 4
7 50	29 12W	15 13	— 6 7
8 0	32 14W	21 48	— 6 9
8 10	35 10W	26 57	— 6 11
8 20	38 10W	32 14	— 6 13
8 30	40 59W	35 26	— 6 14
8 40	44 2W	40 6	— 6 16
8 50	47 27W	46 37	— 6 20
9 0	50 34W	50 18	— 6 22
9 10	54 1W	54 55	— 6 26
9 20	57 42W	59 27	— 6 31
9 30	61 54W	64 15	— 6 38
9 40	66 58W	69 18	— 6 48
9 50	73 56W	74 33	— 7 6
10 0	86 19W	80 1	— 7 45
10 10	120 44W	84 46	— 9 53
10 20	165 4E	84 10	+ 9 20
10 30	135 38E	78 47	+ 7 35
10 40	124 13E	72 24	+ 6 57
10 50	127 36E	66 2	+ 6 40
11 0	112 32E	60 17	+ 6 30
11 10	108 55E	54 57	+ 6 26
11 20	105 27E	50 18	+ 6 22
11 30	102 18E	46 24	+ 6 18
11 40	99 20E	43 16	+ 6 17
11 50	96 32E	40 45	+ 6 14
12 0	93 53E	39 8	+ 6 13
0 10	91 20E	38 9	+ 6 12
0 20	88 55E	37 49	+ 6 12
0 30	86 33E	38 15	+ 6 12
0 40	84 11E	38 53	+ 6 12
0 50	82 20E	41 51	+ 6 14
1 0	80 42E	46 15	+ 6 17

Contact at the horizon.

Table F.

Hours, given place.	Longitude.	Latitude.	Time at place sought.
h. m.	° ′	° ′	h. ′
7 40	24 56W	6 50S	—6 0
7 50	26 44W	13 39	—5 59
8 0	28 52W	16 49	—5 55
8 10	31 9W	18 9	—5 54
8 20	33 3W	19 20	—5 52
8 30	36 1W	17 53	—5 51
8 40	38 29W	17 22	—5 54
8 50	40 55W	17 20	—5 54
9 0	43 33W	15 14	—5 54
9 10	46 14W	12 32	—5 55
9 20	49 1W	8 51	—5 56
9 30	51 51W	4 34	—5 57
9 40	54 46W	0 41N	—5 59
9 50	57 41W	6 8	—6 1
10 0	60 15W	12 20	—6 3
10 10	63 47W	18 53	—6 5
10 20	67 14W	25 31	—6 9
10 30	70 7W	32 9	—6 10
10 40	73 23W	38 24	—6 14
10 50	76 44W	44 15	—6 17
11 0	80 17W	49 58	—6 21
11 10	83 59W	55 13	—6 26
11 20	87 57W	60 10	—6 22
11 30	91 28W	64 48	—6 40
11 40	97 14W	69 2	—6 49
11 50	103 47W	73 36	—7 5
12 0	112 50W	77 49	—7 31
0 10	127 6W	81 42	—8 22
0 20	161 44W	84 40	—10 27
0 30	143 46E	84 16	+10 5
0 40	114 46E	81 2	+8 9
0 50	97 38E	75 9	+7 21
1 0	88 14E	67 50	+6 53

3. *Eclipse of the sun for a particular place.*

Eclipse, or occultation for a particular place.

225. In determining the several circumstances attending an eclipse of the sun in a particular place, we have no occasion for that preparation, which has been explained in the preceding articles: we have a ready means of ascertaining whether the eclipse will or will not be visible; for, if the difference of declination at the instant of the mean conjunction be less than the sum $(P - p) + (\delta + d)$, and the moon nearer the pole than the sun (or than the star when the question is an occultation), it is nearly certain there will be an eclipse, at least if the conjunction does not take place considerably out of the meridian; and in all cases it will be sufficient to compute two triangles to decide the question.

When we have determined that the eclipse will happen, we must search for the time of the middle, known within half an hour or an hour, the longitudes of the two bodies, their latitudes, horary motions, parallaxes, and diameters. For two instants differing by an hour, we must calculate the right ascension of the midheaven M, the nonagesimal degree n, the height h of the pole, of the ecliptic, and the parallax in longitude and latitude, by the formulæ of art. 202, et seq.; substituting there the relative instead of the absolute parallax; the parallax in latitude and longitude being applied according to their signs, and to the places of the moon. We must then compare the position of the moon with that of the body eclipsed, and thence deduce the apparent difference of latitude and longitude for the two instants in question. It is necessary to make these calculations with great precision, but we may neglect the seconds in the subsidiary arcs n and h, which is the principle advantage of the nonagesimal method.

Preparations for observation.

226. This being done, and our object being to prepare ourselves for observation, we may proceed according to the principles of the graphical construction illustrated in art. 209; that is, on the two perpendiculars $A''a''$, $A'''a'''$, (fig. 61,) mark the two apparent latitudes, if the eclipse is of the sun, and the two apparent differences of latitude, if it be an occultation of a star or planet; and through the summits of the two latitudes draw a line, which will be the apparent orbit.

The interval between the two perpendiculars will be the horary motion; it will correspond to 60′ in time, answering to the relative motion of the moon to the sun, or more generally to the relative motion multiplied by the cosine of the latitude of the body eclipsed.

Find on the line AX the place of the apparent conjunction, as we found the centre of the shadow in the eclipse of the moon. Let fall from this point the perpendicular on the apparent orbit to obtain the shortest distance. With the sum of the diameters, mark the points for the beginning and end on the relative orbit; and by means of a square, mark on AX, the points which answer to the beginning and end.

If the eclipse be central or total mark the points of immersion and emersion, with an opening of the compasses equal to the difference of the semidiameters. If it be a planet which is eclipsed, these two phases will differ very little from the two first; and if it be a star, the points of contact will be confounded with those of immersion and emersion.

From the point m, as a centre, describe the lunar disc; and from the point O, as a centre, describe the disc of the body eclipsed, and we shall have the figure of the greatest eclipse.

Draw from the centre O, to the place of commencement C, the radius OC; and through the same point O raise a perpendicular to the ecliptic, which will be the circle of latitude; with the declination of the body calculate the angle of position, placing the circle of declination to the east, if the tangent of the angle of position be positive.

For the time of the beginning compute the angle of the vertical of the sun with the circle of declination, that is, POZ; form the angle POZ on the figure, and OZ will be the vertical.

Now the eclipse ought to commence with the contact at G on the line OC. Place before you the figure of the eclipse in such a manner, that OZ may be a vertical line; and we shall see the position of the point G, at which the eclipse will commence: and thus we shall have the place of the first impression of the lunar disc on the body eclipsed. If it be a planet or a star, of which the disc is very small, through the point C, the centre of the moon, draw the parallel Z′C, which will indicate the point where the eclipse ought to begin.

This is indeed useless for the commencement, because the star or the planet will be seen by the side of the moon, but it is indispensable for the end, in order to know precisely the place of emersion, the planet till then being invisible behind the moon; and it is therefore of importance, in order to ascertain well the instant of emersion, and particularly if it takes place at the enlightened part of the disc. We must perform then, for the emersion of the star, an operation similar to that which we have employed for the beginning of the eclipse of the sun.

If however it is Jupiter that is eclipsed, we need not use the above proceeding, for some one or more of his satellites will commonly emerge before him, and indicate the place of his emersion · and if not, the disc of this planet is such, that if we miss the emersion of his first edge we may get that of the second.

227. In the occultations of the stars and planets, no other considerations than those of immersion and emersion are required; but in eclipses of the sun, we may continually measure the lines of the horns. This line is shortened by refraction; and, in order to compute the quantity, we must know the inclination to the horizon, which we may take at any time from the figure; and the line thus corrected, is a common chord to the lunar and solar discs, from which we may obtain the distance of the centres. The observation is however attended with difficulty, and is scarcely worth attention.

228. An observation more easy to make, and more useful in its results, is of the greatest quantity of the eclipse. Towards the middle of the eclipse, the line of the horns is of a constant magnitude, at least to observation, for one or two minutes, which is plenty of time for observing it very exactly.

Fig. 69. Let then $a\,b$ (fig. 69), the distance of the horns, be the greatest of all those measured during the eclipse, and corrected for refraction; then

$$\frac{\frac{1}{2}a\,b}{\mathrm{S}\,a} = \frac{a\,b}{2\delta} = \sin a\,\mathrm{SL},$$

$$\frac{\frac{1}{2}a\,b}{\mathrm{L}\,a} = \frac{a\,b}{2d} = \sin a\,\mathrm{LS};$$

$\delta \cos a\,\mathrm{SL} + d \cos a\,\mathrm{LS}$ = shortest distance = $(\lambda - p) \cos \mathrm{I}$; thus

$$\frac{\delta \cos a\,\mathrm{SL} + d \cos a\,\mathrm{LS}}{\cos \mathrm{I}} = (\lambda - p) = \text{app. lat.}$$

Calculate now the parallax of latitude p for the middle, then

$$(\lambda - p) + p = \lambda,$$

and we shall have the latitude in conjunction. SL sin I will be the difference of apparent longitudes on the ecliptic; SL sin I—parallax long. will be the true difference of longitude at the instant of the greatest eclipse; we shall thus have the time of the true conjunction by the true relative motions if we have well determined the time of the middle of the eclipse, and by calculating for this instant the true latitude by the tables, we shall know the error of the tables in latitude. It is true that the motion in latitude is only 6″ in one minute of time; and, therefore if we deceive ourselves by a minute as to the time of the middle, we shall not know nearer than within 6″ the errors of the tables in latitude.

If amongst the line of chords measured there are found many that are equal two and two, these will give the time of the middle. We shall also have the time of the middle very nearly by the observations on the beginning and end; and we shall have thus very nearly the errors of the tables.

If the eclipse be annular, the time of the two interior contacts, denominated the formation and breaking of the ring, will give us the difference of the diameters; for the interval of time between the two observations will give us the motion of the centre of the moon on its relative orbit. Let m be the half of this motion, ϵ the shortest distance, then we shall have

$$(\delta - d)^2 = m^2 + \epsilon^2;$$

also, at the instant of the interior contact, the breadth of the ring in the part opposite the contact will be the difference of the two diameters.

If the eclipse be total without duration for more than an instant, the apparent diameters are equal; and this instant will be that of the shortest distance $= (\lambda - p) \cos \mathrm{I} = 0$; whence $\lambda = p$. Calculate therefore for this instant the parallax in latitude, which will be the true latitude, and the parallax in longitude will be the true difference of longitude for this instant; and the augmentation of the diameter will be the difference of the true diameters. We shall have therefore the correction of the three elements, the longitude, the latitude, and the difference of the diameters.

If the sun remains for some time in the shadow, we shall have the apparent relative motion; and if we have deduced the shortest distance, we shall have as above, the difference of the diameters. These observations, however, are very rare; and after all, perhaps they will not admit of great precision.

§ X. *Of Comets.*

Of comets. 229. We have already, in our introductory chapter, made a few observations on this class of celestial bodies, which are occasionally seen in the heavens with ill defined and faint discs, and usually accompanied with a *coma*, or stream of faint light, in the direction of a line drawn from the sun through the comet.

These bodies, as far as regards their change of place in the heavens amongst the fixed stars, resemble the moon and planets; but they differ from them in almost every other respect; they have never, (with one or two exceptions,) been known to return; that is, they do not appear to have definite elliptic orbits: the inclination of their orbits also are inclined to the ecliptic at a great variety of angles, from 0° to 90°; their motions are, some retrograde and some direct:

and the eccentricities of their orbits are so great, that it has been found that they may be very nearly represented by parabolas.

What are called the elements of a comet's orbit, are less than those of a planet's in number, being only five. It is impossible, from the observations made during one appearance of a comet, to compute the major axis of its orbit, and its period with any accuracy: what astronomers endeavour to compute, and what they do with difficulty compute, are, the perihelion distance, its place or longitude, the epoch of that longitude, the longitude of the ascending node, and the inclination of the orbit.

The elements of the orbits of planets are capable of being determined by observations made in the meridian; by longitudes and latitudes computed from right ascensions and declinations; but comets require observations of a different kind, by the rotation of the earth they are brought on the meridian, but generally from their proximity to the sun while they are visible, this does not take place in the night, and it is only then that they can be observed, on account of the faintness of their light. They must therefore be observed out of the meridian; and in that position the difference between the right ascension and declinations, and those of a known contiguous star must be determined.

It is difficult to make these latter observations with accuracy, by reason of the doubtful and ill defined disc of the comet; and a small error in the observations will materially affect the elements of the orbit.

The rigorous solution of the problem of the elements of a comet's orbit requires three observations only. But then the solution is attended with so many difficulties, that in this, as in other like cases, astronomers have sought, by the indirect method of trial and conjecture, to avoid them. If, what generally happens, more than three observations are obtained, the redundant ones are employed in correcting and confirming previous results.

The periodic time, as we have observed, cannot be determined from observations during one appearance of the comet. If known, it can only be so by recognising the comet during its second appearance; and the only mode of recognition is by the identity of the elements of its orbit, with those of the orbit of a comet already observed. If the perihelion distance, the position of the perihelion and of the nodes, and the inclination of the orbit, are the same, or nearly so, we may presume, with considerable probability, that the comet we are observing has been previously in the vicinity of the sun; and that, after moving round by the aphelion of its oval orbit, it has again returned towards its perihelion.

Comets not having been formerly observed with great accuracy, it so happens that, till lately, the period of one alone, that of the comet observed in 1759, 1682, 1607, and 1531, is known to any degree of certainty. Its period being about 76 years: assuming the earth's mean distance to be unity, the perihelion distance of this comet was 0·58, and the major axis of the orbit 35·9. The inequalities which are noted in its period are supposed to arise from the influence of some disturbing forces. This comet was foretold by Dr. Halley; and its appearance in the former of the above years verified the conjecture, and immortalized the name of this celebrated astronomer.

It would carry us far beyond our due limits to enter minutely into all the particulars necessary for illustrating the various methods of computations adopted by astronomers in their inquiries concerning the orbits of comets; and it does not belong to this part of our treatise to discuss the various hypotheses which have been advanced by different philosophers respecting the nature of those bodies, their tails, comas, &c.; we shall therefore conclude this section by giving in the form of a table, the several particulars of one hundred comets, whose elements have been determined with some degree of accuracy; whereby we should be able to ascertain their identity should any of them reappear.

It must not, however, be understood that this table is intended to include all, or even the principal number of the comets belonging to our system; this number being very considerable. From the accounts of the ancient and the more accurate observations of modern astronomers, it has been ascertained that more than 500 comets have been seen since the commencement of our æra. When the attention of astronomers was called to this subject by the expectation of the return of the comet of 1759, no less than seven were observed within the course of as many years; and from 1780 to 1800, about 30 comets were seen, and the elements of their orbits computed. These circumstances therefore, and the probability that most of the comets of small apparent magnitudes were overlooked by the ancients, and moreover that even in the present day many may pass unnoticed in the southern hemisphere, it is reasonable to conclude that the number of these bodies is very considerable, and far beyond any estimation that can be made from the observations we now possess.

Table of the elements of the orbits of 105 comets

Number.	Passage through the perihelion in mean time, Greenwich.	Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	Anno A.C.						
	h. m. ″	s ° ′ ″					
1	539 Oct. 20, 15 0 0	10 13 30 0	0·3412	1s 28° or 70 28°	10° + or −	D	Burckhardt.
	Anno Domini, Old Stile.						
2	837 March	9 19 3 0	0·5800	6s 26° 33 0	10° or 12	R	Pingré.
3	1097 Sept. 21, 21 36 0	11 2 30 0	0·7385	6 27 30 0	73° 30′ 0″	D	Burckhardt.
4	1231 Jan. 30, 7 12 40	4 14 48 0	0·9478	0 13 30 0	6° 5′ 0	D	Pingré.
5	1264 July 6, 8 0 0	9 21 0 0	0·445	5 19 0 0	36 30 0	D	Dunthorn.
	—— July 17, 6 0 40	9 5 45 0	0·41081	5 28 45 0	30 25 0		Pingré.
6	1299 March 31, 7 28 40	0 3 20 0	0·3179	3 17 8 0	68 57 0	R	Pingré.
7	1301 Oct.	9s. or 10s.	0·457	0 15° + or −	70° + or −	R	Pingré.
8	1337 June 2, 6 25 0	1 7 59 0	0·40666	2s 24 21 0	32 11 0	R	Halley.
	June 1, 0 30 40	0 20 0 0	0·6445	2 6 22 0	32 11 0		Pingré.
9	1351 Nov. 26, 12 0 0	2 9 0 0	1·0000			D	Burckhardt. [years.
10	1456 June 8, 22 0 0	10 1 0 0	0·5855	1 18 30 0	17 56 0	R	Pingré, period 75½
11	1472 Feb. 28, 22 23 0	1 15 33 30	0·54273	9 11 46 20	5 20 0	R	Halley.
	1531 Aug. 24, 21 18 30	10 1 39 0	0·56700	1 19 25 0	17 56 0	D	Halley.
12	1532 Oct. 19, 22 12 0	3 21 7 0	0·50910	2 20 27 0	32 36 0	D	Halley.
13	1533 June 16, 19 30 0	4 27 16 0	0·20280	4 5 44 0	35 49 0	R	Douwes.
14	1556 April 21, 20 3 0	9 8 50 0	0·46390	5 25 42 0	32 6 30	D	Halley.
15	1577 Oct. 26, 18 45 0	4 9 22 0	0·18342	0 25 52 0	74 32 45	R	Halley.
16	1580 Nov. 28, 13 44 40	3 19 11 5	0·59553	0 19 7 37	64 51 50	D	Pingré.
	15 0 0	3 19 5 50	0·59628	0 18 57 20	64 40 0	D	Halley.
17	1582 May 7,	8s 5° or 9s 11°	0·23 or 0·04	7s 5° or 21°	59 or 61°	R	Pingré.
18	1585 Sept. 27, 19 20 0	0 8 51 0	1·09358	1s 7° 42′ 30″	6° 4′ 0″	D	Halley.
19	1590 Jan. 29, 3 45 0	7 6 54 30	0·57661	5 15 30 40	29 40 40	R	Halley.
20	1593 July 8, 13 38 40	5 26 19 0	0·08911	5 14 15 0	87 58 0	D	De la Caille.
21	1596 July 31, 19 55 0	7 18 16 0	0·51293	10 12 12 30	55 12 0	R	Halley.
	29, 15 33 40	7 28 30 50	0·549415	10 15 36 50	52 9 45	R	Pingré.
*10	1607 Oct. 16, 3 50 0	10 2 16 0	0·58680	1 20 21 0	17 2 0	R	Halley.
22	1618 Aug. 7, 3 2 40	10 18 20 0	0·51298	9 23 25 0	21 28 0	D	Pingré.
23	1618 Oct. 29, 12 23 0	0 2 14 0	0·37975	2 16 1 0	37 34 0	D	Halley.
24	1652 Nov. 2, 15 40 0	0 28 18 40	0·84750	2 28 10 0	79 28 0	D	Halley.
25	1661 Jan. 16, 23 41 0	3 25 58 40	0·44851	2 22 30 30	32 35 50	D	Halley.
26	1664 Nov. 24, 11 52 0	4 10 41 25	1·025755	2 21 13 55	21 18 40	R	Halley.
27	1665 April 14, 5 15 30	2 11 54 30	0·10649	7 18 2 0	76 5 0	R	Halley.
28	1672 Feb. 20, 8 37 0	1 16 59 30	0·69739	9 27 30 30	83 22 10	D	Halley.
29	1677 April 26, 0 37 30	4 17 37 5	0·28059	7 26 49 10	79 3 15	R	Halley.
30	1678 Aug. 16, 14 3 0	10 27 46 0	1·23801	5 11 40 0	3 4 20	D	Douwes.
31	1680 Dec. 8, 0 1 2	8 22 40 10	0·006030	9 1 57 13	61 22 55	D	Pingré.
	7, 0 4 0	8 22 33 0		9 1 53 0	61 20 20		Halley.
	7, 23 9 0	8 22 44 25	0·006170	9 2 2 0	61 6 48		Halley.
	7, 20 38 39	8 23 26 48	0·006565	9 2 59 9	58 39 50		Euler.
	8, 0 4 0	8 27 43 0	0·005920	9 1 53 0	61 20 20		Newton.
*10	1682 Sept. 4, 7 39 0	10 2 52 50	0·58328	1 21 16 30	17 56 0	R	Halley.
	21 31 0	10 1 36 0	0·58250	1 20 48 0	17 42 0		Halley.
32	1683 July 2, 3 50 0	2 25 29 30	0·56020	5 23 23 0	83 11 0	R	Halley.
33	1684 May 29, 10 16 0	7 28 52 0	0·96015	8 28 15 0	65 48 40	D	Halley.
34	1686 Sept. 6, 14 33 0	2 17 0 30	0·32500	11 20 34 40	31 21 40	D	Halley.
35	1689 Nov. 21, 14 55 40	8 23 44 45	0·016889	10 23 45 20	69 17 0	R	Pingré.

TABLE—*continued.*

Number.	Passage through the perihelion in mean time, Greenwich.	Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	h m ″	s ° ′ ″					
36	1698 Oct. 8, 16 57 0	9 0 51 15	0·69129	8 27 44 15	11 46 0	R	Halley.
37	1699 Jan. 3, 8 22 19	7 2 31 6	0·74400	10 21 45 35	69 20 0	R	De la Caille.
38	1702 March 2, 14 12 19	4 18 41 3	0 64590	6 9 25 15	4 30 0	D	De la Caille.
39	1706 Jan. 19, 4 22 39	2 12 29 10	0·42580	0 13 11 40	55 14 10	D	De la Caille.
	4 56 4	2 12 36 25	0·426865	0 13 11 23	55 14 5	D	Struyck.
40	1707 Nov. 30, 23 29 39	2 19 54 56	0·8597	1 22 46 35	88 36 0	D	De la Caille.
	23 43 6	2 19 58 9	0·85904	1 22 50 29	88 37 40	D	Struyck.
41	1718 Jan. 3, 23 38 39	4 1 30 0	1·02650	4 8 43 0	30 20 0	R	De la Caille.
	4, 1 14 55	4 1 26 36	1·02565	4 7 55 20	31 12 53		Douwes.
42	1723 Sept. 16, 16 10 0	1 12 15 20	0·998651	0 14 16 0	49 59 0	R	Bradley.
43	1729 June 14, 11 6 40	10 22 40 0	4·26140	10 10 32 37	76 58 4	D	De la Caille.
	12, 6 36 2	10 22 16 53	4·0698	10 10 35 15	77 1 58		Douwes.
44	1737 Jan. 19, 8 20 0	10 25 55 0	0·22282	7 16 22 0	18 20 45	D	Bradley.
45	1739 June 6, 9 59 40	3 12 38 40	0·67358	0 27 15 14	55 42 44	R	De la Caille.
46	1742 Jan. 28, 4 33 40	7 7 35 13	0·76568	6 5 38 29	66 59 14	R	De la Caille.
	4 20 50	7 7 33 44	0·765555	6 5 34 45	67 4 11		Struyck.
	27, 4 14 39	7 10 49 23	0·7521	6 9 32 7	61 43 44		Euler.
47	1742 Dec. 30, 20 25 40	3 2 41 45	0·83501	2 8 21 15	2 19 33	D	De la Caille.
	21 15 16	3 2 58 4	0·838115	2 8 10 48	2 15 50		Struyck.
48	1743 Sept. 9, 21 16 18	8 6 33 52	0·52157	0 5 16 25	45 48 21	R	Klinkenberg.
49	1744 Feb. 19, 8 27 0	6 17 12 55	0·22206	1 15 45 20	47 8 36	D	Betts.
50	1747 Feb. 20, 7 10 40	9 7 2 0	2·19851	4 27 18 50	79 6 20	R	De la Caille.
	17, 11 44 38	9 10 5 41	2·29388	4 26 58 27	77 56 55		Chezeaux.
51	1748 April 17, 19 25 0	7 5 0 50	0·84067	7 22 52 16	85 26 57	R	Maraldi.
52	1748 June 7, 1 24 15	9 6 9 24	0·65525	1 4 39 43	56 59 3	D	Struyck.
	New Style.						
53	1757 Oct. 21, 9 46 40	4 2 49 0	0·33800	0 4 4 0	12 48 0	D	Pingré.
54	1758 June 11, 3 17 40	8 27 38 0	0·21535	7 20 50 0	68 19 0	D	Pingré.
*10	1759 Mar. 12, 13 31 40	10 3 16 0	0·58349	1 23 49 0	17 39 0	R	De la Caille.
	13 50 4	10 3 8 10	0·58490	1 23 45 35	17 40 14		Lalande.
	13 48 16	10 3 16 20	0·58360	1 23 49 21	17 35 20		Maraldi.
55	1759 Nov. 27, 0 2 37	1 23 34 19	0·80139	4 19 39 41	79 6 38	D	Pingré.
56	1759 Dec. 16, 21 3 40	4 18 24 35	0·96599	2 19 50 45	4 51 32	R	De la Caille.
57	1762 May 28, 15 17 40	3 15 15 0	1·01240	11 19 20 0	84 45 0	D	Lalande.
	28, 6 51 29	3 14 29 46	1·00985	11 19 2 22	85 3 2		Struyck.
	29, 0 18 28	3 15 22 23	1·01415	11 18 55 31	85 22 21		Maraldi.
58	1763 Nov. 1, 19 43 18	2 24 51 54	0·49876	11 26 23 26	72 40 40	D	Pingré.
	20 58 14	2 25 1 6	0·49820	11 26 27 0	72 28 0		Burckhardt.
59	1764 Feb. 12, 13 42 16	0 15 14 52	0·55522	4 0 4 33	52 53 31	R	Pingré.
60	1766 Feb. 17, 8 40 40	4 23 15 25	0·50533	8 4 10 50	40 50 20	R	Pingré.
61	1766 April 22, 20 46 20	8 2 17 53	0·33274	2 14 22 50	11 8 4	D	Pingré.
62	1769 Oct. 7, 12 20 40	4 24 5 54	0·12376	5 25 0 43	40 37 32	D	Lalande.
	13 36 53	4 24 11 7	0·12272	5 25 6 33	40 48 49		Prosperin.
	14 56 39	4 24 16 0	0·12265	5 25 3 0	40 50 0		Euler.
	15 28 16	4 24 10 51	0·1227	5 25 4 41	40 49 33		Lexell.
	12 34 9	4 24 11 8	0·12328	5 25 2 24	40 48 29		Sir H. Englefield.
	15 42 2	4 24 15 53	0·12275	5 25 6 4	40 46 42		Pingré.
	12 44 38	4 24 11 32	0·12327	5 25 3 40	40 47 56		Legendre.
63	1770 Aug. 14, 0 4 4	11 26 26 13	0·676893	4 12 17 3	1 34 30	D	Pingré.
	13, 12 55 40	11 26 16 26	0·674381	4 12 0 0	1 33 40		Lexell.
	13, 12 37 35	11 26 15 0	0·67435	4 11 54 50	1 34 31		Burckhardt.

TABLE—*continued.*

Number.	Passage through the perihelion in mean time, Greenwich.		Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
		h m "	s ° ' "					
64	1770 Nov. 22,	5 38 40	6 28 22 44	0·52824	3 18 42 10	31 25 55	R	Pingré.
65	1771 April 18,	22 5 7	3 13 28 13	0·90576	0 27 51 0	11 15 20	D	Pingré.
	19,	5 1 21	3 14 2 54	0·90340	0 27 50 27	11 16 0		Burckhardt.
66	1772 Feb. 18,	20 41 15	3 18 6 22	1·01815	8 12 43 5	18 59 40	D	Lalande.
67	1773 Sept. 5,	11 9 25	2 15 35 43	1·13390	4 1 15 37	61 25 21	D	Pingré.
		14 33 48	2 15 10 58	0·10120	4 1 5 30	61 14 7		Burckhardt.
68	1774 Aug. 15,	10 46 15	10 17 22 4	1·4286	6 0 49 48	83 0 25	D	Mechain.
69	1779 Jan. 4,	2 2 40	2 27 13 11	0·71312	0 25 5 51	32 24 0	D	Mechain.
		2 15 10	2 27 13 40	0·7132	0 25 3 57	32 25 30		Chev. D'Angos.
70	1780 Sept. 30,	18 3 30	8 6 21 18	0·09925	4 4 9 19	53 48 5	R	Mechain.
71	1781 July 7,	4 32 0	7 29 11 25	0·77586	2 23 0 38	81 43 26	D	Mechain.
72	1781 Nov. 29,	12 32 26	0 16 3 28	0·96101	2 17 22 52	27 13 8	R	Mechain.
		12 33 25	0 16 3 7	0·96099	2 17 22 55	27 12 4		Legendre.
73	1783 Nov. 15,	5 44 3	1 15 24 46	1·5653	1 24 13 50	53 9 9	D	Mechain.
74	1784 Jan. 21,	4 47 40	2 20 44 24	0·70786	1 26 49 21	51 9 12	R	Mechain.
75	1784 April 9,	21 7 26	10 28 54 57	0·65053	2 26 52 9	47 55 8	R	Chev. D'Angos.
76	1785 Jan. 27,	7 48 44	3 19 51 56	1·143398	8 24 12 15	70 14 12	D	Mechain.
77	1785 April 8,	8 58 52	9 27 29 33	0·427300	2 4 33 36	87 31 54	R	Mechain.
78	1786 July 7,	21 50 52	5 9 25 36	0·41010	6 14 22 40	50 54 28	D	Mechain.
79	1787 May 10,	19 48 40	0 7 44 9	0·34891	3 16 51 36	48 15 51	R	P. de Saron.
80	1788 Nov. 10,	7 25 40	3 9 8 27	1·06301	5 7 10 38	12 28 20	R	Mechain.
81	1788 Nov. 20,	9 4 25	0 23 12 22	0·766911	11 21 42 15	64 52 32	D	Mechain.
82	1790 Jan. 15,	5 5 39	2 0 14 32	0·7581	5 26 11 46	31 54 15	R	De Saron.
83	1790 Jan. 28,	7 36 13	3 21 44 37	1·063286	8 27 8 37	56 58 13	D	Mechain.
84	1790 May 21,	5 46 54	9 3 43 27	0·79796	1 3 11 2	63 52 27	R	Lalande.
85	1792 Jan. 13,	13 35 9	1 6 29 42	1·293	6 10 46 15	39 46 55	D	
86	1795 Dec. 14,	23 17 53	5 13 37 0	0·227	11 29 11 0	24 17 0	D	Zach.
	15,	15 6 18	5 7 37 0	0·258	11 13 23 0	20 3 0		Bouvard.
	14,	18 43 0	5 15 34 0	0·215	0 1 7 0	24 42 0		Prosperin.
87	1796 April 2,	19 47 51	6 12 44 0	1·578	0 17 2 0	64 55 0	R	Olbers.
88	1797 July 9,	2 44 31	1 19 34 48	0·52545	10 29 16 35	50 35 50	R	Bouvard.
89	1797 April 4,	11 32 21	3 14 59 0	0·48476	4 2 9 0	43 52 16	D	Burckhardt.
90	1798 Dec. 31,	12 58 57	1 4 29 48	0·77968	8 9 30 44	42 23 25		Burckhardt.
91	1799 Sept. 7,	5 49 48	0 3 39 12	0·839865	3 9 31 59	50 55 37	R	Mechain.
		5 34 4	0 3 39 10	0·840178	3 9 27 19	50 57 30		Von Zach.
		4 24 39	0 3 36 0		3 9 34 0	50 52 30		Burckhardt.
92	1799 Dec. 25,	21 30 49	6 10 20 12	0·625810	10 26 49 11	77 1 38	R	Mechain.
93	1801 Aug. 8,	13 22 39	6 3 49 0	0·2617	1 14 28 0	21 20 0	R	Burckhardt.
94	1803 Sept. 9,	20 33 54	11 2 8 0	1·0942	10 10 17 0	57 0 0	D	Mechain.
95	1804 Feb. 13,	13 31 7	4 28 44 51	1·07117	5 26 47 58	56 28 40	D	Gauss.
96	1805 Nov. 18,	0 15 39	4 29 0 28	0·37567	11 15 6 51	15 58 12	D	Legendre.
97	1805 Dec. 31,	6 1 36	3 19 23 40	0·89159	8 10 35 24	16 25 25	D	Legendre.
	1806 Jan. 2,	10 35 10	3 19 30 2	0·911786	8 11 28 23	12 43 10	D	Legendre.
	1805 Dec. 31,	6 19 27	3 19 21 55	0·89203	8 10 34 42	16 30 35	D	Bessel.
	31,	18 39 36	3 19 23 13	0·89682	8 10 48 5	15 36 10	D	Bessel.
	31,	6 51 8	3 19 23 39	0·89181	8 10 33 34	16 31 10	R	Legendre.
	31,	6 20 59	3 17 21 50	0·89193	8 10 33 35	16 30 32	D	Gauss.
	31,	8 41 18	3 19 28 54	0·890347	8 10 31 34	16 35 9	R	Bouvard.
98	1806 Dec. 28,	22 2 20	3 4 4 30	1·08193	10 22 18 38	35 4 5	D	Bessel.

TABLE—*continued.*

Number.	Passage through the perihelion in mean time, Greenwich.	Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	h m ″	s ° ′ ″					
	1806 Dec. 28, 22 30 43	3 7 2 3	1·081571	10 22 19 15	35 2 50	D	Burckhardt.
99	1807 Sept. 18, 18 56 0	9 0 56 53	0·647491	8 26 39 40	63 14 1	D	Bouvard.
	18, 19 6 0	9 1 0 13	0·64788	8 26 38 31	63 12 36	D	Gauss.
	18, 5 31 12	9 2 24 0	0·65320	8 26 44 0	62 57 0	D	Beck Colkoen.
	18, 19 49 0	9 0 53 38	0·647617	8 26 29 25	63 16 15	D	Brojelmann.
	18, 20 7 5	9 1 6 8	0·64846	8 26 40 52	63 13 7	D	Oriani.
	18, 15 15 51	9 0 51 5	0·646267	8 26 42 12	63 12 51	D	Ferrari.
	18, 17 38 50	9 0 45 1	0·646213	8 26 39 9	63 12 30	D	Lemaur.
	18, 11 44 26	9 0 5 55	0·646212	8 26 40 26	63 27 14	D	Friesneker.
	18, 15 52 50	9 0 39 22	0·648123	8 26 36 57	63 17 56	D	Santini.
	18, 19 27 55	9 0 59 55	0·649620	8 26 25 3	63 9 57	D	Bowditch.
	18, 20 55 32	9 1 6 53	0·648769	8 26 33 4	63 11 18	D	De Montfort.
	18, 19 49 0	9 0 53 38	0·647617	8 26 29 25	63 16 15	D	Brojelmann.
	18, 19 3 0	9 0 58 22	0·64746	8 26 40 46	63 13 2	D	Cacciatore.
	18, 19 51 7	9 1 6 8	0·64896	8 26 36 52	63 14 28	D	Bessel.
	18, 17 41 43	9 1 53 51	0·64587	8 26 48 9	63 10 11	D	Bessel.
100	1808 July 12, 4 10 49	8 12 38 50	0·60786	0 24 11 15	39 18 59	D	Bessel.
101	1810 Oct. 5, 19 54 11	2 3 9 10	0·96914	10 8 53 4	62 46 17	D	Bessel.
102	1811 Sept. 21, 18 30 0	3 8 34 0	1·76770	4 13 12 0	65 9 0	R	Burckhardt.
	15, 10 0 0	3 18 12 30	1·23370	4 19 0 0	71 50 0	R	Burckhardt.
	12, 9 54 21	2 14 48 14	1·035679	4 20 20 25	73 9 40	R	Bessel.
	12, 5 21 15	2 15 17 34	1·04006	4 20 24 13	75 17 34	R	Gauss.
	12, 4 51 3	2 15 17 34	1·040064	4 20 24 13	73 7 16	R	Gauss.
	12, 3 18 9	2 13 40 45	1·01045	4 20 10 13	72 59 55	R	Olbers.
	Ellipse 10, 0 21 9	2 13 14 35	0·98069	4 21 5 59	73 48 2	R	Gauss.
	Ellipse 12, 6 57 30	2 14 29 40	1·02716	4 20 16 56	72 59 10	R	Flaugergnes.
	Ellipse 12, 5 32 19	2 15 1 9	1·03541	4 20 24 30	73 2 36	R	Bessel.
	12, 7 40 13	2 14 56 0	1·03483	4 20 19 50	72 3 3	R	Bouvard.
	12, 6 0 23	2 15 1 44	1·03539	4 20 21 58	73 4 31	R	Nicolaï.
	12, 9 19 1	2 14 51 58	1·03618	4 20 23 18	73 3 44	R	Piazzi.
	12, 9 15 18	2 14 54 24	1·03620	4 20 23 46	73 3 23	D	Calandrelli
103	1811 Nov. 9, 5 52 47	1 16 10 50	1·58533	3 2 57 1	31 29 28	D	Werner.
	11, 4 46 1	1 17 32 10	1·58892	3 2 53 9	31 32 53	D	Werner.
	11, 13 9 14	1 17 47 0	1·58963	3 2 55 1	31 31 52	D	Werner.
	11, 2 45 9	1 17 29 30	1·58791	3 2 56 13	31 29 14	D	Zach.
104	1812 Sept. 14, 14 17 16	3 2 40 29	0·78212	8 13 36 25	74 1 32	D	Werner.
	14, 20 35 55	3 1 54 45	0·78830	8 13 43 25	73 53 51	D	Werner.
	15, 0 0 0	3 2 54 38	0·77835	8 13 18 50	74 20 30	D	Nicollet.
105	1813 March 5, 16 46 0	2 6 52 30	0·67451	2 17 27 30	27 33 30	R	Werner.

Catalogue of three hundred and ninety stars;

WITH THEIR RIGHT ASCENSION IN TIME AND DEGREES; THEIR DECLINATION; ANNUAL PRECESSION AND PROPER MOTION, BOTH IN RIGHT ASCENSION AND DECLINATION; NUMBERED ACCORDING TO THE CATALOGUES OF FLAMSTEAD AND PIAZZI.

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		Annual proper motion.
Piazzi.	Flamstead.			In time.	In degrees.	Annual preccssion in AR	Annual proper motion in AR	In degrees.	Annual precession.	
				h m ″	° ′ ″	″	″	° ′ ″	″	″
9	88	γ Pegasi	2·3	0 2 57	0 44 15·9	+46·076	−0·03	+14 4 16·6	+20·044	−0·09
42	8	ι Ceti	4	0 9 14	2 18 30·6	45·870	+0·06	− 9 55 58·5	20·030	+0·04
69	..	α Phœnicis	2	0 16 21	4 5 30·9	44·659	−0·24	−43 23 35·8	19·994	−0·01
99	15	κ Cassiopeæ	4	0 21 44	5 26 0·2	49·555	....	+61 49 32·3	19·955	+0·02
123	17	ζ Cassiopeæ	4	0 25 54	6 28 30·7	48·989	+0·10	+52 47 39·4	19·918	0·00
134	30	ε Andromedæ	4	0 28 1	7 0 12·0	47·323	−0·10	+28 13 25·5	19·896	−0·20
136	31	δ Andromedæ	3	0 28 39	7 9 57·3	47·441	+0·35	+29 45 54·2	19·889	−0·09
139	18	α Cassiopeæ	3	0 29 14	7 18 35·7	49·700	−0·05	+55 26 17·6	19·883	−0·07
159	16	β Ceti	2·3	0 33 32	8 23 11·0	45·000	+0·21	−19 5 11·0	19·831	+0·07
182	34	ζ Andromedæ	4	0 36 46	9 11 28·0	47·382	−0·10	+23 10 36·8	19·788	−0·10
185	24	η Cassiopeæ	4	0 37 5	9 16 21·0	50·938	+1·78	+56 45 2·3	19·783	−0·72
194	35	ν Andromedæ	4	0 38 49	9 42 29·4	48·846	+0·80	+39 59 11·8	19·759	−0·10
226	27	γ Cassiopeæ	3	0 44 45	11 11 7·6	52·648	0·00	+59 37 51·4	19·664	+0·04
232	37	μ Andromedæ	4	0 45 51	11 25 29·7	49·049	+0·20	+37 24 41·0	19·648	+0·40
263	1	Polaris	2·3	0 52 25	13 6 19·5	193·942	....	+88 14 24·3	19·524	0·00
264	71	ε Piscium	4	0 52 34	13 8 37·8	46·555	−0·15	+ 6 48 37·5	19·521	+0·06
300	31	η Ceti	3·4	0 58 32	14 38 0·0	45·005	+0·28	−11 14 42·8	19·396	−0·10
301	43	β Andromedæ	2	0 58 34	14 38 33·7	45·501	+0·35	+34 33 23·5	19·395	−0·10
62	37	δ Cassiopeæ	3	1 12 51	18 12 48·8	56·517	+0·64	+59 11 26·2	19·041	−0·16
67	45	θ Ceti	3	1 14 1	18 30 25·8	44·979	−0·19	− 9 13 9·5	19·009	−0·17
98	99	η Piscium	4	1 20 48	20 12 2·4	47·777	+0·07	+14 18 37·3	18·811	−0·06
124	51	r 2Andromedæ	3·4	1 25 47	21 26 48·6	54·041	....	+47 36 33·4	18·658	....
163	52	τ Ceti	3·4	1 34 47	23 41 39·0	43·550	−1·86	−16 59 40·5	18·356	+0·84
184	45	ε Cassiopeæ	3·4	1 40 10	25 2 24·6	62·433	−0·05	+62 40 36·8	18·161	−0·15
192	55	ζ Ceti	3	1 41 35	25 23 51·0	44·289	−0·16	+11 19 42·5	18·108	+0·11
193	2	α Trianguli	3·4	1 41 42	25 25 43·5	50·704	+0·04	−18 35 52·3	18·104	−0·36
202	6	β Arietis	3	1 43 36	25 54 12·6	49·168	+0·14	+19 49 29·0	18·032	−0·23
215	50	f Cassiopeæ	4·5	1 46 39	26 39 40·5	72·806	....	+71 26 37·3	17·914	....
236	57	γ Andromedæ	3·4	1 51 41	27 55 11·5	54·276	+0·26	+41 21 45·5	17·712	−0·11
238	113	α Piscium	5	1 51 42	27 55 39·4	46·305	−0·06	+ 1 47 32·8	17·711	−0·09
253	13	α Arietis	3	1 55 55	28 58 54·0	50·036	+0·20	+22 30 36·5	17·536	−0·20
260	4	β Trianguli	4	1 57 41	29 25 21·0	52·662	+0·33	+34 2 4·6	17·460	−0·07
144	82	δ Ceti	4	2 29 14	37 18 39·0	45·896	+0·07	− 0 32 31·0	15·994	−0· 5
149	83	ε Ceti	4·5	2 29 53	37 28 27·4	43·257	+0·15	−12 43 41·0	15·909	−0·44
150	13	θ Persei	4	2 30 37	37 39 12·0	59·790	+0·67	+48 22 20·7	15·871	−0·10
157	35	Arietis	4	2 31 45	37 56 10·8	52·249	+0·15	+26 50 50·0	15·810	−0·05
161	86	γ Ceti	3	2 32 57	38 14 14·4	49·528	−0·35	+ 2 23 6·5	15·745	−0·20
167	87	μ Ceti	4	2 34 8	38 32 10·5	48·048	+0·06	+ 9 15 44·0	15·702	+0·20
170	89	π Ceti	4	2 34 36	38 39 7·2	42·724	−0·03	−14 42 41·0	15·655	+0·11
178	39	Arietis	4	2 36 2	39 0 31·5	52·836	+0·23	+28 24 26·0	15·574	−0·13
186	41	Arietis	3	2 38 15	39 33 40·5	52·357	+0·20	+26 25 38·0	15·454	−0·13
219	3	η Eridani	3	2 46 39	41 39 59·7	43·733	+0·16	− 9 42 4·0	14·975	−0·30

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		Annual proper motion.
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	
				h m ″	° ′ ″	″	″	° ′ ″	″	″
234	23	γ Persei	3·4	2 50 24	42 35 56·1	+63·829	−0·15	+52 42 39·0	+14·756	−0·05
238	..	θ Eridani præc.	2·3	2 50 41	42 40 15·0	34·135	−0·10	−41 6 44·5	14·739	0·00
244	92	α Ceti	2	2 51 49	42 57 34·3	46·798	−0·08	+ 3 17 48·8	14·670	−0·15
246	25	ρ Persei	3·4	2 52 24	43 6 5·4	56·734	+0·26	+38 3 16·4	14·636	−0·11
249	11	Eridani	4	2 53 34	43 23 39·9	59·760	....	−24 24 54·4	14·566	....
253	..	Persei	4	2 54 42	43 40 36·6	61·844	....	+48 50 10·0	14·498	−0·00
13	12	Eridani	3·4	3 3 24	45 53 44·4	37·774	+0·64	−29 46 56·0	13·951	0·82
22	13	ζ Eridani	4	3 6 7	46 31 53·7	43·558	−0·11	− 9 34 14·5	13·791	−0·02
41	33	α Persei	2·3	3 10 7	47 31 42·4	63·103	−0·21	+49 8 13·0	13·535	−0·02
43	16	Eridani	3·4	3 10 37	47 39 21·9	39·875	....	+22 29 39·0	13·502	....
47	..	e Eridani	4	3 11 56	47 59 13·5	31·711	+4·30	−43 50 34·0	13·417	+0·83
51	..	Camel. 2 Hevel.	4	3 12 59	48 14 51·0	71·125	....	+59 13 40·0	13·349	....
55	1	ο Tauri	4	3 14 4	48 30 58·2	48·206	−0·24	+ 8 16 56·1	13·278	−0·13
63	2	ξ Tauri	4	3 16 20	49 5 10·8	48·418	−0·08	+ 9 1 33·5	13·128	0·00
89	18	ε Eridani	4	3 23 30	50 52 43·9	43·229	....	−10 8 37·0	12·648	....
95	19	Eridani	4	3 24 57	51 14 24·0	39·597	....	−22 18 42·5	12·687	....
106	39	δ Persei	3·4	3 38 45	52 11 12·6	63·073	+0·04	+49 8 2·0	12·290	−0·10
123	..	Persei 31 Hevel.	4	3 31 48	52 57 12·4	55·870	....	+31 38 33·3	12·077	....
134	23	δ Eridani	3·4	3 33 40	53 25 9·3	43·063	−0·23	−10 26 56·2	11·947	+0·60
152	25	η Tauri	3	3 35 37	53 54 16·3	53·046	−0·04	+20 28 31·0	11·810	−0·09
183	..	f Eridani seq. ..	4	3 41 13	55 18 18·7	33·024	....	−38 14 17·0	11·410	....
185	44	ζ Persei	3·4	3 41 35	55 23 50·4	56·035	−0·14	+31 16 37·5	11·383	0·00
196	45	ε Persei	3·4	3 44 29	56 7 7·3	59·722	....	+39 25 5·8	11·175	....
210	34	γ Eridani	2·3	3 48 42	57 10 33·6	41·784	+0·12	−14 5 12·0	10·866	−0·11
218	35	λ Tauri	4	3 49 36	57 24 10·3	49·575	+0·16	+11 54 54·3	10·799	0·00
39	54	γ Tauri	3·4	4 8 25	62 6 22·8	50·802	+0·14	+15 7 57·4	9·378	−0·09
50	4	i Eridani	3·4	4 10 19	62 34 54·6	33·876	....	−34 17 40·5	9·230	....
57	61	δ 1 Tauri	4	4 11 25	62 51 13·2	51·486	+0·10	+17 3 43·6	9·146	−0·04
87	74	ε Tauri	4	4 16 57	64 14 17·1	52·131	+0·08	+18 43 27·5	8·713	−0·13
125	87	Aldebaran	1	4 24 27	66 6 50·4	51·299	+0·04	+16 5 42·0	8·117	−0·21
133	48	ν Eridani	4	4 26 20	66 34 59·4	44·799	−0·02	− 3 46 19·6	7·967	−0·05
144	52	υ 2 Eridani	3	4 27 46	66 56 43·0	34·936	+0·05	−30 58 50·0	7·850	0·00
150	53	Eridani	4	4 29 1	67 15 22·9	41·160	−0·32	−14 42 15·1	7·750	−0·13
166	54	Eridani	4	4 31 42	67 55 28·2	39·227	+0·12	−20 3 52·2	7·534	−0·06
201	1	Orionis........	4	4 38 59	69 44 54·3	48·187	+0·54	+ 6 35 59·5	6·939	−0·07
213	3	Orionis........	4	4 40 34	70 8 29·5	47·744	+0·22	+ 5 15 6·2	6·809	+0·44
252	8	ζ Orionis	4	4 43 50	70 57 31·8	46·707	....	+ 2 6 7·0	6·540	....
235	3	ι Aurigæ	4	4 43 59	70 59 46·8	58·242	0·00	+32 50 6·0	6·527	+0·02
256	7	ε Aurigæ	4	4 47 39	71 54 37·5	64·101	....	+43 30 39·5	6·224	....
262	8	ζ Aurigæ	4	4 48 32	72 7 51·0	62·460	....	+40 46 3·1	6·151	....
283	10	η Aurigæ	4	4 52 31	73 7 39·6	62·656	+0·10	+40 56 51·5	5·818	−012
303	2	ε Leporis	3·4	4 56 59	74 14 54·0	37·961	....	−22 38 55·0	5·451	....
312	67	β Eridani	3	4 58 1	74 30 20·8	44·200	−0·30	− 5 21 22·0	5·355	−0·11
323	69	λ Eridani	4	4 59 34	74 53 40·5	42·939	....	− 9 1 16·5	5·224	....
6	13	Capella	1	5 1 56	75 29 0·6	65·951	+0·12	+45 46 37·5	5·025	−0·44
18	19	Rigel	1	5 4 55	76 13 57·4	43·121	−0·05	− 8 26 36·4	4·770	−0·02
40	20	τ Orionis	4	5 7 54	76 58 30·0	43·589	....	− 7 4 18·0	4·518	....
72	112	β Tauri........	2	5 13 39	76 24 51·9	56·642	−0·03	+28 25 25·5	4·026	0·17

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		Annual proper motion.
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	
				h m ″	° ′ ″	″	″	° ′ ″	″	″
80	24	γ Orionis	2	5 14 24	78 36 8·2	+48·131	−0·17	+ 6 9 21·3	+ 3·961	−0·03
81	28	η Orionis	4	5 14 25	78 36 24·0	45·112	....	− 2 35 33·9	3·960	....
113	9	β Leporis	4	5 19 41	79 55 11·8	38·464	−0·03	−20 55 42·5	3·508	−0·07
126	34	δ Orionis	2	5 21 47	80 26 53·7	45·853	−0·12	− 0 27 32·7	3·326	+ 0·05
130	36	υ Orionis	4	5 22 15	80 33 57·7	43·422	....	− 7 27 38·0	3·286	....
139	11	α Leporis	3	5 23 54	80 58 39·7	39·588	....	−17 58 33·3	3·149	....
140	..	ε Columbæ	4	5 24 6	81 1 43·5	31·812	....	−35 37 39·0	3·125	....
141	39	λ Orionis	4	5 24 7	81 1 54·0	49·427	....	+ 9 47 17·2	3·125	....
151	44	ι Orionis	3·4	5 25 39	81 24 52·0	43·910	....	− 6 3 5·6	2·998	....
152	123	ζ Tauri........	3·4	5 25 41	81 25 22·8	53·623	−0·03	+ 1 0 25·0	2·996	−0·04
160	46	ε Orionis	2·3	5 26 4	81 31 2·1	45·547	−0·14	− 1 20 29·4	2·957	−0·06
172	48	σ Orionis	4	5 28 42	82 10 37·5	45·066	−0·21	− 2 43 36·0	2·728	−0·03
188	50	ζ Orionis	3	5 30 40	82 40 4·0	45·296	−0·12	− 2 3 36·6	2·558	−0·08
196	..	α Columbæ	2	5 32 24	83 6 7·2	32·492	−0·20	−34 11 21·0	2·407	−0·10
219	13	γ Leporis	4	5 36 7	84 1 53·1	37·744	−0·42	−22 31 24·0	2·084	−0·40
234	53	κ Orionis	3	5 38 16	84 34 4·9	42·582	−0·07	− 9 45 4·0	1·808	0·00
262	33	δ Aurigæ	3·4	5 43 4	85 45 57·0	73·780	....	+54 14 57·1	1·480	−0·42
267	..	β Columbæ	3	5 43 54	85 58 44·7	31·561	....	−35 51 4·0	1·406	....
268	58	α Orionis	1	5 44 20	86 5 12·5	48·594	−0·03	+ 7 21 25·0	1·368	+ 0·03
269	34	β Aurigæ	2	5 44 51	86 12 52·9	65·950	−0·03	+44 54 35·3	1·323	+ 0·06
277	37	θ Aurigæ	4	5 46 5	86 31 14·5	61·190	+0·11	+37 11 0·0	1·217	−0·04
281	16	η Leporis	4	5 47 17	86 49 26·7	40·942	....	−14 12 50·5	1·111	....
297	..	γ Columbæ	4	5 50 26	87 36 43·5	31·823	....	−35 18 47·0	0·835	....
18	44	κ Aurigæ	4	6 2 37	90 39 22·9	57·378	−0·30	+29 33 22·3	− 0·230	−0·35
74	13	μ Gemin.	3	6 10 51	92 42 40·9	54·347	+0·13	+22 36 8·5	0·882	−0·16
81	1	ζ Can. maj.	3	6 12 38	93 9 34·5	34·465	−0·20	−29 59 4·0	1·105	−0·25
92	2	β Can. maj.	2	6 13 53	93 28 23·1	39·561	−0·04	−17 52 5·0	1·214	−0·06
95	3	λ Can. maj.	4	6 14 48	93 42 3·0	32·850	−0·34	−33 20 38·5	1·294	−0·03
169	24	γ Gemin.	3	6 26 9	96 32 16·9	51·932	+0·05	+16 33 24·6	2·282	−0·08
204	27	ε Gemin.	3	6 31 37	97 54 16·3	55·403	−0·03	+25 18 54·0	2·757	0·00
205	..	υ Argo navis ..	3	6 31 38	97 54 40·5	27·041	−0·14	−43 1 41·0	2·759	−0·20
217	31	ξ 2 Gemin.	4	6 34 4	98 30 55·2	50·624	....	+13 5 56·5	2·968	....
227	9	Sirius	1	6 36 20	99 4 59·2	40·166	−0·51	−16 27 6·2	3·165	−1·14
248	34	θ Gemin.	4	6 39 36	99 53 52·5	59·425	−0·16	+34 11 12·0	3·446	....
259	13	κ 2 Can. maj. ..	4	6 42 22	100 35 34·5	33·563	−0·10	−32 17 9·5	3·685	+ 0·04
279	16	ο 1 Canis	4	6 45 49	101 27 30·7	37·298	−0·10	−23 56 37·0	3·892	0·00
304	21	ε Canis........	2	6 50 45	102 41 28·9	35·301	−0·05	−28 42 30·3	4·404	+ 0·07
312	43	ζ Gemin.	4	6 52 14	103 3 33·9	53·449	0·00	+20 51 10·0	4·530	−0·05
320	22	σ Canis maj.....	3·4	6 53 45	103 26 18·1	35·794	−0·04	−27 39 27·4	4·659	0·00
328	24	ο 2 Canis	4	6 54 40	103 49 5·4	37·522	−0·01	−23 33 2·0	4·737	+ 0·02
325	23	γ Canis........	4	6 54 42	103 40 36·3	40·666	0·00	−15 20 51·0	4·740	−0·04
57	45	δ Gemin.	3·4	7 8 9	107 2 27·6	53·787	−0·06	+22 20 14·3	5·875	−0·05
68	..	π Argo	3	7 10 5	107 31 1·5	31·738	....	−36 44 46·7	6·033	....
90	60	ι Gemin.	4	7 13 17	108 19 18·0	56·208	−0·27	+28 10 56·4	6·301	+ 0·08
104	31	η Canis min.....	2·3	7 16 10	109 2 42·6	35·542	−0·12	−28 55 18·0	6·544	0·00
106	3	β Canis min.....	3	7 16 17	109 4 24·4	48·904	−0·04	+ 8 40 52·0	6·550	+ 0·04
127	66	Castor præc....	3·4	7 21 49	110 27 7·2	57·392	−0·16	+32 18 45·0	7·006	−0·10
128		seq.	3	7 21 49	110 27 18·0					

Catalogue—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
136	..	σ Argo	4	7 22 53	110 43 10·5	+28·587	−0·43	−42 54 10·5	− 7·092	+0·30
168	10	Procyon	2	7 28 49	112 12 21·7	47·873	−0·71	+ 5 43 38·5	7·576	−0·98
184	77	κ Gemin.	4	7 33 21	113 5 15·0	54·556	−0·16	+24 51 52·7	7·861	−0·02
191	78	β Gemin.	2	7 33 3	113 15 45·6	56·005	−0·72	+28 29 46·8	7·917	−0·11
214	..	C Argo præc.	4	7 33 8	114 31 1·5	32·021	....	−37 29 27·0	8·318	....
230	7	ξ Navis	4	7 40 53	115 13 15·0	37·798	−0·07	−24 22 0·0	8·542	0·00
306	..	ζ Argo	2·3	7 56 33	119 8 19·5	31·605	....	−39 26 46·8	9·761	....
320	15	Navis	3·4	7 59 1	119 45 20·8	38·358	−0·21	−23 44 8·7	9·949	+0·09
28	17	β Cancri	4	8 5 39	121 24 49·6	48·964	−0·13	+ 9 47 30·0	10·448	−0·02
57	1	ο Ursæ maj.	4	8 13 32	123 22 45·0	76·675	....	+61 22 14·0	11·029	....
114	4	δ Hydræ	4	8 27 3	126 45 49·5	47·811	....	+ 6 23 31·7	11·998	....
164	11	ε Hydræ	4	8 36 10	129 2 34·5	47·962	....	+ 7 8 34·0	12·622	....
210	16	ζ Hydræ	4	8 44 49	131 12 13·8	47·783	+0·14	+ 6 41 54·1	13·205	−0·48
212	9	ι Ursæ maj.	3·4	8 45 27	131 21 37·5	63·208	−1·05	+48 48 57·5	13·246	−0·32
222	65	α 2 Cancri	4	8 47 32	131 52 59·4	49·354	....	+12 37 22·0	13·383	....
1	..	λ Argo	2·3	9 0 38	135 9 45·7	33·001	−0·04	−42 37 48·0	14·215	+0·06
29	38	Lyncis	4	9 6 21	136 35 16·0	56·634	+0·28	+37 38 22·8	14·562	−0·20
82	23	h Ursæ	4	9 15 35	138 53 52·5	72·941	−0·09	+63 55 30·0	15·105	−0·04
89	30	α Hydræ	2	9 17 45	139 26 20·2	44·226	−0·15	− 7 47 54·5	15·229	−0·05
98	25	θ Ursæ	3	9 19 23	139 50 52·8	62·902	−0·80	+52 34 45·4	15·322	−0·60
151	14	ο Leonis	4	9 30 27	142 36 53·7	48·331	−0·22	+10 47 43·4	15·928	−0 04
164	17	ε Leonis	3	9 34 28	143 37 2·2	51·477	−0·03	+24 41 18·0	16·138	+0·02
194	24	μ Leonis	4	9 41 21	145 20 21·3	51·806	−0·39	+26 56 31·4	16·489	−0·07
245	30	η Leonis	3·4	9 56 24	149 6 7·5	49·303	−0·05	+17 43 55·0	17·201	−0·02
251	32	Regulus	1	9 57 42	149 25 33·4	48·354	—0·28	+12 56 22·0	17·259	−0·01
20	53	λ Ursæ maj.	3·4	10 4 58	151 14 38·4	55·294	0·00	+43 54 24·5	17·573	−0·05
29	..	q Argo	4	10 6 21	151 35 25·0	37·601	....	−41 8 0·0	17·632	....
38	41	γ Leonis	2	10 8 55	152 13 50·7	49·568	+0·35	+20 50 52·2	17·737	−0·20
45	34	μ Ursæ	3	10 10 21	152 35 22·3	54·468	−0·29	+42 30 0·0	17·795	+0·03
74	42	μ Hydræ	4	10 16 25	154 6 18·0	43·530	−0·21	−15 49 9·2	18·033	−0·06
192	47	ρ Leonis	4	10 22 16	155 34 0·9	47·513	−0·14	+10 19 54·5	18·250	0·00
121	37	Leonis min.	4	10 27 25	156 51 24·0	51·130	−0·03	+33 0 39·3	18·432	+0·01
167	4	ν Hydræ	4	10 39 46	159 56 26·7	44·150	+0·06	−15 8 59·8	18·830	+0·20
207	48	β Ursæ	2	10 49 40	162 25 9·0	55·498	+0·12	+57 27 4·5	19·109	+0·06
209	7	α Crateris	4	10 50 2	162 30 33·0	44·142	−0·59	−17 14 11·2	19·118	+0·06
217	50	α Ursæ	2	10 51 15	162 48 52·2	57·549	−0·24	+62 49 38·4	19·151	0·00
253	52	ψ Ursæ	3·4	10 58 21	164 35 30·0	51·447	−0·08	+45 34 52·8	19·325	−0·07
6	11	β Crateris	3·4	11 1 50	165 27 34·5	44·005	+0·05	−21 44 9·0	19·404	−0·11
10	68	δ Leonis	3	11 3 27	165 51 43·5	47·952	+0·18	+21 37 4·0	19·438	−0·11
13	70	θ Leonis	3	11 3 43	165 55 54·1	47·457	−0·16	+16 31 15·0	19·445	−0·06
28	53	ξ Ursæ maj.	4	11 7 28	166 52 10·0	48·930	−0·52	+32 39 7·5	19·522	−0·64
29	54	ν Ursæ maj.	4	11 7 39	166 54 36·1	49·095	−0·03	+34 11 2·0	19·525	+0·05
38	12	δ Hydræ	3·4	11 9 21	167 20 15·0	44·941	−0·19	−13 41 48·6	19·558	−0·06
42	77	σ Leonis	4	11 10 49	167 42 14·4	46·545	−0·16	+ 7 7 24·5	19·586	−0·05
54	78	ι Leonis	4	11 13 29	168 22 18·0	46·843	−0·00	+11 37 47·5	19·634	−0·03
62	15	γ Hydræ	4	11 14 54	168 43 29·5	44·844	−0·29	−16 35 12·5	19·658	+0·02
76	84	τ Leonis	4	11 17 39	169 24 42·1	46·266	−0·03	+ 3 57 24·5	19·704	−0·02
86	1	λ Draconis	3·4	11 19 22	169 50 27·0	55·960	−0·15	+70 25 59·7	19·731	−0·09

Catalogue—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
103	19	ξ Crateris......	4	11 23 11	170 47 46·5	+44·104	−0·42	−30 45 8·5	−19·788	− 0·11
114	21	θ Hydræ	4	11 26 32	171 38 6·3	45·565	−0·14	− 8 41 47·0	19·832	+ 0·05
150	27	ζ Crateris......	4	11 34 38	173 39 35·1	45·325	−0·06	−17 14 21·5	19·923	− 0·09
152	63	χ Ursæ maj.....	4	11 35 25	173 51 22·5	48·470	−0.36	+48 53 16·6	19·931	− 0·07
159	93	Leonis	4	11 37 39	174 24 45·9	46·774	....	+21 19 49·0	19·950	....
163	94	β Leonis	3	11 38 50	174 42 42·0	46·530	−0·53	+15 41 24·7	19·961	− 0·08
166	5	β Virginis	3·4	11 40 16	175 4 7·8	46·098	+0·76	+ 2 53 30·0	19·972	− 0·30
172	28	β Hydræ	4	11 42 50	175 42 35·5	45·045	−0·20	−32 47 44·0	19·990	− 0·13
174	64	γ Ursæ maj....	2	11 43 14	175 48 37·2	48·088	+0·06	+54 48 23·0	19·992	− 0·03
241	1	α Corvi	4	11 58 7	179 31 50·1	45·940	+0·06	−23 36 44·7	20·045	+ 0·16
248	2	ε Corvi	4	11 59 51	179 57 52·5	46·007	....	−21 30 25·0	20·045	− 0·09
22	69	δ Ursæ maj.....	3	12 5 27	181 21 46·0	45·244	−0·06	+58 8 40·2	20·140	− 0·08
24	4	γ Corvi........	3	12 5 32	181 23 3·3	46·154	−0·32	−16 25 47·0	20·040	+ 0·06
44	15	η Virginis	3·4	12 9 40	182 25 10·2	46·005	−0·05	+ 0 26 47·0	20·028	− 0·04
92	..	μ Centauri	4	12 17 47	184 26 53·4	47·222	....	−37 55 46·0	19·985	....
101	7	δ Corvi........	3·4	12 19 32	184 52 59·4	46·481	+0·07	−15 23 58·6	19·973	− 0·20
123	9	β Corvi........	2·3	12 23 54	185 58 35·1	46·867	....	−22 17 19·5	19·937	....
129	5	κ Draconis	3	12 24 52	186 12 49·0	39·748	−0·50	+70 53 32·7	19·928	− 0·18
157	29	γ 1 Virginis	3	12 31 32	187 52 57·0	46·028	−0·72	− 0 20 59·0	19·856	+ 0·10
220	77	ε Ursæ maj.....	3	12 45 11	191 17 43·2	39·954	+0·24	+57 2 52·5	19·657	− 0·10
223	43	δ Virginis	3·4	12 45 31	191 22 57·9	45·701	−0·65	+ 4 29 16·2	19·651	− 0·02
226	12	Can. venatic ..	2·3	12 46 38	191 39 42·3	42·683	−0·34	+39 24 5·0	19·632	+ 0·04
249	47	ε Virginis	3·4	12 52 13	193 3 17·4	45·046	−0·37	+12 2 16·7	19·528	+ 0·07
273	41	Coma Beren ..	4	12 57 34	194 23 29·2	43·283	....	+28 42 7·0	19·417	....
2	42	Coma Beren ..	4	13 0 15	195 3 46·0	44·259	−0·45	+18 35 27·7	19·357	+ 0·15
45	2	γ Hydræ	3·4	13 8 4	197 1 9·0	48·395	+0·14	−22 6 41·0	19·168	− 0·10
53	..	ι Centauri	3	13 9 24	197 21 2·5	50·299	....	−35 39 8·0	19·142	....
75	67	Spica	1	13 14 40	198 40 6·3	47·156	−0·09	−10 6 44·0	18·991	− 0·03
78	79	ζ Ursæ maj. pr.	3	13 15 51	198 57 27·0	36·366	−0·08	+55 58 25·8	18·958	− 0·01
85	80	G Ursæ maj. ..	4	13 17 10	199 17 46·5	36·178	+0·13	+56 2 1·0	18·919	− 0·18
99	..	D Centauri	4	13 19 30	199 52 33·0	51·407	....	−38 22 5·5	18·851	....
128	79	ζ Virginis	4	13 24 30	201 7 41·1	45·957	−0·49	+ 0 25 55·6	18·699	+ 0·08
197	..	ν Centauri	4	13 37 34	204 23 34·5	53·124	....	−40 40 1·0	18·257	....
198	..	μ Centauri	4	13 37 38	204 24 32·1	53·332	....	−41 28 11·5	18·254	....
209	84	η Ursæ maj.....	3	13 39 39	204 54 33·7	35·836	−0·50	+50 18 59·2	18·181	0·00
210	5	υ Bootis	4	13 39 49	204 57 24·6	43·458	....	+16 47 46·2	18·174	....
240	8	η Bootis	3	13 45 9	206 17 22·5	42·884	−0·10	+19 24 22·0	17·972	− 0·40
293	5	θ Centauri	2	13 54 58	208 41 31·8	52·865	−0·63	−35 22 41·0	17·576	− 0 40
312	11	α Draconis	3·4	13 58 59	209 44 36·6	24·353	−0·15	+65 20 7·7	17·405	+ 0·09
14	98	κ Virginis......	4	14 2 14	210 33 40·8	47·687	−0·18	− 9 20 8·0	17·261	+ 0·13
28	99	ι Virginis......	4	14 5 32	211 23 7·8	46·923	−0·01	− 5 2 20·0	17·113	− 0·47
32	16	Arcturus	1	14 6 32	211 38 6·6	42·137	−1·17	+20 13 48·3	17·067	− 0·96
37	100	λ Virginis	4	14 8 18	212 4 36·7	48·360	−0·08	−12 26 33·0	16·986	+ 0·07
41	19	λ Bootis	4	14 8 47	212 11 31·0	34·553	−0·55	+47 0 44·0	16·964	+ 0·27
42	21	ι Bootis	4	14 9 4	212 16 4·8	32·167	−0·34	+52 17 41·0	16·950	− 0·03
92	23	θ Bootis	3	14 18 23	214 35 41·4	31·028	−0·80	+52 46 48·0	16·501	− 0·54
109	..	η Centauri	3	14 22 52	215 43 4·0	56·280	....	−41 16 0·0	16·275	....
112	25	ρ Bootis	4	14 23 12	215 48 7·0	38·894	−0·07	+31 15 20·0	16·258	+ 0·09

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.								Declination.				
Piazzi.	Flamstead			In time.			In degrees.			Annual precession in AR	Annual proper motion in AR	In degrees.			Annual precession.	Annual proper motion.
				h	m	″	°	′	″	″	″	°	′	″	″	″
136	5	A Ursæ min.	4	14	28	9	217	2	4·5	− 4·609	....	+76	35	6·5	−16·006	....
147	29	π Bootis	4	14	31	20	217	49	52·2	+42·186	....	+17	17	0·4	15·833	....
152	30	ζ Bootis	3·4	14	31	36	217	53	59·4	42·805	0·00	+14	35	40·0	15·818	+0·03
174	109	Virginis	4	14	36	8	219	2	10·0	45·409	....	+ 2	44	40·4	15·570	....
175	26	ε Bootis	3	14	36	5	219	3	43·2	39·316	+0·23	+27	55	28·5	15·565	−0·04
187	9	α 2 Libræ	3	14	39	50	219	57	34·0	49·510	−0·20	−15	12	4·0	15·365	−0·08
197	37	ξ Bootis	3	14	42	10	220	32	28·0	41·285	−0·23	+19	56	17·6	15·234	−0·18
211	..	β Lupi	3	14	45	30	221	22	28·5	58·074	....	−42	18	58·0	15·042	−0·21
216	7	χ Centauri	3	14	46	13	221	33	13·0	57·688	....	−41	17	27·0	15·001	....
240	7	β Ursæ min.	3	14	51	28	222	51	40·0	− 4·779	−0·30	+74	58	20·4	14·694	−0·18
251	20	Libræ	3·4	14	52	23	223	5	55·5	+52·249	−0·14	−24	29	6·5	14·636	+0·08
259	42	β Bootis	3·4	14	54	25	223	36	8·1	33·915	−0·20	+41	11	11·0	14·517	−0·02
26	27	β Libræ	2·3	15	6	15	226	33	55·0	48·222	−0·30	− 8	38	4·7	13·782	+0·07
29	49	δ Bootis	3·4	15	7	26	226	51	32·4	36·119	+0·10	+34	4	8·0	13·707	−0·07
31	..	δ Lupi	4	15	8	18	227	4	31·8	58·288	....	−39	54	36·0	13·652	....
34	..	φ 1 Lupi	4	15	9	9	227	17	24·0	56·517	....	−35	31	27·0	13·597	....
73	51	μ Bootis	4	15	16	56	229	13	59·1	34·113	−0·30	+38	5	11·2	13·090	+0·16
86	3	β Coronæ bor.	4	15	19	35	239	53	45·0	37·229	....	+29	48	13·0	12·913	+0·15
92	12	ι Draconis	3	15	20	30	230	7	25·0	29·716	....	+59	40	14·0	12·852	+0·15
95	13	γ 2 Ursæ min.	4	15	21	9	230	17	10·8	− 3·031	....	+72	32	44·0	12·808	....
98	..	γ Lupi	4	15	21	51	230	28	3·0	59·206	....	−40	28	49·0	12·760	....
106	37	Libræ	4	15	23	15	230	48	57·0	48·575	+0·33	− 9	22	5·5	12·606	−0·30
111	38	γ Libræ	4	15	24	21	231	5	25·5	+49·933	+0·14	−14	6	38·0	12·590	+0·03
117	13	δ Serpentis	3	15	25	15	231	18	48·0	42·908	−0·06	+11	13	2·0	12·530	+0·07
121	5	α Coronæ bor.	2	15	26	15	231	33	17·7	37·875	−0·10	+27	23	48·0	12·463	−0·10
152	7	ζ Coronæ bor.	4	15	31	51	232	57	39·0	33·824	....	+37	17	37·0	12·075	−0·40
163	24	α Serpentis	2·3	15	34	25	233	36	22·2	44·012	−0·10	+ 7	3	53·7	11·894	+0·05
174	5	λ Lupi	4	15	38	17	234	34	19·5	56·621	....	−33	0	16·2	11·620	....
178	34	μ Serpentis	4	15	39	11	234	47	56·1	46·814	−0·04	− 2	48	24·1	11·555	0·00
182	35	κ Serpentis	4	15	39	44	234	56	5·4	40·436	−0·06	+18	46	8·0	11·516	−0·10
187	37	ε Serpentis	3·4	15	40	51	235	12	48·6	44·545	+0·12	+ 5	5	24·4	11·437	−0·01
206	6	π Scorpii	3·4	15	46	46	236	41	42·6	54·012	+0·10	−25	31	30·8	11·005	−0·20
217	..	η Lupi	4	15	46	54	236	43	42·0	59·015	....	−37	48	35·0	10·997	....
219	41	γ Serpentis	3·4	15	47	13	236	48	20·1	41·099	+0·35	+16	19	27·8	10·985	−1·31
225	7	δ Scorpii	3	15	48	32	237	7	59·4	52·827	0·00	−22	2	20·5	10·879	−0·10
248	..	θ Lupi	4	15	53	29	238	22	26·4	58·523	....	−36	14	37·0	10·511	....
251	8	β Scorp. præc.	2	15	53	50	238	27	27·6	51·976	+0·03	−19	14	42·0	10·487	−0·10
277	13	θ Draconis	3·4	15	58	10	239	32	19·0	18·136	....	+56	6	12·6	10·162	+0·30
4	14	ν Scorpionis	4	16	0	23	240	5	54·3	51·971	−0·06	−18	55	42·3	9·993	−0·08
21	1	δ Ophiuchi	3·4	10	3	52	240	58	7·5	46·981	−0·15	− 3	10	3·0	9·728	−0·08
41	2	ε Ophiuchi	3·4	16	7	45	241	56	15·6	47·308	+0·15	− 4	11	33·5	9·430	+0·03
50	20	σ Scorpii	4	16	9	3	242	15	49·8	54·322	−0·06	−25	5	55·8	9·329	−0·04
66	20	γ Herculis	3·4	16	13	6	243	16	31·0	39·624	−0·04	+19	37	59·5	9·015	+0·09
73	22	τ Herculis	4	16	13	44	243	26	2·1	26·923	....	+46	47	39·0	8·965	....
84	21	Antares	1	16	17	10	244	17	32·2	54·810	−0·05	−25	58	26·0	8·695	−0·10
100	10	λ Ophiuchi	4	16	20	50	245	12	28·5	45·238	−0·06	+ 2	26	1·3	8·406	−0·06
102	14	η Draconis	3·4	16	21	18	245	19	27·0	11·797	....	+61	58	11·0	8·369	−0·05
103	27	β Herculis	2·3	16	21	37	245	24	20·7	38·671	−0·22	+21	56	6·5	8·343	−0·04

Catalogue—*continued*.

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
113	23	τ Scorpii	3·4	16 23 27	245 51 50·4	+55·651	−0·04	−27 47 10·5	− 8·197	− 0·17
123	13	ζ Ophiuchi	3	16 26 9	246 32 22·7	49·303	+0·16	−10 8 56·3	7·980	+ 0·10
132	35	σ Herculis	4	16 27 40	246 54 52·0	28·901	−0·30	+42 51 25·5	7·860	+ 0·03
165	40	ζ Herculis	3	16 33 45	248 36 10·5	34·374	−0·70	+31 58 26·5	7·368	+ 0·47
173	44	η Herculis	3	16 36 3	249 0 34·0	30·687	−0·24	+39 18 38·5	7·181	− 0·09
184	26	ε Scorpii	3	16 37 14	249 18 32·2	58·620	−0·65	−33 54 53·0	7·083	− 0·18
189	1	μ 1 Scorpii	4	16 33 21	249 35 16·0	60·525	. . . .	−37 41 15·0	6·991	. . . .
193	2	μ 2 Scorpii	3·4	16 38 29	249 42 17·7	60·521	. . . .	−37 39 35·4	6·953	. . . .
233	29	ι Ophiuchi	4	16 44 33	251 8 16·0	42·489	. . . .	+10 30 25·0	6·481	0·00
252	27	κ Ophiuchi	3·4	16 48 12	252 3 7·0	42·752	−0·30	+ 9 41 51·2	6·177	+ 0·08
272	58	ε Herculis	3·4	16 52 38	253 9 34·5	34·378	−0·18	+31 13 48·4	5·807	+ 0·05
302	. .	η Scorpii	4	16 57 51	254 27 50·4	63·994	. . . .	−42 57 27·0	5·369	. . . .
306	35	η Ophiuchi	2·3	16 58 55	254 43 48·6	51·362	−0·03	−15 27 46·0	5·279	+ 0·09
4	21	μ Draconis	4	17 1 12	255 18 0·0	18·587	−0·20	+54 44 19·2	5·087	+ 0·16
29	64	α Herculis	3·4	17 5 32	256 22 57·1	40·926	−0·11	+14 37 47·7	4·720	+ 0·12
35	65	δ Herculis	3·4	17 6 49	256 42 13·2	36·879	−0·20	+25 5 7·4	4·610	− 0·14
36	22	ε Ursæ min. . .	4	17 6 58	256 43 55·0	−99·116	−0·82	+82 20 35·4	4·601	+ 0·01
39	67	π Herculis	3·4	17 8 5	257 1 16·5	+31·268	. . . .	+37 2 34·4	4·502	. . . .
42	22	ζ Draconis	3	17 8 14	257 3 28·0	2·211	−0·32	+65 57 42·3	4·499	+ 0·04
53	42	θ Ophiuchi	3	17 9 44	257 26 5·4	55·045	−0·08	+24 47 4·0	4·361	− 0·07
105	75	ρ Herculis	4	17 16 48	259 11 44·1	30·990	. . . .	+37 20 21·3	3·758	. . . .
106	34	υ Scorpii	3·4	17 17 10	259 17 46·0	60·918	+0·06	−37 7 7·4	3·723	0·00
121	35	λ Scorpii	3	17 20 2	260 0 39·6	60·856	−0·08	−36 56 26·6	3·477	− 0·02
153	55	α Ophiuchi	2	17 25 39	261 24 48·6	41·538	+0·09	+12 43 3·0	2·993	+ 0·18
155	23	β Draconis	2	17 25 55	261 28 45·6	20·217	−0·27	+52 27 18·7	2·971	0·00
157	55	ξ Serpentis	4	17 26 8	261 32 7·0	51·420	−0·17	−15 15 29·5	2·951	− 0·15
174	. .	κ Scorpii	3	17 28 40	262 9 58·8	62·041	. . . .	−38 54 34·0	2·732	. . . .
209	60	β Ophiuchi	3	17 33 36	263 23 55·5	44·387	−0·17	+ 4 39 48·1	2·304	+ 0·22
211	85	ι Herculis	4	17 33 49	263 27 12·0	25·302	. . . .	+46 7 11·5	2·285	− 0·02
229	. .	γ Telescop.	4	17 36 15	264 3 46·5	61·015	. . . .	−36 57 42·0	2·073	. . . .
239	62	γ Ophiuchi	4	17 37 52	264 28 1·5	45·037	+0·02	+ 2 47 43·0	1·933	− 0·10
244	86	μ Herculis	4	17 38 38	264 39 28·5	35·468	−0·29	+27 50 48·8	1·866	− 0 84
303	64	ν Ophiuchi	4	17 48 1	267 0 16·0	49·446	. . . .	− 9 44 3·5	1·048	0·00
309	91	θ Herculis	4	17 49 24	267 20 54·0	30·765	−0·17	+37 17 8·1	0·987	+ 0·06
314	92	ξ Herculis	4	17 50 0	267 29 55·5	34·782	. . . .	+29 16 46·7	0·875	0·00
316	32	ξ Draconis	3.4	17 50 5	267 31 1·0	15·280	. . . .	+56 54 29·0	0·868	+ 0·30
322	67	ο Ophiuchi	4	17 50 38	267 39 26·1	44·978	. . . .	+ 2 57 16·0	0·819	. . . .
335	33	γ Draconis	2	17 51 58	267 59 26·4	20·810	−0·31	+51 31 4·5	0·703	− 0·07
343	10	γ 2 Sagittarii . .	4	17 52 57	268 14 30·0	−57·770	−0·04	+30 24 35·5	− 0·615	− 0·15
344	95	Herculis	4	17 53 1	268 15 16·0	+38·075	0·00	+21 36 26·8	− 0·611	+ 0·05
374	72	s 2 Ophiuchi . .	4	17 57 52	269 28 1·0	42·640	. . . .	+ 9 32 52·3	− 0·186	. . . .
388	103	o Herculis	4	17 59 45	269 56 8·1	35·016	. . . .	+28 44 41·6	− 0·023	. . . .
7	13	u 1 Sagittarii . .	4	18 1 48	270 27 3·1	53·745	−0·08	−21 5 45·7	+ 0·158	− 0·09
17	. .	β Telescop.	4	18 4 5	271 1 28·5	61·007	−0·19	−36 48 14·7	0·179	− 0·07
32	19	δ Sagittarii	3·4	18 8 11	272 2 50·4	57·529	+0·10	−29 53 50·5	0·716	− 0·10
46	20	ε Sagittarii	3	18 10 54	272 43 27·0	59·753	−0·15	−34 27 40·7	0·953	− 0·08
48	58	η Serpentis	4	18 10 58	272 44 28·0	47·039	−0·67	− 2 56 16·5	0·959	− 0·68
66	22	λ Sagittarii	3·4	18 15 37	273 54 24·3	55·558	−0·11	−25 31 1·0	1·366	− 0·25

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
143	3	α Lyræ........	1	18 30 10	277 32 29·4	+30·145	+0·23	+38 36 20·8	+ 2·631	+0·25
159	27	φ Sagittarii	4	18 33 9	278 17 23·4	56·197	+0·12	−27 10 50·5	2·890	−0·05
178	23	δ Ursæ min.....	3	18 36 42	279 8 46·0	-283·393	....	+86 33 42·5	3·186	....
215	10	β Lyræ........	4	18 42 42	280 40 24·6	+33·150	−0·13	+33 8 23·2	3·713	−0·25
218	34	σ Sagittarii	3	18 42 51	280 42 52·0	55·844	−0·06	−26 31 47·2	3·727	−0·11
257	38	ζ Sagittarii	4	18 49 52	282 28 9·6	57·381	−0·18	−30 9 1·6	4·328	−0·01
262	13	ε Aquilæ	4	18 50 32	282 38 7·0	44·840	−0·17	+14 48 29·3	4·385	−0·08
266	14	γ Lyræ........	3	18 51 27	282 51 53·1	33·598	+0·07	+32 25 27·8	4·463	−0·09
298	16	λ Aquilæ	3	18 55 38	283 54 26·4	47·772	−0·07	− 5 10 9·8	4·818	−0·01
303	17	ζ Aquilæ	3	18 56 13	284 3 15·0	41·315	−0·14	+13 34 41·5	4·868	−0·11
315	41	π Sagittarii	4	18 57 52	284 27 56·2	53·590	−0·08	−21 19 38·0	5·007	−0·14
35	43	d Sagittarii	4	19 5 55	286 28 51·3	52·741	−0·14	−19 17 42·5	5·687	0·00
90	57	δ Draconis	3	19 12 28	286 6 58·0	0·444	+0·07	+67 18 35·7	6·233	+0·02
91	1	k Cygni	4	19 12 28	287 7 5·1	20·724	+0·02	+53 0 17·0	6·234	+0·14
113	30	δ Aquilæ	4	19 15 24	288 51 10·0	45·108	+0·18	+ 2 43 41·0	6·478	+0·10
142	58	π Draconis	4	19 19 36	289 54 8·1	4·974	....	+65 19 51·2	6·213	....
148	6	Lucida Anseris	4	19 20 22	290 5 42·0	37·523	−0·30	+24 16 10·7	6·887	−0·11
161	6	β Cygni præc ..	3	19 22 39	290 39 49·5	36·227	−0·07	+27 32 50·3	7·074	+0·05
187	39	κ Aquilæ	3·4	19 26 7	291 31 53·5	48·453	+0·10	− 7 27 34·7	7·357	−0·08
223	13	θ Cygni	4	19 31 4	292 46 4·8	24·165	−0·06	+49 45 51·9	7·758	+0·35
226	12	φ Cygni	4	19 31 29	292 52 12·6	35·474	−0·05	+29 42 5·4	7·791	+0·10
264	50	γ Aquilæ	3	19 36 44	294 11 14·4	42·742	+0·08	+10 8 11·4	8·213	+0·04
279	7	δ Sagittæ	4	19 38 28	294 36 59·1	40·072	....	+18 3 1·7	8·350	....
280	18	δ Cygni	4	19 38 43	294 40 19·0	28·018	−0·12	+44 39 0·6	8·370	+0·11
294	53	α Aquilæ	1·2	19 41 1	295 15 20·5	43·350	+0·51	+ 8 21 5·2	8·553	+0·38
303	55	η Aquilæ	4	19 42 16	295 34 9·0	45·852	−0·10	+ 0 30 15·2	8·652	+0·02
324	60	β Aquilæ	3	19 45 29	296 22 18·0	44·150	−0·03	+ 5 55 5·2	8·898	−0·54
10	65	θ Aquilæ	4	20 0 58	300 14 41·7	46·436	−0·09	− 1 24.12·7	10·097	+0·07
54	5	α 1 Capricorni..	4	20 6 33	301 38 15·9	49·987	−0·05	−13 6 51·5	10·515	−0·08
58	6	α 2 Capricorni..	3	20 6 56	301 44 12·6	49·995	+0·04	−13 9 10·2	10·545	+0·25
62	31	ο 2 Cygni......	4	20 7 20	301 49 59·1	28·288	−0·03	+46 8 28·8	10·573	+0·08
83	9	β Capricorni ..	3·4	20 9 45	302 26 25·0	50·672	−0·03	−15 24 3·6	10·753	−0·08
124	37	γ Cygni	3	20 15 3	303 45 44·5	32·214	−0·04	+39 37 24·8	11·141	+0·03
191	2	Delphini	4	20 23 39	305 54 49·5	42·963	−0·10	+10 38. 1·0	11·758	0·00
224	71	Aquilæ........	4	20 28 0	307 0 1·9	46·513	....	− 1 47 36·0	12·064	....
227	6	β Delphini	4	20 28 10	307 2 31·5	42·049	+0·12	+13 54 36·0	12·076	+0·08
254	9	α Delphini	3	20 30 21	307 35 12·4	41·691	−0·09	+15 12 57·5	12·227	+0·10
285	50	Deneb	1	20 34 37	308 39 12·3	30·589	−0·08	+44 34 19·8	12·521	0·00
301	3	Aquarii	4	20 37 10	309 17 35·5	47·573	....	− 5 44 58·7	12·695	....
304	12	γ Delphini seq..	4	20 37 23	309 20 50·4	41·738	....	+15 24 46·0	12·709	....
313	53	ε Cygni	3·4	20 38 7	309 31 47·4	35·883	+0·44	+33 13 46·0	12·759	+0·40
338	3	η Cephei	3·4	20 41 12	310 17 51·0	18·354	+0·20	+61 3 56·3	12·965	+0·81
410	58	ν Cygni	4	20 49 43	312 25 46·9	33·417	+0·02	+40 24 14·2	13·525	+0·15
472	62	ξ Cygni	4	20 57 40	314 24 50·1	32·596	....	+43 8 9·0	14·029	−0·01
35	63	ζ Cygni	3	21 4 25	316 6 23·5	38·176	−0·09	+29 24 49·5	14·445	−0·08
100	1	e Pegasi	4	21 12 50	318 12 33·7	41·423	+0·15	+18 57 22·7	14·946	+0·05
105	5	α Cephei	3	21 13 47	318 26 49·5	21·274	+0·27	+61 44 28·8	15·001	−0·07
118	34	ζ Capricorni....	4	21 15 13	318 48 19·0	51·689	−0·26	−23 16 4·5	15·084	−0·30

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
162	22	β Aquarii......	3·4	21 21 1	320 15 17·7	+47·459	−0·06	− 6 26 33·0	+15·414	−0·15
198	8	β Cephei	4	21 26 0	321 30 14·2	12·310	−0·17	+69 41 2·8	15·688	−0·03
223	40	γ Capricorni ..	3	21 28 59	322 14 51·0	49·894	+0·15	−17 33 26·2	15·850	−0·10
260	8	ε Pegasi	3	21 34 21	323 35 25·0	44·134	+0·11	+ 8 57 55·3	16·133	0·00
269	10	κ Pegasi	4	21 35 35	323 53 57·0	40·571	....	+24 43 57·3	16·196	....
276	49	δ Capricorni....	3	21 35 59	323 59 46·5	49·620	+0·24	−17 1 36·2	16·217	−0·26
368	..	γ Gruis........	4	21 41 46	325 26 11·5	54·990	0·00	−38 17 47·2	16·509	−0·03
387	34	α Aquarii......	3	21 55 30	328 52 36·0	46·244	−0·12	− 1 17 6·1	17·160	−0·05
402	24	ι Pegasi	4	21 57 42	329 25 39·0	41·392	+0·33	+24 22 28·0	17·259	+0·06
1	26	θ Pegasi	4	22 0 6	330 1 39·0	45·096	+0·26	+ 5 13 12·8	17·365	+0·11
6	29	π Pegasi	4	22 1 7	330 16 48·6	39·753	....	+32 12 7·7	17·409	....
26	21	ζ Cephei	4	22 3 56	330 58 52·0	30·911	−0·14	+57 13 7·3	17·525	−0·12
72	48	γ Aquarii......	3·4	22 11 19	332 49 48·3	46·393	−0·11	− 2 23 20·4	17·834	+0·05
95	3	Lacertæ	4	22 15 43	333 55 40·0	35·032	....	+51 13 50·6	18·006	....
111	55	ζ Aquarii......	4	22 18 31	334 37 56·1	46·167	−0·10	− 1 2 17·6	18·113	−0·14
123	17	β Piscis austr. ..	4	22 20 6	335 1 31·0	51·586	+0·03	−33 21 56·7	18·166	−0·01
141	7	Lacertæ	4	22 23 4	335 46 8·7	36·461	−0·05	+49 15 29·0	18·280	0·00
151	62	η Aquarii	4	22 25 4	336 16 7·5	46·172	−0·16	− 1 8 33·8	18·351	−0·13
175	18	ε Piscis austr. ..	4	22 29 33	337 23 30·0	50·123	+0·05	−28 4 52·5	18·522	+0·02
189	42	ζ Pegasi	3·4	22 31 29	337 52 21·7	44·708	+0·08	+ 9 47 32·0	18·570	−0·06
205	44	η Pegasi	3	22 33 38	338 24 36·7	41·893	−0·03	+29 10 46·5	18·639	−0·12
231	48	μ Pegasi	4	22 40 21	340 5 22·2	43·036	....	+23 32 57·1	18·847	....
235	73	λ Aquarii......	4	22 42 10	340 32 34·5	47·026	−0·15	− 8 38 22·6	18·901	−0·04
238	32	ι Cephei	4	22 42 33	340 38 49·5	31·667	−0·24	+65 9 2·9	18·913	−0·11
245	76	δ Aquarii	3·4	22 44 1	341 0 19·0	47·991	−0·17	−16 52 47·7	18·954	−0·12
253	24	Fomalhaut	1	22 46 34	341 38 32·1	49·757	+0·33	−30 40 41·2	19·026	−0·26
284	1	ο Andromedæ ..	4	22 52 45	343 11 7·5	40·915	−0·07	+41 15 14·0	19·189	−0·03
288	53	β Pegasi	2	22 54 5	343 31 25·0	43·114	+0·24	+27 0 5·2	19·222	+0·20
290	54	Markab	1	22 54 48	343 42 5·4	44·595	+0·02	+14 7 57·1	19·240	−0·07
313	88	c 2 Aquarii	4	22 58 46	344 41 28·5	48·177	....	−22 25 14·6	19·334	....
138	16	λ Andromedæ ..	4	23 37 49	351 57 11·1	43·168	+0·27	+45 12 34·9	19·848	−0·42
155	35	γ Cephei	3	23 31 15	352 48 38·2	35·549	−0·24	+76 30 59·0	19·888	−0·11
246	28	ω Piscium......	4	23 49 2	357 15 43·8	45·915	+0·06	+ 5 45 23·0	20·023	−0·18
264	2	g Ceti	4	23 53 29	358 22 16·5	46·201	+0·24	−18 26 54·5	20·037	−0·04
281	21	α Andromedæ ..	1	23 58 4	359 31 6·6	45·922	+0·14	+27 59 9·0	20·045	−0·21
283	11	β Cassiopeæ....	2·3	23 58 35	359 38 43·8	45·813	+0·82	+58 2 45·5	20·045	−0·25

NAUTICAL ASTRONOMY.

The situation of a ship at sea is ascertained by determining its latitude and longitude. For this purpose two methods are employed, the one called the "dead reckoning" depending on the course and distance sailed; the other founded upon astronomical observations, and which may be termed *nautical astronomy;* this last will form the subject of the following article.

The dead reckoning proceeds upon the assumption that the ship's course, (that is the angle which her way makes with the meridian,) and the distance she sails are known, and could these data be accurately obtained, the latitude and longitude might be readily found by trigonometrical calculation; but a variety of circumstances generally operate to render this impossible. The difficulty of steering within some degrees of the intended course, uncertainty respecting lee way, errors arising from the log, the heave of the sea, and the influence of currents, contribute more or less to render the dead reckoning, at best, but a rude approximation to the truth.

It must not, however, be understood that the dead reckoning is without its value; on the contrary, when combined with astronomical observations, it is of considerable utility in detecting the existence and velocity of currents, and is indispensably necessary to fill up the short intervals which may occur in unfavourable weather between celestial observations. But the too general practice of relying exclusively upon it, cannot be sufficiently deprecated; and numerous instances might be adduced of the fatal consequences of this reliance in the loss of vessels, from errors of such magnitude, that they might have been detected by the most superficial knowledge of nautical astronomy, and the aid of even a good common watch.

Preliminary Definitions.

Great circles are those which divide the sphere into two equal parts, or hemispheres, and consequently have the same diameter and centre with each other, and with the sphere.

A great circle is called a *secondary* to another when its plane is perpendicular to that of the first.

The *pole* of a great circle is that point which is every where distant 90° from its circumference.

Lesser circles are those which divide the sphere into two unequal parts.

Every circle is supposed to be divided into 360 parts, called degrees; each degree into 60 parts, called minutes; each minute into 60 seconds, and each second into 60 thirds, or into decimals of a second.

The *distance* of any two points in the heavens is expressed by the number of degrees, &c. comprised between them on the arc of a great circle.

A spherical triangle is formed by the intersection of three great circles.

The horizon of any spot on the globe is an imaginary plane, touching the earth's surface at that spot. This is frequently called the *sensible horizon.* A plumb line is perpendicular to this place.

The rational horizon is an imaginary plane, passing through the centre of the earth parallel to the sensible horizon. It is, therefore, a great circle dividing the sphere into two equal parts.

If we observe the fixed stars, we may remark that some of them appear to rise, attain a certain height, decline, and then set. Others describe concentric circles in the heavens about a certain point. This point in the northern hemisphere is called the *north pole,* and in the southern, the *south pole.*

An imaginary line passing through the centre of the earth, in the direction of these points, is called *the axis of the earth:* about which it makes its diurnal revolution, causing the apparent motions of the fixed stars before described.

The zenith is that point which is every where 90° distant from the horizon. The zenith is, therefore, the pole of the horizon.

The nadir is 180° distant from the zenith.

Vertical, or *azimuth circles,* are secondaries to the horizon, and pass through the zenith and nadir.

The *prime vertical* intersects the meridian at right angles. It is sometimes called the six o'clock line.

Cardinal points are the four points where the meridian and the prime vertical intersect the horizon. Those of the meridian, north and south; those of the prime vertical, east and west.

The altitude of an object is its distance from the horizon measured on a vertical circle.

The zenith distance, or complement of altitude is the distance from the object to the zenith; and is, therefore, what the altitude wants of 90°.

The meridian of any spot is that vertical circle which passes through the poles as well as the zenith.

The meridian altitude is the greatest altitude the object can attain; and is its altitude when upon the meridian, and not under the elevated pole.

The azimuth is an arc of the horizon, comprised between the meridian (either from the north or south) and a vertical circle passing through the object.

The amplitude is an arc of the horizon between the east or west point, and the object when rising or setting.

The equator is a great circle at right angles to the axis of the earth, and dividing the globe or heavens into two hemispheres. It intersects the horizon in the east and west points.

Hour circles are secondaries to the equator, and pass through the poles. When named with reference to the earth they are called meridians.

The declination of a celestial object is its distance from the equator, measured upon a secondary, and is either north or south, according to the position of the object with respect to the equator.

Polar distance is the distance of the object from that pole which is elevated above the horizon.

The latitude of any spot upon the globe is its distance north or south of the equator, measured upon a meridian, and is equal to the altitude of the pole.

Complement of latitude is what the latitude wants of 90°, and is equal to the angle which the equator forms with the horizon.

Parallels of latitude and *parallels of declination* are lesser circles, parallel to the equator.

The longitude of any spot on the globe is an arc of the equator, comprised between the meridian of that spot and the *first meridian:* or the meridian from which the longitude is counted; and is said to be east or west, according as the place is to the east or west of the first meridian.

The ecliptic is that great circle which the sun appears to describe in the heavens among the fixed stars, by reason of the motion of the earth in its orbit.

The obliquity of the ecliptic is the angle which it forms by its intersection with the equator; the axis of the earth not being perpendicular to the plane of the ecliptic but inclined about $23\frac{1}{2}^\circ$.

The equinoctial points are those points where the equator intersects the ecliptic.

The ecliptic is divided into twelve parts, called *signs*; reckoning from the vernal equinoctial point towards the west, each sign into 30°, &c.

The *longitude* of a *celestial* body is its distance upon a secondary, north or south of the ecliptic.

Right ascension is an arc of the equator, intercepted (reckoning according to the order of the signs) between the vernal equinoctial point, or the first point of aries, and a secondary to the equator, passing through the object. This, supposing the equator to be divided into twenty-four hours, may be expressed either in time or space.

The tropics are two lesser circles, parallel to the equator, and are the limits of the path of the sun, north and south; they are consequently distant about $23\frac{1}{2}^\circ$ from the equator. The northern tropic is called the tropic of cancer; the southern, the tropic of capricorn.

The zodiac is a space extending about 8° on either side of the ecliptic.

Of time.

In order to render this subject as intelligible as possible, imagine the earth to be at rest, and the fixed stars and sun to move round it. We may then observe, that the interval between any two successive returns of the same fixed star to the meridian is invariable. This is called a *sidereal day*, and is divided into twenty-four hours, &c.; to show which we will suppose a clock to be constructed.

We may also remark that the sun takes a longer time than a star to complete its circle round the earth, but this time will be found to vary. The *mean* interval is called a *mean* solar day, and is divided by a clock or chronometer into twenty-four hours. Such are the clocks and watches in common use, which are intended to shew *mean solar time.*

Whilst the solar clock passes through twenty-four hours, the sidereal clock will have gone through $24^h\ 3^m\ 56^s{\cdot}55$. Such, therefore, is the ratio of mean solar time to sidereal time. As $24^h\ 3^m\ 56^s{\cdot}55$; $3^m\ 56^s{\cdot}55$; $24^h\ 3^m\ 55^s{\cdot}91$; the quantity to be subtracted from twenty-four hours of sidereal time to convert it into mean solar time; and so in proportion for any other period.

It has been remarked that the motion of the sun is not equable, but that it takes sometimes more and sometimes less than a mean solar day to complete its revolution. When the sun is upon the meridian we say it is *apparent* noon, and all time deduced immediately from the sun, is called *apparent time,* and is that portion of an *actual* revolution which the sun has passed through since it left the meridian.

In order to convert apparent into mean solar time, the *equation of time* must be applied to it; the equation of time being the accumulated errors of the sun, (if the expression may be allowed,) from mean solar time. The result will be the time which should be indicated by a solar clock or chronometer.

To convert degrees into time and the contrary.

As every circle is divided into 360°, and also where time is concerned, into twenty-four hours, the proportion for converting degrees, &c. into time, will be the same in all cases, that is, 15° to one hour; but, it must be carefully noted, whether the circle or arc to be converted into time is that described by the motion of a star, the actual motion of the sun, or the motion of the sun supposed to move equably. In the first case, the result will be sidereal time, in the second, apparent time, and in the last, mean solar time.

To convert degrees, &c. into time and the contrary, a table is generally used, which may be found in various authors; but the following method is more expeditious, and it is always desirable to be as independent of extraneous aid as possible.

As 15° degrees give one hour.
1° degree will give 4^m minutes of time.
1′ minute ...... 4^s seconds, and
1″ second ...... 4^{ths} thirds.

If then the seconds of space be multiplied by four, the result will be *thirds* of time. In like manner the minutes will produce *seconds,* and the degrees *minutes* of time.

Let it be required to convert 57° 49′ 54″ *into time.* The operation will stand thus :

$$\begin{array}{r} 57^\circ\ 49'\ 54'' \\ 4 \\ \hline 3^h\ 51^m\ 19^s{\cdot}36. \end{array}$$

To convert time into degrees, &c. the following method will be found the most convenient. Multiply the time by ten, and add to this half the product; the sum will be the corresponding degrees, &c.

Let it be required to convert $3^h\ 51^m\ 19^s{\cdot}36^{th}$ into degrees, &c.

	h	m	s	th
	3	51	19	36
				10
	38	33	16	0
One half	19	16	38	0

57° 49′ 54″ 0‴ corresponding space.

Having the apparent time at the ship, and the longitude, to find the apparent time at Greenwich.

As all the calculations in the *Nautical Almanac*, with the exception of the eclipses of Jupiter's satellites, are made for the instant of apparent noon at Greenwich; it becomes necessary when any of the data contained in the *Almanac* are required for a given time at the ship, to find in the first place the corresponding time at Greenwich. This may be done in the following manner:

Turn the longitude of the ship into time, and add it to the given time if the longitude be west, but subtract it from the given time if the longitude be east. The result will be the apparent time at Greenwich.

In this operation, if the result of the addition should exceed 24 hours, the excess will be the time from the noon of the following day. And in performing the subtraction, 24 hours must be added, if necessary, to the given time at the ship, and the remainder will be the Greenwich time from the noon of the preceding day. This must be carefully recollected.

Examples.

What is the apparent time at Greenwich, when the sun is upon the meridian in 76° of west longitude.

76° 0′ ship's longitude.
4

$5^h\ 4^m\ 0^s$ longitude west in time.
0 0 0 apparent time at the ship.

5 4 0 apparent time at Greenwich.

Given the ship's longitude on the 15th of January, 105° 13′ west, and the apparent time $6^h\ 5^m$, required the apparent time at Greenwich.

105° 13′ longitude west.
4

$7^h\ 0^m\ 52^s$ longitude west in time.
6 5 0 apparent time at the ship.

13 5 52 apparent time the 15th Jan. at Greenwich.

Given the ship's longitude on the 13th August, 96° 7′ east, and the apparent time $5^h\ 7^m\ 30^s$, required the apparent time at Greenwich.

96° 7′
4

$6^h\ 24^m\ 28^s$ longitude east in time.
5 7 30 apparent time at the ship.

22 43 2 apparent time at Greenwich the 12th Aug.

Given the ship's longitude on the 15th of January, 105° 13′ west, and the apparent time $20^h\ 5^m\ 7^s$, required the apparent time at Greenwich.

$7^h\ 0^m\ 52^s$ longitude west in time.
20 5 7 apparent time at the ship.

3 5 59 apparent time at Greenwich the 16th Jan.

HAVING GIVEN THE APPARENT TIME AT THE SHIP AND THE LONGITUDE, TO FIND THE EQUATION OF TIME, THE SUN'S DECLINATION OR RIGHT ASCENSION.

Find the apparent time at Greenwich, corresponding with the given time at the ship. Take the difference between the required element on the preceding and following noon, from the *Nautical Almanac*. Then say, as 24 hours is to this difference, so is the apparent Greenwich time to the proportional variation from this time, to be added or subtracted from the quantity taken from the *Nautical Almanac* for the preceding noon, according as it increases or diminishes.

Examples.

Let it be required to find the equation of time on the 2d April, 1823, at $4^h\ 5^m$ apparent time.

The difference between the equation of time on the 2d and 3d of April, being $18^s{\cdot}2$, we have $24^h : 18^s{\cdot}2 :: 4^h\ 5^m : 3^s{\cdot}1$, the change in the equation of time, or the difference between apparent and mean time for an interval of $4^h\ 5^m$ of apparent time; which being subtracted (because the equation of time is decreasing) from $3^m\ 48^s{\cdot}6$, the equation at the preceding apparent noon, we have $3^m\ 45^s{\cdot}5$, the equation of time required.

Let the equation of time be required for the 1st of April, 1823, at $19^h\ 55^m$ apparent time.

The diurnal variation in 24 hours being $18^s{\cdot}3$, we have

$24^h : 18^s{\cdot}3 :: 19^h\ 55^m : 15^s{\cdot}2$, the difference for $19^h\ 55^m$; which being subtracted from $4^m\ 6^s{\cdot}9$, the equation of time for apparent noon on the 1st of April gives $3^m\ 51^s{\cdot}7$ for the equation of time required.

The same result might have been obtained, by taking the proportional part of $18''{\cdot}3$ for the difference between $19^h\ 55^m$ and 24 hours; and increasing the equation of time for noon on the 2d of April by that quantity.

The sun's declination and right ascension for any time at Greenwich may be found in a similar manner; by taking the proportional part of the daily variation of declination or right ascension, and adding it to, or subtracting it from the quantity at noon, according as it increases or decreases.

This method is however so tedious, that various tables have been constructed to facilitate the operation. A table of proportional logarithms, which appears to be by far the most practically convenient for this purpose, was published by Captain Basil Hall, of the Royal Navy, and is added at the conclusion of this article. A description of it will be best given in the author's own words:—

"This table is useful for finding the proportional part of the daily variation of the sun's right ascension and declination, and of the equation of time, for any hour after noon. The numbers at the top to 24 are for hours, and at the side for minutes: or, the first are minutes, and those at the side seconds: or, the first are seconds, and those at the side thirds; or, in the left hand column, the equivalent decimals of a second. The top arguments are carried on to 30, as the daily variation of the equation of time exceeds 30$^{\mathrm{s}}$ on some days in December.

"The proportional logarithms in this table are found by subtracting the logarithms of the minutes in the several arguments, from the logarithm of the minutes in 24 hours; viz. 3·15836; so that the proportional logarithm of 24$^{\mathrm{h}}$ becomes 0. For the proportional logarithms above 24, the arithmetical complements of the logarithms of the arguments have been added to 3·15836, which renders it necessary to subtract 1 from the index of the result, for every proportional logarithm taken from this part of the table."

Rule.

"To the proportional logarithm of the daily variation, add the proportional logarithm of the time from noon; the sum is the proportional logarithm of the correction for that interval, to be added or subtracted according as the element employed is increasing or decreasing.

"When one or both the terms are small, and consequently their proportional logarithms great, it is convenient, in order to prevent the answer falling near the beginning of the table where the differences are large, to take 1 or 2 from the index of the result, which will bring the answer out 10 or 100 times too great, this is readily done by the eye.

"The value of the terms of the answer is regulated by that of the arguments used; and though on this point at first an occasional ambiguity appears to occur, it is easily removed by a due consideration of the question."

Examples.

Required the equation of time at Greenwich, the 2d of April, 1823, *at* 4$^{\mathrm{h}}$ 5$^{\mathrm{m}}$ *apparent time.*

The daily variation being 18$^{\mathrm{s}}$·2, seek 18 at the top of the table, and ·2 in the left hand side column. Under the first, and in a line with the second, we find the proportional log. of 18$^{\mathrm{s}}$·2 1201
At the top, and opposite 5$^{\mathrm{m}}$ in the second column from the left 7692

8893

The sum is the proportional log. of the variation, which is found in the table to be 3$^{\mathrm{s}}$·1. Subtracting this from 3$^{\mathrm{m}}$ 48$^{\mathrm{s}}$.6, the equation of time for noon at Greenwich, we have 3$^{\mathrm{m}}$ 48$^{\mathrm{s}}$·5. the equation of time sought.

Required the equation of time on the 1*st of April,* 1823, *at* 19$^{\mathrm{h}}$ 55$^{\mathrm{m}}$ *apparent time, at Greenwich.*

Prop. log. of 18$^{\mathrm{s}}$·3, the variation for 24 hours . . 1178
Time from noon 19$^{\mathrm{h}}$ 55$^{\mathrm{m}}$ P. log............. 810

— 55$^{\mathrm{s}}$·2 P. log............... 1988

m s
Equation of time at noon.......... 4 6·9
Variation for 19$^{\mathrm{h}}$ 55$^{\mathrm{m}}$ .... — 15·2

Equation of time required .... 3 51·7

Required the variation in the equation of time, 23d of July, 1823, *for* 7$^{\mathrm{h}}$ 35$^{\mathrm{m}}$.

Daily diff. 1$^{\mathrm{s}}$·9 prop. log. 1.1015
7$^{\mathrm{h}}$ 35$^{\mathrm{m}}$ prop. log.................. ·5004

Variation required 0$^{\mathrm{s}}$·6 prop. log. . . 1·6019

Required the variation in the equation of time, for 9$^{\mathrm{h}}$ 7$^{\mathrm{m}}$, *the* 20*th Dec.* 1823.

Daily diff. 29$^{\mathrm{s}}$·8 prop. log. 9060
9$^{\mathrm{h}}$ 7$^{\mathrm{m}}$.......... prop. log. 4204

1·3264

Here the daily difference exceeding 24, it is necessary to subtract 1 from the index. The remainder ·3264 is the proportional logarithm of 11$^{\mathrm{s}}$·3 the variation required.

What is the sun's declination, April 10*th,* 1823, *when on the meridian in longitude,* 80° *east.*

h m s
Apparent time at the ship 0 0 0
Longitude east in time 5 22 0

Apparent time at Greenwich, 9th April . . 18 38 0
Difference of declination between 9th and 10th Feb. 22′ 19″ prop. log. 316
18$^{\mathrm{h}}$ 38$^{\mathrm{m}}$ .... prop. log. 1099

Change of declination . . +0° 17′ 19″ prop. log. ·1415
Dec. north at noon on the 9th Feb........... 7 22 53

Declination required 7 40 12 north.

Required the sun's declination for 12*th April,* 1823, *at* 4$^{\mathrm{h}}$ 23$^{\mathrm{m}}$ *apparent time at Greenwich.*

Daily difference, 21′ 54″ prop. log. 398
4$^{\mathrm{h}}$ 23$^{\mathrm{m}}$ .... prop. log. 7384

Change of declination +° 4′ 0″ prop. log. 7782
☉ Dec. at apparent noon 8 29 26 north.

Declination required 8 33 26

Required the sun's declination the same day at 12$^{\mathrm{m}}$ 45$^{\mathrm{s}}$ *apparent time, at Greenwich.*

Daily difference 21′ 54″ prop. log. 398
12$^{\mathrm{m}}$ 45$^{\mathrm{s}}$ .... prop. log. 2746

Change of declination . . + 11″·6 prop. log. 3144
Declination at noon .... 8 29 26 ·0

Declination required . . 8 29 37 ·6

Required the sun's declination for 12*th April,* 1823, *at* 16$^{\mathrm{h}}$ 50$^{\mathrm{m}}$ *apparent time, the ship being in longitude* 86° *east.*

Longitude east.............. 86°
4

Longitude in time east........ 5$^{\mathrm{h}}$ 44$^{\mathrm{m}}$
Apparent time at the ship...... 16 50

Apparent time at Greenwich . . 11 6

Daily difference of declination 21′ 54″ prop. log... 398
$11^h\ 6^m$.......... prop. log... 3349

Change of declination + 10 8 prop. log. 3747
Dec. at noon, north .. 8 29 26

Declination required 8 39 34

What is the sun's right ascension on the 12th of April, 1823, at $16^h 50^m$ apparent time at Greenwich.

	h	m	s	
⊙'s right asc. for 12th April,	1	20	27·5	
13th	1	24	8·1	
Difference..		4	40·6	p. log. 7097
	h	m		
	16	50		p. log. 1540
		m	s	
Increase of right ascension .. +		3	17	p. log. 8637
	1	20	27·5	
Right ascension required..	1	23	44·5	

TO FIND THE MOON'S RIGHT ASCENSION AND DECLINATION FOR ANY GIVEN TIME, UNDER ANY GIVEN MERIDIAN.

Convert the longitude of the ship into time, and find the apparent time at Greenwich corresponding to that of the ship.

As the moon's right ascension and declination are given in the *Nautical Almanac* for noon and midnight, find by proportion the variation in right ascension or declination, for the excess of the given Greenwich time beyond the preceding noon or midnight.

If the moon's motion in right ascension and declination were equable, the quantity thus found applied to that of the preceding noon or midnight would be the right ascension or declination required. But as this is not the case, it becomes necessary to employ the "*Table of second differences,*" given at the end of the *Nautical Almanac,* in the following manner:

Take out of the *Nautical Almanac* the two right ascensions or declinations preceding and following the given Greenwich time, and take the successive differences between the first and second, the second and third, and the third and fourth. The quantities thus obtained are called *first* differences. From the first differences find also the *second* differences, of which take the mean.

In this operation, if the first differences increase and then decrease, or the contrary, half the difference of the two second differences must be taken for the mean.

Having thus found the mean of the second differences, enter the table with it and the given time, and take out the corresponding equation. If the mean of the second differences exceed the limits of the table, the equation may be taken out by two such entries as together make up the mean of the second differences.

If the first of the first differences be *greater* than the third of the first differences, *add* this equation to the proportional part before found; otherwise, subtract it. The result will be, the true variation of the moon's right ascension or declination for the excess of the given time beyond the preceding noon or midnight; and this being applied by addition or subtraction to the right ascension or declination for such noon or midnight, as it may be increasing or decreasing, the sum, or difference, will be the moon's correct right ascension or declination for the given time.

EXAMPLES.

Required the moon's right ascension, Jan. 19th, 1823, at 5^h apparent time.

		° ′ ″	1st diff. ° ′ ″	2d diff. ′ ″	Mean. ′ ″
☽'s AR Jan.	18th, at midnight	11 31 24	6 13 48	12 55	13 51
Ditto	19th, at noon	17 45 12	6 26 43	14 47	
Ditto	19th, at midnight	24 11 55	6 41 30		
Ditto	20th, at noon	30 53 25			

As 12^h : 6° 26′ 43″ :: 5^h to the approximate motion in AR ....″ .... 2° 41′ 7·9″
Equation from the table corresponding to.... 5^h and 12′ 0″ = 87·5
5^h and 1 50 = 13·4

Equation of 2d difference ....100·9 .... 1 40·9

Correct motion in AR 2 39 27
AR for preceding noon.... 17 45 12

☽'s AR required 20 24 39

Required the moon's declination, Jan. 16th, 1823, at 18^h apparent time.

18^h apparent time will be 6^h apparent time beyond midnight.

				1st diff.	2d diff.	Mean.
☽'s Declination, Jan.	16th, at noon,	S.	4° 11′ 22″	2° 52′ 16″	2′ 23″	1′ 32″
Ditto	16th, at midnight,	S.	1 19 6	2 54 39	0 41	
Ditto	17th, at noon,	N.	1 35 33	2 55 20		
Ditto	17th, at midnight,	N.	4 30 53			

	°	′	″
As 12^h : 2° 54′ 39″ :: 6^h to the approximate change of declination	1	27	19·5
Equation corresponding to 6^h and 1′ 32″		—	11·3
Correct change of declination	1	27	8·2
Declination for preceding midnight, south	1	19	6·0
True declination, north	0	8	2·2

Required the moon's declination, March 19*th,* 1823, *at* 19^h *apparent time.*

19^h apparent time will be 7^h apparent time after midnight.

				1st diff.	2d diff.	
☽'s Declination, March	19th, at noon	N	26° 42′ 26″	0° 24′ 14″	0° 14′ 8″	Mean. 30′ 0″
Ditto	19th, at midnight,		26 18 12	0 10 6	1 14 9	
Ditto	20th, at noon,		25 28 18	1 14 15		
Ditto	20th, at midnight,		24 14 3			

		°	′	″
As 12^h : 10′ 6″ :: 7^h to the approximate change of declination		0	5	33
Equation corresponding to 7^h and 12′ =	87″·5			
7^h and 12′ =	87 ·5			
7^h and 6′ =	43 ·8			
	218 ·8 =	—	3	39
Correct change of declination		0	1	54
Declination for preceding midnight		26	18	12
Declination required		26	16	8

The preceding examples will probably be found sufficient; but the reader is recommended to peruse with attention the directions given in the *Nautical Almanac,* for the use of the table of second differences.

TO CONVERT APPARENT TIME INTO MEAN TIME, AND THE CONTRARY.

If the apparent time given be that at the ship, convert this into the apparent time at Greenwich. Find the corresponding equation of time, and apply it with its proper sign as directed in the *Nautical Almanac* to the apparent time at the ship. The result will be the mean solar time. If, on the contrary, the time given be mean time, and it is desired to convert it into apparent time, the equation of time, for the time at Greenwich, must be applied to the mean time at the ship with a contrary sign.

EXAMPLE.

Being in long. 105° 13′ *west, and the apparent time the* 15*th Jan. at the ship* $20^h\ 5^m\ 7^s$ *required the mean time.*

h	m	s	
7	0	52	Long. west in time.
20	5	7	Apparent time at the ship.
3	5	59	Apparent time at Greenwich, 16th Jan.
20	5	7	Apparent time at the ship.
+	9	59·2	Equation of time for noon, 16th Jan.
+	0	2·6	Variation for $3^h\ 6^m$.
20	15	8·8	Mean time required.

Had it been required to convert $20^h\ 15^m\ 8^s{\cdot}8$ mean time into apparent time, the equation for noon of the 16th at Greenwich, with its proportional variation, would have been subtracted instead of being added; and the result would have been $20^h\ 5^m\ 7^s$ of apparent time.

OF THE CORRECTIONS TO BE APPLIED TO THE ALTITUDE OBSERVED AT SEA, IN ORDER TO OBTAIN THE TRUE ALTITUDE.

The dip or depression of the horizon.

In observing an altitude with the sextant, or reflecting circle, the image of the object is made to coincide with the visible horizon; but as the eye is elevated above the surface of the sea, the visible horizon will be below the true horizontal plane.

Let AC (fig. 1. plate 1, Nautical Astronomy,) or CB be the radius of the earth; AE the height of the eye above the level of the sea, EH the horizon. Then E being the place of the observer, EH is the horizontal plane; and the angle HEB will be the *dip* or depression of the horizon, which may be thus found.

The angle HEB is equal to the angle BCE; and in the right angled triangle CBE, the difference of the squares of CE (= the rad. of the earth + the height of the eye above the level of the sea,) and CB is equal to the square of BE; the square root of which therefore gives BE. Then in the same triangle, CE : rad. :: BE to the sine of BCE (=HEB) the depression of the horizon.

The depression thus obtained is the true depression, but this must be lessened by the amount of terrestrial refraction, which is very uncertain. About $\frac{1}{8}$ or $\frac{1}{10}$ of the whole quantity is usually allowed.

A table, No. 1, is added of the *dip* or apparent depression of the horizon for different elevations of the eye, allowing $\frac{1}{10}$ for terrestrial refraction, the manner of using it is sufficiently obvious. The dip must be always subtracted from the observed altitude.

The height of the eye above the sea should be carefully measured at the time the altitude is taken; for a neglect of this precaution might easily occasion an error of one minute, according as the observation is made on the windward or leeward side of the vessel.

Refraction.

The rays of light proceeding from a celestial body having to pass through the atmosphere, are bent in the direction of the vertical from their rectilinear course, and the object in consequence appears to have a greater altitude than it really has. This effect, called *refraction,* is greatest in the horizon, and becomes nothing in the zenith.

When the temperature of the atmosphere is increased, the refraction decreases; and when the density of the air becomes greater, (indicated by an increase in the height of the mercury of the barometer,) the refraction suffers an increase. The corrections for temperature and density have hitherto been little attended to in nautical astronomy; but on some occasions they become too considerable to be neglected. In the *Nautical Almanac* is given a table of refractions with corrections for the barometer and thermometer, and such ample directions for their application, as to render the additional trouble of employing them very trifling.

When the corrections for the depression of the horizon and for refraction have been applied to the observed altitude, the result will be the altitude of the object above the sensible horizon.

Semidiameter.

If the altitude obtained by the preceding corrections be that of the limb of the sun or moon, we have to add or subtract the semidiameter, to obtain the true altitude of the centre. This is readily done for the sun by using the semidiameter taken from the *Nautical Almanac,* and found by proportion, if necessary, for the given time; but for the moon some further consideration is necessary.

When the moon is in the horizon, her distance from the observer is greater than when in the zenith by the semidiameter of the earth. Consequently her apparent semidiameter will be least in the horizon, and greatest in the zenith. This increase is called the augmentation of the moon's semidiameter, and may be thus found.

The distance of the moon being about 60 semidiameters of the earth, the moon's horizontal semidiameter will be increased one sixtieth part in the zenith, and the sine of this quantity (or, as the arc is small, the quantity itself) will vary directly as the sine of the altitude; therefore, to the logarithm of one sixtieth part of the moon's semidiameter in the horizon, add the logarithmic sine of the altitude. The sum will be the logarithm of the augmentation to be added to the horizontal semidiameter before found.

The semidiameter of the moon is given in the *Nautical Almanac* for noon and midnight at Greenwich, from which it may be readily found by proportion for any other time or meridian.

A small table, No. 2, is added, from which the augmentation may be taken out with the given horizontal semidiameter and altitude. The proportional parts may be readily found by inspection.

Of parallax.

Having thus obtained the true altitude of the object above the sensible horizon, the last step is to determine its altitude when referred to the rational horizon, or its situation as seen from the centre of the earth, to which point all observations are reduced.

The *horizontal parallax* is the angle under which the semidiameter of the earth, or its radius at the place of observation, would be seen from the celestial body when in the horizon.

The *parallax in altitude* is the angle subtended by this same radius, at any elevation of the object, and is the quantity to be added to the altitude before found, to reduce it to what it would be at the centre of the earth. The effect of parallax is greatest in the horizon, and is nothing in the zenith.

Let CE (plate 1, fig. 2,) be the semidiameter of the earth, ☽E*h* the sensible horizon, HC the rational horizon. Then the angle E☽C is the horizontal parallax, and the angle C*m*E (=GE*m*,) is the parallax for the altitude *m*E☽; or the angle by which the object ☽ would appear lower when viewed from the sensible horizon, than it would if seen from the centre of the earth C.

In the triangle ☽EC, we have ☽C : sin of ☽EC (or its equal the sine of ☽EZ) :: EC : sin. E☽C. And in the triangle *m*EC, we have *m*C (=☽C) : sin. *m*EC (= sin. *m*EZ) :: EC : sin. E*m*C. The sine of the horizontal parallax varies therefore inversely as the sine of the zenith distance, or directly as the cosine of the altitude.

The moon's horizontal parallax is given in the *Nautical Almanac* for every noon and midnight at Greenwich, and may be found by simple proportion for any other time. But the parallax there found is the equatorial horizontal parallax; and as the earth is a spheroid, having the equatorial to the polar diameter about as 306 to 305, the polar horizontal parallax will be less than the equatorial in the same proportion, and consequently the horizontal parallax for any given latitude, will vary between these limits.

A table, No. 3, is added, containing the correction on this account to be taken out with the equatorial horizontal parallax, given in the *Nautical Almanac* and the latitude; which correction being subtracted from the equatorial parallax before found, the remainder will be the horizontal parallax for the place and time of observation, which is to be employed in the following manner for finding the parallax in altitude.*

Having the horizontal parallax, to find the parallax in altitude.

To the log. sine of the horizontal parallax add the log. cosine of the altitude. The sum rejecting ten from the index, will be the log. sine of the parallax in altitude, to be added to the altitude above the sensible horizon before found. Or,

To the prop. log. of the horizontal parallax add the logarithmic secant of the altitude; the sum will be the proportional logarithm of the parallax in altitude: or,

From the proportional log. of the horizontal parallax, with 10 for its index, subtract the cosine of the altitude. The remainder will be the proportional log. of the parallax in altitude.

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* The same table contains also the correction of the proportional log. of the equatorial parallax, which will be found the most convenient if proportional logarithms are employed.

Either of the two latter methods is preferable to the former, where there is a common table of proportional logarithms; as the secant or cosine of the altitude may be taken to the nearest minute only; but for the sine of the parallax a table of sines, &c. to seconds is required.

Examples.

The observed altitude of Sirius being 20°, the barometer 29.5 inches, thermometer 70°, and height of the eye 22 feet, required the true altitude.

		°	′	″
Observed altitude		20	0	0
Depression of the horizon table 1	—		4	30
		19	55	30
Refraction	—		2	39·4
Correction for barometer	—			2·65
Ditto for thermometer	—			6·44
True altitude		19	52	41·50

In the above example, if the corrections for the barometer and thermometer had been omitted, the result would have been erroneous 9 seconds.

The observed altitude of the sun's lower limb on the 3d of Feb. 1823, being 14°, the height of the eye 22 feet, barometer 29 inches, and thermometer 67° required the true altitude of the sun's centre.

		°	′	″
Observed altitude of L. L.		14	0	0
Depression of the horizon	—		4	30
		13	55	30
Refraction	—		3	51·15
Correction for barometer	—			7·70
Ditto for thermometer	—			7·97
Parallax (table 4,)	+			8·50
Sun's semidiameter on 3d Feb.	+		16	14·80
True altitude of sun's centre		14	7	46·48

In this example, the error, from neglecting the corrections for the barometer and thermometer, would have been 14″·64.

Being in latitude 45° north, and longitude 140° west on the 3d Feb. 1823, at 18^h apparent time, the altitude of the moon's lower limb was observed to be 15°. The height of the eye being 22 feet, the barometer 28·7 inches, and thermometer 40°, required the true altitude of the moon's centre.

	h	m	
Longitude of the ship in time, west	9	20	Feb. 3d.
Apparent time at the ship, add	18	0	
Apparent time at Greenwich	3	20	Feb. 4th.

	°	′	″		
☽'s hor. eq. par. Feb. 4, at $3^h 20^m$ from *Naut. Alm.*		54	14·3		
Reduction for lat. 45° from table		—	6·0		
True horizontal parallax		54	8·3	P. L.	10·5218
	15	7		cosine	9·9847
Parallax in altitude	0	52	15·5	P. L.	0·5371

		°	′	″
Observed altitude ☽'s L. L.		15	0	0
Depression of horizon *	—		4	30
		14	55	30
Refraction	—		3	35·4
Barometer	—		0	9·33
Thermometer	+		0	4·39
Horizontal semidiameter for 4th Feb. at $3^h 20^m$ from the *Nautical Almanac*	+		14	47·50
Augment. of semidiameter from (table 2)	+		0	8·00
True altitude of ☽'s centre above the sensible horizon		15	6	45·16
Parallax in altitude	+		52	15·50
True altitude of ☽'s centre		15	59	0·66

Having given the true altitude of the centre of the sun or of the moon, to find the apparent altitude.

In clearing the distance of the sun and moon from the effects of parallax and refraction, it is necessary to have the apparent altitudes of their centres. For this purpose, the usual method has been to apply the semidiameter to the apparent altitude of the limb, but as the semidiameter is affected by refraction as well as the limb, this method is erroneous, and if the altitude should be small, the error might sensibly affect the accuracy of the result. This has been noticed in a very useful article, "*On the methods of correcting lunar observations,*" published in the Journal of Science of the Royal Institution, under the title of *Astronomical and Nautical Collections;* and which, though without the name of the author, there is reason to believe is the work of Dr. Thomas Young.

To find the apparent altitude of the centre of the sun or moon from the true altitude.

Subtract the parallax in altitude, and add the refraction answering to the remainder. Apply the corrections for the barometer and thermometer before found, in obtaining the true altitude *with the signs changed.* The result will be the apparent altitude of the centre.

Taking the last example.

		°	′	″
The true altitude of the ☽'s centre was		15	59	0·66
Subtract th parallax in altitude	—		52	15·50
		15	6	45·16
Refraction for 15° 6′ 45″	+		3	32·70
Barometer	+		0	9·33
Thermometer	—		0	4·39
Apparent altitude		15	10	22·8

But the apparent altitude of the centre may be more readily obtained, from having the observed altitude of the limb, by means of table 5, taken from the article in the journal of science before alluded to.

The correction given in this table is the effect of refraction at different altitudes in lessening the true vertical semidiameter either of the sun or moon, which for this purpose may be considered as the same

		°	′	″
Let the observed altitude of the ☽'s L. L. be as in the former example		15	0	0
Depression of the horizon	−		4	30
		14	55	30
Horizontal semidiameter	+		14	47·5
Correction from table 5	−		0	4·0
Augmentation of semidiameter	+		0	8·0
Apparent altitude of ☽'s centre		15	10	21·5

which is nearly the same as before.

From this the true altitude may be readily deduced, and with sufficient accuracy for every nautical purpose.

		°	′	″
Apparent altitude of ☽'s centre		15	10	21·50
Refraction	−		3	32·00
Barometer	−		0	9·33
Thermometer	+		0	4·39
		15	6	44·56
Parallax in altitude	+		52	15·50
True altitude of ☽ s centre		15	59	0·06

As this method requires only one entry in the table of refractions, it is preferable to that before given, which was principally intended to shew the order in which the corrections should be applied.

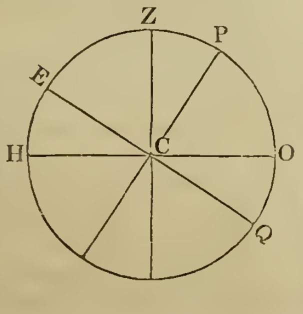
Let the observed altitude of the sun's lower limb on the 3d of February, 1823, be 14° 0′ 0′; height of the eye 22 feet, barometer 29 inches, and thermometer 67, required the apparent and true altitude of the sun's centre.

		°	′	″
Observed altitude		14	0	0
Depression of horizon	−		4	30
Semidiameter	+		16	14·8
Correction from preceding table			−	4
Apparent altitude of sun's centre		14	11	40·8
Refraction	−		3	46·54
Barometer			−	7·70
Thermometer			−	7·97
		14	7	38·59
Parallax			+	8·50
True altitude of sun's centre		14	7	47·09

Of latitude.

The latitude of a place has been defined to be equal to the altitude of the pole above the horizon; and this is equal to the distance of the equator from the zenith.

Let P be the pole, HO the horizon, EQ the equator, and Z the zenith; the angles ZCO and PCE are equal, both being right angles.



Take away the common angle ZCP, and the remaining angle ZCE is equal to the angle PCO. But the angle ZCE is the distance of the equator from the zenith, which is therefore equal to the latitude.

Having the meridian altitude of a celestial body and its declination, to find the latitude.

Definition.—The zenith distance is said to be north or south, according as the zenith is north or south of the object when on the meridian.

Rule.—Find the zenith distance. If the zenith distance and the declination are of the same name, their sum will be equal to the latitude; but if of different names, their difference will be the latitude of the same name with the greater.

In fig. 3, plate 1, let *s* Z be the zenith distance which is north, and E *s* the declination, also north; their sum EZ is equal to the north latitude.

In fig. 4, *s* Z, the zenith distance, is south; and E *s*, the declination, north; the difference EZ is equal to the latitude.

In fig. 5, E *s*, the declination is south, and *s* Z, the zenith distance north; the difference EZ is the latitude, which is also north.

Examples.

The observed meridian altitude of Aldeberan, Jan. 1st, 1823, was 56° 30′, the zenith being north of the star, and the height of the eye 15 feet, required the latitude.

			°	′	″
Observed altitude			56	30	0
Depression of the horizon		−		3	42
Apparent altitude			56	26	18
Refraction		−		0	38
True meridian altitude			56	25	40
True zenith distance,	north		33	34	20
Declination*	north		16	8	44
Latitude	north		49	43	4

Let the observed meridian altitude of Sirius be 80° 4′, the zenith being south of the star, and the height of the eye 20 feet, required the latitude.

			°	′	″
Observed meridian altitude			80	4	0
Depression of the horizon		−		4	17
Apparent altitude			79	59	43
Refraction		−		0	10
True meridian altitude			79	59	33
True zenith distance,	south		10	0	27
Declination	south		16	28	9
Latitude	south		26	28	36

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* At the end of the *Nautical Almanac* a table is given of the right ascensions and north polar distances of the principal fixed stars, with their annual variation in right ascension and polar distance, by means of which either may be computed for any given time, and the declination readily obtained. In finding the latitude at sea, it is not necessary to notice the barometer and thermometer, as the corrections for density and temperature would be an unnecessary refinement.

The observed meridian altitude of the sun's lower limb, on the 5th of Dec. 1823, *was* 25° 29′, *the zenith being to the north of the sun, the ship's longitude* 73° *east, and the height of the eye* 22 *feet; required the latitude.*

		°	′	″
Observed altitude of ⊙'s L. L.		25	29	0
Depression of the horizon	−		4	30
Apparent altitude of ⊙'s L. L.		25	24	30
Refraction	−		2	2
Parallax	+		0	8
⊙'s semidiameter	+		15	54
True altitude of ⊙'s centre		25	38	30
Zenith distance, north		64	21	30
⊙'s declination, south		22	18	14
Latitude, north		42	3	16

The ship's longitude being 73°, or $4^h\ 52^m$ east, the corresponding time at Greenwich will be $19^h\ 8^m$ of the 4th of Dec. for which time the sun's declination was computed as before explained.

The meridian altitude of the planet Jupiter was observed on the 5*th of Feb.* 1823, *to be* 50° 25′, *the zenith being to the north of the planet; the ship's longitude* 24° *west, and the height of the eye* 18 *feet, required the latitude.*

	h	m
Time of the observation at the ship nearly[*]	6	24
Longitude west 24° in time	1	36
Greenwich time nearly	8	0

	°	′
Jupiter's dec. at Greenwich, 1st Feb. north	18	44
Jupiter's dec. 7th Feb. north	18	49
Variation of declination in 6 days	0	5

Hence, by proportion, the declination of Jupiter, the 5th of Feb. at 8^h is north 18° 47′ 37″.

		°	′	″
Observed altitude of Jupiter		50	25	0
Depression of the horizon	−		4	4
Apparent altitude		50	20	56
Refraction	−		0	48
True meridian altitude		50	20	8
Zenith distance, north		39	39	52
Declination, north		18	47	37
Latitude, north		58	27	29

To find the latitude from the meridian altitude of the moon.

Find in the *Nautical Almanac* the time of the moon's passing the meridian of Greenwich on the given day. Take also the difference between it and the time of the transit on the following day; then say, as 24 hours, augmented by this difference, is to the difference of the daily transit, so is the ship's longitude in time, to a fourth proportional; which, added to the time of the moon's transit at Greenwich on the given day, if the longitude be west, or subtracted from it if east, the sum or difference will be nearly the apparent time of the moon's passage over the meridian of the ship. To this apply the longitude in time as before directed, in order to obtain the corresponding time at Greenwich.

Compute the moon's declination, semidiameter, and horizontal parallax, from the *Nautical Almanac* for the Greenwich time thus obtained; applying, if great accuracy be required, the equation of second differences to the proportional variation of the declination for the excess of the given time beyond noon or midnight, as the omission of this correction may occasion an error in the latitude of about $2\frac{1}{2}$ minutes. Employ the horizontal parallax and semidiameter in the manner before directed, to find the parallax in altitude, and the augmentation of the moon's semidiameter, in order to obtain the true altitude of the moon's centre, and thence the zenith distance; with which, and the declination, proceed as before to find the latitude.

EXAMPLE.

Feb. 19*th*, 1823, *the ship being in longitude* 40° *west, the observed meridian altitude of the moon's lower limb was* 55° 6′, *the zenith north of the moon, and the height of the eye* 16 *feet; required the latitude.*

	h	m	s
The moon passed the meridian of Greenwich, 19th of February, 1823	6	56	0
Ditto 20th of February,	7	59	0
Difference of the ☽'s passage	1	3	0
As $25^h\ 3^m : 1^h\ 3^m :: 2^h\ 40^m$:	0	6	42
To which add	6	56	0
Apparent time of ☽ s passing the meridian of the ship	7	2	42
Ship's longitude west	2	40	0
Corresponding time at Greenwich	9	42	42

					1st diff.	2d diff.	Mean.
☽'s declination, February	18th,	at midnight,	N	25° 55′ 46″	42′ 31″	26′ 9″	15′ 54″
Ditto	19th,	at noon,	N	26 38 17	16 22	5 39	
Ditto	19th,	at midnight,	N	26 54 39	10 43		
Ditto	20th,	at noon,	N	26 43 56			

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[*] This will be nearly the same as the time of the planet's passing the meridian of Greenwich, and may be taken from the *Naut. Alm.*

As 12^h : 16′ 22″ : : $9^h 43^m$ to the approximate change of declination........	13′ 15″		
Equation corresponding to 9^h 43^m and 12′ = 55″			
9 43 and 1·15″ = 6			
61 =	+ 1 1		
Correct change of declination	0 14 16		
Declination for preceding noon	26 38 17		
☽'s declination at the time of observation, north	26 52 33		
Moon's horizontal parallax, 19th of February, at 9^h 43^m from *Nautical Almanac*,	59′ 32″	P. L.	10·4805
☽'s altitude	55° 18′	cosine	9·7553
☽'s parallax in altitude	33′ 54″	P. log.	·7252

	°	′	″
Observed altitude ☽'s lower limb	55	6	0
Depression of the horizon	—	3	50
	55	2	10
Refraction	—	0	41
☽'s Horiz. sem. from *Naut. Alm.*	+	16	12
Aug. of semidiameter, (table 2.)	+	0	14
	55	17	55
☽'s parallax in altitude......	+	33	54
True meridian alt. of ☽'s centre	55	51	49
Zenith distance, north	34	8	11
Declination north	26	52	35
Latitude, north	61	0	46

To find the latitude by an altitude, observed upon the meridian below the pole.

Let *r*, (fig. 6, plate 1,) be the star upon the meridian, H *r* its altitude; then the latitude HN is equal to H *r* + *r* N. But N *r* is the polar distance of the star; therefore, to the polar distance, add the meridian altitude, the sum will be the latitude.

Examples.

March 1st, 1823, *the observed altitude of the pole star, when on the meridian below the pole, was* 30° 7′, *the height of the eye being* 18 *feet, required the latitude.*

	°	′	″
Observed meridian altitude	30	7	0
Depression of the horizon	—	3	50
	30	3	10
Refraction	—	1	40
Star's true altitude	30	1	30
Polar distance, March 1st, 1823..	+ 1	38	2
Latitude, north	31	39	32

June 1st, 1823; *the observed altitude of Capella, when on the meridian below the pole, was* 10° 2′, *the height of the eye being* 17 *feet; required the latitude.*

	°	′	″
Observed meridian altitude........	10	2	0
Depression of the horizon	—	3	57
	9	58	3
Refraction..........	—	5	21
Star's true altitude	9	52	42
Polar distance, June 1st, 1823 ..	+ 44	11	33
Latitude, north ..	54	4	15

June 30*th*, 1823; *the observed altitude of the sun's lower limb, when on the meridian below the pole, was* 7° 5′, *the ship being in* 12° *west longitude, and the height of the eye* 20 *feet; required the latitude.*

	h	m
Apparent time at the ship....	12	0
Longitude 12° west in time ..	0	48
Time at Greenwich....	12	48

	°	′	″
Observed meridian altitude L. L. ..	7	5	0
Depression of the horizon	—	4	17
	7	0	43
Refraction	—	7	26
Parallax	+	0	9
	6	53	26
☉'s semidiameter	+	15	45
True altitude of ☉'s centre ..	7	9	11
Sun's declination 1st of July, 1823, at 12^h 48^m, Greenwich time, 23° 8′ 17″ north, therefore the polar distance is 66° 51′ 43″	+ 66	51	43
Latitude, north	74	0	54

N.B. The altitude, when upon the meridian and below the pole, is the least altitude of the object.

To find the latitude by means of two observations of the sun's altitude, the time elapsed between them, and the declination at each altitude.

The general utility of this problem has induced several eminent mathematicians to attempt easy methods for its solution. But as these have been but approximations, the inconvenience arising from possible errors and a variety of cases, have been more than equal to any advantage gained by the apparent shortness of the computation.

M. Delambre has examined the various methods which have appeared for the solution of this problem; and has, in consequence, been led to reject all the indirect processes, and to prefer the rigorous solution by spherical trigonometry, as being superior both in brevity, and in the accuracy of the result.

Mr. Ivory, whose merits as a mathematician are too well known to need any comment, has given in the *Phil. Mag.* for August, 1821, a method, which he conceives to be more simple and easier in practice than that of M. Delambre. But, as this requires as many references to tables, and a correction on account of the change in the declination of the sun, we have been induced to prefer the direct process.

In the *Astronomical and Nautical collections* before mentioned, under the head of "Parallax" is contained the detail of Delambre's method, of which we shall avail ourselves; adding such remarks as may appear necessary to remove any practical difficulty.

In fig. 7. plate 1, let A and B be the two places of the sun at the times of observation; P the pole, Z the zenith. Then ZB and ZA will be the zenith distances; PA and PB the polar distances, and APB the interval between the observations converted into degrees, &c. In the triangle APB we have the two sides PA and PB, and the included angle, to find the third side AB. In the same triangle APB, the three sides are now given to find either of the angles at A or B, suppose the angle BAP, and in the triangle AZB, we have the three sides to find the angle BAZ.

The difference between the angles PAB and ZAB, (or sometimes their sum between the tropics) will be the angle PAZ, subtended by PZ, the compliment of latitude.

Lastly, in the triangle PAZ, the two sides AP and AZ, and the included angle PAZ, are given to find the third side PZ, the compliment of latitude.

Rules for double altitudes.

From the observed altitudes, find the true altitudes of the sun's centre. Correct the lesser altitude for the change of the ship's place during the interval, in the following manner:—

Find the angle between the ship's course and the sun by compass, at the time of the observation of the lesser altitude. With this angle as a course, and the distance sailed in the interval, enter a traverse table, and find the difference of latitude answering to the distance, which will be the correction of the altitude.

If the lesser altitude be observed in the forenoon, this correction is to be added to it, if the angle between the ship's course and the sun is less than 8 points; but if the angle is greater than 8 points, the correction is to be subtracted from the lesser altitude. If the lesser altitude be observed in the afternoon, the correction is to be subtracted from it, should the angle between the ship's course and the sun be less than 8 points; but if more, the correction is to be added to the lesser altitude.

The change of the ship's longitude in time between the observations must be applied to the elapsed time, by adding it to the interval when sailing eastward, or by subtracting it when sailing westward.

Having thus corrected the lesser altitude and the interval, find the declinations for the times of observation, and thence the polar distances; then,

1. Add together the logarithmic sine of half the sum of the polar distances $\frac{1}{2}$ (PA+PB), and the sine of half the interval converted into space, that is, $\frac{1}{2}t$; and the sum will be the sine of half the distance AB.

2. In the triangle APB, as the sine of AB is to the sine of the opposite angle APB (= t), so is the sine of one of the polar distances, for instance, PB, to the sine of the opposite angle PAB.

3. In the triangle AZB, we have the three sides ZA, ZB, and AB, to find the angle BAZ. For this purpose, add together the two zenith distances ZA, ZB, and the distance AB; and from the half sum subtract in succession the two sides containing the required angle ZA and AB: add together the sines of the two remainders, and the arithmetical compliments of the sines of the sides containing the required angle ZA and AB; half the sum of these four logarithms will be the sine of half the angle sought BAZ.

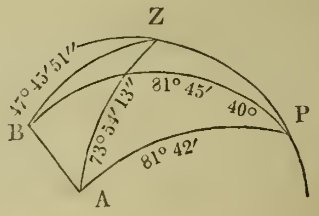
4. The difference of BAZ and PAB, or sometimes the sum between the tropics, will be the angle PAZ, subtended by the compliment of latitude PZ, from the sun's place A. To find the latitude, take out the logarithmic cosines and sines of the sides containing the colatitude, that is, of AP and AZ; and under the sines set down the cosine of the included angle PAZ. Add them separately together, and find the corresponding natural numbers, the sum of which will be the natural sine of the latitude; and its logarithm, of course, the logarithmic sine. But if the angle PAB lies without BAZ, and their sum exceeds a right angle, the cosine becomes negative, and the difference of the natural numbers must be taken for the natural sine of the latitude; and if there is any doubt in the computer's mind, it will be easy to try both suppositions.

Note.—If the declination is very small, it may sometimes be more convenient to find the angle PAB, from the three sides of the triangle, as in the 3d precept, instead of by the 2d.

EXAMPLES.

Let the two zenith distances corrected be ZA = 73° 54′ 13″ ZB = 47° 45′ 51″, *the declinations* 8° 18′ *and* 8° 15′ *north, and the interval of time three hours; or* PA = 81° 42′, PB = 81° 45′, *and* APB = 45°.

Note.—It will be found convenient, and tend much to prevent mistakes, to draw a rough figure, and mark the sides given, thus—



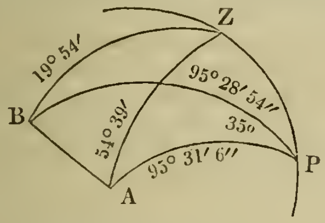
	°	′	′	
PA =	81	42	0	
PB =	81	45	0	
	163	27	0	
1. $\frac{1}{2}$ (PA + PB) =	81	43	30	..sine....9·9954547
$\frac{1}{2}t$ =	22	30	0	..sine....9·5828397
$\frac{1}{2}$ AB =	22	15	11·3	..sine....9·5782944
			2	
AB =	44	30	22·6	

		°	′	″		
2.	AB =	44	30	22·6	sine co.ar.	0·1542898
	APB =	45	0	0	..sine....	9·8494850
	PB =	81	45	0	..sine....	9·9954822
	PAB =	86	38	58	..sine....	9·9992570
3.	ZB =	47	45	51		
	ZA =	73	54	13	sine co. ar.	0·0173686
	AB =	44	30	23	sine co. ar.	0·1542898
	sum..	166	10	27		
	$\frac{1}{2}$ sum	83	5	13·5		
	$\frac{1}{2}$ sum − ZA =	9	11	0·5	sine	9·2030232
	$\frac{1}{2}$ sum − AB =	38	34	50·5	sine	9·7949174
						2)19·1695990
$\frac{1}{2}$	BAZ =	22	36	26·6	sine....	9·5847995
				2		
	BAZ =	45	12	53		
	PAB =	86	38	58		
	PAZ =	41	26	5		

4.	PA cosine	9·1594354	PA sine....	9·9954271
	ZA cosine	9·4428780	ZA sine....	9·9826314
			PAZ cosine..	9·8748934
	·040023 Log.	8·6023134	Log...	9·8529519
	·712774			

·752797 Log. 9·8766779 = log. sine 48° 50′, the latitude required.

Let the two zenith distances corrected be ZA = 54° 39′ ZB = 19° 59′. *The declinations* 5° 31′ 6″, *and* 5° 28′ 54″ *south, and the interval of time* 2^h 20^m ; *or* PA = 95° 31′ 6″, PB = 95° 28′ 54″, and APB = 35°.



		°	′	″		
	PA =	95	31	6		
	PB =	95	28	54		
		191	0	0		
1.	$\frac{1}{2}$ (PA + PB) =	95	30	0	sine	9·99799
	$\frac{1}{2}$ t =	17	30	0	sine	9·47814
	$\frac{1}{2}$ AB =	17	25	0	sine	9·47613
				2		
	AB =	34	50	0		
2.	AB =	34	50	0	sine co. ar...	0·24322
	APB =	35	0	0	sine	9·75859
	PB =	95	28	54	sine	9·99801
		88	21	10	sine	9·99982
	PAB =	91	38	50	(the supplement.)	

		°	′	″		
3.	ZB =	19	59	0		
	ZA =	54	39	0	sine co. ar...	0·08851
	AB =	34	50	0	sine co. ar...	0·24322
	sum	109	28	0		
	$\frac{1}{2}$ sum	54	44	0		
	$\frac{1}{2}$ sum − ZA =	0	5	0	sine	7·16269
	$\frac{1}{2}$ sum − AB =	19	54	0	sine	9·53196
						2)17·02638
	$\frac{1}{2}$ BAZ =	1	52	5	sine	8·51319
				2		
	BAZ =	3	44	10		
	PAB =	91	38	50		
	PAZ =	87	54	40		

4.	PA cosine	8·98288	PA sine........	9·99798
	ZA cosine	9·76235	ZA sine........	9·91149
	·055620 Log.	8·74523	PAZ cosine......	8·56169
	·029591 Log		....	8·47116

·026029 Log. 8·4154575 = log. sine 1° 29′ 28″ the latitude required.

Note. With that zenith distance which was taken furthest from the meridian, suppose PA, the co-latitude ZP, and the polar distance PA, the angle ZPA may be computed by precept 3d; and this, converted into time, is the apparent time from noon, when the observation was taken; to which the equation of time being applied, the error of the watch will be known.

To find the latitude from the altitudes of two fixed stars observed at the same time, having given their polar distances, and their difference of right ascension.

This method of determining the latitude has many advantages over that last described; for, no inaccuracy can arise from an erroneous estimate of the change of the ship's place between the observations as they are completed at the same time; and the risk of losing the second observation of the sun from unfavourable weather is avoided. The difficulty of seeing the horizon of the sea, at night, is the inconvenience to which this method is liable; but this may be obviated by using either of the instruments, which will be described, for taking altitudes at sea when the horizon is not visible.

The solution of this problem depends on the same principles as that of the last. The polar distances of the two stars, and their difference of right ascension are given, being the two sides of a spherical triangle and the included angle, to find, in the first place, the third side AB. But as the difference of the polar distances of the two stars may be considerable, it becomes necessary to compute the side AB, more rigorously than in the case of the sun, in which the difference of declination being small, the 1st precept given, although an approximation was sufficiently accurate.

In making the observations, take the altitude of one star, and note the time by a watch; then the altitude of the other star, noting the time, and then again the altitude of the second star with the corresponding time. From the change of altitude of the second star, and the interval, its altitude for the instant when the first star was observed, may be known by simple proportion.

In order to compute the distance of the two stars, proceed as follows.

Rule.

Take out the logarithmic cosines and sines of the two polar distances PA and PB, and under the sines set down the cosine of the difference in right ascension of the two stars converted into degrees, &c. or the angle APB. Add them separately together, and find the corresponding natural numbers, the sum of which, or their difference, if either polar distance exceeds 90°, will be the natural cosine of the distance of the two stars AB; and its logarithm, of course, the logarithmic cosine.

The remainder of the operation for finding the latitude, is precisely the same as in the last problem, commencing with the second precept.

To insure greater accuracy, stars, when near the zenith, should not be used.

Example.

Let the corrected zenith distance of Capella be 29° 14′ 24″ = ZA, *and that of Sirius be* 72° 5′ 48″ = ZB. *The polar distance of Capella* PA = 44° 11′ 39″, *that of Sirius* PB = 106° 28′ 40″, *and the difference of their right ascensions* $1^h\ 33^m\ 45^s$ *or* APB = 23° 26′ 11″, *required the latitude.*

1. PA cosine .. 9·8555080 PA sine.. 9·8432902
PB cosine .. 9·4527728 PB sine.. 9·9817868

·203367 Log .. 9·3082808 APB cosine 9·9626071

·613316.................... Log.. 9·7876841

Diff. ·409949 Log .. 9·6127298 = log. cosine 65° 47′ 54″ = AB.

2. AB = 65° 47′ 54 sine″ co. ar. 0·0399536
APB = 23 26 11 sine ......9·5995891
PB = 106 28 40 sine ......9·9817868

24 43 04 ...... sine 9·6213295

PAB = 155 16 56 (the supplement.)

3. ZB = 72 5 48
ZA = 29 14 24 sine co. ar. 0·3111631
AB = 65 47 54 sine co. ar. 0·0399536

sum 167 8 06

½ sum 83 34 03
½ sum − ZA = 54 19 39 sine ...... 9·9097504
½ sum − AB = 17 46 09 sine ...... 9·4845601

2)19·7454272

½ BAZ = 48 14 29 ...... sine 9·8727136
2

BAZ = 96 28 58
PAB = 155 16 56

PAZ = 58 47 58

4. PA cosine 9·8555080 PA sine .... 9·8432902
ZA cosine 9·9408059 ZA sine .... 9·6888369

·625619 Log. 9·7963099 PAZ cos .... 9·7143594

·176395..................Log. .. 9·2464865

·802014 Log. 9·9041820 = Log. sine 53° 19′ 22″, the latitude required.

In the *Nautical Almanac* for 1825, Dr. Brinkley has given a new method of solving the preceding problem, from which we have taken the liberty of extracting the distances of those stars most conveniently situated for the purpose, and have added their difference of right ascension in space. These data will much facilitate the operation of finding the latitude as the distance AB, and the difference of right ascension APB need not be computed.

Table I.—*For north latitude.*

Pairs of stars.	North polar distance for 1822.	Change in 10 years.	Distance (AB) for 1822.	Change in 10 years.	Diff. of right ascension (APB.)	Change in 10 years.
Capella ...... Sirius	° ′ ″ 44 11 39 106 28 39	″ − 46 + 44	° ′ ″ 65 47 54	′ ″ +0 3	° ′ ″ 23 26 11	″ −270
Capella ...... Procyon	44 11 39 84 19 33	− 46 + 85	51 6 59	0 0	36 36 23	−189
Procyon ...... Spica Virginis	84 19 33 100 13 41	+ 85 + 190	87 32 33	+0 2	86 27 41	− 1
Procyon ...... Arcturus	84 19 33 69 53 10	+ 85 + 190	96 48 13	+0 2	99 23 29	− 63
Arcturus .... Antares	69 53 10 116 1 33	+ 190 + 86	56 1 25	−0 7	32 44 10	+139
Antares ...... *a* Lyræ	116 1 33 51 22 33	+ 86 − 30	71 41 14	+0 4	33 6 21	−244
Spica Virginis Antares	100 13 41 116 1 33	+ 190 + 86	45 54 15	+0 1	45 39 57	+ 78
Antares ...... Fomalhaut	116 1 33 120 33 48	+ 86 − 191	82 50 50	+0 2	97 19 31	− 48
a Lyræ ...... Fomalhaut	51 22 33 120 33 48	− 30 − 191	91 25 20	+0 3	64 13 10	+196
Arcturus .... *a* Lyræ	69 53 10 51 22 33	+ 190 − 30	59 0 34	+0 24	65 50 31	−105

Table II.—*For south latitude.*

Pairs of stars.	South polar distance for 1822.	Change in 10 years.	Distance (AB) for 1822.	Change in 10 years.	Diff. of right ascension (APB)	Change in 10 years.
Archernar .... Fomalhaut	° ′ ″ 32 35 24 59 26 12	″ + 186 + 191	° ′ ″ 39 7 20	′ ″ −0 1	° ′ ″ 40 49 23	″ +260
Rigel .... Sirius	81 35 10 73 31 21	+ 49 − 44	23 40 35	+0 5	22 49 42	− 40
Canopus...... Procyon	37 19 36 95 40 27	− 16 − 85	60 8 28	−0 10	17 29 33	+270
Sirius .... *β* Argus	73 31 21 21 0 48	− 44 − 148	57 43 43	−0 7	38 27 15	−277
a Crucis ...... Spica Virginis	27 53 12 79 46 19	− 200 − 190	52 58 54	+0 1	14 45 15	− 12
Spica Virginis *a* Centauri	79 46 19 29 53 20	− 190 − 163	51 40 18	+0 2	18 4 30	+190
Spica Virginis Antares	79 46 19 63 58 27	− 190 − 86	45 54 15	+0 1	45 39 57	+ 78

Having the time the sun's diameter takes to pass a horizontal line, to find the latitude of the place of observation.

This problem, which may perhaps sometimes be found useful, is taken from "Selections from the additions that have been occasionally annexed to the *Nautical Almanac.*"

Rule.

From the proportional logarithm of the observed time, the index being increased by 10, subtract the proportional logarithm of the time the sun's diameter takes to pass the meridian, found by doubling the time of the semidiameter set down in the *Nautical Almanac,* the remainder is the logarithmic sine of the angle of position.

To the logarithmic cosine of this angle add the logarithmic cosine of the sun's declination; the sum, rejecting 10 from the index, is the logarithmic sine of the latitude exactly, if the observation was made of the sun's rising through the horizon, otherwise only nearly to be corrected as follows:

Add together the logarithmic sine of the altitude, the logarithmic sine of the declination, and the logarithmic secant of the latitude; the sum, rejecting 20 from the index, is the logarithmic sine of the first correction to be added, if the latitude and declination are both of the same name, otherwise subtracted.

Add together the logarithmic tangent of the latitude, and the logarithmic versed sine of the altitude, the sum rejecting 10 from the index is the logarithmic sine of the second correction to be always subtracted.

Example 1.

Suppose December 21st, 1774, the time of the sun's diameter rising through the horizon of the sea was observed to be $7^{m}\ 20^{s}$; *to find the latitude of the ship.*

Observed time $7^{m}\ 20^{s}$ prop. log............11·3899
Time of ⊙'s diam. passing mer. $2^{m}\ 22^{s}$·2 p.l. 1·8804

Angle of position, 18° 51′½ sine .... 9·5095
Cosine angle of position ....9·97608
Cosine declination, 23° 28′ ....9·96251

Latitude, 60° 15′ sine ....9·93859

Example 2.

Suppose the same day the sun was observed to take 5^{m} 18^{s} *in rising through a space equal to his diameter, the mean altitude being* 5°. *To find the latitude.*

Observed time, $5^{m}\ 18^{s}$ prop. log............ 11·5310
Time of ⊙'s diameter, passing mer. prop. log. 1·8804

Angle of position, 26° 34′ sine .... 9·6506
Cosine angle of position.........9·95154
Cosine declination.............9·96251

Approx. lat. 55° 7′ 50″ sine....9·91405

Sine alt..........8·9403....V. S. alt.... 7·5804
Sine dec. ....9·6001....tang. lat. ..10·1569

Sec. lat. ....10·2428 2d cor. 18′ 50″.. 7·7373

1st cor. 3° 28′ 50″ sine 8·7832

	°	′	″
Approx. lat.	55	7	50
1st cor.	3	28	50
	51	39	0
2d cor.	—	18	50
Latitude............	51	20	10

To find the latitude by the altitude of the pole star when not on the meridian.

This method of finding the latitude is so convenient that it deserves particular attention. It is not confined to any precise hour, as is the case with meridian altitudes, but may be employed during the whole night whenever the pole star is visible; for even though the horizon should not be seen, the instrument, which will be described, for taking altitudes under such circumstances may be used; and by employing the mean of several observations, will be found, on most occasions, sufficiently accurate.

The importance of repeatedly ascertaining the latitude on approaching the English channel at night is well known to every seaman; and, we believe, the usual practice in such a case, particularly in rough weather, is to lay-to till the next morning. We have however known an officer of the British navy, with that confidence and decision which superior science confers, run his frigate boldly up the channel at the rate of eleven knots an hour, in a dark squally night, which allowed only occasional glimpses of the pole star between flying clouds, and find himself off Plymouth the next morning at day break.

The following method, which is due to M. Littrow, is equally valuable for its simplicity and accuracy. The data required are the observed altitude of the pole star, and the corresponding apparent time at the ship, from which the latitude may be found in the following manner:

With the apparent time of observation and the longitude, find the corresponding time at Greenwich; for which take the right ascension of the sun from the *Nautical Almanac,* and add it to the apparent time at the ship. The sum rejecting 24 hours, if necessary, will be the sidereal time, or the right ascension of the meridian.

From the right ascension of the meridian subtract the right ascension of the pole star; the remainder will be the distance of the pole star from the meridian in sidereal time.

Enter the annexed table, and seek this time in one of the columns, entitled "Distance from the meridian," and take out the corresponding first correction, subtracting the annual variation if necessary; take out also the logarithm of A.

Apply the first correction with the sign at the top of the column, in which the distance from the meridian was found, to the true altitude of the pole star.

To the logarithm of A add the logarithmic tangent of the altitude; the sum will be the logarithm of a further correction in seconds; which being always added to the number before found, the result will be the latitude.

Note.—The sun's right ascension, for the corresponding time at Greenwich may be taken from the *Almanac* with sufficient accuracy for the present purpose by estimation.*

Example.

The 23d of June, 1824, in longitude 30° west, the following altitudes were taken of the pole star, the height of the eye being 20 feet; required the latitude.

	h m s		Observed altitude. ° ′ ″		
Mean time	8 34 24	....	50 38 20		
	8 39 0	....	50 40 20		
	8 40 44	....	50 22 10		
Mean..........	8 38 3	Mean	50 33 37		
Longitude in time, add	2 0 0	Dep. of Horizon..	— 4 17		
Mean time at Greenwich	10 38 3		50 29 20		
Equation time	— 1 51	Refraction ..	— 0 48		
Apparent time at Greenwich	10 36 12	True altitude	50 28 32	tangent ..	10·0989
⊙'s right ascension................	6 10 0	1st correction	+ 53 14	log. A ..	1·7685
Right ascension of the meridian	16 46 12		51 21 46		
Right ascension of pole star	0 58 1		+ 1 14 = 74″	Log.	1·8674
✳'s distance from the meridian	15 48 11	Latitude required	51 23 0		

25th of June, 1824, in longitude 16° west, the altitude of the pole star was observed to be 47° 5′ at 12ʰ 14ᵐ of mean time, the height of the eye being 20 feet; required the latitude.

	h m s		Observed altitude. ° ′ ″		
Mean time	12 14 0	....	47 5 0		
Longitude in time, add	1 4 0	Dep. of the horizon	— 4 17		
Mean time at Greenwich	13 18 0		47 0 43		
Equation of time	— 2 17	Refraction ..	— 0 54		
Apparent time at Greenwich	13 15 43	True altitude	46 59 49	tangent ..	10·0455
⊙'s right ascension	6 19 0	1st correction....	— 15 43	log. A ..	1·9098
Right ascension of the meridian	19 34 43		46 44 6		
Right ascension of the pole star	0 58 1		+ 1 30 = 90″	log.	1·9553
✳'s distance from the meridian......	18 36 42	Latitude required	46 45 36		

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* The formula from which the above method is taken, is so admirably adapted to determine the latitude on shore with very great precision and convenience, that we shall here subjoin it.

Let a = the co-latitude sought.
z .. the zenith distance.
p .. the polar distance.
t .. the hour angle. Then
$a = z + p \cos t - \mathrm{A}\ \mathrm{cotang.}\ z + \mathrm{B}.$
Where $\mathrm{A} = \frac{1}{2} p^2 \sin^2 t \sin 1''.$
and $\mathrm{B} = \frac{1}{3} p^3 \sin^2 t \ \mathrm{cosine}\ t \sin^2 1''.$

Note.—In the second and third quadrant of t, the value of B is negative.

Table for finding the latitude by the altitude of the pole star.

Distance from the Meridian.				1st Correction.	Annual variation.	Log. of A″.
−	+	+	−		−	
h m	h m	h m	h m	° ′ ″	″	
0 0	12 0	12 0	24 0	1 37 48	19·45	
10	50	10	50	1 37 42	19·41	9·2009
20	40	20	40	1 37 26	19·37	9·8021
30	30	30	30	1 36 58	19·28	0·1529
40	20	40	20	1 36 19	19·15	0·4009
50	10	50	10	1 35 30	18·97	0·5921
1 0	11 0	13 0	23 0	1 34 28	18·80	0·7475
10	50	10	50	1 33 17	18·54	0·8777
20	40	20	40	1 31 54	18·30	0·9897
30	30	30	30	1 30 21	17·99	1·0871
40	20	40	20	1 28 38	17·62	1·1733
50	10	50	10	1 26 45	17·26	1·2503
2 0	10 0	14 0	22 0	1 24 42	16·83	1·3195
10	50	10	50	1 22 29	16·41	1·3819
20	40	20	40	1 20 7	15·92	1·4387
30	30	30	30	1 17 36	15·42	1·4903
40	20	40	20	1 14 55	14·90	1·5377
50	10	50	10	1 12 7	14·36	1·5809
3 0	9 0	15 0	21 0	1 9 9	13·74	1·6205
10	50	10	50	1 6 5	13·15	1·6569
20	40	20	40	1 2 52	12·51	1·6899
30	30	30	30	0 59 32	11·84	1·7205
40	20	40	20	0 56 6	11·17	1·7483
50	10	50	10	0 52 33	10·45	1·7735
4 0	8 0	16 0	20 0	0 48 54	9·73	1·7965
10	50	10	50	0 45 10	8·98	1·8173
20	40	20	40	0 41 19	8·22	1·8361
30	30	30	30	0 37 26	7·45	1·8527
40	20	40	20	0 33 27	6·66	1·8675
50	10	50	10	0 29 25	5·85	1·8803
5 0	7 0	17 0	19 0	0 25 19	5·04	1·8915
10	50	10	50	0 21 10	4·21	1·9007
20	40	20	40	0 16 59	3·38	1·9081
30	30	30	30	0 12 46	2·54	1·9139
40	20	40	20	0 8 32	1·70	1·9181
50	10	50	10	0 4 16	0·85	1·9207
6 0	6 0	18 0	18 0	0 0 0	0·00	1·9215

This Table is calculated for 1824: for any following year the annual variation must be subtracted from the first correction, and the quantity A diminished $\frac{2}{100}$ parts for each minute of diminution in the polar distance of the Pole Star. The variation of A is, however, so small that for many years it may be neglected at sea.

Of longitude.

The longitude of a place has been defined to be an "arc of the equator comprised between the meridian of that place and the first meridian." This arc may be expressed either in degrees or in time. To find the difference of longitude therefore between any two places, is to find their difference of time. When the sun is on the meridian of any place in west longitude, he has evidently passed the first meridian, and it will there be afternoon; and on the contrary, when it is apparent noon in east longitude, the sun will not have arrived at the first meridian, where, consequently, it must be before noon.

The essence of this problem then, so extremely important in nautical astronomy, consists in finding what o'clock it is at two places at the same instant.

However simple this may at first sight appear, it is attended with so many practical difficulties, that large rewards have been, and still are offered for its more perfect attainment.

The time at the ship may be readily and accurately obtained by altitudes of the sun, or of stars, taken when at a distance from the meridian. The difficulty therefore consists in finding the time for the same instant at Greenwich.

The obvious mode of effecting this, it should seem, would be to take a chronometer from Greenwich, which being previously set to the time at that place, would always indicate it when required by simple inspection. Nothing then would be necessary but to determine the time at the ship, and note the corresponding time shewn by the chronometer; the difference of which being converted into degrees, would give at once the longitude of the ship.

But though chronometers have attained a very great degree of perfection, they are still subject to so many possible variations, besides being liable to accidents, that though they are of the utmost importance at sea, it would not be safe to place an implicit reliance upon them, without the occasional check of some method less under the influence of casual derangements.

Suppose some instantaneous phenomenon were known to take place at a particular moment of Greenwich time. If this phenomenon could be observed from the ship and the time noted, the corresponding time at Greenwich being known, the longitude of the ship would be obtained. Thus, if it be known that the beginning of an eclipse of the moon is to take place at 6^{h} of Greenwich time, and it is observed at the ship to commence at 11 o'clock, the difference 5 hours, or 75 degrees, will be the longitude of the ship east of Greenwich.

The eclipses of Jupiter's satellites, also being calculated for the meridian of Greenwich, may be employed for the same purpose. For the Greenwich time at which the eclipse will take place being known, and the time at which it was observed at the ship, the longitude may be deduced in the same manner as by an eclipse of the moon.

Both these methods of determining the longitude are however of little value. An eclipse of the moon happens but seldom, and then it is very difficult, on account of the penumbra which surrounds the shadow of the earth, to determine the precise time of its commencement; and though Mr. Lecount, a midshipman in the British navy, has succeeded in observing the eclipses of Jupiter's satellites at sea, and has thus proved the possibility of that which was before doubtful; yet, as these eclipses are invisible, a part of the year, during the vicinity of Jupiter to the sun, they are little adapted to supply the every day exigencies of the navigator.

As the distance of the moon from the fixed stars which are situated near her path, varies about half a degree in one hour, her change of place is sufficiently rapid to become perceptible in a very small portion of time. In the *Nautical Almanac* are given the distances of the moon from the sun, and from certain fixed stars for every third hour of Greenwich time, whence the time answering to any intermediate distance may be found by common proportion. Here then the *Nautical Almanac* acts the part of an observer at Greenwich; and if the distance of the moon from the sun, or from any one of these fixed stars, be observed at the ship, and the time noted, the time at Greenwich corresponding to the same distance may be had from the *Nautical Almanac;* and the difference of time between the ship and Greenwich will thus be known. This is called the method of determining the longitude by lunar distances; and it is the most important of any that has yet been devised.

Simple as the lunar method appears from what has been said, it has not been found so free from difficulties in practice. Tables of the moon's motion were necessary, by means of which her distance from the fixed stars might be calculated with the necessary degree of precision. Instruments of corresponding accuracy of construction, and which might be used with facility at sea, were requisite; and when the observations were obtained, the observed distance was not the true distance, or that which it would be if taken at the centre of the earth; and it became necessary to have recourse to calculation to free the observed distance from the effects of parallax and refraction. These difficulties however have been happily surmounted. The tables of the moon's motion are now so perfect, that the greatest error of computation is less than the unavoidable errors of observation. The skill of our artists in the construction and division of instruments is well known, and formulæ have been invented for shortening the labour of computation. Every effort of science and of art has been directed to facilitate the practice of this important problem, and with such success, as to leave scarcely any thing further to be desired.

The following problems preparatory to finding the longitude, by means of chronometers, should be thoroughly understood.

To reduce any portion of sidereal to mean time, and the contrary.

It has been observed that in order to convert 24 hours of sidereal time into mean solar time, $3^m\ 35^s{\cdot}91$, must be subtracted; and as the sidereal day exceeds the solar day by $3^m\ 56^s{\cdot}55$ of sidereal time this quantity must be added to 24 hours of mean solar time to convert it into sidereal time, and so in proportion for any other period.

Table 6, is for converting sidereal into mean solar time, and Table 7 for the reverse.

EXAMPLES.

Reduce $10^h\ 56^m\ 17^s$ *of sidereal time into mean time.*

			m	s
From Table 6............	for 10^h....		1	38·30
	50^m....		0	8·19
	6^m....		0	0·98
	17^s....		0	0·05
Difference of sidereal and mean time....			1	47·52
Sidereal time ..		10^h	56^m	17^s
Mean time required		10	54	29·48

Bring $10^h\ 54^m\ 29^s{\cdot}48$ *of mean solar time into sidereal time.*

			m	s
From Table 7............	for 10^h....		1	38·56
	50^m....		0	8·21
	4^m....		0	0·66
	$29^s{\cdot}48$....		0	0·07
Difference of mean and sidereal time....			1	47·50
Mean time		10^h	54	29·48
Sidereal time required ..		10	56	17·00

The preceding problems relate to *portions* of sidereal and mean time.

To determine the mean time from the time shewn by a sidereal clock, or watch, and the contrary.

A sidereal clock shews $0^h\ 0^m\ 0^s$ when the first point of Aries is on the meridian; and as the earth revolves on its axis in 24 hours of sidereal time, the time shewn by the sidereal clock at any particular instant, indicates the right ascension in time of the meridian, and consequently the right ascension of the sun, or of any celestial body which may then be upon the meridian.

The time shewn by the sidereal clock, when the sun was upon the meridian at the preceding apparent noon, was the sun's right ascension at that moment. Now if this right ascension, or sidereal time, be subtracted from the given sidereal time, or right ascension of the meridian, (increased if necessary by 24 hours,) the remainder will be the portion of sidereal time elapsed since the last apparent noon. And this being converted into mean time will give the portion of mean time since apparent noon, to which applying the equation of time for the preceding noon, the result will be the mean time required.

If on the contrary it were required to find the time shewn by a sidereal clock, or the right ascension of the meridian for any given instant of *mean* time, the reverse of the preceding operation must be pursued. To the given mean time apply the equation of time for the preceding noon, but with an opposite sign; the result will be a portion of mean time which has elapsed since the sun was upon the meridian. This being converted into sidereal time, and the right ascension of the sun for the preceding noon added to it, will give the corresponding sidereal time, or the right ascension of the meridian.

EXAMPLES.

Required the mean time, the 21st April, 1823, *when the sidereal clock shewed* $10^h\ 5^m\ 10^s$.

	h	m	s
Sidereal time, or right ascension of the meridian......................	10	5	10
Subtract ⊙'s right ascension for 21st April at noon....................	1	53	44·9

	h	m	s
Sidereal time elapsed since apparent noon	8	11	25·1
Acceleration of sidereal on mean solar time for 8^h 11^m 25^s from table 6	0	1	20·5
Mean time elapsed since apparent noon	8	10	4·6
Equation of time for apparent noon	— 0	1	15·6
Mean time from mean noon, commonly called the hour of the day	8	8	49·0

At what hour of mean time will Regulus be upon the meridian the 1st March, 1823.

	h	m	s
Right ascension of Regulus, or sidereal time	9	58	56·4
Subtract ⊙'s right ascension for 1st March at noon	22	46	41·9
Sidereal time elapsed since apparent noon	11	12	14·50
Acceleration of sidereal on mean solar time for 11^h 12^m $14{\cdot}50^s$ from Table 6	—	1	50·14
Mean time elapsed since apparent noon	11	10	24·36
Equation of time for apparent noon	+	12	43·40
Regulus on the meridian mean time	11	23	7·76

What is the right ascension of the meridian, or the sidereal time at 11^h 3^m $7{\cdot}76^s$ of mean time on the 1st March, 1823.

	h	m	s
Mean time from mean noon	11	23	7·76
Equation of time for preceding apparent noon	—	12	43·40
Mean time elapsed since apparent noon	11	10	24·36
Excess of sidereal time on 11^h 10^m 24^s of mean time	+	1	50·12
Sidereal time elapsed since apparent noon	11	12	14·48
Add the ⊙'s right ascension for 1st March at noon	22	46	41·90
Right ascension of the meridian or sidereal time	9	58	56·38

To find the longitude by means of a Chronometer.

The chronometer, is an instrument so delicate in its construction, and is consequently so liable to be easily deranged, that great care should be taken to guard against every circumstance unfavourable to its accurate performance. Perhaps the best position for a chronometer would be to place it upon a shelf or tray, suspended freely from the deck of the ship. From this it should never be removed; but in observing, a watch with seconds should be used, which should be compared with the chronometer both before and after the observations. Care should be taken not to place a compass needle, or magnet, near the chronometer as it might impart magnetism to the balance and completely spoil the instrument. If it should be found necessary to move the chronometer, all sudden motion should be avoided, and particularly any circular horizontal movement, as this would accelerate or retard the vibrations of the balance. Attention should be paid to wind up the chronometer carefully, and at the same hour every day, in order that the same portion of the main spring may be constantly employed, and thus the prevalence of any partial irregularity be avoided.

It is necessary before a voyage to know the *rate* of the chronometer; that is, how much it gains or loses in a day. From this, and its ascertained difference from the time at Greenwich, at a certain date the Greenwich time may be obtained in any longitude, by multiplying the rate by the number of days that have elapsed since the error of the chronometer for Greenwich was ascertained, and applying this together with the error to the time shewn by the chronometer. Great care then is necessary in finding the rate of a chronometer, and this is best done in a fixed observatory, but as chronometers have not yet attained perfection, the rate may vary and we shall subsequently give some instructions as to the mode of examining it.

Having the observed altitude of the sun, the declination and the latitude of the place of observation, to find the hour angle or apparent time from noon, and thence if required the mean time.

Rule.

Correct the observed altitude for the depression of the horizon, refraction, parallax, and semi-diameter to obtain the true altitude of the sun's centre, from which find the zenith distance. With the apparent time at the ship nearly, and the estimated longitude, find the corresponding Greenwich time, and to this compute the sun's declination in order to obtain the polar distance. The polar distance, the co-latitude and the zenith distance will form the three sides of a spherical triangle, in which it is required to find the angle at the pole.

Add together the zenith distance, the polar distance, and the co-latitude, and take half the sum. From the half sum subtract the zenith distance (the side opposite to the required angle), then add together the complement arithmetical of the sines of the polar distance and the co-latitude (the sides containing the required angle), the sine of the half sum and the sign of the difference of the half sum, and the zenith distance: half the sum of these four logarithms will be the logarithmic cosine of half the hour angle sought, and this being doubled and converted into time will be the hour angle which, as the sun was taken, is the apparent time from noon.

If the altitude was taken before noon the horary angle must be subtracted from 24 hours, and the equation of time being applied for the corresponding Greenwich time, (if the mean time be required) the result will be the mean time at the place of observation.

Example.

On the 2d June, 1823, the true altitude of the sun's centre was 30° 2′, when the chronometer showed 5^h 1^m 0^s *the latitude being 40° 5′ north, and the sun's declination*

22° 9′ 17″ north. The chronometer on the 20th May preceding was 45ˢ slow for Greenwich time and its rate, losing 2ˢ·1 daily. Required the apparent time at the ship and the longitude.

		h	m	s
Time by chronometer		5	1	0
Slow the 20th April	+		0	45
Loss from the 20th April to the 2d June	+		0	25·2
Mean time at Greenwich		5	2	10·2
Corresponding equation of time	+		2	31·4
Apparent time at Greenwich		5	4	41·6

	°	′	″		
Zenith distance	59	58	0		
Polar distance	67	50	43	co. ar. sine	0·0333098
Co-latitude	49	55	0	co. ar. sine	0·1162768
Sum	177	43	43		
½ Sum	88	51	51	sine	9·9999146
½ sum—zen. dist.	28	53	51	sine	9·6841666
					2)19·8336678
	34	20	17·6	cosine	9·9168339
			2		
Hour angle	68	40	35·2		
			4		
	4ʰ	34	42·3	app. time from noon	

	h	m	s
Apparent time from noon	4	34	42·3
Apparent time at Greenwich	5	4	41·6
Longitude *west* in time	0	29	59·3
Multiply by			10
	4	59	53·0
½ the product	2	29	56·5
Longitude west in degrees, &c.	7°	29	49·5

Having given the observed altitude of a fixed star, its declination and the latitude, to find the apparent time and the longitude of the ship.

RULE.

Find the hour angle or the distance of the star from the meridian as in the last example. If the star is to the *west* of the meridian, add to this the star's right ascension, but if to the east of the meridian subtract it, and the result will be the right ascension of the meridian. With the mean time at Greenwich, given by the chronometer, find the apparent time at Greenwich, and the sun's right ascension for that time. Subtract the sun's right ascension from the right ascension of the meridian, and the remainder will be the apparent time at the ship, the difference between which, and the apparent time at Greenwich deduced from the chronometer, will be the required longitude in time.

EXAMPLE.

28th May, 1823, in latitude 0° 50′ north, the mean of several observed altitudes of the star Antares, when eastward of the meridian, was 30° 42′, the height of the eye being 16 feet, and the mean of the corresponding times by the chronometer 9ʰ 35ᵐ 43ˢ, which chronometer was too fast 20th April at Greenwich 1ᵐ 5ˢ at noon, and gaining daily 5ˢ·4. Required the apparent time and the longitude of the ship.

		h	m	s
Time by the chronometer		9	35	43
Fast 20th April at Greenwich	—		1	5
Gain since the 20th April	—		3	27·5
Mean time at Greenwich		9	31	10·5
Equation of time 28th May at 9ʰ 31ˢ	+		3	9·2
Apparent time at Greenwich		9	34	19·7

		°	′	″
Mean of observed altitudes		30	42	0
Depression of the horizon	—		3	50
		30	38	10
Refraction	—		1	39
Star's true altitude		30	36	31

	°	′	″		
Zenith distance	59	23	29		
Polar distance	116	1	37	co. ar. sine	0·0464395
Co-latitude	89	10	0	co. ar. sine	0·0000459
Sum	264	35	6		
½ Sum	132	17	33	sine	9·8690669
½ Sum—zen. dist.	72	54	4	sine	9·9803665
					2)19·8959188
	27	29	32	cosine	9·9479594
			2		

	h	m	s	
Star's dist. from mer. 54 59 4=	3	39	56·4	of sid. time
Star's right ascension	16	18	31·2	
Right ascension of the meridian	12	38	34·8	
☉'s right ascension 28th May at 9ʰ 34ᵐ 20ˢ of apparent time	4	19	17·0	
Apparent time at the ship	8	19	17·8	
Apparent time at Greenwich	9	34	19·7	
Longitude west in time	1	15	1·9	

In the "Tables requisite to be used with the nautical Ephemeris" a shorter method is given for calculating the hour angle, which will be found very convenient when the observer has tables of natural sines and of the "Logarithmic rising;"* it is as follows:

If the sun's declination and the co-latitude of the

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* The logarithmic rising is the logarithmic versed sine of the hour angle, but with an index of 4, in consequence of the natural sines being considered as whole numbers instead of decimals as is usual, in which case the index would have been 9. The logarithmic versed sine may therefore be used instead of logarithmic rising.

ship be one north and the other south, take their difference, but if they be both north or both south, take their sum for the sun's meridianal altitude. If that sum be greater than 90° take it from 180°.

From the natural sine of the meridianal altitude, subtract the natural sine of the true altitude at the time of observation, find the logarithm of the remainder, and add it to the logarithmic secant of the latitude and the logarithmic secant of the declination, their sum will be the "logarithmic rising," and the time corresponding to it will be the hour angle in time from the meridian: or if the object be the sun, the apparent time from the nearest noon, consequently if the observation be made in the morning, the time thus found must be taken from 24 hours, and the remainder will be the apparent time from the noon of the preceding day.

Taking the first Example.

	°	′	″		
Lat. north	40	5	0	secant	0·11628
Co-latitude	49	55	0		
Declin. nor.	22	9	17	secant	0·03330
Merid. altit.	72	4	17	nat. sine 95144	
True altit.	30	2	0	nat. sine 50050	
				Difference 45094 log.	4·65412
Apparent time ..	4^{h}	34^{m}	$42^{s}\cdot2$	Log. rising ..	4·80370

To find the rate of a chronometer.

The best method of determining the rate of a chronometer, is by comparing it with a good astronomical clock at a fixed observatory. The rate has hitherto been usually given by the maker, but a depot has lately been established at the Royal observatory at Greenwich, on the recommendation of the Board of Longitude, for the chronometers belonging to the Navy, where their rates are determined by the Astronomer Royal. This depot is also open to all chronometer makers, on application at the Admiralty, and superior artists are thus furnished with an opportunity of bringing the merits of their work before the public; and of obtaining from the Board of Longitude a reward for any improvement, *in principle,* attended by practical advantage.

But, however well the rate of a chronometer may be ascertained in the first instance, it is liable to vary from many causes, and particularly unless great care be taken, from the motion in transporting it from the observatory to the ship. Its rate should therefore be examined whenever an opportunity occurs. The most simple method of doing this, when the ship is likely to be stationary for some time, is to determine the error of the chronometer by means of numerous observations of the sun's altitude and the corresponding times, taken when the sun is not less than about 3 hours from the meridian. Having found the error of the chronometer from mean time, this error may be compared with that determined at an interval of not less than a week, and the daily rate may thence be found by simple proportion. The observations may be continued daily, but the rate should not be deduced from a less interval than a week, otherwise it might be too much affected by the unavoidable errors of observation.

If the vessel should be in harbour for some time, observations of the altitude of the sun may be made by means of an artificial horizon, and the error of the chronometer, and consequently its rate, thence determined with much greater accuracy than it could be by employing the horizon of the sea. In using the artificial horizon (which is a surface of quicksilver protected from the wind by a covering of plane glass) the images of the sun's limb are brought into contact, and it will conduce much to accuracy to observe the upper and lower limb alternately, and to take the mean. The result (to which the sextant error if any, must be applied) will be double the apparent altitude of the centre. From the half of this, the difference between the parallax and refraction being subtracted, the remainder will be the true altitude of the sun's centre.

The best method when on shore of determining the rate of a chronometer, is by a portable transit instrument. For this purpose it is by no means necessary that the instrument should be accurately in the meridian, but then the observations must be confined to the stars. The axis, however, should be carefully levelled, and the middle wire (which must be truly vertical) adjusted to a certain mark previous to each observation. It is not *absolute*, but *comparative* time that is required. Now if the time of the passage of a certain star * should be observed every night by the chronometer, and the chronometer keep mean time, should lose $3^{m}\ 55^{s}\cdot91$ between any two successive transits of the same star. If the loss is less than this, the chronometer is *gaining* the difference daily: if on the contrary, the loss is more than $3'\ 55\cdot91''$, the difference is the rate which the chronometer *loses* daily. The observations should be continued for at least a week, and the mean taken for the rate. It must be remarked that this rate is for a sidereal day, but may be readily found for a solar day, if thought necessary, by simple proportion.

Having the apparent time from noon, or hour angle, at the ship, and the estimated longitude, to find the true and the apparent altitude of the sun's centre.

This and the following problems may be found useful for computing the altitude of the sun, the moon, or a star, when circumstances have not permitted them to be obtained by observation at the time of taking lunar distances.

RULE.

Find the apparent time at Greenwich, for which compute the sun's declination, with the declination and co-latitude find the meridian altitude as directed in the last article. Then add together the logarithmic rising of the hour angle, or time from noon, the logarithmic cosine of the latitude of the ship and the logarithmic cosine of the sun's declination. The sum, abating 20 from the index, is the logarithm of a number, which subtracted from the natural sine of the meridian altitude, leaves the natural sine of the true altitude at the time required. From the true altitude the apparent altitude may be deduced in the manner before explained.

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* The reader is supposed to be acquainted with the mode of observing the transit of a star. The telescope is usually furnished with 3 or 5 wires, and the time of the star passing each being noted, the mean of the whole will give the time of its passing the middle wire. It is a good practical check to examine whether the mean of any two wires, equally distant on each side of the centre, gives the same result as the middle wire.

EXAMPLE.

Required the true altitude of the sun's centre the 2d June, 1823, *in latitude* 40° 5′ *north, and longitude by estimation* 7° 29′ 49″ *west, at* 4ʰ 34ᵐ 42ˢ *apparent time.*

	h	m	s
Apparent time at the ship	4	34	42
Longitude in time west +	0	29	59
Apparent time at Greenwich	5	4	41

	h	m	s		
Apparent time	4	34	42	Log. rising	4·80370
Latitude north	40	5	0	cosine	9·88372
Declin. north	22	9	17	cosine	9·96670
Co-latitude	49	55	0	45094 log.	4·65412
Mer. altitude	72	4	17	N. S. 95144	
True alt.	30	2	0	Nat. Sine. 50050	
Refraction	+	1	40		
Parallax	−	0	8		
App. alt.	30	3	32		

Having the apparent time at the ship, and the longitude by estimation, to find the time and apparent altitude of the moon or of a star.

RULE.

Find the apparent time at Greenwich, and for this compute the sun's right ascension, and the right ascension and declination of the moon, or of the star.

To the apparent time at the ship counted from the preceding noon, add the sun's right ascension, the sum will be the right ascension of the meridian.

The difference between this and the right ascension of the moon, or of a star, is the horary angle, or distance of time from the moon or the star's passing the meridian.

With this hour angle, the declination, and the latitude of the ship, compute the true altitude as in the last problem. From this the apparent altitude may be deduced as before directed.

EXAMPLE.

Required the true and apparent altitude of the moon's centre, at 7ʰ 5ᵐ *apparent time January* 11*th,* 1819, *in latitude* 46° 24′ *north and longitude* 50° 16′ *west.*

	h	m	s
Apparent time at the ship	7	5	0
Longitude west in time	3	21	4
Apparent time at Greenwich	10	26	4

	h	m	s
Sun's right ascension for 10ʰ 26ᵐ 4ˢ	19	30	37
Apparent time at the ship	7	5	0
Right ascension of the meridian	2	35	37
☽'s right ascension for 10ʰ 26ᵐ 4ˢ	7	55	32
Horary angle or ☽'s distance from the meridian	5	19	55

Hour angle	5ʰ	19ᵐ	55ˢ	Log. rising	5·91698
Latitude	49°	24′	0″	cosine	9·83861
☽'s Declin. north	25	52	20	cosine	9·95413
				512532 log.	5·70972
☽'s Meridian altitude	66	28	20	N. S.	916867
☽'s true altitude	23	51	0	nat. sine	404335
Parallax in altitude	−	49	16		
	23	1	44		
Refraction	+	2	14		
☽'s apparent altit.	23	3	58		

What was the true and the apparent altitude of Antares the 28*th May,* 1823, *in latitude* 0° 50′ *north, and longitude* 18° 40′ 30″ *west at* 8ʰ 19ᵐ 18ˢ *apparent time?*

	h	m	s
Apparent time at the ship	8	19	18
Longitude west in time	1	15	2
Apparent time at Greenwich	9	34	20
⊙'s right ascension, 23d May, 1823, at 9ʰ 34ᵐ 20ˢ	4	19	17
Apparent time at the ship	8	19	18
Right ascension of the meridian	12	38	35
Right ascension of Antares	16	18	31
Hour angle or star's distance from the meridian	3	39	56

Hour angle	3	39	56	log. rising	4·62959
Lat. north	0	50	0	cosine	9·99999
Dec. south	26	1	37	cosine	9·95355
				38294 log.	4·58313
Mer. alt.	63	8	23	N. S. 89211	
Star's true alt.	30	36	31	N. S. 50917	
Refraction	+	1	39		
Star's app. alt.	30	38	10		

The following method of computing the altitude will, probably, be found convenient, as requiring only the common logarithmic tables.

Take out the logarithmic cosines and sines of the complement of latitude and of the polar distance, and under the sines set down the logarithmic cosine of the hour angle. Add them separately together and find the corresponding natural numbers, the sum of which, or their difference, should the polar distance exceed 90°, will be the natural sine of the altitude, and its logarithm of course the logarithmic sine.

Examples.

Let the ship's latitude be 40° 5′ *north, the sun's declination* 22° 9′ 17″ *north, and the apparent time, or hour angle* $4^h\ 34^m\ 42^s$ = 68° 40′ 35″. *Required the true altitude.*

	°	′	″				
Co-lat. ..	49	55	0	cosine	9·80882	sine ..	9·88372
Polar dist.	67	50	43	cosine	9·57647	sine ..	9·96671
Hour angle	68	40	35	....		cosine	9·56066
	,24283			log.	9·38529		
	,25768			....		log.	9·41109
Sum........	,50051			log.	9.69941	log. sine	30 2′

the altitude required.

The ship's latitude being 0° 50′ *north, the declination of Antares* 26° 1′ 37″ *south, and the hour angle, or distance of the star from the meridian,* $3^h\ 39^m\ 56^s$ = 54° 59′ 4″; *required the true altitude.*

Co-latit. .	89	10	0	cosine	8·16268	sine	9·99995
Polar dist.	116	1	37	cosine	9·64226	sine	9·95356
Hour ang.	54	59	4	....		cosine	9·75876
	,00638			log.	7·80494		
	,51555			....		log.	9·71227
Difference	,50917			log	9·70686	log. sine	of

30° 36′ 30″ the altitude.

To find the longitude by lunar observations.

The method of finding the longitude at sea by lunar distances is the most perfect of any at present known. The rate of a chronometer may change, or it may suffer derangement from accidental causes, but the time marked by the varying distance of the moon from the sun, or from the fixed stars, is liable to no irregularities but such as are subject to laws well ascertained, and the effect of which can be computed, and nothing is wanting to the accuracy of the result but good instruments for taking the distances, and careful observers.

The operation depends upon the solution of two spherical triangles, in one of which three sides are given to find an angle, and in the other, two sides and the included angle to find the third side.

In plate 1, fig. 8, let Z be the zenith, Z ☽, the apparent zenith distance of the moon, Z ⊙, that of the sun and ☽ ⊙, the apparent distance of their centres.

Now as the parallax of the moon is always greater than the refraction, the true altitude of the moon, or that seen from the centre of the earth, will be greater than her apparent altitude, and the parallax of the sun being less than the refraction, his true altitude will be less than the apparent altitude. Let *m* be the true place of the moon, and *s* that of the sun. Then Z *m* will be the moon's true zenith distance, Z *s* the true zenith distance of the sun, and *m s*, the true distance of their centres.

In the triangle Z ☽ ⊙, we have given the three sides, viz. the two apparent zenith distances, and the apparent distance of the centres to find the angle ☽ Z ⊙; or the angle at the zenith. And then in the triangle *m* Z *s*, are given the two sides Z *m* and Z *s*, (the true zenith distances) and the included angle *m* Z *s*, from which the side *m s*, the true distance of the centres may be computed.

Three observers are usually employed in finding the longitude by the lunar method, one to take the distance of the limbs of the sun and moon, or of the limb of the moon from a star, and two others to take their altitudes at the same instant. A fourth observer should be added to mark the time by a chronometer, and to note it down together with the corresponding observed distance and altitudes. Several observations should be thus made in succession as quickly as can be done consistent with accuracy, and the means being taken we shall have the observed distance of the limbs of the sun and moon, their apparent altitudes and the corresponding time by the chronometer.

If the sun, or a star, be at a sufficient distance (about three hours) from the meridian, the apparent time at the ship may be computed from the altitude corresponding to the distance. But if it is too near the meridian, a watch is indispensable. The error of the watch should be inferred from altitudes taken specially for that purpose, as near as may be to the time when the distances are observed, and the observed time by the chronometer, after having applied the error, must be reduced to the meridian, under which the lunar distances are taken, by adding to it the difference of longitude in time since the observation of the altitudes, if the ship has made east longitude, or subtracting it if she has been sailing westward.

But as there may not be always a sufficient number of observers, the following method given in Norie's *Navigation* will be found very convenient where there are only two.

Let the observations be taken in the following order, noting the times by a watch: 1. the altitude of the sun or star; 2. the altitude of the moon; 3. any number of distances; 4. the altitude of the moon; 5. the altitude of the sun or a star. Now add together the distances, and the times when they were taken, each of which, being divided by the number observed, will give the mean time and distance: then to reduce the altitudes to the mean time, say, as the difference of times between the observations is to the difference of their altitudes, so is the difference between the time that the first altitude was taken and the mean time, to a fourth number; which added to, or subtracted from the first altitude, according as it is increasing or decreasing, will give the altitude reduced to the mean time.

Of the oblique semi-diameter.

As the vertical semi-diameters of the sun and moon are diminished by refraction, it becomes necessary to find their semi-diameters in the direction of the observed distance in order to obtain the apparent distance of their centres. Dr. Thomas Young in his comparative view of the principal methods of correcting lunar observations before mentioned, has given two very convenient tables for this purpose which we have taken the liberty to copy, and we shall add an account of them nearly in the author's own words.

"The observed distance requires to be corrected, when the altitudes are small for the effect of refraction on the oblique semi-diameters, which may amount to several seconds, especially in low latitudes, where the ecliptic approaches most frequently to a vertical position. The magnitude of this correction depends on the angle of the triangle concerned at the luminary in question, which is required to be known when we employ the tables of this correction which have been published. It will, however, be more convenient to have an easy method of deducing the correction more immediately from the elements observed; and for this purpose the minutes and seconds may be neglected, the nearest degrees only being employed for entering the tables. The first table affords by adding together the numbers taken from it with different arguments, a logarithm of the sine of half the angle at the luminary, multiplied by the cosine of the altitude, and the second table being entered with this argument (A), and with the altitude of the luminary gives the correction which is to be subtracted from the semi-diameter to obtain the oblique semi-diameter.

"The arguments of the first table are d, the distance, or its supplement, h, the half sum of the distance and both altitudes and $h—s$, the same half sum diminished by the altitude of the *opposite* luminary."

USE OF THE TABLES.

EXAMPLE I.

Let h, = 32°, $h—s$ = 21°, d = 44°, and the altitude 11°. Entering table 1 in the column "For h," opposite 32° we find 93, and opposite 21°, in the column $h—s$ on the left hand, we have 55; again with 44° in the column h, we find 16. The sum of these three numbers is 164, with which entering table 2, in the column "sum of A," and under 11° we find O, shewing that the correction required is less than one second.

EXAMPLE II.

Let the ☽'s altitude = 41°, *the sun's altitude* = 19°, *and their distance* 104°, *required the correction for the oblique semidiameters.*

First, h, =82°, $h—s$ = 63°, d =76°, (the supplement to 104°), and the ☽'s altitude 41°. Then 14+95+1 = 110, with which, entering table 2, under 41°, we find 0, implying that no correction is necessary to the ☽'s semi-diameter.

Next for that of the sun, h = 82°, $h—s$ = 41°, d = 76° and the ⊙'s altitude 19°. Then 14+82+1 =97; with which, entering table 2, under the altitude 19°, we find $1''$ the quantity to be subtracted from the ⊙'s semi-diameter.

EXAMPLE III.

Let the ☽'s altitude be 8°, *that of the* ⊙ 36°, *and the distance* 31°. Then for the correction of the ☽'s semi-diameter we have, h= 37°, $h—s$ = 1°, d=31°, and the ☽'s altitude 8°, which from table 1 we obtain 90+924+29=1043 or 43 rejecting the 1000, and from table 2, with 43, and the ☽'s altitude 8°, we get $10''$, the correction to be subtracted from the ☽'s semi-diameter.

For the sun we have h = 37°, $h—s$ = 29°, d=31°, and the ⊙'s altitude 36°, with which we get from table 1, 90+69+29=188, and from table 2, $1''$ for the diminution of the ⊙'s semi-diameter.

Having the observed distance of the limb of the moon from that of the sun, or from a fixed star, together with their apparent altitudes, the apparent time at the ship, the latitude and the estimated longitude to find the true distance of their centres.

RULE.

With the apparent time at the ship and the estimated longitude, find the apparent time at Greenwich, for which take from the *Nautical Almanac* the horizontal semidiameters of the sun and moon, and the ☽'s horizontal parallax.

Find the apparent and the true altitudes of the centres in the manner which has been before fully explained.

To the observed distance apply the semi-diameters of the sun and moon, that of the moon being previously corrected for augmentation and both reduced to the oblique semi-diameters. The result will be the apparent distance of the centres.

Add together the apparent distance and the apparent zenith distances, and subtract each of the two latter from the half sum, noting the remainders.

Add together the sines of these two remainders, the arithmetical complements of the sines of the apparent zenith distances, and the sines of the true zenith distances.

From half the sum of these six logarithms subtract the sine of half the difference of the true zenith distances, and the remainder is the tangent of an arc.

Subtract the sine of this arc from the said half sum, and the remainder is the sine of half the true distance.

EXAMPLE.

Let the latitude be 35°; the observed distance of a star from the moon's nearest limb, 30° 57′ 12″; the observed altitude of the lower limb, 8° 13′ 50″; and that of the star, 35° 43′ 50″; the height of the eye being 16 feet, the barometer 28·7 inches, and Fahrenheit's thermometer 78°; the moon's horizontal semi-diameter at the time being 16′ 22″. and the horizontal parallax, 60′ 0″. Required the true distance.

		°	′	″
Observed altitude of ☽'s L. L.		8	13	50
Depression of the horizon (table 1.)	−		3	50
		8	10	0
☽'s horizontal semi-diameter	+		16	22
Augmentation (table 2.)	+		0	2
Diminution of vertical semi-diameter	−		0	11
Apparent altitude, ☽'s centre		8	26	13
Refraction	−		6	17
Barometer	+		0	35
Thermometer	+		0	56
Parallax in altitude	+		59	18
☽'s true altitude		9	20	45

		°	′	″
☽'s semi-diameter		0	16	24
Correction for obliquity, (tables 1 & 2.)	−		0	10
Oblique semi-diameter		0	16	14
Observed distance		30	57	12
Apparent distance of ✱ from ☽'s centre		31	13	26

		°	′	″
Star's observed altitude		35	43	50
Depression of horizon	−		3	50
Star's apparent altitude		35	40	0
Refraction			1	19
Barometer	+		0	3·5
Thermometer	+		0	4·5
Star's true altitude		35	38	49

☽'s equal parallax	60′ 0″		
Reduction for latitude (table)	− 4		
Horizontal parallax	59 56	P. log.	10·4776
☽'s altitude 8° 20′		cosine	9·9954
Parallax in altitude, 59′ 15″		p log.	·4822

	°	′	″		
Apparent distance	31	13	26		
☽'s apparent zenith distance	81	33	47	co. ar. sine	0·0047256
✱'s apparent zenith distance	54	20	0	co. ar. sine	0·0902179
sum	167	7	13		
½ sum	83	33	36		
1st remainder	1	59	49	sine	8·5421554
2d remainder	29	13	36	sine	9·6886564
☽'s true zenith distance	80	39	15	sine	9·9941966
✱'s true zenith distance	54	21	11	sine	9·9098894
				sum	38·2298413
				½ sum	19·1149206
½ diff. true zenith distances	13	9	2	sine	9·3570017
Tangent of arc	29	47	57		9·7579189
Same arc	29	47	57	sine	9·6963226
½ true distance	15	11	58	sine	9·4185980
True distance	30	23	56		

The preceding method is that given by Dr. Kelly, in his excellent *Practical Introduction to Spherics.* It has been preferred to that of Borda, from sines being employed instead of cosines; as the former in consequence of their increasing instead of decreasing, are perhaps more readily taken from a table than the latter. It is the *direct* method of solution, and the result is perfectly accurate, if the data are correct.

It may here be proper to urge the necessity of an habitual attention to those commonly supposed minor corrections, which have hitherto been neglected, especially those depending on the barometer and thermometer; the omission of which would, in the present instance, have occasioned an error of upwards of one minute in the moon's distance, corresponding to more than 30 minutes of longitude.

As the direct method requires the logarithmic sines, &c. to be taken out to every second, various approximate methods of solution have been devised. Most of these, however, have depended upon tables calculated for the purpose, and many are very inconvenient in practice. The two following are free from these objections, and possess the peculiar, and very desirable, advantage of the result being but little influenced by errors in the observed altitudes. The author of the first is not known to the writer of this article; but the second method may be found in the Journal of the Royal Institution, among the valuable astronomical and nautical collections, which are believed to be due to Dr. Thomas Young.

In the first method, the author observes that it is preferable to attend less to abbreviate the methods of computation, than to investigate a solution, in which considerable errors in the data may produce but a small error in the required distance. When either of the luminaries has a small elevation, its altitude will be affected by the uncertainty of refraction, and the inaccuracy necessarily attending altitudes observed at sea. The sum of these errors, when they tend the same way, may be supposed to amount to at least one minute in altitude; which, in many cases, according to the common rules for computing the true distance will produce an error of about 30 minutes in the longitude. Perhaps the only method of determining the distance, so as not to be affected by the errors of altitude, is that, by first finding the angles at the sun and moon, and by the help of them, the corrections of distance for parallax and refraction.

Rule.

1. Add together the moon's apparent zenith distance, the sun's apparent zenith distance, and the apparent distance of their centres. From the half sum subtract the moon's zenith distance, and call it the first remainder. From the half sum subtract the star's zenith distance, and call it the second remainder. Add the arithmetical complements of the sines (or the co-secants) of the sun's zenith distance, and the apparent distance of the centres, to the sines of the half sum, and of the first remainder: half the sum of these four logarithms is the cosine of half the angle at the sun

2. Add the arithmetical complements of the sines (or the co-secants) of the moon's zenith distance, and the apparent distance of the centres, to the sines of the half sum, and of the second remainder. Half the sum of these four logarithms is the cosine of half the angle at the moon.

3. To the cosine of the angle at the moon add the logarithm of the difference between the moon's refraction, and her parallax in altitude in seconds. The result will be, the logarithm in seconds of the first correction, which is to be subtracted from the observed distance, if the angle at the moon is acute, but added if it is obtuse.

4. To the cosine of the angle at the sun, add the logarithm of the difference between the sun's refraction and parallax in seconds. The sum is the logarithm in seconds of the second correction. To be added when the angle at the sun is acute, but subtracted when obtuse.

5. To the sine of the angle at the moon, add the beforementioned logarithm of the difference between the moon's parallax and refraction, and multiply the sum by 2. From this subtract the logarithm of twice the distance of the centres in seconds increased by the radius. The result will be the logarithm in seconds of a third small correction, which is also to be added.

Note.—The third correction is generally very minute, except when the distance is small, and the angle at the moon nearly a right angle.

It is sufficient to use the zenith distances, and the apparent distances to the nearest minute only, as a small error in the angles at the sun and moon will very little affect the corrections of distances. If a star is observed, the only change necessary is to employ the refraction of the star, instead of the difference between the refraction and parallax of the sun.

Examples.

Taking, as in the last example, the moon's zenith distance, 81° 34′; *the star's zenith distance,* 54° 20′; *and the apparent distance,* 31° 13′ 26″; *the difference between the* ☽*'s parallax in altitude and refraction,* 54′ 32″; *and the star's refraction,* 1′ 11″; *to find the true distance.*

For ☽'s angle.			°	′	For ✱'s angle.	
10·00472	sin. co. ar.....	☽'s zen. dist	81	34		
		✱'s zen. dist	54	20	sine co. ar.	10·09022
10·28523	sine co. ar.....	Distance	31	14	sine co. ar.	10·28523
		Sum..........	167	8		
9·99726	sine..........	½ sum	83	34	sine	9·99726
		1st rem.	2	0	sine	8·54282
9·68875	sine..........	2d rem.	29	14		18·91553
19·97596					cosine ..	9·45776
9·98798	cosine					

½ angle..13° 25′
2
26 50 ☽ s ∠ sine 9·65456
cos.......9·95052
3272″ log...3·51481 .... 3·51481
−2920″ log...3·46533 13·16937
2
26·33874
Dist. × 2 = 224880″ log.... 15·35195
+ 9″·7 log.... 0·98679

½ angle..73° 20′
2
146 40 ✱'s ∠
cos. .... 9·92194
71″ log. .... 1·85126
− 59″·3 log. 1.77320

Apparent distance	31°	13′	26″
First correction........	—	48	40
Second correction	— 0		59·3
Third correction	+ 0		9·7
True distance......	30	23	56·4

This result is the same as that obtained by the *direct* method.

Let the moon's apparent zenith distance be 80° 22′ ; *the star's apparent zenith distance,* 78° 43′ ; *and the apparent distance of the centres,* 43° 36′ ; *the difference between the moon's parallax in altitude and refraction,* 58′ 10″, (= 2910″) ; *and the star's refraction,* 4′ 40″, (= 280″) ; *required the true distance.*

For ☽'s angle.		° ′	For ✱'s angle.	
10·00617 sine co. ar.	☽'s apparent zenith distance ..	80 22		
	✱'s apparent zenith distance ..	78 43	sine co. ar.	10·00848
10·16139 sine co. ar.	Distance	43 36	sine co. ar.	10·16139
	Sum	202 41		
9·99145 sine	½ sum	101 20	sine	9·99145
	1st rem.	20 58	sine	9·55367
9·58497 sine	2d rem.	22 37		
				19·71499
19·74398				
			cosine	9·85749
9·87199 cosine				

½ angle.. 41° 52′
2

83 44′ ☽'s ∠ sine.... 9·99740
cos..... 9·03805
2910″ log. .. 3·46389.......... 3·46389
−318″ log. .. 2·50194 13·46129
2
26·92258
Dist. × 2 = 313920″ log. .... 15·49679
+ 26″·6 .... 1·42579

½ angle.. 43° 55′
2

87 50 ✱'s ∠
cos.... 8·57757
280″ log. .. 2·44716
+10″·5 log. .. 1·02473

	°	′	″
Apparent distance	43	36	0
First correction	—	5	18
Second correction	+	10	5
Third correction	+	26	6
True distance	43	31	19·1

The second method which is taken from the Journal of the Royal Institution, is as follows :

Rule.

1. Add together the apparent altitude of the star, the apparent altitude of the moon, and the apparent distance ; and from half the sum subtract the apparent altitude of the star, and call the result the remainder.

2. To the proportional logarithm of the moon's horizontal parallax, add the logarithmic secant of the moon's altitude ; the sum will be the proportional logarithm of the parallax in altitude.

3. Add together the logarithmic secant of the half sum, the sine of the apparent distance, the co-secant of the remainder, the constant logarithm 9·6990, and the proportional logarithm of the horizontal parallax. The sum will be the proportional logarithm of the diminution of parallax ; the difference between which, and the parallax in altitude, is the *parallactic correction,* which is to be subtracted from the observed distance, if the parallax in altitude is the greater ; but if otherwise, to be added.

4. To the logarithmic sine of the moon's altitude, add the proportional logarithm of the parallactic correction, and subtract the sum from the proportional logarithm of the horizontal parallax. The difference will be the tangent of an angle, which is the refractional distance for the moon. Consider, as zenith distances, both this angle, and the difference between it and the apparent distance ; and find the corresponding refractions in a table of refraction. These will be the first and second refractional corrections ; the first of which is to be applied to the apparent distance, with a contrary sign to that of the parallactic correction, and the second to be added when the refractional distance for the moon is less than the observed distance ; otherwise the second refractional correction must be subtracted.

5. Add together, the proportional logarithm of the diminution of parallax, that of the sum of the parallactic correction and the parallax in altitude, the logarithmic tangent of the apparent distance, and the constant logarithm 1·5870. The sum will be the proportional logarithm of the correction for obliquity which is always additive.

Example.

Taking, as in the last example, the apparent distance, 43° 36′; *the apparent altitude of the moon,* 9° 38′; *that of the star,* 11° 17′; *the moon's horizontal parallax,* 54′ 42″; *and its proportional logarithm,* ·5173 : *to find the true distance.*

1	2	3	4	5
	Parallax in altitude.	Parallactic correction.	Refractional correction.	Cor. for obliquity.
✳'s apparent altitude 11° 17′ ″		const. log. . 9·6990		const.log. 1·5870
☽'s apparent altitude 9 38	secant . . . . 0·0612	. . . .	sine . . . . 9·2236	. . . .
Apparent distance . . 43 36	. . . .	sine . . . . . . 9·8386	. . . .	Tangent 9·9788
Sum 64 31				
½ sum 32 15	. . . .	secant . . . . 0·0728	. . . .	. . . .
Remainder . . 20 58	. . . .	co-secant . . 0·4462	. . . .	. . . .
Horizontal parallax 54 42	P. log. . . . . 0·5173	P. log. . . . . 0·5173	. . . .	. . . .
Parallax in altitude. .	53′55″P.log. 0·5785			
	Dimin. of parallax	48′ 1″P.log. 0·5739	. . . .	P. log. . . 0·5739
	Parallax in altitude	53 55		
Diff. = parallactic correction −		5 54	P. log. . . . . . . 1·4844	
Parallactic correction + Parallax in altitude		59 49	. . . .	P. log. . . 0·4784
			Sum 0·7080	
			Hor.par.P.log. 0·5173	
Refractional distance for ☽ . . . . 32° 48′			Tang . . . . 9·8093	
Apparent distance 43 36				
Difference . . 10 48				26″ P.log. 2·6181

For zenith distance 32° 48′ tab. refraction = 37″·5 } Refractional corrections.
. . . . 10 46 . . . . = 11·0 }

		°	′	″
Apparent distance		43	36	00
Parallactic correction . .	−		5	54
1st refractional correction	+		0	37·5
2d refractional correction	+		0	11·0
Correction for obliquity . .	+		0	26·0
True distance		43	31	20·5

Having the true central distance between the moon and the sun, or a fixed star, and the apparent time at the ship; to find the longitude.

Rule.

Among the true distances of the moon's centre from the sun, or fixed stars, in the *Nautical Almanac,* find those two distances, which are next less and greater than the true given distance, which true distance place under it. Take the difference between the true distance and the first of the two distances taken from the *Almanac,* and also the difference between the two distances. From the proportional logarithm of the first difference subtract the proportional logarithm of the second difference; the remainder will be the proportional logarithm of a portion of time; which, added to the time corresponding to the first of the two differences taken from the *Nautical Almanac,* will give the apparent time of observation at Greenwich.

Take the difference between this time and the apparent time of observation at the ship, which, being converted into degrees, will give the longitude of the ship; east, if the time at the ship be greater than that at Greenwich, otherwise west.

Examples.

Let the true distance of the moon's centre from α Aquilæ, on the 1st of March, 1823, *be* 82° 25′ 41″; *the corresponding apparent time at the ship being* 12ʰ 4ᵐ 16ˢ : *required the longitude.*

	° ′ ″	Diff. ° ′ ″	
True distance	82 23 41	0 59 4	P. log. 4839
Distance at night, from *Naut. Alm.*	83 34 35		
Dist. at 15ʰ from Do.	82 14 6	1 0 39	P. log. 4724

	h	m	s		
Time from midnight	2	55	18....	P. log...0115	
	12	0	0		
App. time at Greenwich	14	55	18		
App. time at the ship....	12	4	16		
Longitude west in time..	2	51	2	= 30° 13′ 15″	

The true distance of the moon's centre from Aldebaran the 20th of March, 1823, was 35° 50′ 24″; the corresponding apparent time at the ship being 7ʰ 5ᵐ 21ˢ. What was the ship's longitude?

	°	′	″	Diff.	
True distance	35	50	24	° ′ ″ 0 5 51	P. log. 1·4881
Dist. at 6ʰ from *Naut. Almanac*	35	44	33		
Dist. at 9ʰ from Do.	37	26	15	1 41 42	P. log. 0·2480
	h	m	s		
Time from midnight....	0	10	21....		P. log. 1·2401
	6	0	0		
App. time at Greenwich	6	10	21		
App. time at the ship ..	7	5	21		
Longitude east in time ..	0	55	0	= 13° 45′ 0″	

VARIATION OF THE COMPASS.

To find the variation of the compass by observed magnetic amplitude of the sun, or of a star.

The solution of this problem depends upon finding the hypothenuse of a right angled spherical triangle, in which one side (the declination of the sun or star) and an angle (the co-latitude) are given.

RULE.

(From Kerrigan's *Young Navigator's Guide.*)

From the logarithmic sine of the declination, the index being increased by 10, subtract the logarithmic cosine of the latitude, the remainder is the logarithmic sine of the true amplitude, to be reckoned north or south according to the name of the star's declination; then, if the true and observed amplitudes be both north, or both south, their difference is the variation; but if one is north and the other south, their sum is the variation. And to know if it be east or west, let the observer look directly towards the compass representing the true amplitude; then, if the observed amplitude is to the left of the true, the variation is easterly; but if to the right, it is westerly.

Note.—The magnetic bearing of the object must be taken when its apparent altitude is equal to the depression of the horizon.

EXAMPLES.

The latitude being 18° 40′ north, the star Sirius was observed to set 24° 10′ to the southward of the west point of the compass. Required the variation.

Declination of Sirius..	16° 27′ S	sine ..	19·45206
Latitude............	18 40	cosine	9·97653
True amplitude....W	17° 23′ S	sine ..	9·47553
Magnetic amplitude W	24 10 S		

Variation ...... 6 47 which is east, because the observed amplitude is to the left hand of the true.

In latitude 48° 20′ north, the star Rigel was observed to set 9° 50′ to the northward of the west point of the compass; required the variation.

Declination of Rigel	8° 25′ S	sine ..	19·16545
Latitude	48 20	cosine	9·82269
Star's true amplitude W	12° 43′ S	sine ..	9·34276
Magnetic amplitude .. W	9 50 N		

Variation ...... 22 33 which is west, because the observed amplitude is to the right hand of the true amplitude.

To find the variation of the compass, having the true altitude of the sun or of a fixed star, and its magnetic azimuth, the latitude of the ship and the declination being known.

To compute the true azimuth, we have given the three sides of a spherical triangle; viz. the co-latitude, the zenith distance, and the polar distance, to find the angle at the zenith between the zenith distance and the co-latitude.

RULE.

Add together the polar distance, the zenith distance, and the co-latitude, and take half the sum; the difference between which and the polar distance call the remainder.

To the arithmetical complements of the sines (or the co-secants) of the zenith distance and co-latitude, add the sines of the half sum and of the remainder. Half the sum of these four logarithms will be the cosine of half the true azimuth to be reckoned from the north in north latitude, but from the south in south latitude.

Then if the true and observed azimuths are on the same side of the meridian, their difference is the variation; but if on different sides, the sum is the variation. Now, if the observed azimuth is to the left of the true, the variation will be easterly; but if the right, the variation will be westerly.

EXAMPLE.

In latitude 48° 50′ north, the true altitude of the sun's centre was 22° 2′; the declination being 10° 12′ south, and its magnetic bearing north 161° 32′ east. Required the variation.

Polar distance	100 12		
☉'s zenith distance ..	67 58	co. ar. sine..	0·03294
Co-latitude	41 10	co. ar. sine..	0·18161
sum	209 20		
½ sum......	104 40	sine	9·98561
Remainder..	4 28	sine	8·89142
			19·09158
	69° 26′	cosine..	9·54579
	2		
☽'s true azimuth ....N	138 52 E		
Magnetic or observed azimuth ....N	161 32 E		

Variation ...... 22 40 which is west, because the observed azimuth is to the right of the true.

OF THE INSTRUMENTS EMPLOYED IN NAUTICAL ASTRONOMY.

THE SEXTANT.

The sextant is an instrument so well known, that a detailed account of it would be superfluous; as an examination of the instrument itself for a few minutes will probably convey a better idea of its construction, than the most elaborate description.

The object is reflected from the index-glass to the horizon-glass, and from thence to the eye. The horizon-glass is half silvered, and a telescope being properly directed towards it, and parallel to the sextant, the observer is enabled to see the horizon of the sea through the unsilvered part of the glass, at the same time that he views the reflected image of the object in that part which is silvered. By moving the index, and consequently the index-glass, the reflected image appears to change its place, and may be made apparently to coincide with the horizon of the sea. Dark glasses are placed so that they may be interposed between the index-glass and the horizon-glass, in observations of the sun; and there is a screw at the back of the sextant acting upon the telescope, by which it may be brought nearer to, or removed further from the instrument, for the purpose of rendering the reflected image more or less bright.

The glasses of a sextant should be perfectly plane, and the surfaces of each parallel. In order to examine this, place the dark glasses before the horizon-glass so as to prevent seeing through it, and interpose as many dark glasses as may be found necessary to weaken sufficiently the reflected image of the sun. Place the index at zero. Direct the telescope to the sun, and make the image as distinct as possible, by moving the eye piece, when the image should appear neat and well defined: shift the index to 120°, keeping the sun in the telescope during the motion of the index. If now the image of the sun appears sharp, round, and well defined, the surfaces of each glass are parallel; but if the edge appears indistinct in any part, and a projection like a faint second image is seen, the glasses are bad, and the instrument should be rejected.

Adjustments of the sextant.

The adjustments of a sextant are to set the index and horizon glasses perpendicular to the plane of the instrument, and their planes parallel to each other when the index division is at O, upon the arch. Also to set the axis of the telescope parallel to the plane of the instrument.

To set the index-glass perpendicular to the plane of the sextant.

Move the index forward to about the middle of the arch; then holding the instrument with the limb from the observer, look obliquely into the index-glass, so as to see the arch by reflection; and remark if the reflected and the true arch appear in the same plane. If they do, the index-glass is perpendicular to the plane of the instrument; but if otherwise, it is to be rectified by means of the screws at the back of the glass.

To set the horizon-glass perpendicular to the plane of the sextant.

Screw on the dark glass to the eye-piece of the telescope, and directing it to the sun, make the two images pass over each other. If they do not appear perfectly to coincide in passing, turn the screw, which is under the horizon-glass, till this is effected. The same adjustment may be made by means of a star.

To find the index error.

When the index is at O, the images seen by reflection and by direct vision should coincide. In the best sextants there is no adjustment for this purpose, but the error is to be found and to be applied to the observed angle. The best method of determining the index error is to measure the diameter of the sun by moving the index forward and backward on the arch. Half the difference of these diameters will be the index-error, which must be subtracted from the observations when the diameter taken forward is the greater, otherwise it must be added. This correction should be frequently examined.

To set the axis of the telescope parallel to the plane of the sextant.

If the axis of the telescope is not parallel to the plane of the sextant, it may occasion a considerable error when the angle is large. In order to make this adjustment, place the wires which are in the telescope parallel to the plane of the instrument. Then select two well defined objects, of which the best are the moon and a star, the angle between which must not be less than 90 degrees. Bring their images exactly into contact at the wire which is nearest to the sextant; then, by altering a little the position of the instrument, bring them upon the wire furthest from the sextant. If the images still continue in contact, no adjustment is required; but if they appear to separate at the wire which is furthest from the instrument, it is a proof that the object end of the telescope is too near to the sextant, and it must be altered by means of the screws in the ring which holds the telescope.

By repeating this operation a few times, the contact will remain perfect upon both wires, and consequently the axis of the telescope will be parallel to the plane of the sextant.

USE OF THE SEXTANT.

(From Norie's *Navigation.*)

To observe the angular distance between the moon and the sun.

"Screw on the telescope, and place the wires parallel to the plane of the instrument; then turn down one or more of the dark glasses according to the brightness of the sun; place the index at O on the limb, and if the sun's image be very bright, turn up the dark glasses before the index-glass, and with the screw raise the telescope to the transparent part of the horizon-glass. Having done this, hold the sextant so that the plane may be in the right line joining the two objects. If the sun be to the right hand of the moon, the sextant is to be held with its face upwards; but if it be to the left hand, the face is to be held downwards. With the instrument in this position, look directly at the moon through the telescope, and move the index forward till the sun's image comes nearly in contact with the moon's nearest limb. Then fix the index, by the screw under the sextant, and make the contact perfect by means of the adjusting screw; at the same time move the sextant slowly, making the axis of the telescope the centre of motion, by which the objects will pass each other, and the contact be more accurately made. The index will then point out the observed distance of the nearest limbs of the sun and moon.

It will perhaps be more easy for those who are not accustomed to make observations, to find the distance nearly; and setting the index forward to it, to look directly towards the moon, holding the instrument as before; the sun will then appear nearly in contact with it, and is to be made perfect by the method above mentioned. In the *Nautical Almanac* the distances of the sun and moon are set down for the beginning of every third hour of time at Greenwich, on such days as the moon is not less than 35 degrees, nor more than 125 degrees from the sun, and may be found for any intermediate time by proportion. From these distances, their distance at the time of observation may be computed roughly thus:—Find the time at Greenwich nearly; then find by the *Nautical Almanac* the distance nearly at that time, from which subtract 30 minutes, the sum of the semidiameters nearly, and the remainder will give the distance of their nearest limbs at the time of observation.

It will save some trouble, and serve the purpose of finding the reflected image of the sun or moon in the horizon-glass, if you only set the index to the central distance as set down in the *Nautical Almanac* for the nearest three hours without correcting it, to the intermediate time by a rough computation.

To observe the distance between the moon and a star.

Turn down the lightest skreen before the horizon-glass, and direct the telescope to the star, holding the sextant in its proper position; so that if seen edgewise, it may appear to pass through the moon and star, with its face upwards or downwards, according as the star is to the eastward or westward of the moon; then move the index forward till the reflected image of the moon is seen in the telescope. By moving the instrument slowly up and down, the moon will appear to rise and fall by the star; the enlightened limb is then to be brought in contact with the star by means of the adjusting-screw; and the number of degrees and minutes pointed out by the vernier will give the observed distance between the objects.

If the distance between the moon and one of the stars set down in the *Nautical Almanac* for finding the longitude is to be observed, their distance may be roughly calculated as before directed, to which set the index: then look through the telescope, and direct the sight to the star, which is generally a bright one, and lies in a line nearly perpendicular to the horns of the moon either to the eastward or westward, as denoted in the *Almanac;* and holding the sextant in the plane of the two objects, give it a slow motion round the axis of the telescope; and if the moon's image comes into the field of the telescope, it is a proof the right star has been taken, as no other in that direction will correspond to it in distance.

The comparative lightness of the sextant has caused it frequently to be preferred to the reflecting circle, but whenever great accuracy is desired, the latter is far superior; for the sextant having but one vernier, any error which arises from eccentricity remains undiscovered, whilst in the circle this error is corrected by taking a mean of the readings of the several verniers.

Description of Troughton's reflecting circle.

This instrument, the production of one of the most celebrated of the English artists, combines considerable strength with lightness. It is composed of a circular rim screwed to a cross bar frame, to the middle of which is fixed a hollow centre A, (plate 2, fig. 1.) upwards of two inches long, and projecting from the back of the frame. This hollow centre receives an axis, to one extremity of which is firmly fixed the index, having three branches at equal distances from each other, and to the other extremity of the axis is attached the index-glass. Each of the three branches terminates in a vernier, and one of them carries the usual apparatus for quick and slow motion, which of course governs also the other two.

On the back of the circle on that side where the index-glass is placed, is a kind of secondary frame BB, attached to the hollow centre, near the index-glass, and also by the pillars to the main frame.

Upon this secondary frame are firmly fixed the horizon-glass and the telescope; on the same frame are also placed two sets of coloured glasses, one between the index and horizon glasses, and the other beyond the horizon-glass. The distance between the two frames affords a relief for the dark glasses to be turned down round a joint, so as to place them out of the way when they are not wanted. The telescope screws into a collar, the square stem of which having its four angles worked into a screw, passes through a barrel, which connects the main and secondary frames. By means of a finger screw upon the stem, the distance of the telescope from the frame of the instrument, may be varied at pleasure; and the two objects observed, may be made of equal brightness, even without removing the eye from the telescope. The collar into which the telescope screws has the usual adjustment for placing the line of collimation parallel to the plane of the circle. A handle, D, appears on the divided side of the instrument, and is attached to a brass tube, which bends over the edge of the circle, and is fixed to the instrument on the same side as the glasses. E, is another handle screwed fast to the frame on the back of the instrument. By means of these handles the circle may be conveniently held in any position.

The telescope being fixed near one edge of the circle, and the horizon-glass near the opposite edge, the rays of light from the object may fall upon the index-glass on either side of the horizon-glass; by which means the great advantage is obtained, of observing on both sides of zero.

It has been before remarked, that in the sextant, from its having only one vernier, any error arising from eccentricity is uncorrected; but in the circle, by taking a mean of the readings of the three verniers, not only is this error removed, but any errors of division are reduced to about one-sixth part. Also, when the angle measured is 120 degrees, in a backward and forward observation, the three readings will just occupy the 720 degrees of this circle.

In Troughton's circles, as they are generally made, the degrees are subdivided into three parts, of 20 minutes each, and each vernier subdivides again to 20 seconds.

The following printed instructions for its use are generally given with the instrument.

Directions for observing with Troughton's reflecting circle.

"Prepare the instrument for observation by screwing the telescope into its place, adjusting the drawer to focus, and the wires parallel to the plane exactly as you do with a sextant: also set the index forwards to the rough distance of the sun and moon, or moon and star; and holding the circle by the short handle, direct the telescope to the fainter object, and make the contact in the usual way. Now read off the degree, minute, and second, by that branch of the index to which the tangent screw is attached; also the minute and second shewn by the other two branches; these give the distance taken on three different sextants; but as yet it is only to be considered as half an observation. What remains to be done is to complete the whole circle, by measuring that angle on the other three sextants. Therefore set the index backwards nearly to the same distance, and reverse the plane of the instrument, by holding it by the opposite handle, and make the contact as above, and read off as before what is shewn on the three several branches of the index. The mean of all six is the true apparent distance, corresponding to the mean of the two times at which the observations were made.

"When the objects are seen very distinctly, so that no doubt whatever remains about the contact in both sights being perfect, the above may safely be relied on as a complete set; but if, from the haziness of the air, too much motion, or any other cause, the observations have been rendered doubtful, it will be adviseable to make more: and if at such times so many readings should be deemed troublesome, six observations, and six readings may be conducted in the following manner. Take three successive sights forwards exactly as is done with a sextant; only take care to read them off on different branches of the index: also make three observations backwards, using the same caution; a mean of these will be the distance required. When the number of sights taken forwards and backwards are unequal, a mean between the means of those taken backwards and those taken forwards, will be the true angle.

"It need hardly be mentioned that the shades, or dark glasses apply like those of a sextant for making the objects nearly of the same brightness; but it must be insisted on, that the telescope should on every occasion be raised or lowered by its proper screw for making them perfectly so.

"The foregoing instructions for taking distances apply equally for taking altitudes by the sea or artificial horizon, they being no more than distances taken in a vertical plane. Meridian altitudes cannot however be taken backwards and forwards in the same day because there is not time: all therefore that can be done is, to observe the altitude one way, and use the index error; but even here you have a mean of that altitude, and this error taken on three different sextants. Both at sea and land, where the observer is stationary, the meridian altitude should be observed forwards one day, and backwards the next, and so on alternately from day to day; the mean of latitudes, deduced severally from such observations, will be the true latitude; but in these there should be no application of index error; for that being constant, the result would in some measure be vitiated thereby.

"When both the reflected and direct images require to be darkened, as is the case when the sun's diameter is measured, and when his altitude is taken with an artificial horizon, the attached dark glasses ought not to be used; instead of them, those which apply to the eye end of the telescope will answer much better, the former having their errors magnified by the power of the telescope, will, in proportion to this power, and those errors, be less distinct than the latter.

"In taking distances when the position does not vary from the vertical above thirty or forty degrees, the handles which are attached to the circle are generally most conveniently used; but in those which incline more to the horizontal, that handle which screws into a cock on one side, and into the crooked handle on the other, will be found more applicable.

"When the crooked handle happens to be in the way of reading one of the branches of the index, it must be removed for the time, by taking out the finger screw which fastens it to the body of the circle.

"If it should happen that two of the readings agree with each other very well, and the third differs from them, the discordant one must not on any account be omitted, but a fair mean must always be taken.

"It should be stated, that when the angle is about thirty degrees, neither a distance of the sun and moon, nor an altitude of the sun with the sea horizon, can be taken backwards; because the dark glasses at that angle prevent the reflected rays of light from falling on the index-glass; whence it becomes necessary, when the angle to be taken is quite unknown, to observe forwards first where the whole range is without interruption; whereas, in that backwards, you will lose sight of the reflected image about that angle. But in such distances where the sun is out of the question, and when his altitude is taken with an artificial horizon, (the shade being applied to the end of the telescope) that angle may be measured nearly as well as any other; for the rays incident on the index-glass will pass through the transparent half of the horizon-glass, without much diminution of their brightness.

"The advantages of this instrument, when compared with the sextant, are chiefly these:—The observations for finding the index error are rendered useless; all knowledge of that being put out of the question, by observing both forwards and backwards. By the same means, the errors of the dark glasses are also corrected; for if they increase the angle one way, they must diminish it the other way by the same quantity. This also perfectly corrects the errors of the horizon-glass, and those of the index-glass very nearly. But what is of still more consequence, the error of the centre is perfectly corrected, by reading the three branches of the index; while this property combined with that of observing both ways, probably reduces the errors of dividing to one-sixth part of their simple value. Moreover, angles may be measured as far as 150 degrees, consequently the sun's double altitude may be observed when his distance from the zenith is not less than fifteen degrees; at which altitude, the head of the observer begins to intercept the rays of light incident on the artificial horizon; and, of course, if a greater angle could be measured, it would be of no use in this respect.

"This instrument, in common with the sextant, requires three adjustments. First, the index-glass perpendicular to the plane of the circle. This being done by the maker, and not liable to alter, has no direct means applied to the purpose: it is known to be right when, by looking into the index-glass, you see that part of the limb which is next you reflected in contact with the opposite side of the limb, as one continued arc of a circle: on the contrary, when the arc appears broken, where the reflected and direct parts of the limb meet, it is a proof that it wants to be rectified. The second is, to make the horizon-glass perpendicular. This is performed by a capstan screw, at the lower end of the frame of that glass; and is known to be right, when, by a sweep with the index, the reflected image of any object will pass exactly over or cover the image of that object seen directly. The third adjustment is for making the line of collimation parallel to the plane of the circle. This is performed by two small screws, which also fasten the collar into which the telescope screws, to the upright stem on which it is mounted: this is known to be right, when the sun and moon, having a distance of 130 degrees, or more, their limbs are brought in contact, just at the outside of that wire which is next to the circle; and then examining if it be the same, just at the outside of the other wire: its being so is the proof of adjustment."

A DESCRIPTION OF DOLLOND'S IMPROVED REFLECTING CIRCLE.

This instrument is represented at plate 2, fig. 2. It consists of two concentric circles; the one moving within the other, both in the same plane, and nearly in contact. The inner circle has three flat radii, two of which carry at their extremities the telescope and the horizon-glass. The outer circle is divided into degrees, &c. as near as possible to its exterior edge, and numbered in the usual manner. The interior circle has also every tenth degree numbered upon it on both sides of zero to 180, but without any other divisions. A vernier, A, is fixed to the interior circle opposite to its zero, for the purpose of reading off the degrees, &c. on the exterior circle. Near the telescope an apparatus for quick and slow motion is attached to the interior circle. An arm carrying the index-glass, and a vernier, B, moves freely round upon its own centre, and is furnished with an apparatus for slow motion, and a clamp, by means of which it may be attached to the exterior circle. The divisions upon the exterior circle may be read off by means of this vernier, but the figures corresponding to those divisions are concealed, whilst those of the interior circle are visible.

The following observations upon this instrument, and precepts for its use, are given in the maker's own words:

"The reflecting circle being constructed upon the same principles as the sextant, requires the same process for making the observations. The circle having the advantage of enabling the observer to take the angle on what is usually termed the off arc; that is, he may first take the angle precisely in the same manner as with the sextant; and then, by reversing the face of the instrument, take the angle on the off arc, or the other side of the zero. By this method he will correct any error that may arise from a want of parallelism in the glasses, or in the adjustment of the verniers to the zero; and by adding the angles so taken to each other, and dividing them by two, the correct angle will be obtained.

"There are two verniers on each index, which read from the centre division according to the arc upon which the angle is taken, and the one may be read backward as a check upon the other

"There is another and a greater advantage to be derived from the circular construction of the instrument, which is the power of repeating the observation, and thereby correcting the errors arising from the eccentricity of the centering or dividing.

"In the instrument here described there is a very considerable improvement for facilitating the operation of repeating the observations; it principally consists of the two sets of figures, one upon the inner circle, and the other upon the outer or divided circle; the first set are for reading off the angle, which is taken with the index B; and the second for reading off the number of the degrees, &c. which are to be divided by the number of repetitions.

"The method of observing with the instrument is as follows:

"If it is to be used as a sextant, the vernier A, which is attached to the inner circle, must be set to a primary division, and it will be the most correct to set it to 720,* for which purpose there is as a rough guide a line upon the outer chamfered edge of the vernier A, which corresponds with the zero of that vernier. Having adjusted this vernier correctly to 720, the angle must be taken by moving the index B forward, in the same manner as with the sextant, and the angle contained between the two objects will be seen upon the circle by the vernier belonging to that index, and the figures on the inner circle will be the guide for reading off. The index error should in this case be found as with the sextant, by taking a small angle on each side the zero. If there should be time for a second observation, the same angle may be taken with the face of the instrument reversed, by moving the index B the contrary way to the former, or upon the off arc: in this case the index error need not be attended to, as the half of the two angles will be the correct angle.

"The circle may also be used for taking a series of observations, by setting the vernier A to 720, and proceeding in the first instance by moving the index B forward, and then the inner circle, telescope, &c. and so on, each time reading off the angle, or noting the number of observations; and dividing the amount of the degrees, &c. passed through by the number of the observations; and if the number of the angles taken are equal to or nearly so, to the whole circle, it is evident that the errors of division and centering will have been taken into the account.

"When the instrument is used as a repeating instrument, the vernier A should be very correctly adjusted to 720; and the index B moved forward until the two objects, the angle of which is to be taken, are brought into contact; read off, and note the amount, then leaving the index B fast, and unclamping the telescope, &c. move the whole of those parts which are attached to the inner circle forward, until the figures on the inner circle shew the amount of the first angle, on the contrary side of their zero, bring the objects into contact, and it will here be evident that the space passed over, will be equal to twice the angle. Note down two, and without disturbing the last adjustment move the index B forward twice the angle, and bring the objects again into contact; then move the telescope, &c. forward as before, and note down two in addition to the former. In this mode of observing, if the face of the instrument is continued always one way, it will be requisite to present the telescope, first to the left hand object, and secondly, to the right; but if the face is reversed alternately, the telescope should be presented to the same object. Proceed with the observations, and each time the telescope, &c. is moved forward, note two; and when the vernier A has either approached or exceeded the whole circle, or 720, read off by that vernier, and divide the amount by the number of the angles taken, and the quotient will be the correct angle; it will here be clearly evident, that all errors of division, centering, and observation, will have been taken into the account, and therefore the angle will have been correctly obtained."

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* In the plate the index A is by a mistake set to 360; this does not affect the reasoning, as it may be equally considered the zero.

This instrument, (as well as Troughton's circle,) is furnished with a stand, which will be found very convenient in making observations on shore.

"To place the circle upon the stand, unscrew the brass nut which fastens the ball or handle, holding the instrument in the left hand with the face downwards, so that it may not slip from the handle; take off the handle carefully, and the part upon which the handle fitted will fit the hole in the end of the bent arm, near to the quick adjusting screw. The collar that goes on before the brass nut which held the handle in its place, must then be put on the same place, and the counterpoise screwed after it, which answers the purpose of the nut; it should not be screwed tight, only sufficiently so as to prevent shake, and allow the circle to turn pleasantly."

Description of an instrument for taking altitudes at sea, when the horizon is not visible.

This instrument, which is represented at plate I. fig. 9. is a common Hadley's octant of wood, having a level, A, fixed above the horizon-glass, parallel to the plane of the instrument, in such a manner that the bubble may be seen by reflexion through the tube and the fluid of the level. A projecting frame of brass, B, is also fixed to the instrument, and a part of it forms a white image, or line, in the horizon-glass. The image of the white line is always visible, and that of the bubble of the level may be seen bisected by the white line when the instrument is held vertically in the requisite position.

This very ingenious arrangement is due to Mr. Robinson, mathematical instrument maker, of Devonshire-street, Portland-place. He generally attaches the level to an octant of wood, which is sufficiently accurate for the purpose, as the angle cannot be expected to be obtained by this method nearer than to about two minutes of the truth. But it is evident that the same contrivance may be applied to a sextant.

As this instrument may be employed in the same manner as a sextant, the adjustments are similar; that is to say, the index and the horizon glasses must be placed perpendicular to the plane of the instrument.

To find the index error.

There is no telescope used with this instrument, but merely a plain tube. Screw this tube into its place, and setting the index near O, direct the tube towards the horizon of the sea, and move the instrument vertically, till the bubble of the level appears bisected by the white line. Move the index now a little backwards or forwards, till the horizon of the sea seen by reflexion appears to coincide with the white line, taking care that it at the same time bisects the bubble of the level. The index ought now to shew, (counting *backwards* from O) the depression of the horizon, known from the height of the eye and table 1. If the reading is less than the depression, the difference between them must be added, but if otherwise subtracted from the observed altitude. But should the reading be *forward*, or in the order of the degrees from O, the sum of the reading and the depression is the index error to be subtracted from the observed altitude. The mean of a number of observations should be used.

Manner of using the instrument.

Place the index at O, and keeping the plane of the instrument in a vertical position, look at the sun or star, and cause the white line to bisect it. Then move the index gradually, keeping the reflected image in view, and always bisected by the line till the bubble of the level is seen, when the index must be clamped, and gradually moved by the tangent screw till the bubble and the reflected image of the object are bisected at the same instant by the white line, when the degrees, &c. are to be read off. In this manner several observations should be made, and the mean taken; and we may here caution the observer against being discouraged by the differences he will perceive in his readings, and which may sometimes amount to 10 or 12 minutes; notwithstanding which, by taking the mean of four or six observations, the result will seldom differ more than two or three minutes from the truth. To this mean the index error and other corrections, with the exception of that for the depression of the horizon are to be applied. In observing the sun, it is preferable to take the centre instead of the limb. When altitudes of stars are taken, the level and white line must be illumined by a lanthorn held by an assistant; the light employed should be no more than sufficient to render the level and line visible.

An instrument for the same purpose was contrived some time since by Captain Kater for the arctic expedition, and was found to answer very well, though the use of it was attended with some difficulties. This instrument is made by Mr. Dollond. It consists of a circle six inches diameter divided into degrees, and is furnished with a vernier reading off to minutes. At right angles to the index, and near the vernier, a level is fixed; and upon the centre of the instrument is placed a small plane mirror, which is moveable about an axis, the mirror continuing perpendicular to the plane of the circle. To the back of the circle a telescope is fixed, furnished with cross wires, and a skreen is so placed as to shield the eyes from the sun. To use this instrument, direct the telescope to the sun or star; (if to the former, previously putting on a dark glass,) and keep the intersection of the cross wires on the centre of the object. Move the index till the level is nearly horizontal, and turn the mirror till the image of the bubble is seen in it by the left eye, whilst the sun or star is viewed with the right. Clamp the index, and move it by its tangent screw till the bubble of the level which is very active, is seen in the mirror to fluctuate backwards and forwards, whilst the cross wires are kept as steadily as possible upon the centre of the sun or star. The degrees, &c. shewn by the index are then to be registered, and the mean of several observations is to be taken. The index error may be found by taking the depression of the horizon of the sea, and inferring the error in the same manner as for the instrument last described. With this instrument, the altitude of the sun has been taken at sea, when it was so much obscured by fog as not to be visible by reflexion; and the latitude deduced from the observations made with it seldom differed more than two minutes from the truth.

In concluding this article, we may remark that care has been taken to give all that is necessary, and to avoid every thing superfluous. From among a variety of methods which tend only to embarras the student, those have been selected which are the most accurate, and at the same time practically convenient. No other tables are necessary but the few which follow, and a common table of logarithms to minutes. In the direct method, however, of computing lunar distances, the logarithms are required to single seconds; but unless the altitudes can be obtained very correctly, the approximate method given is perhaps to be preferred.

Among the works which may be consulted on nautical astronomy, may be mentioned Ducom's *Cours d' Observations Nautiques*, and Kerrigan's *Young Navigator's Guide*, both of which contain tables of logarithmic sines, &c. conveniently arranged, so that they may be readily obtained to single seconds. Kelly's *Introduction to Spherics, Tables requisite to be used with the Nautical Ephemeris*, and Professor Lax's Tables, by the aid of which last all the problems of nautical astronomy may be solved in a very neat and expeditious manner.

Proportional Logarithms to twenty-four hours, for correcting the daily differences.

Tenths.	M.or S.	H.orM 0	...1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	0		1·3802	1·0792	9031	7782	6812	6021	5351	4771	4260	3802	3388	3010	2663	2341	2041
	1	3·1584	1·3730	1·0756	9007	7764	6798	6009	5341	4762	4252	3795	3382	3004	2657	2336	2036
	2	2·8573	1·3660	1·0720	8983	7745	6784	5997	5331	4753	4244	3788	3375	2998	2652	2331	2031
	3	2·6812	1·3590	1·0685	8959	7728	6769	5985	5320	4744	4236	3781	3369	2992	2646	2325	2027
	4	2·5563	1·3522	1·0649	8936	7710	6755	5973	5310	4735	4228	3773	3362	2986	2641	2320	2022
	5	2·4594	1·3455	1·0615	8912	7692	6741	5961	5300	4726	4220	3766	3355	2980	2635	2315	2017
.1	6	2·3802	1·3388	1·0580	8889	7674	6726	5949	5290	4717	4212	3759	3349	2974	2629	2310	2012
	7	2·3133	1·3323	1·0546	8865	7657	6712	5937	5279	4708	4204	3752	3342	2968	2624	2305	2008
	8	2·2553	1·3259	1·0512	8842	7639	6698	5925	5269	4699	4196	3745	3336	2962	2618	2300	2003
	9	2·2041	1·3195	1·0478	8819	7622	6684	5913	5259	4691	4188	3738	3329	2956	2613	2295	1998
	10	2·1584	1·3133	1·0444	8796	7604	6670	5902	5249	4682	4180	3730	3323	2950	2607	2289	1993
	11	2·1170	1·3071	1·0411	8773	7587	6656	5890	5239	4673	4172	3723	3316	2945	2602	2284	1989
.2	12	2·0792	1·3010	1·0378	8751	7570	6642	5878	5229	4664	4164	3716	3310	2939	2596	2279	1984
	13	2·0444	1·2950	1·0345	8728	7552	6628	5867	5219	4655	4156	3709	3304	2933	2591	2274	1979
	14	2·0122	1·2891	1·0313	8706	7535	6614	5855	5209	4646	4149	3702	3297	2927	2585	2269	1974
	15	1·9823	1·2833	1·0280	8683	7518	6601	5843	5199	4638	4141	3695	3291	2921	2580	2264	1969
	16	1·9542	1·2776	1·0248	8661	7501	6587	5832	5189	4629	4133	3688	3284	2915	2575	2259	1965
	17	1·9279	1·2719	1·0216	8639	7484	6573	5821	5179	4620	4125	3681	3278	2909	2569	2255	1960
.3	18	1·9031	1·2663	1·0185	8617	7467	6559	5809	5169	4611	4117	3674	3271	2903	2564	2249	1955
	19	1·8796	1·2607	1·0154	8595	7451	6546	5797	5159	4603	4110	3667	3265	2897	2558	2244	1951
	20	1·8573	1·2553	1·0122	8573	7434	6532	5786	5149	4594	4102	3660	3258	2891	2553	2239	1946
	21	1·8361	1·2499	1·0091	8552	7417	6519	5774	5139	4585	4094	3653	3252	2886	2547	2234	1941
	22	1·8159	1·2446	1·0061	8530	7401	6505	5763	5129	4577	4086	3646	3246	2880	2542	2229	1936
	23	1·7966	1·2393	1·0030	8509	7384	6492	5752	5120	4568	4079	3639	3239	2874	2537	2224	1932
.4	24	1·7782	1·2341	1·0000	8487	7368	6478	5740	5110	4559	4071	3632	3233	2868	2531	2219	1927
	25	1·7604	1·2289	9970	8466	7351	6465	5729	5100	4551	4063	3625	3227	2862	2526	2214	1922
	26	1·7434	1·2239	9940	8445	7335	6451	5718	5090	4542	4056	3618	3220	2856	2520	2209	1918
	27	1·7270	1·2188	9911	8424	7319	6438	5707	5081	4534	4048	3611	3214	2850	2515	2203	1913
	28	1·7112	1·2139	9881	8403	7302	6426	5695	5071	4525	4040	3604	3208	2845	2510	2198	1908
	29	1·6960	1·2090	9852	8382	7286	6413	5684	5061	4517	4033	3597	3201	2839	2504	2193	1904
.5	30	1·6812	1·2041	9823	8361	7270	6399	5673	5052	4508	4025	3590	3195	2833	2499	2188	1899
	31	1·6670	1·1993	9794	8341	7254	6385	5662	5042	4499	4017	3583	3189	2827	2493	2183	1894
	32	1·6532	1·1946	9765	8320	7238	6372	5651	5032	4491	4010	3577	3183	2822	2488	2178	1890
	33	1·6399	1·1900	9737	8300	7222	6359	5640	5023	4483	4002	3570	3176	2816	2483	2173	1885
	34	1·6269	1·1852	9708	8280	7206	6346	5629	5013	4474	3995	3563	3170	2810	2477	2169	1880
	35	1·6143	1·1806	9680	8259	7190	6333	5618	5004	4466	3987	3556	3164	2804	2472	2164	1876
.6	36	1·6021	1·1761	9652	8239	7175	6320	5607	4994	4457	3979	3549	3158	2798	2467	2159	1871
	37	1·5902	1·1716	9625	8219	7159	6307	5596	4985	4449	3972	3542	3151	2793	2461	2154	1866
	38	1·5786	1·1671	9597	8199	7143	6295	5585	4975	4440	3964	3535	3145	2787	2456	2149	1862
	39	1·5673	1·1627	9570	8179	7128	6282	5574	4966	4432	3957	3529	3139	2781	2451	2144	1857
	40	1·5563	1·1584	9542	8159	7112	6269	5563	4956	4424	3949	3522	3133	2776	2446	2139	1852
	41	1·5456	1·1540	9515	8140	7097	6256	5552	4947	4415	3942	3515	3127	2770	2440	2134	1848
.7	42	1·5351	1·1498	9489	8120	7081	6243	5541	4937	4407	3934	3508	3120	2764	2435	2129	1843
	43	1·5249	1·1455	9462	8101	7066	6231	5531	4928	4399	3927	3502	3114	2758	2430	2124	1839
	44	1·5149	1·1413	9435	8081	7051	6218	5520	4919	4390	3920	3495	3108	2753	2424	2119	1834
	45	1·5052	1·1372	9409	8062	7035	6205	5509	4909	4382	3912	3488	3102	2746	2419	2114	1829
	46	1·4956	1·1331	9383	8043	7020	6193	5498	4900	4374	3905	3481	3096	2741	2414	2109	1825
	47	1·4863	1·1290	9357	8023	7005	6180	5488	4891	4366	3897	3475	3089	2736	2409	2104	1820
.8	48	1·4771	1·1249	9331	8004	6990	6168	5477	4881	4357	3890	3468	3083	2730	2403	2099	1816
	49	1·4682	1·1209	9305	7985	6975	6155	5466	4872	4349	3883	3461	3077	2724	2398	2095	1811
	50	1·4594	1·1170	9279	7966	6960	6143	5456	4863	4341	3875	3455	3071	2719	2393	2090	1806
	51	1·4508	1·1130	9254	7948	6945	6131	5445	4853	4333	3868	3448	3065	2713	2388	2085	1802
	52	1·4424	1·1092	9228	7929	6930	6118	5435	4844	4325	3860	3441	3059	2708	2382	2080	1797
	53	1·4341	1·1053	9203	7910	6915	6106	5424	4835	4316	3853	3435	3053	2702	2377	2075	1793
.9	54	1·4260	1·1015	9178	7892	6900	6094	5414	4826	4308	3846	3428	3047	2696	2372	2070	1788
	55	1·4180	1·0977	9153	7873	6885	6081	5403	4817	4300	3839	3421	3041	2691	2367	2065	1784
	56	1·4102	1·0939	9129	7855	6871	6069	5393	4808	4292	3831	3415	3035	2685	2362	2061	1779
	57	1·4025	1·0902	9104	7836	6856	6057	5382	4799	4284	3824	3408	3028	2679	2356	2056	1775
	58	1·3949	1·0865	9079	7818	6842	6045	5372	4789	4276	3817	3401	3022	2674	2351	2051	1770
	59	1·3875	1·0828	9055	7800	6827	6033	5362	4780	4268	3809	3395	3016	2668	2346	2046	1766
.10	60	1·3802	1·0792	9031	7782	6812	6021	5351	4771	4260	3802	3388	3010	2663	2341	2041	1761

TABLE—*continued.*

Tenths.	M.or S.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
	0	1761	1498	1249	1015	792	580	378	185	0000	9823	9652	9488	9330	9178	9031
	1	1756	1493	1245	1011	788	577	375	182	9997	9820	9650	9486	9328	9176	9028
	2	1752	1489	1241	1007	785	573	371	179	9994	9817	9647	9483	9325	9173	9026
	3	1747	1485	1237	1003	781	570	368	175	9991	9814	9644	9480	9323	9171	9024
	4	1743	1481	1233	999	777	566	365	172	9988	9811	9641	9478	9320	9168	9021
	5	1738	1476	1229	996	774	563	361	169	9985	9808	9638	9475	9318	9166	9019
.1	6	1734	1472	1225	992	770	559	358	166	9982	9805	9636	9472	9315	9163	9016
	7	1729	1468	1221	988	767	556	355	163	9979	9802	9633	9470	9312	9161	9014
	8	1725	1464	1217	984	763	552	352	160	9976	9799	9630	9467	9310	9158	9012
	9	1720	1460	1213	980	759	549	348	157	9973	9797	9627	9464	9307	9156	9009
	10	1716	1455	1209	977	756	546	345	153	9970	9794	9625	9462	9305	9153	9007
	11	1711	1451	1205	973	752	542	342	150	9967	9791	9622	9459	9302	9151	9004
.2	12	1707	1447	1201	969	749	539	339	146	9964	9788	9619	9456	9300	9148	9002
	13	1703	1443	1197	965	745	535	335	144	9961	9785	9616	9454	9297	9146	8000
	14	1698	1438	1193	962	741	532	332	141	9958	9782	9614	9451	9294	9143	8997
	15	1694	1434	1190	958	738	528	329	138	9955	9779	9611	9448	9292	9141	8995
	16	1689	1430	1186	954	734	525	326	135	9952	9777	9608	9446	9289	9138	8992
	17	1685	1426	1182	950	731	522	322	132	9949	9774	9605	9443	9287	9136	8990
.3	18	1680	1422	1178	947	727	518	319	129	9946	9771	9603	9440	9284	9133	8988
	19	1676	1417	1174	943	724	515	316	125	9943	9768	9600	9438	9282	9131	8985
	20	1671	1413	1170	939	720	511	313	122	9940	9765	9597	9435	9279	9128	8983
	21	1667	1409	1166	935	716	508	309	119	9937	9762	9594	9433	9277	9126	8980
	22	1663	1405	1162	932	713	505	306	116	9934	9760	9592	9430	9274	9124	8978
	23	1658	1401	1158	928	709	501	303	113	9931	9757	9589	9427	9271	9121	8976
.4	24	1654	1397	1154	924	706	498	300	110	9928	9754	9586	9425	9269	9119	8973
	25	1649	1393	1150	920	702	495	296	107	9925	9751	9583	9422	9266	9116	8971
	26	1645	1388	1146	917	699	491	293	104	9922	9748	9581	9419	9264	9114	8969
	27	1641	1384	1142	913	695	488	290	101	9919	9745	9578	9417	9261	9111	8966
	28	1636	1380	1138	909	692	484	287	98	9916	9742	9575	9414	9259	9109	8964
	29	1632	1376	1134	905	688	481	283	94	9913	9740	9572	9411	9256	9106	8961
.5	30	1627	1372	1130	902	685	478	280	91	9910	9737	9570	9409	9254	9104	8959
	31	1623	1368	1126	898	681	474	277	88	9907	9733	9567	9406	9251	9101	8957
	32	1619	1364	1123	895	677	471	274	85	9905	9731	9564	9403	9249	9099	8954
	33	1614	1359	1119	891	674	468	271	82	9902	9728	9561	9401	9246	9096	8952
	34	1610	1355	1115	887	670	464	267	79	9899	9725	9559	9398	9243	9094	8950
	35	1605	1351	1111	883	667	461	264	76	9896	9723	9556	9396	9241	9092	8947
.6	36	1601	1347	1107	880	663	458	261	73	9893	9720	9553	9393	9238	9089	8945
	37	1597	1343	1103	876	660	454	258	70	9890	9717	9551	9390	9236	9087	8942
	38	1592	1339	1099	872	656	451	255	67	9887	9714	9548	9388	9233	9084	8940
	39	1588	1335	1095	868	653	447	251	64	9884	9711	9545	9385	9231	9082	8938
	40	1584	1331	1092	865	649	444	248	61	9881	9708	9542	9383	9228	9079	8935
	41	1579	1327	1088	861	646	441	245	58	9878	9706	9540	9380	9226	9077	8933
.7	42	1575	1322	1084	858	642	438	242	55	9875	9703	9537	9377	9223	9075	8931
	43	1571	1318	1080	854	639	434	239	52	9872	9700	9534	9375	9221	9072	8928
	44	1566	1314	1076	850	635	431	235	48	9869	9697	9532	9372	9218	9070	8926
	45	1562	1310	1072	847	632	427	232	45	9866	9694	9529	9369	9216	9067	8924
	46	1558	1306	1068	843	628	424	229	42	9863	9691	9526	9367	9213	9065	8921
	47	1553	1302	1064	839	625	421	226	39	9860	9689	9523	9364	9211	9062	8919
.8	48	1549	1298	1061	836	621	418	223	36	9858	9686	9521	9362	9208	9060	8916
	49	1545	1294	1057	832	618	414	220	33	9855	9683	9518	9359	9206	9057	8914
	50	1540	1290	1053	828	614	411	216	30	9852	9680	9515	9356	9203	9055	8912
	51	1536	1286	1049	824	611	408	213	27	9849	9677	9513	9354	9201	9053	8910
	52	1532	1282	1045	821	608	404	210	24	9846	9675	9510	9351	9198	9050	8907
	53	1528	1278	1041	817	604	401	207	21	9843	9672	9507	9349	9196	9048	8905
.9	54	1523	1274	1037	814	601	398	204	18	9840	9669	9505	9346	9193	9045	8902
	55	1519	1270	1034	810	597	394	201	15	9837	9666	9502	9343	9191	9043	8900
	56	1515	1265	1030	806	594	391	197	12	9834	9663	9499	9341	9188	9041	8898
	57	1510	1261	1026	803	590	388	194	9	9831	9661	9496	9338	9186	9038	8895
	58	1506	1257	1022	799	587	384	191	6	9824	9658	9494	9336	9183	9036	8893
	59	1502	1253	1018	795	583	381	188	3	9826	9655	9491	9333	9181	9033	8891
.10	60	1498	1249	1015	792	580	378	185	0	9823	9652	9488	9330	9178	9031	8888

TABLE I.

Depression of the horizon.

Height.	Depression.		Height.	Depression.	
Feet.	′	″	Feet.	′	″
1	0	58	19	4	11
2	1	21	20	4	17
3	1	40	21	4	23
4	1	56	22	4	30
5	2	9	23	4	36
6	2	21	24	4	42
7	2	33	26	4	52
8	2	44	28	5	5
9	2	53	30	5	15
10	3	2	35	5	39
11	3	10	40	6	4
12	3	19	45	6	27
13	3	27	50	6	46
14	3	36	60	7	25
15	3	42	70	8	1
16	3	50	80	8	34
17	3	57	90	9	6
18	4	4	100	9	35

TABLE II.

Augmentation of moon's semi-diameter.

Altitude	Aug.
°	″
0	0
5	1
10	3
15	4
20	6
25	7
30	8
35	9
40	10
45	11
50	12
55	13
60	14
70	15
80	15
90	16

TABLE III.

Diminution of equatorial parallax

Subtract	Equatorial Parallax.					Subtract
	53′	55′	57′	59′	61′	
1″	Lat. 18°	18°	17°	17°	17°	1″
2	26	25	25	24	24	2
3	32	31	31	30	30	3
4	38	37	36	36	35	4
5	43	42	41	41	40	5
6	49	48	47	46	45	6
7	55	54	52	51	50	7
8	61	59	58	56	55	8
9	69	66	66	62	60	9
10	79	74	71	68	66	10
11			83	71	74	11
12					88	12

For the proportional logarithm.

Lat....	0°	11°	19°	25°	30°	34°	39°	43°	47°	51°	56°	60°	61°	71°	79°	90°
Add....	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	

TABLE IV

Sun's parallax in altitude.

Altitude.	Parallax.
0°	9″
10	9
20	8
30	8
40	7
50	6
55	5
60	4
65	4
70	3
75	2
80	2
85	1
90	0

TABLE V.

Vertical semi-diameter.

Altitude.	Correction.
5°	25″
6	19
7	14
8	11
9	9
10	8
11	7
12	6
13	5
14	4
15	4
16	3
18	3
20	2
30	1
45	1

TABLE VI.

For reducing sideral to mean time.

Hours.	Min.	Sec.	Min.	Seconds.	Sec.	Seconds.
1	0	9·83	1	0·16	1	0·00
2	0	19·16	2	0·33	2	0·01
3	0	29·49	3	0·49	3	0·01
4	0	39·32	4	0·66	4	0·01
5	0	49·15	5	0·82	5	0·01
6	0	58·98	6	0·98	6	0·02
7	1	8·81	7	1·15	7	0·02
8	1	18·64	8	1·31	8	0·02
9	1	28·47	9	1·47	9	0·02
10	1	38·30	10	1·64	10	0·03
11	1	48·13	11	1·80	11	0·03
12	1	57·96	12	1·97	12	0·03
13	2	7·78	13	2·13	13	0·04
14	2	17·61	14	2·29	14	0·04
15	2	27·44	15	2·46	15	0·04
16	2	37·27	16	2·62	16	0·04
17	2	47·10	17	2·78	17	0·05
18	2	56·93	18	2·95	18	0·05
19	3	6·76	19	3·11	19	0·05
20	3	16·59	20	3·28	20	0·05
21	3	26·42	30	4·91	30	0·08
22	3	36·25	40	6·55	40	0·11
23	3	46·08	50	8·19	50	0·14
24	3	55·91	60	9·83	60	0·16

TABLE VII. *For reducing mean to sideral time.*

Hours.	Min.	Sec.	Min.	Seconds.	Sec.	Seconds.
1	0	9·86	1	0·16	1	0·00
2	0	19·71	2	0·33	2	0·01
3	0	29·57	3	0·49	3	0·01
4	0	39·43	4	0·66	4	0·01
5	0	49·28	5	0·82	5	0·01
6	0	59·14	6	0·99	6	0·02
7	1	8·99	7	1·15	7	0·02
8	1	18·85	8	1·31	8	0·02
9	1	28·71	9	1·48	9	0·02
10	1	38·56	10	1·64	10	0·03
11	1	48·42	11	1·82	11	0·03
12	1	58·28	12	1·97	12	0·03
13	2	8·13	13	2·14	13	0·04
14	2	17·99	14	2·30	14	0·04
15	2	27·85	15	2·46	15	0·04
16	2	37·70	16	2·63	16	0·04
17	2	47·56	17	2·79	17	0·05
18	2	57·42	18	2·96	18	0·05
19	3	7·27	19	3·12	19	0·05
20	3	17·13	20	3·28	20	0·05
21	3	26·98	30	4·93	30	0·08
22	3	36·84	40	6·57	40	0·11
23	3	46·70	50	8·21	50	0·14
24	3	56·55	60	9·86	60	0·16

Correction for the oblique semi-diameter. **TABLE I.** *For argument* A.

For *h*		*h* − *s* *d*		For *h*		*h* − *s* *d*		For *h*		*h* − *s* *d*	
°	A	°	A	°	A	°	A	°	A	°	A
89	924	1	176	59	71	31	29	29	94	61	6
88	954	2	146	58	72	32	28	28	95	62	5
87	972	3	128	57	74	33	26	27	95	63	5
86	984	4	116	56	75	34	25	26	95	64	5
85	994	5	106	55	76	35	24	25	96	65	4
84	2	6	98	54	77	36	23	24	96	66	4
83	9	7	91	53	78	37	22	23	96	67	4
82	14	8	86	52	79	38	21	22	97	68	3
81	19	9	81	51	80	39	20	21	97	69	3
80	24	10	76	50	81	40	19	20	97	70	3
79	28	11	72	49	82	41	18	19	98	71	2
78	32	12	68	48	83	42	17	18	98	72	2
77	35	13	65	47	83	43	17	17	98	73	2
76	38	14	62	46	84	44	16	16	98	74	2
75	41	15	59	45	85	45	15	15	98	75	2
74	44	16	56	44	86	46	14	14	99	76	1
73	47	17	53	43	86	47	14	13	99	77	1
72	49	18	51	42	87	48	13	12	99	78	1
71	51	19	49	41	88	49	12	11	99	79	1
70	53	20	47	40	88	50	12	10	99	80	1
69	55	21	45	39	89	51	11	9	99	81	1
68	57	22	43	38	90	52	10	8	100	82	0
67	59	23	41	37	90	53	10	7	100	83	0
66	61	24	39	36	91	54	9	6	100	84	0
65	63	25	37	35	91	55	9	5	100	85	0
64	64	26	36	34	92	56	8	4	100	86	0
63	66	27	34	33	92	57	7	3	100	87	0
62	67	28	33	32	93	58	7	2	100	88	0
61	69	29	31	31	93	59	7	1	100	89	0
60	70	30	30	30	94	60	6	0	100	90	0

Correction for the oblique semi-diameter. **TABLE II.** *Diminution of the semi-diameter. Argument* A (*h*) + A (*h*—*s*) + A (*d*).

	Altitude.													
Sum of A	5°	6°	7°	8°	9°	10°	11°	12°	14°	16°	18°	20°	30°	45°
0″	25″	19″	14″	11″	9″	8″	6″	5″	4″	3″	3″	2″	1″	1″
20	24	18	14	11	9	7	6	5	4	3	2	2	1	0
40	23	17	13	10	8	7	6	5	4	3	2	2	1	0
60	21	16	12	9	8	6	5	5	3	3	2	2	1	0
70	20	15	12	9	8	6	5	5	3	3	2	2	1	0
80	19	14	11	8	7	6	5	4	3	2	2	2	1	0
90	17	13	10	8	7	6	5	4	3	2	2	2	1	0
100	16	12	9	7	6	5	4	4	3	2	2	1	1	0
110	14	10	8	6	5	4	3	3	2	2	1	1	1	0
120	11	9	7	5	4	3	2	2	2	1	1	1	0	0
130	9	7	5	4	3	3	2	2	1	1	1	1	0	0
135	7	6	4	3	2	2	2	1	1	1	1	0	0	0
140	6	5	4	3	2	2	1	1	1	1	1	0	0	0
145	5	4	3	2	2	1	1	1	1	0	0	0	0	0
150	3	3	2	2	1	1	1	1	0	0	0	0	0	0
155	3	2	2	1	1	1	1	0	0	0	0	0	0	0
160	1	1	1	0	0	0	0	0	0	0	0	0	0	0
170	0	0	0	0	0	0	0	0	0	0	0	0	0	0
178	1	1	1	0	0	0	0	0	0	0	0	0	0	0
180	2	1	1	1	1	1	0	0	[illegible]	0	0	0	0	0
182	3	2	2	1	1	1	1	0	0	0	0	0	0	0
184	4	3	2	2	1	1	1	1	1	1	0	0	0	0
186	5	4	3	2	2	2	1	1	1	1	1	1	0	—
188	7	6	4	3	2	2	2	1	1	1	1	1	1	—
190	9	7	5	4	3	3	2	2	1	1	1	1	1	—
191	10	8	6	4	4	3	3	2	2	1	1	1	1	—
192	11	9	7	5	4	4	3	3	2	2	1	1	1	—
193	12	9	7	5	5	4	3	3	2	2	2	1	1	—
194	14	10	8	6	5	4	4	3	2	2	2	2	1	—
195	15	11	9	6	6	5	4	4	3	2	2	2	—	—
196	17	13	10	7	6	6	5	4	3	3	2	2	—	—
197	19	14	11	8	7	6	5	5	3	3	2	2	—	—
198	21	16	12	9	8	7	6	5	3	3	3	—	—	—
199	23	17	13	10	8	8	6	5	4	—	—	—	—	—
200	25	19	14	11	9	—	—	—	—	—	—	—	—	—
Alt.	5°	6°	7°	8°	9°	10°	11°	12°	14°	16°	18°	20°	30°	45°

PHYSICAL ASTRONOMY.

The object of the philosopher in the investigation of nature is to arrange and classify facts and phenomena, with a view to trace the agency of their remote, or, at least, their proximate causes, and ascend as high as the imperfection of human means of observation, and the limited powers of the human intellect will allow us in the scale of generalization.

To beings endowed with more perfect faculties, and more comprehensive intelligence than ourselves, much of that complication we observe in natural phenomena would disappear: many effects, which seem to us independent of each other, and linked by no natural connection, would be in their eyes collateral results of one and the same principle: much, that to us seems fortuitous would to them appear predisposed and regularly arranged. The laws of nature would at once be reduced in number and enlarged in extent; and that higher order of generalization which would consist in classifying together laws of the same kind, and referring them to others yet more universal, would exercise *their* power and constitute *their* science. That such would be the case with more perfect beings, our own experience, limited as it is, amply shews. The man who has learnt to regard the fall of a leaf and the precession of the equinoxes as results equally certain and unavoidable of a law capable of being stated in three lines, and understood by a child of ten years old, has made already a considerable step in this way—the patient exercise of his natural reason has stood him in the stead of a sharper intellect; and secrets which an angel might penetrate perhaps at a glance, become revealed to man by the slow, yet sure, effects of persevering thought.

The progress of modern science has done more than the keenest metaphysical reasoning, and has given us the most convincing proofs of the agency of one general and intelligent cause throughout the whole system of nature. When we see on all sides phenomena grouping themselves under laws intelligible and simply expressed, which are themselves subordinate to others, yet more simple and extensive; when we see every anomaly which threatened destruction to a theory, becoming, in the progress of our knowledge, its firmest support; every inequality disappearing when viewed from a higher level; every exception proving a rule of greater generality; all, in short, conveying more and more towards order and simplicity the more severely we scrutinize it; it is impossible not to allow that that last great step, which unites all the phenomena of the universe under one general head, and refers them to one all-pervading agency—however inconceivably remote, and surpassing probably the utmost limits of the human intellect to comprehend, if explained, would still be but the continuation and final completion of a chain of reasoning whose first links we hold within our grasp—the consummation of a process actually begun—the termination of a career into which we are fairly entered.

It is difficult to avoid such contemplations at the outset of an essay on physical astronomy; they crowd upon us; and in rejecting them we should reject the noblest use of the sublimest of sciences. For scarcely in any are the phenomena presented by nature more various and more complicated; in none is the generalization so complete, the final result so simple, or the object more imposing

From what has just been said, it may be gathered that the object of physical astronomy is to reduce, under general laws, the motions and phenomena of the heavenly bodies, and investigate their causes; to trace the history of what has already happened in our own system, and to ascertain what changes the causes demonstrably in action (unless interfered with by others we have no knowledge of,) will superinduce in the course of ages, and thus to appreciate the stability of the present order of creation. In a more limited and practical point of view, the physical astronomer is called on to furnish formulæ deduced from theory for determining the state of the system at any assigned instant; and adapted for the purposes of the observer, so as to serve as a basis for the construction of tables; and to descend, by the application of his general principles, to those more refined inequalities which, owing to their minuteness, or the length of their periods, would escape or mislead the observer unassisted by theory.

There is one feature in physical astronomy which renders it remarkable among the sciences, and has been the chief, if not the only, source of the perfection it has attained. It is this—that the fundamental law embracing all the minutiæ of the phenomena so far as we yet know them, presents itself at once, on the consideration of broad features and general facts, deduced by observations even of a rude and imperfect kind, in such a form as to require no modification, extension, or addition when applied in minute detail. In other sciences, when an induction of a moderate extent has led us to the knowledge of a law which we conceive to be general, the further progress of our inquiries frequently obliges us either to limit its extent or modify its expression. To those who are familiar with the history of chemistry, instances of this will present themselves at every turn. In physical optics, the general representation of all the series of polarised tints and the colours of natural bodies by a certain universal scale—the Cartesian law of refraction when applied to the extraordinary ray in crystallized media, and even to the ordinary, if the reports of some recent experiments are to be relied on—together with innumerable other laws, simple, natural, and resting on extensive inductions, have all been either overset, extended, or materially modified by the progress of the science.

In physical astronomy, however, when taken in that limited acceptation, which restricts it to the explanation of the planetary motions, our first conclusion is our last. The law on which all its phenomena depend, flows naturally and easily from the simplest among them, as presented by the rudest observation; and, in point of fact, such has really been the order of investigation in this science. The rude supposition of the uniform revolution of the moon in a circle about the earth as a centre, led Newton at once to the true law of gravity, as extending from the earth to its companion. The uniform circular motions of the planets about the sun, in times following the progression assigned by observation in Kepler's rule, confirmed the law, and extended its influence to the boundaries of our system. Every thing more refined than this—the elliptic motions of the planets and satellites — their mutual perturbations — the slow changes of their orbits and motions, denominated secular variations—the deviation of their figures from the spherical form—the oscillatory motions of their axes, which produce nutation and the precession of the equinoxes—the theory of the tides, both of the ocean and the atmosphere, have all in succession been so many trials for life and death in which this law has been, as it were, pitted against nature; trials, whose event no human foresight could predict, and where it was impossible even to conjecture what modifications it might be found to need. Even at this moment, if among the innumerable inequalities of the lunar or planetary motions any one, however small, should be discovered decidedly not explicable on the hypothesis of a force varying as the inverse square of the distance, that hypothesis must be modified till it accounts for it. It is hardly necessary to add, however, that in the present state of science, this is a case not to be contemplated.

Still, these are refinements. The deviations of the planetary orbits from circles are small, their deviations from ellipses excessively minute: the lunar orbit alone presents results of perturbation so large as to strike us at once with the appearance of irreconcilable anomalies, but it is only by a refinement of calculation that we can trace them all to the laws of gravity; but the motions of comets put the truth and generality of this law to a severe and rude test, by giving it a trial under the greatest varieties of distance, position, and velocity of motion, and instancing its influence on matter of a rarity almost spiritual, and differing so utterly from that of which our planet consists, as scarcely to authorize the admittance of any property in common.

The above observations have been made in conformity with the general language of natural philosophers, and the customary acceptation of the term physical astronomy, and are, no doubt, strictly applicable when we confine ourselves to the celestial phenomena of our own immediate system, and the motions of those larger masses of matter of which planets and their satellites consist. The cautious philosopher however will still regard it as worthy inquiry, whether, at enormous distances, like those of the fixed stars, or at such comparatively microscopic intervals as those we are ordinarily conversant with on the surface of our planet, the rigorous law of a force as the inverse square of the distance may not suffer some modification. An emanation, like light, traversing in succession every part of space, may be conceived to go on without acceleration or retardation, without loss or change, to the remotest regions; but an active and immediate intercourse carried on between points infinitely distant, is not only incapable of demonstration, but, could it be proved, must, I suppose, be referred to direct spiritual agency. At the same time, it is worthy attention how strict and indissoluble a bond gravity establishes between remote objects—to see this in its real light, we must compare it with the most effectual of our ordinary means of transmitting power. If the earth and sun were connected by a rod of cast iron, in one piece, an impulse or pull, however violent and sudden, applied at the sun, would not begin to be felt at the earth till after a lapse of eighteen months from the moment of its communication, while a change in the sun's attraction, such as might arise from a sudden alteration of its figure or density, would demonstrably* affect our planet in an instant of time many thousand times less than the least interval perceptible to our senses.

The subsistence of sidereal clusters, in which the compression or crowding of the stars is carried to the extent we have instances of in many parts of the heavens, seems hardly compatible with a gravitating force, unopposed by some principle of conservation, unless we suppose them in a state rapidly verging to a catastrophe. On the other hand, with regard to small distances, we have no distinct proof, that within a few inches, or even miles, from a material point, the law of gravity may not begin to deviate appreciably from the Newtonian law. The experiments of Maskelyne and Cavendish, which may perhaps be adduced as supporting its rigorous application, are far too gross, and differ too widely in their results, to be cited in so delicate a matter, besides which, their results, as applied to such an inquiry, are affected with an unknown element, the mean density of the earth. At much closer intervals we are certain of the existence of attractive and repulsive forces following a widely different law; and by what imperceptible gradations these shade into that of gravity, or whether they are to be regarded as distinct from it in their nature and origin, is a point whose consideration seems reserved for a much higher state of science than we can boast of having yet attained.

But it is quite sufficient for the purposes of physical astronomy to know, that as far as the motions of the great masses of matter connected with our system either in the heavens or on our globe are concerned, observation and theory present no difference capable of being made an objection to the strict expression of Newton's law, and we shall therefore wave all further discussion of the subject, and proceed to the object of the present essay, in which the reader will be presumed acquainted with the general *facts* of astronomy, with the principles of mechanical philosophy, and so much of analysis, of the differential and integral calculus, plane and spherical trigonometry, as shall render it unnecessary for us to interrupt the general chain of our reasoning to demonstrate such theorems, &c. as we shall have occasion to call to our aid.

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* Laplace, *Système du Monde*, p. 286.

SECTION I.

ON THE CIRCULAR AND ELLIPTIC MOTIONS OF THE PLANETS AND SATELLITES.

By observing the places of the sun, moon, and planets in the heavens at different times, and measuring their angular diameters, we have learnt, that, provided certain excessively small inequalities, to be hereafter considered, are disregarded, their motions are all compatible with the supposition of the planets revolving about the sun in elliptic orbits of small eccentricity, and having the sun's centre in one focus; each orbit lying wholly in a fixed plane peculiar to itself, and slightly inclined to the ecliptic, or plane, in which the sun appears to revolve about the earth. We learn, moreover, that if the apparent motion of the sun be transferred in a contrary direction to the earth, and the sun be supposed at rest in space, the motion so assigned to the earth will be such as to include it in the expression of the same law. The supposition then, of the sun at rest, and the earth in motion, being agreeable to this general analogy, and supported by incontrovertible arguments drawn from the great magnitude of the former in comparison with the latter body, is assumed as a demonstrated truth. Observation moreover assisted, it is true, by calculation, but independent of all theory, (i.e. of all reasoning from causes,) has taught us the truth of the following remarkable laws, in which also the earth is included among the planets.

The areas described about the sun's centre by the radius vector of any one of the planetary orbits (or the line drawn from the sun to the place of the planet,) are proportional to the times of their description.

Fig. 1. Let S be the sun, and APP′ part of the orbit of any planet. Then, A being assumed as a point of departure, the area ASP is to the area ASP′ as the time of the planet's describing the arc AP of its orbit is to the time of its describing AP′.

The squares of the periodic times of different planets, (or of the times of a complete revolution of each about the sun,) are as the cubes of their mean distances from the sun, or of the greater semiaxes of their respective ellipses.

The periodic time of the earth is 365·2564, and that of Mars 686·9796. The greater semiaxis of the earth's orbit being 1, that of Mars is 1·523693; and we may easily satisfy ourselves, by executing the computation, that $(365{\cdot}2564)^2 : (686{\cdot}9796)^2 :: 1^3 : (1{\cdot}523693)^3$.

These three laws, viz. 1st. The elliptic motion of the planets about the sun as a focus: 2dly. The proportionality of the areas to the times: and 3dly. The law of the periodic times, which have immortalized the name of Kepler, and whose discovery, and the manner of it, afford at once matter of humiliation and triumph to the human intellect, were all deduced immediately from observation, as insulated, and, for aught their discoverer knew, unaccountable facts. It shall now be our business to demonstrate their mutual dependence, and to shew how the general law of attraction may be derived from them most simply.

The analogy observed between the motions of the other planets and of the earth, affords a reasonable presumption of their being masses of matter subject to the same mechanical laws of rest, impulse, and resistance, as that of which our own planet consists. Moreover, from what we know of the constitution of our own atmosphere, and its rapid diminution of density as we recede from the earth, we have every reason to believe, that the immense space in which their revolutions are performed, is either completely void, or at least free from any material substance capable of sensibly resisting or impeding their motions, or preventing any external impulse they may receive from acting on them with its full effect. Setting out with these assumptions (the strict truth of which will be best tried by the conclusions they will lead to,) it is obvious that as the planets, instead of moving continually forward in straight lines, as masses of inert matter would do if projected in space and left to themselves, are, in fact, constantly deviating from this rectilinear progression—they must be under the perpetual influence of some agency external to themselves, which, (by the second law of motion,) can be no other than that of a mechanical force acting in a direction inclined to that in which they move at any instant.

The enormous distance at which the planets are from the sun, and their own minuteness, compared with it, permit us at present to regard them as points; and it will be shewn hereafter that this supposition introduced here merely for simplification, is strictly legitimate. Let us then consider the motion of a material point perpetually deflected from a straight line by the action of an external force; and to this end let us conceive the curve OPQ described by the planet to be replaced by a polygon of an infinite number of sides OP, PQ, &c. and setting out from O, let it describe the chord OP in the first instant of time dt. Fig. 2. In an equal subsequent instant it would, if left to itself, go on describing PR equal to OP, and in the same straight line. But since we have regarded the curve as replaced by an elementary polygon, we must (on the principles of the differential calculus) conceive the deflecting force to act by interrupted impulses at the angles of that polygon. Let the first impulse therefore be conceived to take place at P. Then, since the material point P, in virtue of the motion inherent in it at P, would have described PR in the instant dt; but, in virtue of that motion, combined with the new motion it receives at P, does actually describe PQ in the same time, that new motion (by the composition of motions,) must be such as, *alone*, would carry it from P over a space Pv equal and parallel to RQ; and as the change of motion takes place in the direction in which the moving force acts, Pv must be the direction of the deflecting force. Prolong Pv indefinitely, and take in it any point S; join OS, SQ, SR. Then, since OP = PR, the area OSP = SPR = SPQ, because QR is parallel to PS.

A force may be conceived to *tend to* any point in the line of its direction. We see, therefore, that any point to which the force acting at P tends is characterised by this remarkable property, that the areas described about it in equal evanescent instants on either side of P are equal. This property belongs to every point in the line PvS, but (as is obvious) to no point situated out of that line; and any point possessing this property may be regarded (at least for that moment) as a point of tendency, or centre of the force acting on the body at P.

Now, as we have seen, it is matter of observation that each planet describes areas proportional to the times, and consequently equal areas in equal infinitely small times before and after any given instant, about a fixed point in the system coincident with the sun's centre. This point, therefore, possesses at all times and in all positions of the planets, the property above demonstrated to belong to a point situated in the direction of the deflecting force; or, in other words, the forces deflecting the planets in their orbits are invariably directed to the centre of the sun.

The moon, (neglecting periodical inequalities,) describes about the centre of the earth, and the satellites of Jupiter and Saturn about their respective primaries, areas proportional to the times of their description. The forces, therefore, which deflect them in their orbits, are directed (small causes of inequality being neglected) to the centres of the earth, of Jupiter, and of Saturn respectively.

Having ascertained the directions of the forces which deflect the planets from their rectilinear paths, and retain them in their orbits, we come now to estimate their intensity, and investigate the laws of their action. In order to this, we shall find it more simple to abandon the supposition of the interrupted impulsive action of the deflecting force, and consider the body P as deflected from the *tangent* PR, and describing not the chord PQ, but the infinitesimal arc, the force being supposed to act during the whole time $d\,t$. This time, however, being infinitely small, the force may be regarded as constant; and since the angle PSQ between its first and last directions is also evanescent, it must be considered as acting constantly in a direction parallel to PS or QR. If then we take F to represent the force at P, and $g = 32^{\text{ft.}}1908$ (or double the space through which a heavy body falls in the first second at the earth's surface,) and suppose unity to represent the force of gravity, we shall have by mechanics,

$$\text{RQ} = \tfrac{1}{2} g \,.\, \text{F}\, d\,t^2$$

and*

$$\text{F} = \frac{2\text{QR}}{g\,.\,d\,t^2}.$$

If, therefore, we know by observation the nature of the orbit, and the velocity of the body at any point P, we may thence calculate the magnitude of the deflection QR produced in any very minute time $d\,t$: and thus the intensity of the deflecting force will become known. Let us, for instance, take the case of the moon; and, supposing her orbit a circle, with the earth in the centre, let us inquire the actual magnitude of QR, the deflection from the tangent produced in some extremely minute portion of time as 1″, by the force retaining it in its orbit.

Call the mean radius of the moon's orbit R, her period (in seconds of mean time) T; then will her velocity (being equal to the circumference of her orbit, divided by the number of seconds in the time of one revolution,) be represented by $\frac{2\pi \text{R}}{\text{T}}$, where $\pi = 3{\cdot}14159$, &c.; and this is the actual length of the arc described in 1″. Now, since (neglecting higher powers of the arc than the square)

$$\text{versed sine} = \frac{(\text{arc})^2}{\text{diameter}};$$

and since QR or Pv is in this case equal to the versed sine of PQ, we must have,

$$\text{QR} = \frac{1}{2\text{R}} \times \left(\frac{2\pi\text{R}}{\text{T}}\right)^2 = \frac{2\pi^2\text{R}}{\text{T}^2};$$

and, finally, (since $d\,t = 1''$)

$$\text{F} = \frac{4\pi^2}{g} \times \frac{\text{R}}{\text{T}^2};$$

and this is the general expression for the force perpetually urging a body to the centre of a circle. To reduce it to numbers in the case of the moon, we have

$\text{R} = 238783^{\text{m}} = 39165700 \times g$ (since $g = 32^{\text{ft.}}1908$)

and $\frac{\text{R}}{g} = 39165700$

moreover,

$\text{T} = 27^{\text{d.}}32167 = 2360592''$.

Calculating from which data, we find

$$\text{F} = 0{\cdot}00028394 = \frac{1}{3522}.$$

So that the force by which the moon is retained in its orbit, is about 3522 times feebler than that of gravity at the earth's surface.

When we observe a tendency in all bodies at the earth's surface to approach or fall towards its centre, and if hindered from approaching, still to *press* towards that point, we express these phenomena, by saying, that they are attracted towards the earth. At all moderate elevations above its surface, and in the same geographical situation this tendency seems invariable; but at great elevations, the delicate indications of modern instruments will detect a decrease in its energy;* and indeed the gradual enfeebling of attraction towards any body by an increase of distance, is not only a natural supposition of itself, but is borne out by the strong analogy of magnetic and electrical attractions. At vast elevations then, like that of the moon, there is reasonable ground to expect a considerable diminution of attraction; and if the force by which the moon be retained in her orbit be nothing more than this same attraction modified by the remoteness of the two bodies, we see that an increase of the distance to about 60 times the earth's radius from its centre, is sufficient to weaken it more than 3500 times. As the distance then from the earth's centre increases, the attraction diminishes in a much more rapid progression. What the exact nature of this progression is, we must satisfy ourselves by other phenomena; but even from the rude calculation already made, (in which every correction has been neglected) we may perceive, that a law of decrease as the squares of the distances (the next in simplicity to the distances themselves,) has a *primâ facie* probability. In fact, $60^2 : 1^2 :: 3600 : 1$.

Having only one attendant satellite, however, we

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[*] QR here represents the deflection from the *tangent*, and is only half the length of QR in the last figure, which represents the deflection from the preceding chord prolonged.

[*] At an elevation of a mile above the surface of the earth, the intensity of gravity is diminished $\frac{1}{1977{\cdot}291}$; and a pendulum clock, beating seconds at the level of the sea, would lose 21·898 seconds a day at this altitude, a quantity not to be overlooked. Any traveller having leisure, and the proper apparatus, might try the experiment in the barrack on Mont Cenis, or at the Hospice of St. Bernard.

have no means of obtaining any further verification of such a law, in this way; but if we regard the earth as well as the other planets, as so many satellites of the sun, we have here ample room to satisfy ourselves, having a progression of no less than eleven distances, from that of Mercury to that of Uranus, on which to ground the assumption of a law. And here we have the advantage of dispensing altogether with numerical calculation, and substituting in its stead the third law of Kepler: for if we call R and r the radii of the circles described by any two planets round the sun, T and t their periodic times, F and f the forces retaining them in their orbits, we have (as a result of observation,)

$$\frac{T^2}{t^2}=\frac{R^3}{r^3},$$

which, combined with the equations,

$$F=\frac{4\pi^2}{g}\times\frac{R}{T^2}, \qquad f=\frac{4\pi^2}{g}\times\frac{r}{t^2}$$

which give

$$\frac{F}{f}=\frac{R}{r}\cdot\frac{t^2}{T^2},$$

we shall find

$$\frac{F}{f}=\frac{r^2}{R^2}.$$

Thus, then we encounter the same rate of diminution in the attractive tendencies of each of the planets towards the sun, which the lunar motions had given reason to surmise in the case of the earth and moon; but in the case now under consideration, the verification is much more satisfactory, and the numerical coincidences, when the calculations are gone through, complete; the third law of Kepler, on which the whole is founded, being almost rigorously exact.

We may now, with great confidence, presume the inverse proportion of the squares of the distances to be the law of variation of that force which retains the bodies of our system in their orbits; but previous to assuming its generality, it will be right to compare the force retaining the moon in its path round the earth, and that deflecting the earth in its orbit about the sun. Calling R, r, the respective radii of the earth's and moon's orbits, and T, t, their periodic times, we have still

$$\frac{F}{f}=\frac{R}{r}\cdot\left(\frac{t}{T}\right)^2.$$

Now we have

$$\frac{r}{R}=\frac{60{\cdot}23799}{23405}; \quad \frac{T}{t}=\frac{365{\cdot}25638}{27{\cdot}32167}$$

So that, executing the calculation,

$$\frac{F}{f}=2{\cdot}17399.$$

The sun then, although more than 380 times the distance of the moon, exerts a force of more than double the intensity on the earth compared with the earth's attraction on the moon. At equal distances, then, the forces exerted by the sun and earth would be in the ratio of

$$2{\cdot}17399\times\left(\frac{R}{r}\right)^2 : 1 \text{ or } 328196 : 1.$$

This enormous difference in the attractive energies of the two bodies, must evidently be owing to some equally striking difference in the bodies themselves; and when we consider the immense magnitude of the sun (in comparison with our planet) we shall not be at a loss to what cause to assign it. Whatever be the cause of attraction, we may fairly conclude that, if it be the result of a force inherent in matter, two equal and similar bodies (i. e. each containing the same quantity of attracting matter,) placed close together, will each attract a third placed at a distance, with equal forces; and both together, with double the force of either separately, and pursuing the same idea—that 328196 such bodies as the earth, if placed close together, and forming one mass in the place of the sun, would attract as the sun actually does: in other words, that the sun only attracts other bodies with more energy than the earth, by reason of its being a greater body, and containing a greater quantity of attracting or *gravitating* matter.

By such reasonings we are led to assume, as a general law, that similar and equal particles of matter, however situated in space, attract, or tend to each other with a force directly proportional to their *masses* or quantities of gravitating substance, and inversely proportional to the squares of their distances from each other; and having arrived at this law by the steps described, we must now proceed to verify its rigorous exactness, by applying it in succession to the phenomena as presented by nature in our system, which will be the object of the following sections.

SECTION II.

On the attractions of spherical bodies

The earth, sun, and planets, as well as their satellites, being shewn by observation to be spherical bodies of great magnitude, it becomes necessary to examine, *in limine*, whether the law of attraction above stated be compatible with this fact—in other words, whether from a knowledge that the gross attractions of the whole masses follow that law of decrease, we can argue that the attraction of each elementary molecule follows the same.

Let BDCE be the attracting sphere, m the body attracted, which at present we will suppose to be a single particle, taking its mass as unity. Suppose mBAC the axis of the sphere passing through the molecule m, and let M, M′ be two equal and similar molecules, similarly, but oppositely situated with respect to the axis. Each of these molecules will attract m with a force represented by $\frac{M}{(Mm)^2}$, but the directions of their attractions not coinciding, we must resolve them into others, whose effects may directly assist or counteract each other. Draw MPM′ (which will of course be perpendicular to mAC), and if we take M$m=f$, and MP $=\rho$, we shall have $\frac{M}{f^2}$ to represent the force of M on m, which, reduced to the directions mC and PM, give the partial forces $\frac{M}{f^2}\times\frac{mP}{mM}$ and $\frac{M}{f^2}\times\frac{MP}{mM'}$, that is, (if we call AP, x, and Am, a) Fig. 3.

$$\frac{M(a-x)}{f^3} \text{ and } \frac{M\cdot\rho}{f^3}.$$

The partial forces of M′ are represented by the same quantities, but the latter of them, acting in the direction PM′ destroys the partial force of M in the opposite direction PM, while the former conspires with the corresponding force of M, and doubles its effect, so that we have

$$\frac{2\,(a-x)\cdot \mathrm{M}}{f^3}$$

for the attraction of this pair of molecules in the direction m A; and the sum of all such pairs throughout the sphere being found by the ordinary rules of the integral calculus will express the whole attraction of the sphere. The simplest way will be to regard the molecule M as a parallelopiped included, first, between two consecutive positions of the plane DE perpendicular to the axis Am, separated from each other by the interval $d\,x$; 2dly. Between two cylindrical surfaces, having for their bases the circle MM′, whose radius is PM $(=\rho)$ and the same circle in its consecutive position, when its radius varies from ρ to $\rho+d\rho$; and 3dly. Between two consecutive positions of the plane PMm, assumed during its rotation about Pm as an axis. In virtue of this, if we put the angle FPM $=\theta$ the dimensions of the molecule in a direction perpendicular to PM will be $\rho\,d\,\theta$, and its dimensions in other two directions being respectively $d\,x$ and $d\,\rho$, we have

$$\mathrm{M}=\rho\,d\rho\,.\,d\theta\,.\,dx\,;$$

so that the whole attraction (A) will be expressed by the triple integral

$$\iiint\frac{2\,(a-x)\,.\,\rho\,d\rho\,.\,d\theta\,.\,dx}{f^3}.$$

The variables ρ, θ, and x, are here independent; and it is therefore indifferent with which we begin, we will commence with θ, because f being $=\sqrt{(a-x)^2+\rho^2}$ is independent on θ. Thus we have

$$\mathrm{A}=\iint\frac{2\,(a-x)\,\rho\,d\rho\,.\,dx}{f^3}\,(\theta+\mathrm{Const.})$$

This integral must be extended only from $\theta=o$ to $\theta=\pi$, or over only *half* the circumference of the circle MM′, otherwise the attraction of each molecule M, M′ (having been grouped in pairs,) would be repeated twice over. Then we have

$$\mathrm{A}=\iint\frac{2\,\pi\,(a-x)\,dx\,.\,\rho\,d\rho}{\left\{(a-x)^2+\rho^2\right\}^{\frac{3}{2}}}.$$

If we now perform the integration relative to ρ regarding x as constant, we get

$$\mathrm{A}=\int 2\,\pi\,(a-x)\,dx\,.\left\{\mathrm{Const.}-\frac{1}{\sqrt{(a-x)^2+\rho^2}}\right\}$$

But the integral in this case is to be extended from $\rho=o$ to $\rho=\mathrm{PD}=\sqrt{r^2-x^2}$, r being the radius of the sphere, so that it becomes

$$\mathrm{A}=\int 2\,\pi\,(a-x)\,dx\,.\left\{\frac{1}{a-x}-\frac{1}{\sqrt{(a^2+r^2)-2\,a\,x}}\right\}$$

$$=2\,\pi\,x+\pi\int\frac{(a-x)\,d\,.\,(a-x)}{\sqrt{2\,a\,(a-x)-(a^2-r^2)}}$$

$$=\mathrm{Const.}+2\,\pi\,x+\frac{2\,\pi}{3}\left\{2\,a^2-r^2-a\,x\right\}.\frac{\sqrt{a^2+r^2-2\,a\,x}}{a^2}$$

This integral must be extended from $x=\mathrm{AC}=-r$ to $x=\mathrm{AB}=+r$, when it finally becomes, after all reductions,

$$\mathrm{A}=\frac{4}{3}\cdot\frac{\pi\,r^3}{a^2}.$$

Now, $\frac{4}{3}\pi\,r^3$ represents the mass of the sphere, which being called S, we have

$$\mathrm{A}=\frac{\mathrm{S}}{a^2},$$

an equation which shews, that *the attraction of the sphere is expressed by the whole mass, divided by the square of the distance of its centre from the attracted molecule, and is therefore precisely the same as if the whole sphere were condensed into its centre.*

The hypothesis, then, which refers the observed attractions of the great masses composing our system to the effect of the mutual attraction of their ultimate molecules, varying according to the same law, has nothing in it incompatible with mathematical reasoning; but it is a very remarkable coincidence that this should so happen, as the only mathematical laws of attraction which would lead to a similar conclusion, are that of nature, and that in which the force is directly proportional to the distance, or one resulting from the combination of these two laws.

Let us next examine the case when the attracted body is also a sphere of sensible magnitude. S and s being the two spheres, and a the distance of their centres, it has been shewn that S will attract every molecule of s with the same force as if it were condensed into its centre. Now the mass of a single molecule having been regarded as unity, S, the mass of the first sphere will be proportional to, and represent, the number of molecules it consists of. The attraction then of s on S will be the same as if the latter sphere were removed, and in its centre a single molecule placed, endowed with an attractive energy S times as great as that of any molecule, such as S actually consists of. But the attraction of s on one molecule of the last named kind has been shewn to be $\frac{s}{a^2}$, therefore its attraction on S being S times as forcible, will therefore be represented by $\frac{\mathrm{S}\times s}{a^2}$

This expression represents the absolute force with which the two spheres tend to each other, or the number of pounds, grains, or other units, which must be opposed to it in order to hinder their approach. This in mechanical language, is called the *moving force;* and we therefore see, that *the moving force with which two homogeneous spheres attract each other, is as the product of their masses directly, and the square of the distances of their centres inversely.*

The moving force then is, of course, the same on each sphere; and in consequence of the equality of action and re-action, (which always refers to *moving force,*) it ought to be so. Were the spheres allowed to approach each other, however their velocities would obviously be different, the greater moving slower than the less. In fact, the accelerating force on any body being equal to the moving force divided by the mass moved, we have

$$\text{accelerating force on S}=\frac{s}{a^2}$$

$$\text{accelerating force on } s=\frac{\mathrm{S}}{a^2}.$$

Suppose now the two spheres at liberty in space, and moving, in consequence of their mutual attraction and any projectile force. If to both of them we apply an accelerating force $\frac{s}{a^2}$ equal to that exerted by s, but in an opposite direction, or towards S, the sphere S will be urged by forces destroying each other, and will therefore either remain at rest, or move uniformly in a right line: but the sphere s will now be urged by an accelerating force equal to $\frac{S+s}{a^2}$. Now, it is shown in mechanics, that the application of a common accelerating force to all the bodies of a system, does not alter their relative motions. Hence, if we refer the motions of our spheres, not to a fixed point in space, but to the centre of one of them S; or take that centre as the origin of our co-ordinates, we must then put $\frac{S+s}{a^2}$ for the accelerating force animating the other.*

Let us now consider the attractions of spheres not homogeneous, but composed of concentric strata varying in density according to any law of the distance from their centres. There is every reason to suppose this the actual constitution of the sun and planets; and it therefore becomes necessary to examine this case. Now, any stratum of infinitesimal thickness dr, may be regarded as the difference of two spheres s, and $s + ds$, homogeneous, and of the same density as the stratum, their radii being r and $r + dr$. The attraction of s on a molecule equal to 1 placed at a distance a, is $\frac{S}{a^2}$, and that of $s + ds$ is $\frac{s+ds}{a^2}$: consequently the attraction of the stratum is $\frac{s+ds}{a^2} - \frac{s}{a^2} = \frac{ds}{a^2}$; and therefore the same as if the stratum were collected in its centre. As this is true of every stratum separately, whatever be its density, and their attractions do not interfere, it will be true of all together; so that, whether the sphere be homogeneous, or composed of concentric layers, or strata of different density, the same property still holds good; and all we have demonstrated in the case of homogeneous spheres, remains true in this.

SECTION III.

Theory of elliptic motion.

We are now enabled to enter on the general theory of the planetary motions, but we will still confine ourselves to a case of comparative simplicity. The vast mass of the central body of our system, compared with those which circulate round it, permits us to regard their motions as influenced by it alone, and to neglect, in a first approximation, all the minute effects arising from the mutual attractions of the planets and satellites on each other. The case then we propose to consider in this section is that of the sun and a single planet, or a primary and one of its satellites.

Let M represent the mass of the sun or central body, and m that of the planet; and, fixing the origin of the co-ordinates in the centre of M, let x, y, z, represent the co-ordinates of m. Also let r be *the radius vector*, or line joining the centres of the two bodies; so that $r^2 = x^2 + y^2 + z^2$, and t the time (in seconds of mean solar time,) elapsed since any fixed epoch.

The accelerating force of M on m is $\frac{M}{r^2}$, and that of m on M $\frac{m}{r^2}$; and since we regard M as fixed, the latter quantity must be added to the force animating m, according to the observation made in the last section; so that the *relative* accelerating force acting on m in the direction of the radius vector m M will be $\frac{M+m}{r^2}$ which being resolved into forces in the directions m P, PQ, QM, of the three co-ordinates, becomes multiplied by $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$, respectively, and produces the partial forces Fig. 4.

$$P = \frac{(M+m)\,x}{r^3},\ Q = \frac{(M+m)\,y}{r^3},\ R = \frac{(M+m)\,z}{r^3}.$$

The effect of an accelerating force P acting, during an instant of time dt, on a body in a direction parallel to any given axis, that of the x, is to produce a variation in its velocity in that direction, which is to the variation gravity on the earth's surface would produce in the same time, as the force P is to the accelerating force of gravity which we will represent by unity. Now gravity producing the variation $g \,.\, dt$ in that time, the variation produced by the force P, will be $P \,.\, g\,dt$; or, if instead of taking one foot, as we have hitherto done, for the unit of linear measure, we take g $(= 32^{\text{ft}}\cdot1908)$ for our standard unit, simply $P\,dt$. But to this the negative sign should be prefixed, as the force P tends to *diminish* the co-ordinate x. Again, the velocity in the direction x being $\frac{dx}{dt}$, its variation is $d\,\frac{dx}{dt}$; we have therefore $d\,\frac{dx}{dt} = -P\,dt$; or, supposing

$$R = \frac{M+m}{r^3}$$

and writing for P its value Rx

$$d\frac{dx}{dt} + Rx = o; \qquad (1)$$

similarly,

$$d\frac{dy}{dt} + Ry = o; \qquad (2)$$

and

$$d\frac{dz}{dt} + Rz = o; \qquad (3)$$

These equations contain the whole theory of the planetary motions, neglecting their mutual perturbations; and if, instead of supposing $R = \frac{M+m}{r^3}$, we suppose it equal to $\frac{\phi\,(r)}{r}$, the same equations will express the motion of a point m about a centre of force M, attracting it with a force represented by any function $\phi\,(r)$ of the distance Mm.

If we eliminate R from the two first of these equations, by multiplying the first by y, and the second by $-x$, and adding, we get

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* If, however, we suppose the sphere S forcibly retained in its place by some external agency *not* acting on s, the case will be different; and $\frac{S}{a^2}$ will continue to represent the accelerating force on S.

$$y\,d\,\frac{d\tau}{d\,t} - x\,d\,\frac{dy}{d\,t} = o$$

that is,

$$\frac{y\,d^2x - x\,d^2y}{d\,t} = \frac{(y\,d\,x - x\,d\,y)\;d^2\,t}{d\,t^2}$$

or

$$\frac{y\,d^2x - x\,d^2y}{y\,d\,x - x\,d\,y} = \frac{d^2\,t}{d\,t}$$

Each member of this equation being a complete differential, because $y\,d^2x - x\,d^2y = d\,.\,(y\,d\,x - x\,d\,y)$, we get by integration

$$y\,d\,x - x\,d\,y = h\,d\,t; \tag{4}$$

and similarly,

$$z\,d\,x - x\,d\,z = h'\,d\,t; \tag{5}$$

$$z\,d\,y - y\,d\,z = h''\,d\,t; \tag{6}$$

If, now, we multiply the first of these three equations (in which h, h', and h'', represent arbitrary constant quantities) by z, the second by $-y$, and the third by x, and add all together, we get

$$h\,z - h'\,y + h''\,x = o; \tag{7}$$

which is the equation of a plane passing through the origin of the co-ordinates. Consequently, the curve described by the body is one of simple curvature, and its plane passes through the centre of attraction.

For simplicity, let us suppose the plane of the orbit to be coincident with that of the x and y, we have then $z = o$, and our equations are reduced to

$$d\,\frac{d\,x}{d\,t} + \mathrm{R}\,x\,d\,t = o$$

$$d\,\frac{dy}{d\,t} + \mathrm{R}\,y\,d\,t = o$$

$$y\,d\,x - x\,d\,y = h\,d\,t$$

Fig. 5.

The area of the elementary sector $\mathrm{M}\,m\,m'$, described by the radius vector $\mathrm{M}\,m$ in the instant $d\,t$, is equal to

$$\begin{aligned} &\mathrm{M}\,m\,m'\,\mathrm{P}' - \mathrm{M}\,m'\,\mathrm{P}' \\ &= \mathrm{M}\,m\,\mathrm{P} - \mathrm{M}\,m'\,\mathrm{P}' + \mathrm{P}\,m\,m'\,\mathrm{P}' \\ &= -d\cdot\frac{xy}{2} + y\,d\,x = \frac{y\,d\,x - x\,d\,y}{2} \end{aligned}$$

If, then, we call A the area described by the radius vector since the commencement of the time t, we have $y\,d\,x - x\,d\,y = 2\,d\,\mathrm{A}$, and consequently

$$2\,d\,\mathrm{A} = h\,d\,t;\ \mathrm{A} = \frac{h}{2}\,t; \tag{8}$$

this equation expresses the proportionality between the areas and times in Kepler's law; and since the process by which it is deduced, is independent of any particular value of R, R having been eliminated to obtain it, the analytical demonstration here given applies generally for all possible laws of central force.

In order next to investigate the nature of the curve described, we must eliminate t, which will be easiest done after a transformation of the co-ordinates. Let then the angle $\mathrm{AM}\,m$ described by the radius vector, since the origin of the time t, be called θ, and we have $m\,\mu = r\,d\,\theta$, and the elementary sector $m\,\mathrm{M}\,m' = \frac{r \times r\,d\,\theta}{2}$, or $d\,\mathrm{A}\,\frac{1}{2}\,r^2\,d\,\theta$, so that

$$r^2\,d\,\theta = h\,d\,t; \tag{9}$$

Now, since $r^2 = x^2 + y^2$, we have

$$r\,d\,r = x\,d\,x + y\,d\,y$$

$$r\,d^2\,r + d\,r^2 = (x\,d^2\,x + y\,d^2\,y) + (d\,x^2 + d\,y^2)$$

so that (putting $d\,s = m\,m' = \sqrt{d\,x^2 + d\,y^2}$)

$$x\,d^2\,x + y\,d^2\,y = r\,d^2\,r + d\,r^2 - d\,s^2.$$

But $\quad d\,s^2 - d\,r^2 = m\,m'^2 - m'\,\mu^2 = m\,\mu^2 = r^2\,d\,\theta^2$

whence, $\quad x\,d^2\,x + y\,d^2y = r\,d^2\,r - r^2\,d\,\theta^2.$

This premised, since our equations (1) and (2) give

$$o = d^2\,x - \frac{d\,x}{d\,t}\,d^2\,t + \mathrm{R}\,x\cdot d\,t^2$$

$$o = d^2\,y - \frac{dy}{d\,t}\,d^2\,t + \mathrm{R}\,y\cdot d\,t^2$$

if we multiply the first of them by x, and the second by y, and add, we have

$$o = (x\,d^2\,x + y\,d^2\,y) - \frac{x\,d\,x + y\,d\,y}{d\,t}\,d^2\,t + \mathrm{R}\,r^2\,d\,t^2 \tag{10}$$

but if we take the logarithmic differential of the equation $h\,d\,t = r^2\,d\,\theta$, and suppose $d\,\theta$ constant, (which we are at liberty to do, having as yet taken no differential constant,) we get

$$\frac{d^2\,t}{d\,t} = 2\cdot\frac{d\,r}{r}.$$

So that, making this substitution for $\frac{d^2t}{d\,t}$, and for $x\,d^2\,x + y\,d^2\,y$, writing its value $r\,d^2\,r - r^2\,d\theta^2$, and $r\,d\,r$ for $x\,d\,x + y\,d\,y$, we get

$$o = r\,d^2\,r - r^2\,d\,\theta^2 - 2\,d\,r^2 + \mathrm{R}\,r^2\,d\,t^2,$$

in which we have put for R its value $\frac{\mathrm{M} + m}{r^3}$, and for $d\,t$, $\frac{r^2\,d\,\theta}{h}$, it becomes

$$o = r\,d^2\,r - r^2\,d\,\theta^2 - 2\,d\,r^2 + \frac{\mathrm{M} + m}{h^2}\cdot r^3\,d\,\theta^2$$

Put $r = \frac{1}{u}$, and since $d\,r = -\frac{d\,u}{u^2}$, and $d^2\,r = -\frac{d^2\,u}{u^2} + \frac{2\,d\,u^2}{u^3}$, it becomes

$$\frac{d^2\,u}{d\,\theta^2} + u - \frac{\mathrm{M} + m}{h^2} = o; \tag{11}$$

This equation (being the simplest case of an equation of the second order of a linear form,) is immediately integrable, and gives

$$u = f\cos\,(\theta + g) + \frac{\mathrm{M} + m}{h^2}; \tag{12}$$

whence,

$$r = \frac{\dfrac{h^2}{\mathrm{M} + m}}{1 + \dfrac{h^2\,f}{\mathrm{M} + m}\cdot\cos\,(\theta + g)}; \tag{13}$$

f and g being two arbitrary constants.

In any conic section, if we call a the semiaxis major, $a\,(1 - e^2)$ the semiparameter, and $\theta + g$ the angle included between the radius r and the vertex nearest the focus from which r is supposed to take its origin, we have

$$r = \frac{a\,(1 - e^2)}{1 + e\cdot\cos\,(\theta + g)} \tag{14}$$

Consequently we see that the curve described by the body must necessarily be a conic section, having the body M in the focus, and the relation between the arbitrary constants f, h, and the axis and semiparameter of this conic section will be (if we call p the semiparameter, or put $p = a\,(1 - e^2)$

$$f = \frac{e}{a\,(1 - e^2)} = \frac{e}{p}; \qquad (15)$$

$$h = \sqrt{a\,(1 - e^2)\,(M + m)} = \sqrt{p\,(M + m)}\,; \qquad (16)$$

If we suppose the angle θ to commence from the vertex nearest the focus, or from the *nearer apside*, or *perihelion* of the orbit, we have $g = o$, and

$$r = \frac{a\,(1 - e^2)}{1 + e\,.\cos\theta} = \frac{p}{1 + e\cos\theta}; \qquad (17)$$

If the value of e be less than unity, the conic section described is an ellipse, if equal and a infinite, a parabola, if greater, and a negative, an hyperbola.

But to complete the theory of the planetary motions, it is necessary to know, not only the nature of the orbit, in general, but also whereabouts in it the body will be at any moment assigned. For this purpose we must obtain a finite equation involving t, and either r or θ, or some functions of them. Now, the equation (9) gives $h\,d\,t$, or

$$\sqrt{a\,(1 - e^2)\,(M + m)} \times d\,t = r^2\,d\,\theta$$

and substituting for r, its value in equation (17)

$$d\,t = \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{M + m}} \cdot \frac{d\,\theta}{(1 + e\,.\cos\theta)^2} \qquad (18)$$

and

$$t + C = \frac{a^{\frac{3}{2}}\,(1 - e^2)^{\frac{3}{2}}}{\sqrt{M + m}} \cdot \int \frac{d\,\theta}{(1 + e\,.\cos\theta)^2}$$

To integrate this, take another variable v, such, that

$$\cos\theta = \frac{\cos v - e}{1 - e\,.\cos v}; \qquad (19)$$

whence,

$$\sin\theta = \frac{\sqrt{1 - e^2}\,.\sin v}{1 - e\,.\cos v}$$

$$d\,.\cos\theta = -\,d\,v\,.\frac{(1 - e^2)\,.\sin v}{(1 - e\,.\cos v)^2}$$

$$d\,\theta = -\frac{d\,.\cos\theta}{\sin\theta} = \frac{d\,v\,.\sqrt{1 - e^2}}{1 - e\,.\cos v}$$

$$1 + e\,.\cos\theta = \frac{1 - e^2}{1 - e\,.\cos v}$$

and substituting these expressions in the value of t above given, it will be found to reduce itself to the following very simple form:

$$t + C = \frac{a^{\frac{3}{2}}}{\sqrt{M + m}} \int d\,v\,(1 - e\,.\cos v)$$

so that, making

$$\frac{a^{\frac{3}{2}}}{\sqrt{M + m}} = \frac{1}{n}; \qquad (20)$$

and taking the integral to commence, when $v = o$, or $\theta = o$, that is, from the instant of the body leaving the lower apside,

$$n\,t = v - e\,.\sin v; \qquad (21)$$

This equation fixes the relation between t and v; but that between v and θ may be expressed more conveniently for the purposes of calculation than by equation (19), as follows. By the equation last mentioned, we get

$$1 - \cos\theta = \frac{(1 + e)\,(1 - \cos v)}{1 - e\,.\cos v}$$

$$1 + \cos\theta = \frac{(1 - e)\,(1 + \cos v)}{1 - e\,.\cos v}$$

whence,

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{1 + e}{1 - e} \cdot \frac{1 - \cos v}{1 + \cos v}$$

or,

$$\tan\tfrac{1}{2}\,\theta = \sqrt{\frac{1 + e}{1 - e}} \cdot \tan\tfrac{1}{2}\,v; \qquad (22)$$

Finally, we obtain immediately the relation between r and v, by substituting, in equation (17) for $\cos\theta$, its value in (19), when we find

$$r = a\,(1 - e\,.\cos v); \qquad (23)$$

These equations comprise the whole theory of the motions of bodies in conic sections. Equation (17) exhibits the relation between r and θ, or the polar equation to the curve; while (21), (22), and (23), express relations between the time, t, the *true anomaly* (as it is called) θ, and the *radius vector*, r, respectively, and an auxiliary angle v, to which the name of the *eccentric anomaly* has been given. The quantity $n\,t$, when reduced into angular measure, by multiplying it by 180°, and dividing it by $\pi = 3{\cdot}14159$, &c. is called also the *mean anomaly*. The eccentric anomaly may be exhibited geometrically, as follows:

Fig. 6.

On the major axis AB of the ellipse, let a semicircle A m' B be described, and draw the ordinate P $m\,m'$ through m. Then, AM m will be the true anomaly of the body m, and AC m' the eccentric. For, if we take CP $= x$,

$$\cos AC\,m' = -\frac{C\,P}{C\,m'} = -\frac{x}{a}$$

$$\cos AM\,m = -\frac{MP}{M\,m} = -\frac{a\,e + x}{a + e\,x}$$

because, by the property of the ellipse, M $m = a + e\,x$. Hence, we have

$$\frac{1 - \cos AM\,m}{1 + \cos AM\,m} = \frac{(1 + e)\,(a + x)}{(1 - e)\,(a - x)}$$

$$= \frac{1 + e}{1 - e} \times \frac{1 + \frac{x}{a}}{1 - \frac{x}{a}}$$

$$= \frac{1 + e}{1 - e} \cdot \frac{1 - \cos AC\,m'}{1 + \cos AC\,m'}$$

and, consequently,

$$\tan\tfrac{1}{2}\,AM\,m = \frac{1 + e}{1 - e} \times \tan\tfrac{1}{2}\,AC\,m'$$

Let T represent the time of one entire revolution in the ellipse, or the periodic time, then, as θ increases from 0 to 360° (or $2\,\pi$) t increases from 0 to 360 ($2\,\pi$) also. Consequently, by (21) we get

$$n\,T = 2\,\pi; \qquad T = \frac{2\,\pi}{n}$$

or

$$T = \frac{2\,\pi\,.\,a^{\frac{3}{2}}}{\sqrt{M + m}}; \qquad (24)$$

The periodic times, then, of several bodies revolving about the same central body, are, in the sesquiplicate ratio of the major axes of their orbits, (or mean distances from the central body,) directly, and in the subduplicate ratio of the sum of the masses of the revolving and central body inversely. The mass of the sun being enormously great compared with those of the planets, we may neglect m in comparison with M; and M being the same for the whole system, we have $T \propto a^{\frac{3}{2}}$; $T^2 \propto a^3$ which is no other than Kepler's third law.

The periodic times of the planets being very exactly known, we might expect to find in equation (24) the means of ascertaining the masses of the planets, supposing that of the sun, and any one of them, known. Thus, if m, m', be the masses of two planets, and T, T′, their periodical times, we have

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{a'}{a}\right)^3 \times \frac{M+m}{M+m'} \qquad (25)$$

On applying calculation, however, all we learn from this relation is, that the resulting masses are so small, as to be incapable of accurate determination by this method; their values, as deduced from it, being materially affected by the small uncertainties still prevailing as to the lengths of the periods, and by the mutual perturbations of the sun and planets. There is a case, however, where it may be used with advantage, viz. in that of a planet accompanied by a satellite. If we call M the sun, and m the planet, and neglect the mass of the latter in comparison with the former, and that of the satellite in comparison with the primary, we have at once

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{a'}{a}\right)^3 \times \frac{M}{m}; \quad \frac{m}{M} = \left(\frac{a'}{a}\right)^3 \times \left(\frac{T}{T'}\right)^2$$

Thus, in the case of the earth, we have

$$\frac{a'}{a} = \frac{60{\cdot}23799}{23405}; \quad \frac{T}{T'} = \frac{365{\cdot}25638}{27{\cdot}32167}$$

So that, by executing the computations, we find

$$\frac{m}{M} = 0{\cdot}00000304697 = \frac{1}{328196}$$

To find v in terms of t, or to calculate the eccentric (and thence the true) anomaly at any given instant, we must resolve the transcendental equation,

$$n\,t = v - e\,.\,\sin v$$

This can only be done in a series; and fortunately, in the case of the planets, e is so small, that a series ascending by powers of e will converge sufficiently. Now we have

$$v = n\,t + e\,.\,\sin v$$

and, since e is small, and sin v necessarily less than 1, $n\,t$ itself expresses the value of v within a limit less than e, and is therefore a first approximation. Again, if in sin v we write its value for v or $(n\,t + e\,.\,\sin v)$, we get

$$v = n\,t + e\,.\,\sin(n\,t + e\,.\,\sin v)$$

So that,

$$v = n\,t + e\,.\,\sin n\,t$$

is an approximation carried one step farther, or to the first power of e. Let this be again substituted for e, and we have

$$v = n\,t + e\,.\,\sin\{n\,t + e\,.\,\sin n\,t\}$$

But (neglecting the squares and higher powers of e) we have

$$\sin\{n\,t + e\,.\,\sin n\,t\} = \sin n\,t + e\,.\,\sin n\,t\,.\,\cos n\,t = \sin n\,t + \frac{e}{2}\sin 2\,n\,t$$

So that we find

$$v = n\,t + e\,.\,\sin n\,t + \frac{e^2}{2}\sin 2\,n\,t \qquad (26)$$

If we again repeat the process, we get an approximation, pushed to the third power of e, and so on, as far as we please.

For numerical calculation, however, the equation $v = n\,t + e\,.\,\sin v$ furnishes the readiest solution; as we have only to reduce e into seconds, (taking 57° 17′ 44″·8 for the arc equal to radius or 1, or adding the logarithm 5·3144251 to the logarithm of e, which gives at once that of the number of seconds e is equal to,) and assuming $n\,t$ (the given mean anomaly) for a first approximation, correct it successively, as in the following example.

Required Jupiter's eccentric anomaly corresponding to 53° of mean anomaly.

Here $n\,t = 53°$, and in Jupiter's orbit we have

$$\begin{array}{lr}
e = 0{\cdot}048077 & \log..8{\cdot}6819374 \\
 & 5{\cdot}3144251 \\
\hline
e = 9916''{\cdot}6\,; & \log..3{\cdot}9963625 \\
\text{Take } v = 53°......\log\sin 53° & 9{\cdot}9023486 \\
\hline
e\,.\,\sin v = \;2°\,12'\,0'' = 7919''{\cdot}8\,; & 3{\cdot}8987111 \\
\quad n\,t = 53 & \\
\hline
\end{array}$$

$v = 55\;\;12\;\;0$ Corrected value, with which resuming the process.

$$\begin{array}{lr}
 & \log\sin v = 9{\cdot}9144221 \\
 & e......3{\cdot}9963625 \\
\hline
2°\,15'\,43'' = 8143''{\cdot}0....\log.... & 3{\cdot}9107846 \\
53° \qquad = n\,t & \\
\hline
\end{array}$$

55 15 43.... Second corrected value.

Another repetition of the very same process gives $v = 55°\,15'\,49''{\cdot}1$, which is true within $0''{\cdot}2$.

But even this process, simple as it is, becomes tedious for the orbit of Mercury, and those of the new planets, Pallas and Juno, in which the value of e is not *very* small; and here we must have recourse to the well known method of trial and error, which may be applied in this case as follows:—Having assumed by estimation a value of v, (neglecting minutes and seconds,) by noticing whether the term $e\,.\,\sin v$ is additive or subtractive, and increasing or diminishing $n\,t$ accordingly; calculate the value of $v - e\,.\,\sin v$ for that and the next *subsequent* degree; and let the values so found be called V, and V′: should either of these be exactly equal to the proposed value of $n\,t$, the corresponding value of v will be the truth; but as this will probably never happen, we have only to say,

$$V' - V : V' - n\,t :: -3600'' : x = -\frac{3600''}{V' - V} \times (V' - n\,t)$$

which correction being applied with its proper sign to the *latter* of the two assumed values of v, will give an approximate value. Let the value of $v - e\,.\,\sin v$ be again computed with this value of v, and call the result V″. This will always be found *very* nearly equal to $n\,t$; but if not exactly so, the correction

$$-\frac{3600}{V' - V}(V'' - n\,t)$$

must be computed and applied to the new value, and so on.

For instance: Let $n\,t = 332°\,28'\,65''$, and $e = 50600'' = 14°\,3'\,20''$. Here sin $n\,t$ is negative; so that $e\,.\,\sin n\,t$ is subtractive, and v must be less than $n\,t$. Take then for the two values of v, 325° and 326° respectively, and compute as follows:—

$$\sin 325^\circ - 9{\cdot}7585913 \text{ (neg)}$$
$$\log e'' \ldots . 4{\cdot}7041513$$
$$4{\cdot}4627426 \text{ (neg)}$$
$$e \,.\, \sin 325^\circ = -\, 29023''{\cdot}0 = -\, 8^\circ\, 3'\, 43''{\cdot}0$$

$$\sin 326^\circ - 9{\cdot}7475617 \text{ (neg)}$$
$$4{\cdot}7041513$$
$$4{\cdot}4517130 \text{ (neg)}$$
$$e \,.\, \sin 326^\circ = -\, 28295{\cdot}2 = -\, 7^\circ\, 51'\, 35''{\cdot}2$$

Hence

$$V = 333^\circ\, 3'\, 43''{\cdot}0 \qquad V' = 333^\circ\, 51'\, 35''{\cdot}2$$
$$V' - V = 2872''{\cdot}2 \qquad V' - n\,t = -\, 4960{\cdot}2$$

$$\begin{array}{ll} \text{Log } 3600 \ldots\ldots\ldots\ldots & 3{\cdot}5563025 \\ \text{Log } 2872{\cdot}2 \ldots\ldots\ldots & 3{\cdot}4582147 \\ \hline & 0{\cdot}0980878 \\ \text{Log } 4960{\cdot}2 \ldots\ldots\ldots & 3{\cdot}6954992 \\ \hline \text{Log } 6217{\cdot}1 \ldots\ldots\ldots & 3{\cdot}7935870 \end{array}$$

$$v = 326^\circ - 6217''{\cdot}1 = 324^\circ\, 16'\, 23''$$

Taking this for a new value of v, we find, by another repetition of the process,

$$V'' = 332^\circ\, 28'\, 49''{\cdot}6\,; \quad V'' - n\,t = -\, 5''{\cdot}4$$

and, $$-\frac{3600}{V' - V} \times (V'' - n\,t) = +\, 6''{\cdot}768$$

so that $$v = 324^\circ\, 16'\, 29''{\cdot}768$$

which is true to the hundredth of a second; and this case is nearly the worst that can occur in the theory of the planets.

If we substitute the value of v given in equation (26) in the expression for r, $r = a\,(1 - e \,.\, \cos v)$; (23), and develope in powers of e, we obtain r in a series of powers of e, and cosines of $n\,t$, and its multiples,

$$r = a \left\{ 1 - e \,.\, \cos n\,t + \frac{e^2}{2}(1 - \cos 2\,n\,t) - e^3 \times \&\text{c.} \right\};$$

Finally, if we resolve the equation

$$\tan \tfrac{1}{2}\,\theta = \sqrt{\frac{1+e}{1-e}} \,.\, \tan \tfrac{1}{2}\,v$$

we get

$$\theta = v + e \,.\, \sin v + \frac{e^2}{4} \,.\, \sin 2\,v + \&\text{c.}$$

wherein, if we write for v its value, we shall obtain

$$\theta = n\,t + 2\,e \,.\, \sin n\,t + \frac{5}{4}\,e^2 \,.\, \sin 2\,n\,t + \&\text{c.}$$

To carry these series to the higher powers of e, and to ascertain their law, we must have recourse to particular theorems for facilitating such developements, but these will suffice for our present purpose; and the reader who wishes to proceed farther in the investigation, will find in the second book of the *Mécanique Céleste*, arts. xx. xxi. xxii. every information he can desire.

We shall content ourselves here with merely setting down the formulæ, which are as follows:—

$$\begin{aligned} v = n\,t + e \,.\, \sin n\,t &+ \frac{e^2}{1\,.\,2\,.\,2} \,.\, 2 \,.\, \sin 2\,n\,t \\ &+ \frac{e^3}{1\,.\,2\,.\,3\,.\,2^2}\left\{ 3^2 \,.\, \sin 3\,n\,t - 3 \,.\, \sin n\,t \right\} \\ &+ \frac{e^4}{1\,.\,2\,.\,3\,.\,4\,.\,2^3}\left\{ 4^3 \,.\, \sin 4\,n\,t - 4 \,.\, 2^3 \,.\, \sin 2\,n\,t \right\} \\ &+ \frac{e^5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,2^4}\left\{ 5^4 \,.\, \sin 5\,n\,t - 5 \,.\, 3^4 \sin 3\,n\,t + \frac{5\,.\,4}{1\,.\,2} \,.\, \sin n\,t \right\} \\ &+ \&\text{c.}; \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{r}{a} = 1 + \frac{e^2}{2} - e \,.\, \cos n\,t &- \frac{e^2}{2} \cos 2\,n\,t \\ &- \frac{e^3}{2\,.\,4}\left\{ 3 \,.\, \cos 3\,n\,t - 3 \,.\, \cos n\,t \right\} \\ &- \frac{e^4}{2\,.\,4\,.\,6}\left\{ 4^2 \,.\, \cos 3\,n\,t - 4 \,.\, 2^2 \,.\, \cos 2\,n\,t \right\} \\ &- \frac{e^5}{2\,.\,4\,.\,6\,.\,8}\left\{ 5^3 \,.\, \cos 5\,n\,t - 5 \,.\, 3^3 \,.\, \cos 3\,n\,t + \frac{5\,.\,4}{1\,.\,2} \,.\, \cos n\,t \right\} \\ &- \&\text{c.}; \end{aligned} \tag{28}$$

$$\theta = v + 2\,\epsilon \,.\, \sin v + \frac{2\epsilon^2}{2} \,.\, \sin 2\,v + \frac{2\epsilon^3}{3} \,.\, \sin 3\,v + \&\text{c.}; \tag{29}$$

where $\epsilon = \dfrac{e}{1 + \sqrt{1 - e^2}}$ and is, of course, a fraction smaller than e. Also, if we neglect powers of e higher than the fifth,

$$\left.\begin{aligned} \theta = n\,t + \left\{ 2\,e - \frac{1}{4}\,e^3 + \frac{5}{96}\,e^5 \right\} \,.\, \sin n\,t + \left\{ \frac{5}{4}\,e^2 - \frac{11}{24}\,e^4 \right\} \sin 2\,n\,t \\ + \left\{ \frac{13}{12}\,e^3 - \frac{43}{64}\,e^5 \right\} \sin 3\,n\,t + \frac{103}{96}\,e^4 \,.\, \sin 4\,n\,t + \frac{1097}{960}\,e^5 \,.\, \sin 5\,n\,t \end{aligned}\right\}; \tag{30}$$

SECTION IV.

On the velocities of the planetary motions, and the determination a priori of the elements of their orbits.

The angular velocity of a body is measured by the angle which it appears to describe in any very small time to the eye of a spectator. In fact, if we call θ the angle, and t the time, we have

$$\text{Angular velocity about the sun} = \frac{d\,\theta}{d\,t}$$

Now, by our 9th equation, we have $r^2\,d\,\theta = h\,d\,t$, and, consequently, $\frac{d\theta}{d\,t} = \frac{h}{r^2}$; that is to say, the angular velocity in any orbit is inversely as the square of its distance from the centre; and this law is general for all central forces.

The paracentric velocity is the approach to, or recess from, the centre; and is measured by $\frac{d\,r}{d\,t}$.

Now, $\frac{d\,r}{d\,t} = \frac{d\,r}{d\,\theta} \cdot \frac{d\,\theta}{d\,t}$, but

$$\frac{d\,r}{d\,\theta} = \frac{a\,(1 - e^2)\,.\,e \sin\theta}{(1 + e\,.\cos\theta)^2} = \frac{e\,.\sin\theta}{a\,(1 - e^2)\,.\,r^2}$$

and, consequently,

$$\frac{d\,r}{d\,t} = \frac{h\,e\,.\sin\theta}{a\,(1 - e^2)\,r^4} = \sqrt{\frac{\text{M} + m}{a\,(1 - e^2)}} \cdot \frac{e\,.\sin\theta}{r^4}; \qquad (31)$$

To complete our knowledge of the body's motion, we must inquire its linear velocity at any instant. To this end, if we call V the velocity, $\text{V} = \frac{d\,s}{d\,t}$, $d\,s$ being the element of the curve. Now, first, we must remark, that if we write for $d\,t$ its value $\frac{y\,d\,x - x\,d\,y}{h}$, this gives

$$\text{V} = h\,.\,\frac{d\,s}{y\,d\,x - x\,d\,y}; \qquad (32)$$

But $\frac{y\,d\,x - x\,d\,y}{d\,s}$ expresses the length of a *perpendicular dropped, or a tangent to the curve from the origin of the co-ordinates.* Thus we see, that *the velocity is inversely as the perpendicular so let fall, and directly as the quantity* h, *or the area described in a given time.*

Moreover, we have $d\,s^2 = d\,r^2 + r^2\,d\,\theta^2$; so that, writing for $d\,r$ its equal $-\frac{d\,u}{u^2}$, or $-r^2\,d\,u$,

$$\left(\frac{d\,s}{d\,t}\right)^2 = r^2\left(\frac{d\theta}{d\,t}\right)^2 \left\{1 + r^2 \,.\left(\frac{d\,u}{d\,\theta}\right)^2\right\}$$

Now, by reason of the equation $r^2\,d\,\theta = h\,d\,t$, we have

$$r^2\left(\frac{d\,\theta}{d\,t}\right)^2 = \frac{h^2}{r^2}$$

Again, if we differential the equation (12), we find

$$\begin{aligned}\left(\frac{d\,u}{d\,\theta}\right)^2 &= f^2\,.\sin(\theta + g)^2\\ &= f^2 - f^2\,.\cos(\theta + g)^2\\ &= f^2 - \left(u - \frac{\text{M} + m}{h^2}\right)^2, \quad \text{(by equation 12)}\\ &= \frac{e^2}{p^2} - \left(u - \frac{1}{p}\right)^2, \quad \text{(by equations 15, 16)}\\ &= \frac{e^2 - 1}{p^2} + \frac{2}{p\,r} - \frac{1}{\;^{2}}\end{aligned}$$

writing for u its value $\frac{1}{r}$. Consequently, we find

$$\begin{aligned}1 + r^2\left(\frac{d\,u}{d\,\theta}\right)^2 &= \left(\frac{2}{p\,r} - \frac{1 - e^2}{p^2}\right).\,r^2\\ &= \frac{r^2}{p}\left(\frac{2}{r} - \frac{1}{a}\right)\end{aligned}$$

because $p = a\,(1 - e^2.)$ Hence, we finally obtain

$$\left(\frac{d\,s}{d\,t}\right)^2 = \frac{h^2}{p}\left(\frac{2}{r} - \frac{1}{a}\right)$$

or

$$\text{V}^2 = (\text{M} + m)\,.\left(\frac{2}{r} - \frac{1}{a}\right); \qquad (33)$$

On the determination of the elements of the planetary orbits, *a priori*, there is no occasion to enter into any very extensive discussion, in a practical point of view. Since, however, it is a subject generally touched upon in astronomical works, and is not without its interest, when we consider what changes in our own system may have taken place, or may yet take place from the action of violent causes, we shall devote a part of our space to its consideration.

Let us suppose, then, a body of a given mass, to be launched in space from a given point, with a velocity and direction also given; and to be attracted by another body, whose mass is also given with a force inversely as the square of the distance: it is required to determine the form, magnitude, and position of the conic section it will describe.

The plane passing through the attracting body and the primitive direction of projection will, of course, be that in which the orbit will lie, there being no force to draw the body out of this plane. Taking then the central body for the origin of two co-ordinates, x and y, lying in this plane, and retaining all the other denominations of the foregoing pages, and considering, as the unit of velocity, that with which a body would describe a space equal to 32·1908 feet in one second, we shall have, by (29),

$$v^2 = (\text{M} + m)\left(\frac{2}{r} - \frac{1}{a}\right)$$

and, consequently,

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\text{M} + m} \qquad (34)$$

Now, by hypothesis, the masses of the central and revolving body are given: and the distance from the centre, as well as the velocity with which the latter is projected. If, then, we suppose the quantities in this equation to correspond to the point of projection, r is this distance, and V, M, m, are known; so that we have at once the value of a, the semiaxis major of the orbit.

This result is a remarkable one. It shews us, that the major axis of the orbit is independent of the angle of inclination to the radius at which the original projection takes place: in other words, that any number of bodies (of equal, or of exceedingly small magnitudes compared with the central one,) launched from one point with equal velocities, but in any different directions, will all describe orbits having equal major axes. Another result, not less curious, follows from this—that they will all describe conic sections of the same nature, that is, all ellipses, or all parabolas, or all hyperbolas—for the nature of the conic section depends only on the algebraic sign of its major axis.

Thus, if a be positive, or if V, the velocity of projection, be *less* than $\sqrt{\frac{2\,(\mathrm{M}+m)}{r}}$, the orbit will be an ellipse. If equal, a will be infinite $\left(\frac{1}{a}=o\right)$, and the orbit will be a parabola; but if greater, a is negative, and the orbit will be an hyperbola

In the two latter cases, the bodies will never return to their original point of departure, but in the former they will do so: and since the periodic time depends only on the major axes and masses, if their masses be either all equal, or all extremely small, they will all return in the same time, and a collision will take place; after which it is impossible to say what will happen. If, on the other hand, there exists any sensible proportion between the revolving and central bodies, and any considerable inequality in their masses, their periods will be unequal, and each may perform its orbit undisturbed.

This is supposing their mutual attractions to be neglected. In fact, however, at the instant of their departure, these may be incomparably greater than that of the central body, and will then materially change their velocities, unless the latter be so great as speedily to carry them beyond the sphere of their mutual influence. If a small portion of the earth, for instance, were suddenly projected from its surface, the attraction of the earth on it would, at the moment of its departure, exceed that of the sun in the ratio of 3522 : 2·17399, or upwards of 1600 : 1. So that, in the first instants of its motion, it would move as if influenced by the earth alone. But this effect would diminish rapidly; by the time the projectile had reached the distance of the moon, the sun's action would already have a preponderance (as we have seen in section I,) in the ratio of 2·17399 : 1, and it would depend entirely on the relative velocity of projection, whether such a space could be described in a time small enough to escape the influence of the earth or not.

It has been a matter of some speculation, whether the small planets between Mars and Jupiter may not have had their origin in the destruction, by violence, of some larger mass once revolving in the situation they now occupy. The very considerable approximation of their periodic times, which, in the case of Ceres and Pallas is singularly near (within $\frac{1}{300}$ of the whole period,) and the equally remarkable fact of the mutual intersections of their orbits falling all in the same part of the heavens, (in a general way,) have given rise to this surmise; and it has even been conjectured, that an explosive rupture of a former planet may have scattered its fragments far and wide over our system, and produced these singular bodies. There is no limit to conjecture; but if any such event have taken place, we are forced to conclude, that the mass of the ruptured planet must have been very small, or the fragments must have collapsed by their mutual attraction; or, at least, their velocities would have been so materially modified by it, as to obliterate all traces of their once having had a common velocity about the sun. The smallness of the ruptured mass renders the supposition of an explosion less revolting; and we know, at least from observation, that the fragments (if such) are extremely minute.

If the orbit be a circle, we have $r=a$, and $\frac{2}{r}-\frac{1}{a}=\frac{1}{r}$; so that, for the velocity in a circle, we have $\mathrm{V}^2=\frac{\mathrm{M}+m}{r}$, $\mathrm{V}=\frac{\sqrt{\mathrm{M}+m}}{\sqrt{r}}$; that is to say, the velocity in different circles is in the sub-duplicate ratio of the sum of the masses, or the absolute force, as it is sometimes called, directly, and of the radii inversely. Moreover, if we denote by V the velocity in a circle of the radius r, or $\mathrm{V}=\sqrt{\frac{\mathrm{M}+m}{r}}$, we have

$$\frac{v^2}{\mathrm{V}^2}=\frac{\frac{2}{r}-\frac{1}{a}}{\frac{1}{r}}=2-\frac{r}{a}=\frac{2\,a-r}{a}$$

or

$$v^2 : \mathrm{V}^2 :: 2\,a-r : a.$$

Fig. 7. Now, if APM be an ellipse, S, H, the foci, $\mathrm{AM}=2\,a$ and $\mathrm{SP}=r$, we have $\mathrm{HP}=2\,a-r$; so that

$$v : \mathrm{V} :: \sqrt{\mathrm{HP}} : \sqrt{\mathrm{AR}}$$

by which property the velocity in a conic section may be immediately compared with that in a circle at the same distance.

Hence, when SP = AC, or at the extremities of the conjugate axis, the velocity is equal to that in a circle.

In the parabola, we have $\frac{v^2}{\mathrm{V}^2}=2$, or $v=\mathrm{V}\,.\,\sqrt{2}$, so that in this curve the velocity bears a constant ratio to that in a circle at the same distance, $\sqrt{2} : 1$.

In the hyperbola, HP increases without limit, and the velocity bears continually a greater and greater ratio to that in a circle.

As the velocity depends only on the distance and major axis of the conic section, and not at all on its form, we may conceive the conjugate axis so diminished that the conic section shall pass into a straight line. In this case, the extremity of the axis will coincide with the focus, and the velocity at any distance r will be that acquired by falling from a distance $2\,a$ from the centre to the distance r. The expression for this velocity is therefore still the same with that in the conic section. Hence, "The velocity in a conic section at any point is that which would be acquired by falling freely towards the centre, from a distance equal to the longer axis, to that point." In the parabola the longer axis is infinite, and the velocity at any point is, therefore, that acquired by falling from an infinite distance. In the hyperbola the axis is negative, and even an infinite fall is not sufficient to give a body all the velocity requisite for the description of this curve.

We have then the following expressions:—

$$\left.\begin{array}{ll}\text{Velocity in a conic section, semiaxis} = a, \text{ distance} = r\,; & v = \sqrt{(\mathrm{M}+m)\left(\frac{2}{r}-\frac{1}{a}\right)} \\ \text{———————circle, whose radius is } r\ldots\ldots\ldots\ldots\ldots & \mathrm{V} = \sqrt{\frac{\mathrm{M}+m}{r}} \\ \text{———————parabola at any distance } r,\ldots\ldots\ldots\ldots\ldots & v = \sqrt{\frac{2\,(\mathrm{M}+m)}{r}}\end{array}\right\} \quad (35)$$

Let us now consider the effect on the form of the conic section and on the position of its major axis, arising from a change not in the velocity, but in the *angle of projection*, i. e. the angle made with the radius vector by the direction in which the body is projected.

Fig. 7. APM being any conic section, and S, H, its foci, SY, SZ, perpendiculars on a tangent at P, the angle of projection SPy, which we will call A, is equal to HPZ, and therefore

$$\mathrm{SY} = \mathrm{SP}\,.\,\sin \mathrm{A}, \text{ and } \mathrm{HZ} = \mathrm{HP}\,.\,\sin \mathrm{A}.$$

consequently,

$$\mathrm{SY} \times \mathrm{HZ} = \mathrm{SP}\,.\,\mathrm{HP}\,.\,\sin \mathrm{A}^2;$$

but, by the property of the conic sections, SY × HZ $= \mathrm{CD}^2$; and therefore $\mathrm{CD}^2 = \mathrm{SP}\,.\,\mathrm{HP}\,.\,\sin \mathrm{A}^2$. Now, we have $\mathrm{SP} = r$, $\mathrm{HP} = 2\,a - r$, $\mathrm{CD}^2 = a^2\,(1 - e^2)$;

hence, $$a^2\,(1 - e^2) = r\,(2\,a - r)\,.\,\sin \mathrm{A}^2$$

and $$a\,(1 - e^2) = r \times \left(2 - \frac{r}{a}\right).\sin \mathrm{A}^2.$$

But $a\,(1 - e^2)$ is the semiparameter of the conic section. Moreover, calling v the velocity in the curve, and V that in a circle; and denoting by n the ratio $\left(\frac{v}{\mathrm{V}}\right)^2$ we have already seen, that $\left(2 - \frac{r}{a}\right) = n^2$.

Hence, we have

$$a\,(1 - e^2) = r \times n^2\,.\,\sin \mathrm{A}^2; \qquad (36)$$

and therefore is given, when the angle and distance of projection are given, and also the velocity. When the angle of projection varies, other circumstances remaining, we see hence that the parameter varies as the square of its sine.

The eccentricity is easily found; for we have

$$e = \sqrt{1 - \frac{r}{a}\left(2 - \frac{r}{a}\right).\sin \mathrm{A}^2}$$

Now $\frac{r}{a} = 2 - \frac{r\,v^2}{\mathrm{M}+m}$; but since $v^2 = n^2\ \mathrm{V}^2 = n^2\,.\,\frac{\mathrm{M}+m}{r}$, we have $\frac{r\,v^2}{\mathrm{M}+m} = n^2$, and $\frac{r}{a} = 2 - n^2$; $2 - \frac{r}{a} = n^2$. so that we get by substitution

$$\begin{aligned} e &= \sqrt{1 - n^2\,(2 - n^2)\,.\,\sin \mathrm{A}^2} \\ &= \sqrt{\cos \mathrm{A}^2 + \sin \mathrm{A}^2\,(1 - n^2)^2}\,; \end{aligned} \qquad (37)$$

We see, therefore, that the ratio e of the eccentricity to the semiaxis, or the figure of the ellipse, or hyperbola, depends solely on the angle of projection and the ratio of the velocity of projection to that in a circle at the same distance; and if this latter ratio remain the same, the distance may be varied to any extent without changing the figure of the conic section.

It only remains to determine its position, or the angle made by the greater axis (or line of apsides) with the distance SP. Now, if we call ASP, θ, we have

$$r = \frac{a\,(1 - e^2)}{1 + e\,.\cos\theta};$$

$$\cos\theta = \frac{1}{e} \times \left\{\frac{a\,(1 - e^2)}{r} - 1\right\}$$

in which, substituting for e and $a\,(1 - e^2)$, their values before found, we shall obtain

$$\cos\theta = \frac{n^2\,.\,\sin \mathrm{A}^2 - 1}{\sqrt{\cos \mathrm{A}^2 + (1 - n^2)^2\,.\,\sin \mathrm{A}^2}} \qquad (38)$$

on which value we may make the same remark as on that of e, and in both which it will be recollected, that $n^2 = \frac{r\,v^2}{\mathrm{M}+m}$.

Section V.

On certain peculiar cases of the celestial motion; viz. when the orbit is of very great, or very small eccentricity. Of circular and parabolic motion.

The circumstances of circular motion are too simple to need much consideration. The velocity is uniform and equal to that which would be acquired by falling down half the radius with the force at the circumference continued uniformly. This is evident, if we recollect that the velocity in a circle, by what has been just proved, is $\sqrt{\frac{\mathrm{M}+m}{r}}$, but that acquired by the action of a constant force F, acting through a space $\frac{1}{2}\,r$, is given by the equation $v^2 = 2\,\mathrm{F}\,s = \mathrm{F}\,r$ in the present case $= \frac{(\mathrm{M}+m)\,r}{r^2} = \frac{\mathrm{M}+m}{r}$. The periodic time is equal to the circumference divided by this constant velocity, or to $\frac{2\,\pi\,r^{\frac{3}{2}}}{\sqrt{\mathrm{M}+m}}$, as we have before shewn by a more general reasoning.

In a conic section of so small an eccentricity that its square may be neglected, it may be worth while to recapitulate some of the chief formulæ, developed to the first power. We have then

$$\left.\begin{aligned} r &= a\,(1 - e\,.\cos\theta) \\ &= a\,(1 - e\,.\cos v) \\ &= a\,(1 - e\,.\cos n\,t) \end{aligned}\right\}; \qquad (39)$$

$$\left.\begin{aligned} \theta &= v - e\,.\sin v \\ \theta &= n\,t + 2\,e\,.\sin n\,t \end{aligned}\right\}; \qquad (40)$$

$$v = n\,t + e\,.\sin n\,t; \qquad (41)$$

$$n\,t = \theta - 2\,e\,.\sin\theta; \qquad (42)$$

Let us consider the motion in a parabola, and since in this case we have $a = \infty$, $e = 1$, and $p = a\,(1 - e^2)$, if we call the perihelion distance D, we have $\mathrm{D} = \frac{1}{2}\,p$, and $p = 2\,\mathrm{D}$. Then will the equations (17) and (18) become

$$r = \frac{2\,\mathrm{D}}{1 + \cos\theta} = \frac{\mathrm{D}}{\left(\cos\frac{\theta}{2}\right)^2}; \qquad (43)$$

$$d\,t = \frac{(2\,\mathrm{D})^{\frac{3}{2}}}{\sqrt{\mathrm{M}+m}} \cdot \int \frac{d\,\theta}{(1+\cos\theta)^2}$$

To integrate the latter, put $\tan\frac{\theta}{2} = x$, and we have

$$\tan\theta = \frac{2\,x}{1-x^2}$$

$$\cos\theta = \frac{1-x^2}{1+x^2}$$

$$d\,\theta = \frac{2\,d\,x}{1+x^2}$$

which substituted, give

$$d\,t = \frac{(2\,\mathrm{D})^{\frac{3}{2}}}{\sqrt{\mathrm{M}+m}} \times \left(\frac{d\,x}{2} + \frac{x^2\,d\,x}{2}\right)$$

$$t = \frac{\mathrm{D}^{\frac{3}{2}} \cdot \sqrt{2}}{\sqrt{\mathrm{M}+m}} \left(x + \frac{x^3}{3}\right)$$

or,

$$t = \frac{\mathrm{D}^{\frac{3}{2}} \cdot \sqrt{2}}{\sqrt{\mathrm{M}+m}} \times \left\{\tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\}; \qquad (44)$$

This equation gives $\tan\frac{\theta}{2}$, and consequently θ, by the resolution of a cubic equation of the form

$$x^3 + 3\,x = \mathrm{A}$$

where $\mathrm{A} = \frac{3\,t\,.\,\sqrt{\mathrm{M}+m}}{\mathrm{D}^{\frac{3}{2}}\,.\,\sqrt{2}}$. This cubic, it is easily shewn, has but one real root; and since the co-efficient of x is an absolute number, and therefore x a function of A only, its root may be found in any proposed case by a table of single entry. In fact, suppose we have formed such a table, containing the values of x (or, for greater convenience, of $2 \times \text{arc}\,(\tan = x)$ or θ) for every value of t on the supposition of $\mathrm{D} = 1$, then will this table serve for all cases, and for every other value of D; for we have only to multiply t by $\mathrm{D}^{-\frac{3}{2}}$; and calling the product T, look out in the table for the value of θ corresponding to the time T. A comet, describing a parabola, whose perihelion distance is 1, will describe 90° of anomaly from the perihelion in about 109 days. Hence, the use of the *table of a comet of* 109 *days* given in works on astronomy, Lalande's *Astron.* 2d edit. vol. iii. p. 335; Delambre's *Astronomy*, vol. iii. p. 434, from which we extract it as subjoined.

Table of a comet of 109 *days.*

Days	True anomaly.	Days	True anomaly.	Days	True anomaly.	Days	True anomaly.	Days	True anomaly.	Days	True anomaly.
0	0° 0′ 0″0	35	43°58′34″8	70	71°51′23″2	105	88°20′42″9	140	98°56′22″5	175	106°20′14″2
1	1 23 37·4	36	44 59 57·5	71	72 27 4·4	106	88 42 42·6	141	99 11 12·9	176	106 30 59·4
2	2 47 11·9	37	46 0 26·6	72	73 2 13·4	107	89 4 26·0	142	99 25 54·3	177	106 41 39·4
3	4 10 40·4	38	47 0 2·2	73	73 36 51·0	108	89 25 53·4	143	99 40 26·9	178	106 52 14·1
4	5 34 0·1	39	47 58 44·8	74	74 10 57·8	109	89 47 5·1	144	99 54 50·9	179	107 2 43·5
5	6 57 8·0	40	48 56 34·5	75	74 44 34·3	110	90 8 1·3	145	100 9 6·4	180	107 13 7·7
6	8 20 1·3	41	49 53 32·0	76	75 17 41·2	111	90 28 42·4	146	100 23 13·4	181	107 23 26·9
7	9 42 37·2	42	50 49 37·6	77	75 50 18·9	112	90 49 8·7	147	100 37 12·2	182	107 33 41·1
8	11 4 53·0	43	51 44 51·8	78	76 22 28·1	113	91 9 20·3	148	100 51 2·8	183	107 43 50·2
9	12 26 45·9	44	52 39 15·2	79	76 54 9·3	114	91 29 17·6	149	101 4 45·4	184	107 53 54·6
10	13 48 13·4	45	53 32 48·2	80	77 25 23·1	115	91 49 0·7	150	101 18 20·1	185	108 3 54·1
11	15 9 13·1	46	54 25 31·6	81	77 56 10·1	116	92 8 30·0	151	101 31 47·1	186	108 13 48·8
12	16 29 42·5	47	55 17 25·8	82	78 26 30·6	117	92 27 45·7	152	101 45 6·5	187	108 23 38·8
13	17 49 39·4	48	56 8 31·5	83	78 56 25·3	118	92 46 48·0	153	101 58 18·2	188	108 33 24·2
14	19 9 1·5	49	56 58 49·4	84	79 25 54·6	119	93 5 37·3	154	102 11 22·6	189	108 43 5·0
15	20 27 46·8	50	57 48 20·1	85	79 54 59·1	120	93 24 13·6	155	102 24 19·6	190	108 52 41·3
16	21 45 53·4	51	58 37 4·3	86	80 23 39·1	121	93 42 37·2	156	102 37 9·4	191	109 2 13·1
17	23 3 19·4	52	59 25 2·7	87	80 51 55·4	122	94 0 48·4	157	102 49 52·1	192	109 11 40·4
18	24 20 3·1	53	60 12 15·9	88	81 19 48·1	123	94 18 47·3	158	103 2 27·8	193	109 21 3·5
19	25 36 2·9	54	60 58 44·8	89	81 47 17·9	124	94 36 34·2	159	103 14 56·6	194	109 30 22·2
20	26 51 17·3	55	61 44 29·9	90	82 14 25·2	125	94 54 9·2	160	103 27 18·5	195	109 39 36·7
21	28 5 45·1	56	62 29 32·1	91	82 41 10·3	126	95 11 32·7	161	103 39 33·8	196	109 48 46·9
22	29 19 25·0	57	63 13 52·0	92	83 7 33·7	127	95 28 44·7	162	103 51 42·4	197	109 57 53·1
23	30 32 15·8	58	63 57 30·3	93	83 33 35·9	128	95 45 45·4	163	104 3 44·6	198	110 6 55·1
24	31 44 16·7	59	64 40 27·7	94	83 59 17·2	129	96 2 35·2	164	104 15 40·3	199	110 15 53·1
25	32 55 26·7	60	65 22 45·0	95	84 24 38·0	130	96 19 14·0	165	104 27 29·7	200	110 24 47·1
26	34 5 45·2	61	66 4 22·8	96	84 49 38·7	131	96 35 42·2	166	104 39 12·8		
27	35 15 11·4	62	66 45 22·0	97	85 14 19·8	132	96 51 59·8	167	104 50 49·8		
28	36 23 44·8	63	67 25 43·1	98	85 38 41·6	133	97 8 7·1	168	105 2 20·7		
29	37 31 24·9	64	68 5 26·9	99	86 2 44·4	134	97 24 4·2	169	105 13 45·6		
30	38 38 11·5	65	68 44 34·0	100	86 26 28·7	135	97 39 51·3	170	105 25 4·7		
31	39 44 4·2	66	69 23 5·1	101	86 49 54·7	136	97 55 28·6	171	105 36 17·9		
32	40 49 2·9	67	70 1 1·0	102	87 13 2·8	137	98 10 56·2	172	105 47 25·4		
33	41 53 7·6	68	70 38 22·1	103	87 35 53·3	138	98 26 14·3	173	105 58 27·2		
34	42 56 18·2	69	71 15 9·3	104	87 58 26·5	139	98 41 23·0	174	106 9 23·4		
35	43 58 34·8	70	71 51 23·2	105	88 20 42·9	140	98 56 22·5	175	106 20 14·2		

2dly. Let us now consider the circumstances of the motion of a body in an ellipse of great eccentricity, or approaching very near to a parabola. This is necessary to complete the theory of such comets as are known, or may be suspected, to revolve in very elongated ellipses.

Since e, the ratio of the eccentricity, approaches very near to unity, $1 - e$ must be a very small quantity; and if we put $\beta = \frac{1-e}{1+e}$, we shall have $e = \frac{1-\beta}{1+\beta}$, $1 - e = \frac{2\beta}{1+\beta}$, $1 + e = \frac{2}{1+\beta}$, $(1 - e^2) = \frac{4\beta}{(1+\beta)^2}$; now, if D = the perihelion distance, we have $D = a\,(1 - e)$, consequently

$$r = \frac{D\,(1+e)}{2 + e\cos\theta} = \frac{2\,D}{(1+\beta) + (1-\beta)\,.\cos\theta} = \frac{D}{\frac{1+\cos\theta}{2} + \beta\,.\,\frac{1-\cos\theta}{2}}$$

$$= \frac{\frac{D}{\left(\cos\frac{\theta}{2}\right)^2}}{1 + \beta\,.\,\frac{1-\cos\theta}{1+\cos\theta}} = \frac{D}{\left(\cos\frac{\theta}{2}\right)^2\left(1 + \beta\,.\left(\tan\frac{\theta}{2}\right)^2\right)}$$

and developing this in powers of β, we obtain

$$r = \frac{D}{\left(\cos\frac{\theta}{2}\right)^2}\left\{1 - \beta\,.\left(\tan\frac{\theta}{2}\right)^2 + \beta^2\,.\left(\tan\frac{\theta}{2}\right)^4 - \&c.\right\}; \qquad (45)$$

which is a very simple expression for the radius vector in the elongated ellipse.

To find the time, we must make the same substitution in the expression,

$$t + C = \frac{a^{\frac{3}{2}}\,(1 - e^2)^{\frac{3}{2}}}{\sqrt{M+m}}\,.\int\frac{d\,\theta}{(1 + e\,.\cos\theta)^2}$$

Now $\left\{a\,.\,(1 - e^2)\right\}^{\frac{3}{2}} = \left(\frac{2\,D}{1+\beta}\right)^{\frac{3}{2}}$ and since

$$(1 + e\,.\,\cos\theta)^2 = \frac{1}{(1+\beta)^2}\left\{1 + \beta + (1-\beta)\,.\,\cos\theta\right\}^2$$

$$= \frac{4}{(1+\beta)^2}\left\{\frac{1+\cos\theta}{2} + \beta\,.\,\frac{1-\cos\theta}{2}\right\}^2$$

$$= \frac{4\,.\left(\cos\frac{\theta}{2}\right)^2\left\{1 + \beta\,.\left(\tan\frac{\theta}{2}\right)^2\right\}^2}{(1+\beta)^2}$$

if we put $\tan\frac{\theta}{2} = T$, we shall have $\frac{\frac{1}{2}\,d\,\theta}{\left(\cos\frac{\theta}{2}\right)^2} = d\,T$; and the integral will become $\frac{(1+\beta)^2}{2}\,.\int\frac{d\,T}{(1+\beta\,.\,T^2)^2}$

whence we get after all substitution,

$$t + C = \sqrt{\frac{2\,(1+\beta)}{M+m}}\,.\,D^{\frac{3}{2}}\,.\int\frac{d\,T\,(1 + T^2)}{(1 + \beta\,.\,T^2)^2}$$

and the part of this under the integral sign being developed in powers of β, and the whole integrated, we get

$$t + C = D^{\frac{3}{2}}\,.\sqrt{\frac{2\,(1+\beta)}{M+m}} \times \left\{\left(T + \frac{T^3}{3}\right) - \frac{2}{1}\,\beta\left(\frac{T^3}{3} + \frac{T^5}{5}\right) + \frac{2\,.\,3}{1\,.\,2}\,\beta^2\left(\frac{T^5}{5} + \frac{T^7}{7}\right) - \&c.\right\} \qquad (46)$$

By the aid of this series, we may express very readily the difference between the true anomaly in a very eccentric ellipse, and that in a parabola of equal perihelion distance, for, T still expressing the tangent of half the former true anomaly, and τ that of the latter, if we develope according to powers of β, pursuing the process only as far as β^2, we get

$$t + C = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M+m}}\left\{\left(T + \frac{T^3}{3}\right) + \beta\left(\frac{T}{2} - \frac{T^3}{2} - \frac{2\,T^5}{5}\right) - \beta^2\left(\frac{T}{8} + \frac{3\,T^3}{8} - \frac{2\,T^5}{5} - \frac{3\,T^7}{7}\right)\right\}; \qquad (47)$$

Now, the time being the same both in the parabola and ellipse, this must equal $\frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M+m}}\left\{\tau + \frac{\tau^3}{3}\right\}$, and therefore the series within the brackets on both sides must be equal.

Let

$$\tau + \frac{\tau^3}{3} = a;\ \frac{\tau}{2} - \frac{\tau^3}{2} - \frac{2\,\tau^5}{5} = b;\ -\frac{\tau}{8} - \frac{3\,\tau^3}{8} + \frac{2\,\tau^5}{5} + \frac{3\,\tau^7}{7} = c$$

$$T + \frac{T^3}{3} = A,\ \frac{T}{2} - \frac{T^3}{2} - \frac{2\,T^5}{5} = B;\ -\frac{T}{8} - \frac{3\,T^3}{8} + \frac{2\,T^5}{5} + \frac{3\,T^7}{7} = C$$

then we have
$$A + B\beta + C\beta^2 = a;$$
and the first member of this equation, though apparently a function of β, cannot be really so, as the second does not contain it. Hence, if we call $\triangle$ the difference between the elliptic and parabolic anomaly, or suppose the former anomaly $\theta + \triangle$ the latter θ, since $T = \tan\frac{\theta+\triangle}{2}$, and $\tau = \tan\frac{\theta}{2}$, we may express A, B, and C, thus :—

$$A = a + \frac{da}{d\theta}\cdot\frac{\triangle}{1} + \frac{d^2 a}{d\theta^2}\cdot\frac{\triangle^2}{1\,.\,2} + \&c.$$

$$B = b + \frac{db}{d\theta}\cdot\frac{\triangle}{1} + \frac{d^2 b}{d\theta^2}\cdot\frac{\triangle^2}{1\,.\,2} + \&c.$$

$$C = c + \frac{dc}{d\theta}\cdot\frac{\triangle}{1} + \frac{d^2 c}{d\theta^2}\cdot\frac{\triangle^2}{1\,.\,2} + \&c.$$

but $\triangle$ itself is required to be expressed in terms of β and its powers; and, consequently we may suppose $\triangle = P\beta + Q\beta^2 + \&c.$

If then we make these substitutions, our equation becomes
$$\begin{aligned} a = a + P\frac{da}{d\theta}\beta + \left(Q\frac{da}{d\theta} + \frac{P^2}{2}\cdot\frac{d^2 a}{d\theta^2}\right)\beta^2 + \&c. \\ + b\beta + P\frac{db}{d\theta}\cdot\beta^2 + \&c. \\ + c\beta^2 + \&c. \end{aligned}$$
whence we obtain by equating the co-efficients of β to zero,
$$o = P\frac{da}{d\theta} + b;\ o = Q\frac{da}{d\theta} + \frac{P^2}{2}\cdot\frac{d^2 a}{d\theta^2} + P\cdot\frac{db}{d\theta} + c;\ \&c.$$
Now we have $\frac{da}{d\theta} = \frac{da}{d\tau}\cdot\frac{d\tau}{d\theta}$, and since $a = \tau + \frac{\tau^3}{3}$, $\frac{da}{d\tau} = 1 + \tau^2$, moreover $\frac{d\tau}{d\theta} = \frac{1+\tau^2}{2}$, whence $\frac{da}{d\theta} = \frac{(1+\tau^2)^2}{2}$, and therefore $P = \frac{1}{(1+\tau^2)^2}\left\{-\tau + \tau^3 + \frac{4}{5}\tau^5\right\}$

Again,
$$\frac{d^2 a}{d\theta^2} = \frac{d}{d\theta}\cdot\frac{da}{d\theta} = 2\tau(1+\tau^2)\,.\,\frac{d\tau}{d\theta} = \tau(1+\tau^2)^2$$
$$\frac{db}{d\theta} = \frac{d}{d\theta}\left\{\frac{\tau}{2} - \frac{\tau^3}{2} - \frac{2}{5}\tau^5\right\} = \left\{\frac{1}{2} - \frac{3\tau^2}{2} - 2\tau^4\right\}\cdot\frac{1+\tau^2}{2}$$
whence, after all reductions, we obtain
$$Q = \frac{\frac{3}{4}\tau - \frac{5}{4}\tau^3 + \frac{1}{20}\tau^5 + \frac{167}{140}\tau^7 + \frac{24}{35}\tau^9 + \frac{18}{175}\tau^{11}}{(1+\tau^2)^4}$$
and, finally,
$$\left.\begin{aligned} \triangle = \frac{-\tau + \tau^3 + \frac{4}{5}\tau^5}{(1+\tau^2)^2}\,.\,\beta \\ + \frac{\frac{3}{4}\tau - \frac{5}{4}\tau^3 + \frac{1}{20}\tau^5 + \frac{167}{140}\tau^7 + \frac{24}{35}\tau^9 + \frac{18}{175}\tau^{11}}{(1+\tau^2)^4}\,.\,\beta^2; \end{aligned}\right\} \quad (48)$$

SECTION VI.

On the determination of the planetary orbits *a posteriori*, or from observation.

The determination of the elements of the orbit of a planet, or comet, is justly regarded as one of the most difficult problems of astronomy. By the *elements* of the orbit are meant those constant data which determine the position, form, and magnitude of the conic section described, and the body's place in it at a given epoch, and are six in number, viz.

The inclination of its plane to the ecliptic,
The longitude of its ascending node,
The position of its axis, or the longitude of the perihelion,
The length of its axis,
The eccentricity, or the length of the semiparameter, or the perihelion distance,
The moment of passing the perihelion, or the heliocentric longitude at any assigned time,

which correspond to, and are functions of the six arbitrary constants introduced by the integration of the equations (1), (2), and (3). If we knew, and could subject to calculation, the causes which originally determined the motions of the heavenly bodies, we could assign, *a priori*, by the means already pointed out, the values of these constants from the conditions which subsisted at the commencement of their motions; but as these are, and must ever remain, unknown, we can only discover the values in question, *a posteriori*, by observation of the apparent places of each body at different times.

In this research, the great difficulties arise from our not observing from a fixed station in the centre of our system. To an observer stationed in the sun, the elements would offer themselves with comparative ease.

But the motion and eccentric position of the earth render the whole affair infinitely more complicated; and to analyse the problem, it becomes necessary to take into consideration the relations between the heliocentric and geocentric places of a heavenly body.

To this effect, let S be the sun, E the earth, P the planet, or comet, projected into p upon the ecliptic by the perpendicular Pp. Let S γ be the line of equinoxes, parallel to which draw E γ. Then will

Fig. 8.

$L = \gamma SE$ represent the earth's *heliocentric* longitude.
$l = \gamma Sp$ ——— the planet's heliocentric longitude.
$\lambda = \gamma Ep$ ——— the planet's geocentric longitude.
$b = PSp$ ——— the planet's heliocentric latitude.
$\beta = PEp$ ——— its geocentric latitude.
R = the earth's distance from the sun.
r = the planet's distance from the sun.
ρ = the planet's distance from the earth.
d, δ = the respective *curtate*, or projected distances of the planet from the sun and earth.

It is obvious, then, that we shall have the following relations :—

1st. $Pp = r \,.\, \sin b$, also the same $Pp = \rho \,.\, \sin\beta$, so that,

$$r \,.\, \sin b = \rho \,.\, \sin\beta; \tag{49}$$

2dly.
$$\begin{cases} d = r \,.\, \cos b & (50) \\ \delta = \rho \,.\, \cos\beta & (51) \end{cases}$$

3dly. Since the angle PSE of the triangle PSE is equal to $l - L$, and its sides are R, d, and δ

$$\delta^2 = R^2 + d^2 - 2Rd\cos(l-L); \tag{52}$$

Lastly producing SE to C, the angle $pEC = pE\gamma - CS\gamma = \lambda - L$, but the angle $pEc = 180° - pES$, consequently the equation

$$d^2 = R^2 + \delta^2 - 2R\delta \,.\, \cos pES$$

gives

$$d^2 = R^2 + d^2 + 2R\delta \,.\, \cos(\lambda - L); \tag{53}$$

Thus then we have five distinct and independent relations among the ten quantities L, l, λ, &c. Now, it is evident, that if R and L be given, (or the place of E fixed) the geocentric latitude and longitude, β and λ, will be determined when the place of P is, but this is not determined unless l, d, and b, are so. But when all the five quantities L, R, l, d, b, are given, the five equations above deduced suffice to determine the others, and therefore express all the relations subsisting between the heliocentric and geocentric places. The last may, however, be simplified by substituting for δ^2 its value given in (52), when the whole becomes divisible by 2 R, and gives

$$o = R + \delta \,.\, \cos(\lambda - L) - d \,.\, \cos(l - L); \tag{54}$$

Another equation (included, of course, in the foregoing,) may also be obtained if we consider, that in the triangle PSE, we have

$$r^2 = R^2 + \rho^2 - 2R\rho \,.\, \cos PES$$

but cos PES = cos pES × cos PEp by spherical trigonometry, the plane PEp being at right angles to pES. Hence, we get (writing $\frac{\rho'}{\cos\beta}$,) or $\rho' \,.\, \mathrm{Sec}\,\beta$ for ρ, and $-\cos(\lambda - L)$ for cos pES

$$r^2 = R^2 + \delta^2 \,.\, \mathrm{Sec}\,\beta^2 + 2R\delta \,,\, \cos(\lambda - L); \tag{55}$$

In like manner, eliminating ρ from (49), it becomes

$$r \,.\, \sin b = \delta \,.\, \tan\beta; \tag{56}$$

and putting in (54) for d, its value $r \,.\, \cos b$,

$$o = R + \delta \,.\, \cos(\lambda - L) - r \,.\, \cos b \,.\, \cos(l - L) \tag{57}$$

This equation may also be derived independently, if we drop the perpendicular pq, for we then have

$$\begin{aligned} Sq &= SP \,.\, \cos PSp \,.\, \cos pSq \\ &= r \,.\, \cos b \,.\, \cos(l - L) \end{aligned}$$

and also

$$\begin{aligned} Sq &= SE + Eq = \\ &= R + \delta \,.\, \cos(\lambda - L) \end{aligned}$$

which equated, give the equation in question. We may also derive another, which will be useful in the sequel, by equating two values of the perpendicular itself.

$$\begin{aligned} pq &= Ep \,.\, \sin pEq \\ pq &= Sp \,.\, \sin pSq \end{aligned}$$

which equated, give

$$r \,.\, \cos b \,.\, \sin(l - L) = \delta \,.\, \sin(\lambda - L); \tag{58}$$

Thus we have reduced the five equations to four, 55, 56, 57, 58, which contain only the radius vector of the earth, that of the planet, and its curtate distance from the earth, and the heliocentric and geocentric latitudes and longitudes of the two bodies.

The problem of determining the elements of P's orbit from observation requires these relations to be combined with the conditions of P's motion and of the earth's. Now, if we call ☊ the longitude of P's ascending node, and i the inclination of its orbit, we have, by spherical trigonometry, for the heliocentric latitude.

$$\tan b = \tan i \,.\, \sin(l - \text{☊}) \tag{59}$$

Moreover, if we call π the longitude of the perihelion on the orbit, θ the true anomaly, and ψ the angle ☊ SA, or the distance of the node from the perihelion on the orbit, (see fig. 9) we shall have

Fig. 9.

$$\cos A\text{☊}a = \tan \text{☊}Sa \,.\, \cot \text{☊}SA$$

or

$$\tan(\pi - \text{☊}) = \cos i \,.\, \tan\psi \tag{60}$$

and also

$$\sin b = \sin i \,.\, \sin(\psi + \theta) \tag{61}$$

these seven equations, of which the four first, 55, 56, 57, 58, express the relations between the heliocentric and geocentric places of a body any how situated, the other three, 59, 60, 61, merely express the condition of every point in the orbit lying in a plane passing through the sun. These must be combined with the dynamical results peculiar to the planetary motions, viz. 1st. The proportionality of the areas to the times; and, 2dly, The nature of the orbit, regarded as an ellipse or parabola of unknown position, form, and magnitude; all which are included in the three equations, 21, 22, 23, or others equivalent to them. Thus we have a system of nine equations, representing, in general, the relations between the time elapsed since a given epoch, and the observed geocentric latitude and longitude.

These equations involve, as constant quantities, the elements of the orbit, i, ☊, π, a, e, E, while the variable ones, being either given by observation, or expressed in functions of others that are so, the whole system of equations may be regarded as expressing a known (though a very complicated) relation among the elements; and as each observation affords a similar relation, it is evident that the complete determination of the latter cannot require more than six observations. In fact, however, from the peculiar nature of some of the equations, fewer will suffice, half that number only being necessary.

The complication of the relations in question precludes all idea of a direct solution of the problem, and to apply them to any particular case indirect and approximative ones have been invented, but with every assistance from such simplification; and, after all the force of analysis has been exercised upon it, it still remains a very difficult problem, and one which our limits will by no means permit us to enter upon in its full extent. In fact, the case of an elliptical orbit to be determined is one of very rare occurrence; and we shall therefore content ourselves with pointing out the course to be pursued to obtain the readiest approximation to the elements of parabolic motion. The frequent visits of comets to our system render this an important case. These singular bodies, with one or two exceptions, have always been found to describe parabolas, or ellipses of such extreme eccentricity, as to be undistinguishable from parabolas within considerable distances on either side their perihelion.

Preparatory to this research, we must premise the following remarkable theorem.

In a parabola, if r, r' denote any two distances from the focus, k the chord of the arc they include, and t the time of a body describing that arc, and g a certain constant the same for all parabolas, we shall have

$$g\,t = (r + r' + k)^{\frac{3}{2}} - (r + r' - k)^{\frac{3}{2}}; \qquad (62)$$

To shew this, let θ and θ' be the true anomalies corresponding to the radii r, r', and t being the time of describing the intermediate arc, and ϵ the time elapsed since the perihelion passage to the commencement of the time t, we have, by (44)

$$\epsilon = q\left\{\tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\}$$

$$t + \epsilon = q\left\{\tan\frac{\theta'}{2} + \frac{1}{3}\left(\tan\frac{\theta'}{2}\right)^3\right\}$$

where $q = \dfrac{D^3\sqrt{2}}{\sqrt{M+m}}$. Consequently, we have

$$t = q\,.\left\{\left(\tan\frac{\theta'}{2} - \tan\frac{\theta}{2}\right) + \frac{1}{3}\left(\left(\tan\frac{\theta'}{2}\right)^3 - \left(\tan\frac{\theta}{2}\right)^3\right)\right\}$$

Now, by the property of the parabola, we have also

$$r = \frac{D}{\left(\cos\frac{\theta}{2}\right)^2} \qquad r' = \frac{D}{\left(\cos\frac{\theta'}{2}\right)^2}$$

And if we put $\dfrac{\theta' - \theta}{2} = \phi$, we shall have

$$\sqrt{\frac{r}{r'}} = \frac{r}{\sqrt{r\,r'}} = \frac{\cos\frac{1}{2}\theta'}{\cos\frac{1}{2}\theta} = \frac{\cos(\frac{1}{2}\theta + \phi)}{\cos\frac{1}{2}\theta}$$

or, $$\frac{r}{\sqrt{r\,r'}} = \cos\phi - \tan\tfrac{1}{2}\theta\,.\,\sin\phi$$

whence we get

$$\tan\frac{\theta}{2} = \text{cosec.}\,\phi\left\{\cos\phi - \frac{r}{\sqrt{r\,r'}}\right\} \qquad (63)$$

Similarly, since $\frac{1}{2}\theta = \frac{1}{2}\theta' - \phi$, we have,

$$\sqrt{\frac{r'}{r}} \text{ or } \frac{r'}{\sqrt{r\,r'}} = \frac{\cos(\frac{1}{2}\theta' - \phi)}{\cos\frac{1}{2}\theta'}$$

and therefore

$$\tan\frac{\theta'}{2} = -\text{cosec.}\,\phi\left\{\cos\phi - \frac{r'}{\sqrt{r\,r'}}\right\}$$

but k being the side of a triangle opposite to the angle $(\theta' - \theta)$ or 2ϕ and r and r' the sides adjacent, we have, by trigonometry,

$$\cos\phi = \frac{\sqrt{(r + r')^2 - k^2}}{2\sqrt{r\,r'}}$$

$$\sin\phi = \frac{\sqrt{k^2 - (r - r)^2}}{2\sqrt{r\,r'}}$$

or, if we put

$$\sqrt{(r + r')^2 - k^2} = R \text{ and } \sqrt{k^2 - (r - r)^2} = S$$

$$R = 2\sqrt{r\,r'}\,.\cos\phi \text{ and } S = 2\sqrt{r\,r'}\,.\sin\phi$$

$$\cos\phi = \frac{R}{2\sqrt{rr'}} \qquad \sin\phi = \frac{S}{2\sqrt{rr'}}$$

Hence, we obtain

$$\tan\frac{\theta}{2} = \frac{R - 2r}{S}; \quad \tan\frac{\theta'}{2} = -\frac{R - 2r'}{S}$$

$$\tan\frac{\theta'}{2} - \tan\frac{\theta}{2} = \frac{2(r + r' - R)}{S}$$

also

$$\left(\cos\frac{\theta}{2}\right)^2 = \frac{1}{1 + \left(\tan\frac{\theta}{2}\right)^2} =$$

$$= \frac{S^2}{(R^2 + S^2) - 4\,r\,R + 4\,r^2}$$

but $R^2 + S^2 = (r + r')^2 - (r' - r)^2 = 4\,r\,r'$, so that we get

$$\left(\cos\,.\,\frac{\theta}{2}\right)^2 = \frac{1}{4r}\,.\,\frac{S^2}{r + r' - R}$$

and, in consequence,

$$D = r\,.\left(\cos\frac{\theta}{2}\right)^2$$

$$= \frac{1}{4}\,.\,\frac{S^2}{r + r' - R}$$

Now $(r + r' - R)(r + r' + R) = (r + r')^2 - R^2 = k^2$; so that

$$D = \frac{S^2}{4\,k^2}\,.\,(r + r' + R)$$

and $$D\left\{\tan\frac{\theta'}{2} - \tan\frac{\theta}{2}\right\} = \frac{S}{2}$$

consequently

$$\frac{D^{\frac{3}{2}}\,.\sqrt{2}}{\sqrt{M+m}}\left\{\tan\frac{\theta'}{2} - \tan\frac{\theta}{2}\right\} = \frac{S^2\,.\sqrt{2(r + r' + R)}}{4\,k\,.\sqrt{M+m}};$$

Now, if we put T and T′ for $\tan\dfrac{\theta}{2}$ and $\tan\dfrac{\theta'}{2}$, we have $T'^3 - T^3 = (T' - T)\left\{T'^2 + TT' + T^2\right\}$ and therefore

$$t = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M+m}}(T' - T)\left\{1 + \frac{T^2 + TT' + T'^2}{3}\right\}$$

The part without the brackets we have already considered. Let us next examine that within. Now, if in $T^2 + TT' + T'^2$, we substitute for T and T′, their values $\frac{R - 2r}{S}$ and $\frac{-R + 2r'}{S}$, it will become

$$\frac{R^2 - 2R(r + r') + 4(r^2 - rr' + r'^2)}{S^2}$$

and substituting this, the quantity within the brackets becomes

$$\frac{3S^2 + R^2 + 4(r^2 - rr' + r'^2) - 2R(r + r')}{3S^2}$$

But $R^2 + S^2 = 4rr'$, therefore $3S^2 + R^2 = 12rr' - 2R^2 = 2k^2 - 2r^2 + 8rr' - 2r'^2$. So that our expression reduces itself to

$$\frac{2}{3S^2}\left\{(r + r')^2 + k^2 - (r + r').R\right\}$$

$$= \frac{2}{3S^2}\left\{k^2 + (r + r')(r + r' - R)\right\}$$

and therefore we have, finally,

$$t = \frac{\sqrt{2(r + r' + R)}}{6k.\sqrt{M + m}}\left\{k^2 + (r + r')(r + r' - R)\right\}$$

but $$\sqrt{r + r' + R}.(r + r' - R) =$$

$$= \sqrt{(r + r' + R).(r + r' - R)^2}$$

$$= k.\sqrt{r + r' - R}$$

because $(r + r' + R)(r + r' - R) = k^2$. So that substituting, we get

$$= \frac{k.\sqrt{r + r' + R} + (r + r').\sqrt{r + r' - R}}{6.\sqrt{M + m}}.\sqrt{2}$$

Now $R = \sqrt{(r + r')^2 - k^2}$, so that the two radicals in the above expression are of the form

$$\sqrt{a \pm \sqrt{a^2 - k^2}}$$

putting a for $r + r'$. Hence the usual rule for extracting the root of a binomial surd applies to them, the root being, as it is shewn in all books of algebra,

$$\frac{\sqrt{a + k} \pm \sqrt{a - k}}{\sqrt{2}}$$

Consequently, we have

$$t = \frac{1}{6\sqrt{M + m}}\left\{\begin{array}{l} k(\sqrt{r + r' + k} + \sqrt{r + r' - k}) \\ + (r + r')(\sqrt{r + r' + k} - \sqrt{r + r' - k}) \end{array}\right\}$$

or,

$$6\sqrt{M + m} \times t = (r + r' + k)^{\frac{3}{2}} - (r + r' - k)^{\frac{3}{2}}; \quad (62)$$

which agrees with the proposition as announced, if we take $g = 6\sqrt{M + m}$.

The relation just proved to subsist between the time of describing any parabolic arc, and the rectilinear distances of its two extremities from each other, and from the sun, is very useful, as it enables us from any three of these quantities given, to find the fourth without knowing either the perihelion distance, or the position of the perihelion with respect to the arc described. In an analytical point of view, it is certainly remarkable for the length and complexity of the transformations by which it is obtained from the general principles of parabolic motion. It appears to admit of no simple demonstration, and that above given, tedious as it may seem, can hardly be replaced by one materially shorter.

We shall now proceed to show how this relation may be rendered available in computing the elements of a comet's orbit.

Let x, y, z, be the co-ordinates of the comet's place at the epoch of a first observation, x', y', z', their values after any time t and x'', y'', z'', their values after any other times t' has elapsed. Let X, Y, Z, X', Y', Z', and X'', Y'', Z'', be the corresponding co-ordinates of the earth's place. (If the plane of the ecliptic be chosen for that of the X and Y, we shall have $Z = o$, $Z' = o$, $Z'' = o$.) Also let

$$t' - t = t''$$

then will t'' be the time elapsed between the second and third observations.

Since the comet moves in a plane passing through the sun, we have

$$z = ax + by \quad (64, 1)$$

$$z' = ax' + by' \quad (64, 2)$$

$$z'' = ax'' + by'' \quad (64, 3)$$

from which, if we eliminate a and b; first, from the first and second; secondly, from the first and third; and lastly, from the second and third; we shall get the three following

$$o = x(y''z' - y'z'') + x'(yz'' - y''z) + x''(y'z - yz') \quad (65, 1)$$

$$o = y(z''x' - z'x'') + y'(zx'' - xz'') + y''(xz' - zx') \quad (65, 2)$$

$$o = z(x''y' - x'y'') + z'(xy'' - r''y) + z''(x'y - xy') \quad (65, 3)$$

Now $zy' - z'y$ represents the double area of the projection of the plane triangle included between the radii, r, r' drawn to the place of the comet at the first and second observations, and the chord k of the parabolic arc joining their extremities on the plane of the z and y; similarly, $zy'' - z''y$ represents double the triangle formed by the projections of the radii r, r'' and the chord k' joining their extremities on the same plane; and if we denote by k'' the chord joining the extremities of the radii r', r'', the double of the projection of the triangle included between r', r'', and k'' will be represented by $z'y'' - z''y'$. If, therefore, we call the surfaces of these triangles respectively

$$S, S', S''$$

Since the projection of each is equal to the surface multiplied by the cosine of its inclination to the plane of the z, y, which is the same for all, we shall have the following equation instead of (65,1),

$$xS'' - x'S' + x''S = o \quad (66, 1)$$

and similarly,

$$yS'' - y'S' + y''S = o \quad (66, 2)$$

$$zS'' - z'S' + z''S = o \quad (66, 3)$$

Now, if we retain the denominations of the foregoing pages, and assume the line of the equinoxes S ♈ for the axis of the x, we have

$$x = S\pi = Se + e\pi$$

$$= R.\cos L + \delta.\cos\lambda; \quad (67, 1)$$

$$x' = R'.\cos L' + \delta'.\cos\lambda'; \quad (67, 2)$$

$$x'' = R''.\cos L'' + \delta''.\cos\lambda''; \quad (67, 3)$$

also

$$y = Ee + pq = R.\sin L + \delta.\sin\lambda; \quad (67, 4)$$

$$y' = R'.\sin L' + \delta'.\sin\lambda'; \quad (67, 5)$$

$$y'' = R''.\sin L'' + \delta''.\sin\lambda''; \quad (67, 6)$$

we have also

$$z\ = \delta\,.\tan\beta \qquad (67,7)$$
$$z' = \delta'\,.\tan\beta' \qquad (67,8)$$
$$z'' = \delta''\,.\tan\beta'' \qquad (67,9)$$

and by the substitution of these values we shall obtain equations, in which, instead of the co-ordinates x, y, z, &c. the curtate distances δ, δ', δ'', will be involved, viz.

$$\left.\begin{aligned} o = {} & S''\,(\delta\,.\cos\lambda + R\,.\cos L) \\ & - S'\,(\delta'\,.\cos\lambda' + R'\,.\cos L') \\ & + S\,(\delta''\,.\cos\lambda'' + R''\,.\cos L'') \end{aligned}\right\}; \qquad (68,1)$$

$$\left.\begin{aligned} o = {} & S''\,(\delta\,.\sin\lambda + R\,.\sin L) \\ & - S'\,(\delta'\,.\sin\lambda' + R'\,.\sin L') \\ & + S\,(\delta''\,.\sin\lambda'' + R''\,.\sin L'') \end{aligned}\right\} \qquad (68,2)$$

$$o = S''\,.\,\delta\,.\tan\beta - S'\,.\,\delta'\,.\tan\beta' + S\,.\,\delta''\,.\tan\beta''; \qquad (68,3)$$

R, L, λ, and β, and their accented values, are all quantities given by the solar tables, or by observation at the three assigned instants, so that the only unknown quantities contained in these equations are the three curtate distances, δ, δ', δ'', and the areas S, S′, S″; and by elimination, any one of the three former may be expressed in terms of the latter. Let this process be executed then in succession for each of the quantities δ, δ', δ'', and we shall have the following results:

$$(\,a\,.\,\delta + AR)\,.\,S'' - A'R'\,.\,S' + A''R''\,.\,S = o; \qquad (69,1)$$
$$BR\,.\,S'' - (\,a\,.\,\delta' + B'R')\,S' + B''R''S = o; \qquad (69,2)$$
$$CR\,.\,S'' - C'R'\,.\,S' + (a\,.\,\delta'' - C''R'')\,S = o; \qquad (69,3)$$

where, for brevity's sake, we have made the following substitutions—

$$\left.\begin{aligned} a = {} & \tan\beta\ \ .\sin(\lambda'' - \lambda') \\ & - \tan\beta'\ .\sin(\lambda'' - \lambda) \\ & + \tan\beta''\,.\sin(\lambda' - \lambda) \end{aligned}\right\}; \qquad (70,1)$$

$$A\ \ = \tan\beta\ \ .\sin(L\ - \lambda'') - \tan\beta''\,.\sin(L\ - \lambda'); \qquad (70,2)$$
$$A'\ = \tan\beta'\ .\sin(L' - \lambda'') - \tan\beta''\,.\sin(L' - \lambda'); \qquad (70,3)$$
$$A'' = \tan\beta'\ \ \sin(L'' - \lambda'') - \tan\beta''\,.\sin(L'' - \lambda'); \qquad (70,4)$$

$$B\ \ = \tan\beta''\,.\sin(L\ - \lambda\,) - \tan\beta\ \ .\sin(L\ - \lambda''); \qquad (70,5)$$
$$B'\ = \tan\beta''\,.\sin(L' - \lambda\,) - \tan\beta\ \ .\sin(L' - \lambda''); \qquad (70,6)$$
$$B'' = \tan\beta''\,.\sin(L'' - \lambda\,) - \tan\beta\ \ .\sin(L'' - \lambda''); \qquad (70,7)$$

$$C\ \ = \tan\beta\ \ .\sin(L\ - \lambda'\,) - \tan\beta'\ .\sin(L\ - \lambda\,); \qquad (70,8)$$
$$C'\ = \tan\beta\ \ .\sin(L' - \lambda'\,) - \tan\beta'\ .\sin(L' - \lambda\,); \qquad (70,9)$$
$$C'' = \tan\beta\ \ .\sin(L'' - \lambda'\,) - \tan\beta'\ .\sin(L'' - \lambda\,); \qquad (70,10)$$

In these equations, the right hand members are composed entirely of quantities given either by observation, or by the solar tables; and their calculation, though long, is greatly facilitated by the regularity of their composition. The values of a, A, B, C, &c. obtained, we have, by (69, 1) (69, 2)

$$\delta = -\,\frac{ARS'' - A'R'S' + A''R''S}{a\,S''}$$

$$\delta' = +\,\frac{BRS'' - B'R'S' + B''R''S}{a\,S'}$$

whence we obtain

$$\frac{\delta}{\delta'} = -\,\frac{S'}{S''}\,.\,\frac{A'}{B'} \qquad (71)$$
$$+\,\frac{(B'A - BA')\,.\,RS'' + (B'A'' - B''A')\,R''S}{B'\,\{BRS'' - B'R'S' + B''R''S\}} \times \frac{S'}{S''}$$

So far, we have nothing but rigorous equations, and it does not immediately appear how these can become serviceable in the question before us; but if we consider attentively the composition of the numerator of the fraction which forms the second term of $\frac{\delta}{\delta'}$ here stated, we shall find that the equation last arrived at, affords a means of obtaining a first approximation. Let us consider, first, the term AB′ — BA′ :. If we substitute in this the values of A, B, A′, B′, in equations (70, 2, 3, 5, 6,) execute all the multiplications, strike out such terms as mutually destroy each other, and then reduce as much as possible, we shall find for its value

$$\tan\beta''\left\{\begin{aligned} & \tan\beta\ \ .\,\{\sin(L - \lambda')\,.\sin(L' - \lambda'') - \sin(L' - \lambda')\,.\sin(L - \lambda'')\} \\ & + \tan\beta'\,.\,\{\sin(L - \lambda'')\,.\sin(L' - \lambda) - \sin(L' - \lambda'')\,.\sin(L - L)\} \\ & + \tan\beta''.\,\{\sin(L - \lambda)\,.\sin(L' - \lambda') - \sin(L' - \lambda)\,.\sin(L - \lambda')\} \end{aligned}\right\}.$$

Now, by trigonometry, we have, (since $\sin A\,.\sin B = \tfrac{1}{2}\,\{\cos(A - B) - \cos(A + B)\}$

$$\sin(L - \lambda)\,.\sin(L' - \lambda') = \tfrac{1}{2}\cos\{(L - L') - (\lambda - \lambda')\} - \tfrac{1}{2}\cos\{(L + L') - (\lambda + \lambda')\}$$
$$\sin(L' - \lambda)\,.\sin(L - \lambda') = \tfrac{1}{2}\cos\{(L - L') + (\lambda - \lambda')\} - \tfrac{1}{2}\cos\{(L + L') - (\lambda + \lambda')\}$$

and subtracting, we find

$$\sin(L - \lambda)\,.\sin(L' - \lambda') - \sin(L' - \lambda)\,.\sin(L - \lambda') =$$
$$= \tfrac{1}{2}\cos\{(L - L') - (\lambda - \lambda')\} - \tfrac{1}{2}\cos\{(L - L') + (\lambda - \lambda')\}$$

which another application of the same trigonometrical formula already used converts back again into

$$\sin(L - L')\,.\sin(\lambda - \lambda')$$

a similar transformation applies to the co-efficients of $\tan\beta$ and $\tan\beta'$; so that our expression becomes

$$AB' - BA' = \tan\beta'' \times \left\{\begin{array}{l} \sin(L-L') \,.\, \sin(\lambda' - \lambda'') \,.\, \tan\beta \\ + \sin(L-L') \,.\, \sin(\lambda'' - \lambda) \,.\, \tan\beta' \\ + \sin(L-L') \,.\, \sin(\lambda - \lambda') \,.\, \tan\beta'' \end{array}\right.$$

or

$$AB' - BA' = \sin(L-L') \,.\, \tan\beta'' \,.\, \left\{\begin{array}{l} \tan\beta \,.\, \sin(\lambda' - \lambda'') \\ + \tan\beta' \,.\, \sin(\lambda'' - \lambda) \\ + \tan\beta'' \,.\, \sin(\lambda - \lambda') \end{array}\right\}$$

or, if we put

$$\gamma = \left.\begin{array}{l} \tan\beta \,.\, \sin(\lambda' - \lambda'') \\ + \tan\beta' \,.\, \sin(\lambda'' - \lambda) \\ + \tan\beta'' \,.\, \sin(\lambda - \lambda') \end{array}\right\}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (72)$$

$$AB' - BA' = \gamma \,.\, \sin(L-L') \,.\, \tan\beta'' \qquad (73,\,1)$$

Now, it is evident that γ is a symmetrical function, and is not changed by putting at the same time β' for β, and λ' for λ, and reciprocally; so that, by pursuing a process of reduction exactly similar, we should obtain

$$AB'' - BA'' = \gamma \,.\, \sin(L - L'') \,.\, \tan\beta'' \qquad (73,\,2)$$

$$A'B'' - B'A'' = \gamma \,.\, \sin(L' - L'') \,.\, \tan\beta'' \qquad (73,\,3)$$

and were we to pursue similar processes, combining A and C, B and C, we should get

$$CA' - AC' = \gamma \,.\, \sin(L - L') \,.\, \tan\beta \qquad (73,\,4)$$

$$CA'' - AC'' = \gamma \,.\, \sin(L - L'') \,.\, \tan\beta' \qquad (73,\,5)$$

$$C''A' - A''C' = \gamma \,.\, \sin(L' - L'') \,.\, \tan\beta' \qquad (73,\,6)$$

and

$$BC' - CB' = \gamma \,.\, \sin(L - L') \,.\, \tan\beta \qquad (73,\,7)$$

$$BC'' - CB'' = \gamma \,.\, \sin(L - L'') \,.\, \tan\beta \qquad (73,\,8)$$

$$B'C'' - C'B'' = \gamma \,.\, \sin(L' - L'') \,.\, \tan\beta \qquad (73,\,9)$$

Thus the numerator of the fraction in $\frac{\delta}{\delta'}$ above referred to, becomes

$$\gamma \,.\, \tan\beta'' \,.\, \left\{RS'' \,.\, \sin(L - L') - R''S \,.\, \sin(L' - L'')\right\}; \qquad (74)$$

Now, let us suppose the three observations made at small intervals of time. The supposition almost always holds good in the case of comets, which usually excite sufficient interest about the time of their first appearance, to induce astronomers to observe them as frequently as possible; so that observations separated by an interval of a few days at most may generally be obtained for calculation. On this supposition, then, the variations $\lambda - \lambda'$, $\lambda' - \lambda''$ and $\lambda'' - \lambda$ of the observed geocentric longitudes may be regarded as very small quantities of the first order; so that γ is also a very small quantity of the first order, especially as β (the observed geocentric latitude) is usually below 45°; so that $\tan\beta$, $\tan\beta'$, &c. are less than unity. Moreover, $L - L'$, $L' - L''$, &c. which represent the sun's motion in the intervals between the observations, are also very small quantities of the first order; so that the expression (74) is, on these accounts, a very small quantity of the second order.

Fig. 10. But besides this, since S, S″ represent the rectilinear triangles SPP′ and SP′P″ described about the sun in the times $t' - t$ and $t'' - t'$ (which for the moment we will call θ and $n\,\theta$) they will be nearly proportional to those times, being extremely near to equality with the sectors SP n P′, SP′ n' P″, so that we have very nearly $S'' = n \,.\, S$. Again, the earth's orbit being nearly a circle, and its motion in it nearly uniform, we have

$$R = R'' \text{ and } (L' - L'') = n \,.\, (L - L')$$

whence $(L - L')$ being very small, $\sin(L' - L'') = n \,.\, \sin(L - L')$ If these suppositions were rigorously exact, we should have

$$\begin{aligned} R \,.\, S'' \,.\, \sin(L - L') &= n\,RS \,.\, \sin(L - L') \\ &= R''S \,.\, \sin(L' - L'') \end{aligned}$$

and the quantity within the brackets of (74) would be rigorously zero; and in all cases we see that it must be a very small quantity of the second order at least, so that the whole expression (74) must be a very small quantity of the third order; and that, whether the intervals between the consecutive observations approach to equality or not.*

Let us next consider the denominator of the fraction, and retaining the substitution $n = \frac{\theta''}{\theta} = \frac{t'' - t'}{t' - t}$ it becomes, on the same hypotheses as before

$$B' \,.\, RS\,\left\{n\,B - B'\,(1 + n) + B''\right\}$$

because $S' = S + S''$ nearly. We have, therefore, first to consider the values of $B'' - B'$ and of $B - B'$. Now, if in $B'' - B'$ we substitute the values of B', B'', and reduce as far as possible, we shall find

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* Mr. Littrow, from whose excellent and most useful work on theoretical and practical astronomy (*Theoretische und Practische Astronomie*, von J. J. Littrow, director der Sternwarte und professor der astron. an der K. K. universität in Wien.—Wien, 1821, 2 vols. 8vo.) I have taken this exposé of the determination of a comet's orbit and the example which follows, has merely remarked that the quantity $\frac{\delta}{\delta'}$ is equal to $-\frac{A'}{B'} \times \frac{S}{S''}$ + a remainder, and "this remainder is of the order $AB' - A'B$, and therefore may be neglected in a first approximation, when the intervals are small *and nearly equal*," (vol. ii. p. 129.) This is not, however, a satisfactory view of the subject, and is, in fact, slurring over a very considerable difficulty and evading one of the most difficult and delicate points in the theory of comets. I have therefore judged it better to enter, though somewhat more at length than is absolutely consonant with the nature of this work into this part of the subject, rather than leave a doubt on the mind of the student, or a mystery attached to a fundamental proposition. Meanwhile I am happy to have to call the attention of the lovers of astronomy to the interesting work alluded to, of which a translation by any competent hand would be a real accession to our stock of elementary works.

$$B'' - B' = 2 \tan\beta \,.\, \tan\beta'' \,.\, \sin\frac{L'-L''}{2} \,.\, \left\{\cot\beta \,.\, \cos\left(\frac{L'+L''}{2} - \lambda\right) - \cot\beta'' \,.\, \cos\left(\frac{L'+L''}{2} - \lambda''\right)\right\}$$

and the quantity within the brackets being the difference between two values of the same function at a small interval of time is of the first order, as is also $\sin\frac{L'-L''}{2}$, so that $B''-B'$ is of the second order. Similarly, we have

$$B - B' = -2 \,.\, \tan\beta \,.\, \tan\beta'' \sin\frac{L-L'}{2} \,.\, \left\{\cot\beta \,.\, \cos\left(\frac{L+L'}{2} - \lambda\right) - \cot\beta'' \,.\, \cos\left(\frac{L+L'}{2} - \lambda''\right)\right\}$$

Now, $\sin\frac{L'-L''}{2} = n \,.\, \sin\frac{L-L'}{2}$, and therefore we have

$$n(B-B') + (B''-B') = 2n \,.\, \tan\beta \,.\, \tan\beta'' \,.\, \sin\left(\frac{L-L'}{2}\right) \,.\, \left\{\begin{array}{l} \cot\beta\left(\cos\left(\frac{L'}{2} - \lambda + \frac{L''}{2}\right) - \cos\left(\frac{L'}{2} - \lambda + \frac{L}{2}\right)\right) \\ -\cot\beta''\left(\cos\left(\frac{L'}{2} - \lambda'' + \frac{L''}{2}\right) - \cos\left(\frac{L'}{2} - \lambda'' + \frac{L}{2}\right)\right) \end{array}\right\}$$

Let this be reduced as much as possible ; and, putting M for $\frac{L + 2L' + L''}{4}$, and $(1+n) \,.\, \sin\left(\frac{L-L'}{4}\right)$ for $\sin\left(\frac{L-L''}{4}\right)$ it will become

$$R \,.\, S \times 2n\,(1+n)^2 \,.\, \tan\beta^2 \,.\, \tan\beta''^2 \,.\, \sin\frac{L-L'}{2} \,.\, \sin\frac{L-L'}{4} \times$$
$$\times \left\{\sin(L'-\lambda) \,.\, \cot\beta - \sin(L'-\lambda'') \,.\, \cot\beta''\right\} \times$$
$$\times \left\{\sin(M-\lambda) \,.\, \cot\beta - \sin(M-\lambda'') \,.\, \cot\beta''\right\}$$

This is evidently a quantity of the fifth order, the two last factors of it being each the difference of consecutive values.

Granting then the hypotheses above employed, viz. —1. The strict circularity and uniformity of the earth's motion.—2. The proportionality of the rectilinear, instead of the parabolic, sectors to the times we see that the numerator of the fraction vanishes absolutely, while the denominator on the same hypotheses does not strictly vanish, but only reduces itself to a quantity of the fifth order. Hence, we are entitled (at least in making a first approximation to the elements) to neglect the fraction, and suppose

$$\frac{\delta}{\delta'} = -\frac{S'}{S''} \,.\, \frac{A'}{B'} \,; \qquad (75)$$

and similarly

$$\frac{\delta}{\delta''} = +\frac{S}{S''} \,.\, \frac{A'}{C'} \,; \qquad (76)$$

and

$$\frac{\delta'}{\delta''} = -\frac{B'}{C'} \,.\, \frac{S}{S'} \,; \qquad (77)$$

It is true, from the minuteness of the denominator of the fraction in (71) the deviation from truth of the hypotheses assumed becomes *magnified*. But these hypotheses themselves are so very nearly correct, that unless in extremely unfavourable cases the equations (75) and (76) may safely be employed ; and it will be observed too, that *independent of both hypotheses*, the numerator would be of the fourth order at least, for if we throw it into the form

$$\gamma \,.\, \tan\beta'' \,.\, SS'' \,.\, \left\{R \,.\, \frac{\sin(L-L')}{S} - R'' \,.\, \frac{\sin(L'-L'')}{S''}\right\}$$
$$= \gamma \,.\, \tan\beta'' \,.\, SS'' \,.\, \left\{R \,.\, \frac{\sin(L-L')}{S} - R' \,.\, \frac{\sin(L'-L'')}{S''}\right\}$$
$$+ \gamma \,.\, \tan\beta'' \,.\, SS'' \,.\, (R' - R'') \,.\, \frac{\sin(L'-L'')}{S''}$$

the second term is evidently so ; and since the factor of the first within the brackets is the difference of consecutive values, and therefore a very small quantity,—the first is also.

To apply these equations to the determination of a comet's elements, we will first suppose the sectors P *n* SP′, &c. equal to the plane triangles SPP′, &c. or at least in the same ratio with them. This amounts to substituting $\frac{t'-t}{t''-t'}$ or $\frac{\theta'}{\theta''}$ for $\frac{S'}{S''}$; when we obtain, instead of the equations (69, 1, 2, 3,) Fig. 10.

$$(a\,\delta + AR) \times \theta'' - A'R' \,.\, \theta' + A''R''\,\theta = o \qquad (78,1)$$
$$BR\,\theta'' - (a\,\delta' + B'R')\,\theta' + B''R''\,\theta = o \qquad (78,2)$$
$$CR\,\theta'' - C'R'\,\theta' + (a\,\delta'' - C'R'')\,\theta = o \qquad (78,3)$$

which give the values of $\delta, \delta', \delta''$.

Now we get, by the equation (55)

$$r'^2 = R'^2 + \delta'^2 \,.\, \sec\beta'^2 + 2R'\delta' \,.\, \cos(L'-\lambda') \,; \qquad (79)$$

whence the value of r' is obtained.

r' once obtained, the values of δ, δ', &c. and r' may be corrected by the following process, let τ represent the time between the second observation and any assigned instant. If then x, y, z, represent the coordinates at the first observation, and x'', y'', z'', at the third, we have

$$x = x' - \frac{\theta}{1} \,.\, \frac{dx'}{d\tau} + \frac{\theta^2}{1\,.\,2} \,.\, \frac{d^2x'}{d\tau^2} - \text{\&c.} \qquad (80,1)$$
$$x'' = x' + \frac{\theta''}{1} \,.\, \frac{dx'}{d\tau} + \frac{\theta''^2}{1\,.\,2} \,.\, \frac{d^2x'}{d\tau^2} - \text{\&c.} \qquad (80,2)$$
$$y = y' - \frac{\theta}{1} \,.\, \frac{dy'}{d\tau} + \frac{\theta^2}{1\,.\,2} \,.\, \frac{d^2y'}{d\tau^2} - \text{\&c.} \qquad (80,3)$$
$$y'' = y' + \frac{\theta''}{1} \,.\, \frac{dy'}{d\tau} + \frac{\theta''^2}{1\,.\,2} \,.\, \frac{d^2y'}{d\tau^2} - \text{\&c.} \qquad (80,4)$$

and similar expressions for z, z''. Now, assuming τ as the independent variable, or $d\tau$ constant, we have, by (1), (2), (3),

$$\left.\begin{aligned}\frac{d^2 x'}{d\tau^2} &= -(\mathrm{M}+m)\,.\,\frac{x'}{r'^3}\\ \frac{d^2 y'}{d\tau^2} &= -(\mathrm{M}+m)\,.\,\frac{y'}{r'^3}\\ \frac{d^2 z'}{d\tau^2} &= -(\mathrm{M}+m)\,.\,\frac{z'}{r'^3}\end{aligned}\right\} \qquad (81,\ 1,\ 2,\ 3)$$

The numerical value of M + m is easily found, since it is the same for the whole solar system except in so far as the masses of the planets and comets denoted by m differ, and these bear so minute a proportion to the mass M of the sun, that they may be neglected: let τ denote the period of any one plane, as the earth, and a the semi-major axis of its orbit, then we shall have, by (24)

$$\frac{\mathrm{T}^2}{a^3} = \frac{4\,\pi^2}{\mathrm{M}+m}$$

so that $\frac{\tau^2}{a^3}$ is a quantity very nearly constant for the whole planetary system; and the period of the earth being $365^{\mathrm{d}}.256384$, and the semiaxis of its orbit 1, its value for the earth is $\frac{(365{\cdot}256384)^2}{1^3}$, and the same quantity expresses its value for all the other planets and comets. Hence we have

$$\mathrm{M}+m = \frac{4 \times (3{\cdot}141592)^2}{(365{\cdot}256384)^2} = 0{\cdot}0002959122\,;$$

and $\qquad \log(\mathrm{M}+m) = 6{\cdot}4711628$

To correct the values of δ, δ', &c. we must correct the supposition on which they were obtained, of the proportionality of the rectilinear triangles instead of the parabolic sectors to the times, or the values of S, S′, S″. Now, if c be the inclination of the plane of the orbit to the plane of the x and y, we have

$$\begin{aligned}2\,\mathrm{S}\;.\cos c &= y\,x' - x\,y'\\ 2\,\mathrm{S}'\,.\cos c &= y'x'' - x'y''\\ 2\,\mathrm{S}''.\cos c &= y\,x'' - x\,y''\end{aligned}$$

In the first of these, for y and x write their values as given in (80, 1) and (80, 3), and we find

$$2\,\mathrm{S}\,.\cos c = \frac{\theta}{1}\,.\,\frac{y'\,d\,x' - x'\,d\,y'}{d\,\tau} - \frac{\theta^2}{1\,.\,2}\,.\,\frac{y'\,d^2x' - x'\,d^2y'}{d\,\tau^2} + \frac{\theta^3}{1\,.\,2\,.\,3}\,.\,\frac{y'\,d^3x' - x'\,d^3y'}{d\,\tau^3} - \&c.$$

But if the values of $\frac{d^2\,x'}{d\,\tau^2}$ and $\frac{d^2\,y'}{d\,\tau^2}$ be written in the second term of this, it will vanish. Again, if we differentiate the equations (81, 1) and (81, 2) we get

$$\frac{d^3\,x'}{d\,\tau^3} = -(\mathrm{M}+m)\,.\left(\frac{1}{r'^3}\,.\,\frac{d\,x'}{d\,\tau} - \frac{3\,x'}{r'^4}\,.\,\frac{d\,r'}{d\,\tau}\right)$$

$$\frac{d^3\,y'}{d\,\tau^3} = -(\mathrm{M}+m)\,.\left(\frac{1}{r'^3}\,.\,\frac{d\,y'}{d\,\tau} - \frac{3\,y'}{r'^4}\,.\,\frac{d\,r'}{d\,\tau}\right)$$

so that, by substitution, we obtain

$$\frac{y'\,d^3\,x' - x'\,d^3\,y'}{d\,\tau^3} = -\,\frac{\mathrm{M}+m}{r'3}\,.\,\frac{y'\,d\,x' - x'\,d\,y'}{d\,\tau}$$

If we neglect the higher powers of θ than the third, and put, for the present, p for $\frac{x'\,d\,y' - y'\,d\,x'}{d\,\tau}$ we shall have

$$2\,\mathrm{S}\,.\cos c = p\,\theta\left\{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta^2\right\}; \qquad (82,\ 1)$$

and similarly,

$$2\,\mathrm{S}'.\cos c = p\,\theta'\left\{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta'^2\right\}; \qquad (82,\ 2)$$

$$2\,\mathrm{S}''.\cos c = p\,\theta''\left\{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta''^2\right\}; \qquad (82,\ 3)$$

If we now employ the value of r' as above approximately determined, and substitute it in the right hand members of these equations, we shall obtain, dividing one by the other the *ratios* of S, S′, S″, much more accurately than on the original hypothesis, for we have

$$\frac{\mathrm{S}}{\mathrm{S}''} = \frac{\theta}{\theta''} \times \frac{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta^2}{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta''^2} \qquad (83,\ 1)$$

$$\frac{\mathrm{S}'}{\mathrm{S}''} = \frac{\theta'}{\theta''} \times \frac{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta'^2}{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta''^2} \qquad (83,\ 2)$$

Now, in the equations for determining δ, δ', &c. (69, 1, 2, 3) it is only the ratio of S, S′, S″, that are required, and thus we are enabled to correct the values of their quantities, and thence again that of r'. To carry the process to a greater degree of precision, however, the series expressing the value of 2 S . cos c must be continued further. Without, however, going through the process as here set down we may content ourselves, after computing first values of δ' and r' by the equations (78, 2) and (79) with calculating the quantity

$$\frac{\mathrm{M}+m}{r'^3}\left\{\mathrm{B}'\mathrm{R}'.\,\frac{\theta}{\theta'}\,(\theta'^2 - \theta^2) - \mathrm{BR}\,.\,\frac{\theta''}{\theta'}\,(\theta'^2 - \theta''^2)\right\}; \qquad (84)$$

which must be applied as a correction to the first found value of δ' with its proper sign.

The values of r, r', r'', δ, δ', δ'', so obtained, the remaining unknown quantities may be found as follows:

The equation (58)

$$r\,.\cos b\,.\sin(l - \mathrm{L}) = \delta\,.\sin(\lambda - \mathrm{L})$$

divided by the equation (38)

$$r\,.\cos b\,.\cos(l - \mathrm{L}) = \mathrm{R} + \delta\,.\cos(\lambda - \mathrm{L})$$

member for member, gives

$$\tan(l - \mathrm{L}) = \frac{\delta\,.\sin(\lambda - \mathrm{L})}{\mathrm{R} + \delta\,.\cos(\lambda - \mathrm{L})} \qquad (85)$$

the right hand member consists only of known quantities, so that the value of $l - \mathrm{L}$ is easily found, and thus the heliocentric longitude becomes known at each of the three observations. Again, we have by the equation (56)

$$r\,.\sin b = \delta\,.\tan\beta$$

and by (58)

$$r\,.\cos b = \delta\,.\,\frac{\sin(\lambda - \mathrm{L})}{\sin(l - \mathrm{L})}$$

which, divided member for member, give

$$\tan b = \tan \beta \,.\, \frac{\sin (l - \text{L})}{\sin (\lambda - \text{L})}\,; \qquad (86)$$

and thus the heliocentric latitudes may be computed.

The values of l and b known, the same equations afford a value of r, viz.

$$r = \delta \,.\, \frac{\tan \beta}{\sin b}$$

whence r, r', r'' may be found; and if these values agree with those before determined, it will be a proof of the correctness of the previous work. If l' and l'' be greater than l, the comet's motion is direct; if less, retrograde.

The heliocentric longitudes and latitudes thus obtained at each of the three observations, we easily obtain the inclination and place of the node. In fact, (taking the first and last observations, to embrace a greater arc of the great circle) our equation (40) gives

$$\tan b \;= \tan i \,.\, \sin (l - \text{☊})$$
$$\tan b'' = \tan i \,.\, \sin (l'' - \text{☊})$$

so that we have

$$\frac{\sin (l - \text{☊})}{\sin (l'' - \text{☊})} = \frac{\tan b}{\tan b''}$$

which gives

$$\tan b'' (\sin l \,.\, \cos \text{☊} - \cos l \,.\, \sin \text{☊})$$
$$= \tan b \,(\sin l'' \,.\, \cos \text{☊} - \cos l'' .\, \sin \text{☊})$$

or

$$\sin \text{☊} \left\{ \tan b \,.\, \cos l'' - \cos l \,.\, \tan b'' \right\} =$$
$$= \cos \text{☊} \left\{ \sin l \,.\, \tan b'' - \sin l'' .\, \tan b \right\}$$

and consequently

$$\tan \text{☊} = \frac{\tan b \,.\, \cos l'' - \tan b'' \,.\, \cos l}{\sin l \,.\, \tan b'' - \sin l'' \,.\, \tan b}\,; \qquad (87)$$

and

$$\tan i = \frac{\tan b}{\sin (l - \text{☊})} \qquad (88)$$

If the motion of the comet be *direct*, the value of ☊ given by the above equation, will be the longitude of the *ascending node;* but if retrograde, it is evidently that of the *descending;* and to get that of the ascending node, 180° must be added. The equation (61) or the corresponding one,

$$\tan (l - \text{☊}) = \cos i \,.\, \tan (\psi + \theta) \qquad (89)$$

gives the values of $\psi + \theta$ the longitudes on the orbit reckoned from the node, and these being obtained, $(\psi + \theta, \psi + \theta', \text{ and } \psi + \theta'')$* the angle $\theta'' - \theta$ between the first and last places of the comet at the sun becomes known.

If we denote this by ϕ, we have at once, by the equation

$$\tan \frac{\theta}{2} = \operatorname{cosec} \phi \left\{ \cos \phi - \sqrt{\frac{r}{r''}} \right\} \qquad (90)$$

demonstrated in the foregoing pages (p. 665) the value of θ, or the distance (on the orbit) of the perihelion from the place of the first observation; so that, subtracting the value found, from $\psi + \theta$ before determined, we get ψ, and thence π, the longitude of the perihelion on the ecliptic, by the equation (60)

It only remains to find the perihelion distance and time of perihelion passage. The equation of the parabola

$$\text{D} = r \,.\, \left(\cos \frac{\theta}{2}\right)^2$$

gives the former immediately, r and θ being known, and the latter is found as follows. In the table of a comet of 109 days subjoined, seek the number of days (by interpolation) corresponding to the anomaly θ, and call it M, then, by equation (44) we have, (the perihelion distance of such a comet being unity)

$$\text{N} = \sqrt{\frac{2}{\text{M} + m}} \times \left\{ \tan \frac{\theta}{2} + \frac{1}{3} \,.\, \tan \left(\frac{\theta}{2}\right)^3 \right\}$$

and consequently, if T be the time of the perihelion passage *before* the first observation,

$$\text{T} = \text{N} \times \text{D}^{\frac{3}{2}}$$

If we institute a similar operation for the time T'' of the perihelion passage *before* the third observation, we shall find

$$\text{T}'' = \text{N}'' \,.\, \text{D}^{\frac{3}{2}}$$

Now, we ought to have T'' − T = the interval between the first and last observations. If this be found on trial to be the case, it is a proof of the correctness of the work. The perihelion passage will be before or after the first observation (in the case of direct motion) according as the longitude π of the perihelion is less or greater than the heliocentric longitude l at the first observation; the reverse is the case in retrograde motion.

The chief difficulty in the solution of the problem of a comet's orbit, consists in the determination of the distances from the sun and earth, at the moments of the several observations. Our equations above derived afford a great variety of means for accomplishing this, among which we shall only select one, proposed by Dr. Olbers, which in many cases has particular advantages. It depends on the property already demonstrated of the chords of the parabolic arcs.

Let k represent the chord of the arc included between the two extreme places, then we have

$$k^2 = (x'' - x)^2 + (y'' - y)^2 + (z'' - z)^2$$

or

$$k^2 = r^2 + r''^2 - 2\,(xx'' + yy'' + zz'')$$

In this equation, let the values of x, y, z, x'', y'', z'', in (67, 1, 2, 3, 4, &c.) in terms of R and δ, &c. be substituted; and if we put

$$m = \frac{\text{C}'}{\text{A}'} \times \frac{\theta''}{\theta}$$

so that, by (76) $\delta'' = m\,\delta$

we shall have

$$\begin{aligned} k^2 &= r^2 + r''^2 \\ &- 2\, m\, \delta^2 \,.\, \left\{ \cos (\lambda - \lambda'') + \tan \beta \,.\, \tan \beta'' \right\} \\ &- 2\, m\, \text{R}\, \delta \,.\, \cos (\lambda'' - \text{L}) \\ &- 2\, \text{R}''\, \delta \,.\, \cos (\lambda - \text{L}'') \qquad (91) \\ &- 2\, \text{RR}'' \,.\, \cos (\text{L} - \text{L}'') \end{aligned}$$

This equation combined with the equation (63)

$$6 \,.\, \sqrt{\text{M} + m} \,.\, \theta' = (r + r'' + k)^{\frac{3}{2}} - (r + r'' - k)^{\frac{3}{2}}$$

and the two equations

$$r^2 = \text{R}^2 + \delta^2 \,.\, \sec \beta^2 + 2\, \text{R}\, \delta \,.\, \cos (\lambda - \text{L})$$
$$r''^2 = \text{R}''^2 + m^2\, \delta^2 \,.\, \sec \beta''^2 + 2\, m\, \text{R}''\, \delta \,.\, \cos (\lambda'' - \text{L})$$

gives four equations for determining the four unknown quantities. To resolve them, we must use the well known *rule of false*, one of the simplest and most widely useful rules mathematical science affords. Assume a value for δ, as near the truth as can be conjectured, compute m and thence $m\,\delta$, or δ''; and having used these to obtain k, r, and r'', substitute these in the expression for $g\,\theta'$. Call the result A. Assume then a second value for δ, which call Δ, and executing the same process, call the result B. Then we shall have

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* The reader will not confound θ here with the same letter before used to denote the time.

$$\mathrm{B}-\mathrm{A} : g\,t-\mathrm{A} :: \Delta-\delta : x-\delta$$

where x is the corrected value of δ, and hence we get

$$x=\delta+(\Delta-\delta)\,.\,\frac{g\,t-\mathrm{A}}{\mathrm{B}-\mathrm{A}}$$

and substituting this again, and employing that one of the former values of δ which gave a result nearest to $g\,t$ for a corresponding value, we shall obtain a yet nearer approximation, and so on.

It remains to exemplify the method of computation by an instance. Let us therefore take the following observations of the comet of 1779, taken at Paris with a transit and a repeating circle.

	Mean Time, Paris.	h ′ ″		° ′ ″		° ′ ″
1779.	Aug. 30	11 9 42,	λ =	125 48 39·3	β =	41 53 52·2
	Sep. 2	10 36 8,	λ' =	132 53 48·5	β' =	45 54 48·1
	Sep. 4	10 7 51,	λ'' =	138 56 31·2	β'' =	48 32 27·8

$$\mathrm{L} = 337\ 29\ \ 8{\cdot}7 \qquad \mathrm{R} = 1{\cdot}0087218$$
$$\mathrm{L}' = 340\ 22\ 26{\cdot}9 \qquad \mathrm{R}' = 1{\cdot}0079991$$
$$\mathrm{L}'' = 342\ 17\ 47{\cdot}8 \qquad \mathrm{R}'' = 1{\cdot}0074854$$

The intervals therefore between the first and second, first and third, and second and third observations are respectively* $\theta = 2{\cdot}976690$ days, $\theta' = 4^{\mathrm{d}}{\cdot}957049$, $\theta'' = 1^{\mathrm{d}}{\cdot}980350$

Consequently, we have by the equations expressing the values of A′ and C′ reduced into numbers

$$\frac{\mathrm{C}'}{\mathrm{A}'} = 1{\cdot}18408\,; \quad m = \frac{\delta''}{\delta} = 0{\cdot}787752$$

and substituting this value of m, and the values of λ, λ'', β, β'', R, R″, we get the following equations, noticing that $g = 0{\cdot}103212$

$$r^2 = 1{\cdot}01752 - 1{\cdot}71693\,.\,\delta + 1{\cdot}80493\,.\,\delta^2\,; \qquad r''^2 = 1{\cdot}01503 - 1{\cdot}45724\,.\,\delta + 1{\cdot}41564\,.\,\delta^2$$

$$k^2 = 0{\cdot}00716 - 0{\cdot}04739\,.\,\delta + 0{\cdot}08627\,.\,\delta^2\,; \quad 135{\cdot}84219 = \left(\frac{r+r''+k}{2}\right)^{\frac{3}{2}} - \left(\frac{r+r''-k}{2}\right)^{\frac{3}{2}}$$

These equations are very nearly satisfied by taking $\delta = 0{\cdot}71469$, whence we find

$$r = 0{\cdot}8440\,; \quad r'' = 0{\cdot}8346\,; \quad k = 0{\cdot}1317\,; \quad \delta'' = m\,\delta = 0{\cdot}562998$$

whence we next deduce the heliocentric longitudes and latitudes at the extreme observations from equations (85) and (86)

$$l = 20^\circ\ 37'\ 40''{\cdot}8 \quad \text{and } b = 49^\circ\ 26'\ 23''{\cdot}1$$
$$l'' = \ \ 6\ \ 45\ \ 39\ {\cdot}7 \qquad b'' = 49\ \ 46\ \ 46\ {\cdot}9$$

Since l'' is less than l, the comet's motion is retrograde. Moreover, we have by equations (87) and (88)

$$☊ = 100^\circ\ 51'\ 53''{\cdot}4\,; \quad i = 49^\circ\ 51'\ 7''{\cdot}9$$

we obtain also, by following the process pointed out in equations (89) and (90) $\theta = 12^\circ\ 39'\ 35''{\cdot}6$, $\pi = 4^\circ\ 32'\ 8''{\cdot}2$ for the longitude of the perihelion, and $\mathrm{D} = 0{\cdot}830761$ for the perihelion distance. Consequently the time between the first observation and the perihelion passage is

$$\frac{\mathrm{D}^{\frac{3}{2}}\,.\,\sqrt{2}}{\sqrt{\mathrm{M}+m}}\,.\,\left\{\tan\frac{\theta}{2}+\frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\} = +\ 6{\cdot}97118 =$$

$$= 6^{\mathrm{d}}\ 23^{\mathrm{h}}\ 18'\ 30''$$
$$\text{30th Aug.}\ \ 11\quad 9\quad 42$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$10\quad 28\quad 12\quad \text{Sept. 6, 1799} = \text{time of perihelion passage.}$$

Of the two solutions given above, the latter only supposes the orbit to be determined a parabola; the former may be applied to the determination of elliptic elements, but the latter part of the process from equation (70) must be varied accordingly; but as the various methods of determining and correcting elliptic as well as parabolic elements would lead us a great deal beyond our proposed limits, we must content ourselves with referring to Mr. Gauss's *Theoria Motus Corporum Celestium*, in which the whole subject is treated with the utmost generality. The reader may also consult Sir H. Englefield, *On Comets;* the *Mécanique Céleste* of Laplace, vol. I. c. iv.; Lagrange, *Méc. Analytique* (the new edition.) The German scholar will find it well worth his while to consult the paper of the same eminent writer in the *Berlin Ephemeris* for 1785; and the volume of the same work for 1789, p. 197; the papers of Gauss and Olbers, in the *Monatliche Correspondenz;* as well as the treatise of the latter, *Ueber die leichteste und bequemste Methode den Bahn eines Cometen zu berechnen.*

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* The reader is requested to take the numbers in this example upon Mr. Littrow's authority, from whose work the example is taken.

PART II. SECTION I.

Of the perturbation of the elliptic motions of the heavenly bodies, arising from their mutual attraction

In the preceding investigations we have supposed only two attracting bodies to exist in space; and on this supposition have succeeded in representing all the phenomena of their motions in finite equations, by the aid of which their relative situations may be assigned at any past or future moment. The resulting formulæ reduced into tables, and compared with actual observation, manifest an agreement sufficiently complete to leave no doubt of the correctness of the principles from which they have been deduced, at least, when observations separated only by moderately long intervals of time are compared together. Yet, on descending to a more rigorous nicety, and especially on comparing together observations embracing a very long series of years, it is found that the results of the calculations, founded on the assumption of elliptic motion according to the laws above demonstrated, do not represent the observations perfectly. Minute irregularities are still detected, and the planets are found sometimes a little in advance, and sometimes a little falling short, sometimes a little to the right or left, above or below, their calculated places in their orbits. Moreover, if the elements of the orbits themselves, as deduced from modern observations, be compared with those similarly deduced from ancient ones, they will be found not to correspond exactly, but to differ by small variations, which are however greater, as the observations themselves compared are more distant in their dates. In a word, though the elliptic theory agrees very nearly with observation; yet, to make it tally rigorously with them, it is necessary to introduce modifications of this kind.

1st. The ellipse in which each planet moves, must be conceived to change its eccentricity and position in space, by exceedingly slow gradations—so slow indeed, as to be insensible in a single revolution of the planet, and only discoverable by a comparison, such as we have described, of its nature in past and present ages.

2dly. The planet itself is not always found exactly in its place in the ellipse *even* when so varied; nor indeed is it always found in the exact periphery of the ellipse at all. But if we conceive an imaginary point to describe this ellipse according to the rigorous laws of elliptic motion, the real planet will never be very far distant from it, but will oscillate or revolve round it like a secondary about its primary in an orbit of extremely small dimensions, yet according to laws of a very complicated nature, too complicated indeed for observation alone to unravel.

These variations are much more sensible in the orbit and motions of the moon, where, in fact, they amount to very considerable quantities. If the moon, for instance, at its greatest northern latitude be observed to pass over a certain star, it should continue to do so each revolution, if its motion were strictly conformable to the elliptic hypothesis. So far, however, is this from being the case, that it will be observed to deviate visibly southward every lunation; and after a lapse of nine years and a half, its path, in the same part of the heavens as to longitude, will pass not less than 10° south of the star; after which it will again advance northward, and in nine years and a half more will once more pass over, or at least very near the star. Deviations like these must have some positive and decided cause; as much so as the elliptic motion itself; and if they are to be accounted for on the theory of universal attraction, it is manifest that the same mechanical principles applied in the same way to the case of several bodies abandoned in free space to their mutual attractions, ought to lead us to their explanation.

But the mathematical difficulties to be encountered in this research, are of a much higher order than in the case of two bodies only. *There* we had no difficulty in integrating the differential equations of the problem. Here, on the other hand, the equations are too complicated to allow of their integrals being exhibited otherwise than in series; and even then we have no means of ascertaining their laws. Fortunately, however, this is not necessary; the enormous preponderance of the sun's mass in our system being such, that the fractions, representing those of each of the other planets, are small enough to allow of their squares and products being neglected without any fear of inducing appreciable errors.

It is a well known theorem, which extends to all the applications of mathematical reasoning to natural phenomena, that when several causes of motion act together, if their effects taken singly are of such an order of minuteness, that their squares and products may be disregarded, then their joint effect will be the sum of the effects which would be produced by each acting alone. This principle allows us to regard the disturbance or perturbation of any one planet produced by the joint action of all the others, as the sum of the effects each would produce separately; so that we may simplify the investigation by leaving out of consideration all but a single disturbing planet, and applying the analytical formulæ so investigated to each in succession, we shall thus obtain expressions for the perturbations produced by each; which, added together, give the total perturbation.

Investigation of the forces exerted by one body to disturb the orbit of another revolving about a common central body, and of the differential equations of their motions.

Taking unity for the mass of the central body, and its centre for the origin of the co-ordinates, let m and m' represent the masses of two revolving bodies, and suppose x, y, z, r to denote the three co-ordinates and radius vector of the body m, whose perturbations we would investigate; and x', y', z', r', the corresponding quantities relative to the disturbing body m'. Moreover, let λ represent the distance of the two bodies, m, m', on each other. Then will the several attractive forces exerted by the bodies of the system on each other be as follows :—

1st. The sun (or central body) attracts m and m' with forces represented respectively by $\frac{1}{r^2}$ and $\frac{1}{r'^2}$.

2dly. m attracts the sun with a force $\frac{m}{r^2}$, and it attracts m' with a force $\frac{m}{\lambda^2}$.

3dly. m' attracts the sun with the force $\frac{m'}{r'^2}$, and it attracts m with the force $\frac{m'}{\lambda^2}$.

Fig. 11 These are the forces exerted in the directions of the lines joining the several bodies in the system; but to estimate their effects, they must be reduced to the directions AS, BS, CS, of the three co-ordinates. This done, we shall find

For the forces acting on the sun,

$$\left.\begin{aligned} \text{Force of } m &= -\frac{m\,x}{r^3} \\ \text{Force of } m' &= -\frac{m'\,x'}{r'^3} \end{aligned}\right\} \text{ in the direction AS}$$

and similarly,

$$-\frac{m\,y}{r^3} \text{ and } -\frac{m'\,y'}{r'^3} \text{ in the direction BS}$$

$$-\frac{m\,z}{r^3} \text{ and } -\frac{m'\,z'}{r'^3} \text{ in the direction CS.}$$

So that the three forces acting on the sun, are respectively

$$-\left(\frac{m\,x}{r^3}+\frac{m'\,x'}{r'^3}\right)$$
$$-\left(\frac{m\,y}{r^3}+\frac{m'\,y'}{r'^3}\right)$$
$$-\left(\frac{m\,z}{r^3}+\frac{m'\,z'}{r'^3}\right)$$

Again, the forces acting on m are

$$\text{In the direction AS} \ldots\ldots\ldots\ldots \left\{\begin{aligned} &\text{Force of the sun } = +\frac{x}{r^3} \\ &\text{Force of } m' \ldots\ldots = -\frac{m'\,(x'-x)}{\lambda^3} \end{aligned}\right.$$

$$\text{In the direction BS} \ldots\ldots\ldots \left\{\begin{aligned} &\text{Force of the sun } = +\frac{y}{r^3} \\ &\text{Force of } m' \ldots\ldots = -\frac{m'\,(y'-y)}{\lambda^3} \end{aligned}\right.$$

$$\text{In the direction CS} \ldots\ldots\ldots \left\{\begin{aligned} &\text{Force of the sun } = +\frac{z}{r^3} \\ &\text{Force of } m' \ldots\ldots = -\frac{m'\,(z'-z)}{\lambda^3} \end{aligned}\right.$$

So that the aggregate forces acting on m in these directions are

$$+\frac{x}{r^3} - m' \,.\, \frac{x'-x}{\lambda^3}$$
$$+\frac{y}{r^3} - m' \,.\, \frac{y'-y}{\lambda^3}$$
$$+\frac{z}{r^3} - m' \,.\, \frac{z'-z}{\lambda^3}$$

and in the same manner, the forces acting on m' will be found to be respectively

$$+\frac{x'}{r'^3} - m \,.\, \frac{x-x'}{\lambda^3}$$
$$+\frac{y'}{r'^3} - m \,.\, \frac{y-y'}{\lambda^3}$$
$$+\frac{z'}{r'^3} - m \,.\, \frac{z-z'}{\lambda^3}$$

From these three sets of forces we might determine separately the motions of the sun, of m, and of m'; but as we have chosen to suppose the sun at rest, and fix the origin of our co-ordinates in its centre, we must transfer to both m and m' in a contrary direction the forces which act upon it. This done, the forces animating m in the respective directions AS, BS, CS, will become

$$\frac{(1+m).x}{r^3} + m'\left\{\frac{x'}{r'^3} - \frac{x' \quad x}{\lambda^3}\right\}$$
$$\frac{(1+m).y}{r^3} + m'\left\{\frac{y'}{r'^3} - \frac{y'-y}{\lambda^3}\right\}$$
$$\frac{(1+m).z}{r^3} + m'\left\{\frac{z'}{r'^3} - \frac{z'-z}{\lambda^3}\right\}$$

and if, in these expressions, we exchange the accented letters for the corresponding unaccented ones, and *vice versâ*, (λ only excepted) we shall evidently have the forces acting on m'.

The first terms of these expressions, as will be observed, are the same as if the mass m' of the disturbing planet were nothing, or as if there were no such planet; the other parts, those multiplied by m', express the *disturbing forces*, or the effect of the attraction of m' to derange the orbit of m from its elliptical form. These, it will also be observed, consist each of two parts; the one expressing the action of m' on the sun, and the other its action on m with an opposite sign. It is only therefore in virtue of the difference of the attractions of m' on the sun, and on m that the orbit of the latter is deranged.

These forces are susceptible of a very simple mode of expression; and the equations of m's motion in consequence admit of a very compendious form, if we consider, that owing to the form of the function λ, which is equal to $\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$, we have

$$\left.\begin{aligned} \frac{x'-x}{\lambda^3} &= \frac{d}{dx}.\frac{1}{\lambda} \\ \frac{y'-y}{\lambda^3} &= \frac{d}{dy}.\frac{1}{\lambda} \\ \frac{z'-z}{\lambda^3} &= \frac{d}{dz}.\frac{1}{\lambda} \end{aligned}\right\} \quad \ldots\ldots (92)$$

and that moreover, if we put U for the function $\frac{x\,x'+y\,y'+z\,z'}{r'^3}$, we shall have

$$\frac{x'}{r'^3} = \frac{d\,U}{d\,x}; \quad \frac{y'}{r'^3} = \frac{d\,U}{d\,y}; \quad \frac{z'}{r'^3} = \frac{d\,U}{d\,z}$$

so that if we assume

$$\begin{aligned} \Omega &= U - \frac{1}{\lambda} \\ &= \frac{x\,x'+y\,y'+z\,z'}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} \end{aligned} \quad (93)$$

the disturbing forces will be represented by

$$m'.\frac{d\,\Omega}{d\,x}; \quad m'.\frac{d\,\Omega}{d\,y}; \quad m'.\frac{d\,\Omega}{d\,z}$$

and the equations representing the motion of m, will be (if we put $1+m=\mu$) and suppose $d\,t$ the element of the time invariable.

$$\left.\begin{aligned} o &= \frac{d^2\,x}{d\,t^2} + \mu.\frac{x}{r^3} + m'.\frac{d\,\Omega}{d\,x} \\ o &= \frac{d^2\,y}{d\,t^2} + \mu.\frac{y}{r^3} + m'.\frac{d\,\Omega}{d\,y} \\ o &= \frac{d^2\,z}{d\,t^2} + \mu.\frac{z}{r^3} + m'.\frac{d\,\Omega}{d\,z} \end{aligned}\right\} \quad \ldots\ldots (94)$$

Similarly, if we put $U' = \frac{x\,x'+y\,y'+z\,z'}{r^3}$ and take $\mu' = 1+m'$, and

$$\Omega' = U' - \frac{1}{\lambda} \quad (95)$$

the motion of m will be represented by the equations

$$\left.\begin{aligned} o &= \frac{d^2\,x'}{d\,t^2} + \mu'.\frac{x'}{r'^3} + m.\frac{d\,\Omega'}{d\,x'} \\ o &= \frac{d^2\,y'}{d\,t^2} + \mu'.\frac{y'}{r'^3} + m.\frac{d\,\Omega'}{d\,y'} \\ o &= \frac{d^2\,z'}{d\,t^2} + \mu'.\frac{z'}{r'^3} + m.\frac{d\,\Omega'}{d\,z'} \end{aligned}\right\} \quad \ldots\ldots (96)$$

The joint integration of these six equations would determine the motions of both m and m', but their rigorous integration being impracticable in the present state of analysis, we are driven to have recourse to methods of approximation, of which the principle may be explained as follows :—

Suppose $A = o$ to be any equation, or system of equations, susceptible of rigorous integration, and let the process to be followed for this purpose, consist in deducing in succession from $A = o$, by any analytical artifices, or transformations, the equations, or systems of equations,

$$B = o, \quad C = o, \ldots\ldots\ldots K = o$$

the last of which, $K = o$, is integrable by known methods. Now, suppose, instead of $A = o$, we had proposed the equation

$$A + m' \,.\, a = o$$

where $m' \,.\, a$ is a minute term of the order of the disturbing forces, m' being an extremely small constant quantity, and a any assigned function of the co-ordinates. If then we pursue with this equation the very same process by which $B = o$ was derived from $A = o$, it is clear that we shall obtain an equation which can differ only from $B = o$ by a quantity which vanishes when $m' = o$, and which therefore must be of the form $m' \times b$. So that instead of $B = o$, we have

$$B + m' \times b = o$$

where b is some explicit function of the co-ordinates, their differential co-efficients, and of m', and either developed, or at least developable in a series of ascending powers of m'. In like manner we may deduce equations

$$C + m' \,.\, c = o; \ldots\ldots K + m' \,.\, k = o$$

in the place of the equations

$$C = o, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots K = o$$

In order then to obtain a final equation for the solution of the problem of three bodies, we have only to follow out any system of processes and transformations which, in the case of two only, would prove successful in reducing the differential equations to an integrable form.

This may be accomplished in a great variety of ways, as the equations of undisturbed motion are integrable by a great many different artifices, besides those which we have employed in the former part of this essay; and it is therefore necessary to select among them such as lead to final equations best adapted to the nature of the case under consideration. Now two courses have been adopted by geometers; the former adapted to the theory of the planetary, and the latter to that of the lunar perturbations.

In the theory of the planets, the disturbing force is so extremely minute, that its square and higher powers may be neglected with safety. This simplification being permitted, it becomes practicable (as we shall presently see) so to conduct the investigation as to make the time, or *mean* longitude of the disturbed planet our independent variable, and thus to express the perturbations at once in functions of the time, or of angles proportional to it, a simplification of the utmost moment in the construction of tables. In the more complicated theory of the moon, in which the part of the perturbations depending on higher powers of the disturbing force than the first is very conspicuous, it is no longer permitted to neglect them, and the necessity of preserving, as far as possible, the rigorous expressions of the forces, &c. at least in the *differential* equations, obliges us to employ, not the mean, but the true longitude of the moon for our independent variable, as by this means we are enabled to arrive at final equations perfectly rigorous, and can thus estimate the influence which the neglect of small quantities is capable of producing in their integration, with less likelihood of being misled.

The theory of the moon then differs entirely from that of the planets in its treatment. The general principles of approximation, however, are the same in both. Both theories, as we shall see, lead to final equations of the form

$$\frac{d^2 u}{d t^2} + n^2 u + m' \,.\, k = o \tag{97}$$

where n^2 is constant, and k an explicit function of the several co-ordinates, distances, and angles, of the problem, and m' a very small quantity (which, in the lunar theory, also enters into the composition of k,) of the order of the disturbing forces. To approximate then to the value of u, we first suppose $m = o$, and we get a value of u corresponding to $m' = o$, and which we call its *elliptic value.*

2dly. We deduce from this the *elliptic values* of all the variables which enter into the composition of the term $m' k$, in terms of the independent variable t, whether t represent the mean, or the true longitude. These being substituted in the last term, it will become an explicit function of the independent variable.

3dly. If we now again integrate the differential equation so prepared, the value of u will consist of two parts; the first will be the same as before, viz. the elliptic value, and the second will be a correction which must be of the order of disturbing forces, and will express the perturbation of u with a degree of precision corresponding to their first power.

If the same process of substitution be repeated, and the equation again integrated, another set of terms will be added to the value of u, which carry the approximation a step farther, or to the squares of the disturbing forces; and were the same process continued to infinity, the series of terms so obtained would be a rigorous analytical expression for u.

The final equation (97) is of the form so well known in analysis, under the name of linear differential equations, and (as we have observed) almost all the equations on which the planetary motions depend being of this nature, it will be right to premise some few points relative to their theory, to which to refer hereafter.

It is demonstrated in all works in the differential and integral calculus, and will be so in our article on that subject, that any linear differential equation of the second order is integrable, provided we can find two, or even one particular value capable of satisfying it when deprived of its last term. Thus, if

$$\frac{d^2 u}{d t^2} + M \,.\, \frac{d u}{d t} + N u + \Pi = o \tag{98}$$

be any such equation, M and N being functions of t, and if u' and u'' be any two functions of t which satisfy the equation

$$\frac{d^2 u}{d t^2} + \mathrm{M} \, . \, \frac{d u}{d t} + \mathrm{N} \, u = o$$

then the complete integral will be

$$\mathrm{C}' u' + \mathrm{C}'' u'' - u' \int d \frac{u''}{u'} \int \frac{\Pi \, d t^2}{u' \, d \, \frac{u''}{u'}} \tag{99}$$

Now, the last term of this, being integrated by parts, becomes

$$u' \left\{ \frac{u''}{u'} \int \frac{\Pi \, d t^2}{u' \, d \, \frac{u''}{u'}} - \int \frac{\Pi \, d t^2 \times u''}{u'^2 \, d \, \frac{u''}{u'}} \right\}$$

$$= u'' \int \frac{\Pi \, d t^2}{u' \, d \, \frac{u''}{u'}} + u' \int \frac{\Pi \, d t^2}{u'' \, d \, \frac{u'}{u''}}$$

Consequently, the complete integral of the equation (98) will be

$$u = \mathrm{C}' u' + \mathrm{C}'' u'' - \left\{ u' \int \frac{\Pi \, d t^2}{u'' \, d \, \frac{u'}{u''}} + u'' \int \frac{\Pi \, d t^2}{u' \, d \, \frac{u''}{u'}} \right\} \tag{100}$$

Suppose we have the equation

$$\frac{d^2 u}{d t^2} + n^2 u + \Pi = o \tag{101}$$

then u' and u'', the particular integrals of $\frac{d^2 u}{d t^2} + n^2 u = o$ offer themselves readily, being no other than $\sin n t$ and $\cos n t$; and if these be substituted in the general expression above given, we get at once

$$u = \mathrm{C}' \, . \, \cos n t + \mathrm{C}'' \, . \, \sin n t$$
$$+ \frac{1}{n} \left\{ \cos n t \int \Pi \, d t \, . \, \sin n t - \sin n t \int \Pi \, d t \, \cos n t \right\}; \tag{102}$$

and if $n = 1$, or in the case of

$$\frac{d^2 u}{d t^2} + u + \Pi = o \tag{103}$$

the integral is

$$u = \mathrm{C}' \, . \, \cos t + \mathrm{C}'' \, . \, \sin t$$
$$+ \cos t \, . \int \Pi \, d t \, . \, \sin t - \sin t \int \Pi \, d t \, . \, \cos t \tag{104}$$

These values of u are rigorous, whatever be the value of Π, and independent of any approximation, as it is easy for the reader to satisfy himself by substituting them in the differential equations from which they were deduced, when the whole will be found to vanish, independent of any particular value of Π, or any supposition made as to its magnitude. The equation (102) may however be obtained perhaps easier as follows.

Let $\frac{d^2 u}{d t^2} + n^2 u = - \Pi$ be multiplied by $d t^2 \, . \, \cos n t$ and it becomes

$$d^2 u \, . \, \cos n t + n^2 u \, d t^2 \, . \, \cos n t = - \int \Pi \, d t^2 \, . \, \cos n t$$

and integrating,

$$d \, u \, . \cos n t + n \, u \, d t \, . \sin n t = - \int \Pi \, d t^2 \, . \cos n t$$

If we again multiply this by $\frac{1}{\cos n t^2}$ we get

$$\frac{d u}{\cos n t} + n \, u \, d t \, . \, \frac{\sin n t}{(\cos n t)^2} = - \frac{d t}{(\cos n t)^2} \int \Pi \, d t \, . \cos n t$$
$$= - \frac{d \, . \, \tan n t}{n} \int \Pi \, d t \, . \cos n t$$

and again integrating,

$$\frac{u}{\cos n t} = - \frac{1}{n} \int d \, . \tan n t \int \Pi \, d t \, . \cos n t$$
$$= \frac{1}{n} \left\{ - \tan n t \int \Pi \, d t \, . \cos n t + \int \Pi \, d t \, . \sin n t \right\}$$

by integrating by parts. Hence we have

$$u = \frac{1}{n} \left\{ \cos n t \int \Pi \, d t \, . \sin n t - \sin n t \int \Pi \, d t \, . \cos n t \right\} \tag{105}$$

or, expressing the arbitrary constants by C′ and C″

$$u = C' . \cos n\,t + C'' . \sin n\,t$$
$$+ \frac{1}{n}\left\{ \cos n\,t \int \Pi\, d\,t . \sin n\,t - \sin n\,t \int \Pi\, d\,t . \cos n\,t \right\}$$

as before.

If Π be an explicit function of t, this value of u is always assignable, at least by quadratures, and, whenever the integrations can be executed, in finite terms. In the lunar and planetary theories, Π is always reducible to a series of sines or cosines of the form $\overset{\cos}{\underset{\sin}{.}}\,(A\,t + B)$. Let us therefore consider this case more closely. Now any term of Π, such as $a \times \sin (A\,t + B)$ will introduce into u the term

$$\frac{a}{n}\left\{ \cos n\,t \int \sin n\,t . \sin (A\,t + B)\, d\,t - \sin n\,t \int \cos n\,t . \sin (A\,t + B)\, d\,t \right\}$$

But we have

$$\sin n\,t . \sin (A\,t + B) = \tfrac{1}{2}\left\{\cos (\overline{A - n} . t + B) - \cos (\overline{A + n} . t + B)\right\}$$

whence

$$\int d\,t . \sin n\,t . \sin (A\,t + B) = \frac{\sin (\overline{A - n} . t + B)}{2\,(A - n)} - \frac{\sin (\overline{A + n} . t + B)}{2 . (A + n)}$$

and similarly,

$$\int d\,t . \cos n\,t . \sin (A\,t + B) = - \frac{\cos (\overline{A + n} . t + B)}{2 . (A + n)} - \frac{\cos (\overline{A - n} . t + B)}{2\,(A - n)}$$

So that the term introduced into u will become

$$a . \frac{\cos n\,t . \sin (\overline{A - n} . t + B) + \sin n\,t . \cos (\overline{A - n} . t + B)}{2\,n\,(A - n)}$$
$$- a . \frac{\cos n\,t . \sin (\overline{A + n} . t + B) - \sin n\,t . \cos (\overline{A + n} . t + B)}{2\,n\,(A + n)}$$
$$= a . \sin (A\,t + B)\left\{\frac{1}{2\,n\,(A - n)} - \frac{1}{2\,n\,(A + n)}\right\}$$
$$= \frac{a . \sin (A\,t + B)}{A^2 - n^2}$$

Similarly, if $a . \cos (A\,t + B)$ were any term of Π, the corresponding term in u would be $\dfrac{a . \cos (A\,t + B)}{A^2 - n^2}$

Π therefore consisting of a series of terms, such as

$$\Pi = a . \overset{\cos}{\underset{\sin}{.}}\,(A\,t + B) + a' . \overset{\cos}{\underset{\sin}{.}}\,(A'\,t + B') + \&c. \qquad (106)$$

we shall have

$$u = \frac{a}{A^2 - n^2} . \overset{\cos}{\underset{\sin}{.}}\,(A\,t + B) + \frac{a'}{A'^2 - n^2} . \overset{\cos}{\underset{\sin}{.}}\,(A'\,t + B') + \&c. \qquad (107)$$
$$+ C . \cos n\,t + C' . \sin n\,t$$

C and C′ being two arbitrary constants.

Terms of the form $a . \overset{\cos}{\underset{\sin}{.}}(A\,t + B)$ being of perpetual occurrence in physical astronomy, it is necessary to designate them and their several parts by names. The whole term is called an equation, or inequality: the part $(A\,t + B)$ within the sign sin or cos is called the *argument;* and the co-efficient a, the maximum. The *period* of the inequality, or the time (in units of time such as t consists of) which it occupies in passing through all its gradations of magnitude and sign, is equal to $\dfrac{360}{A}$, A being expressed in degrees. Hence, the period of an inequality is longer or shorter, according as the co-efficient of the time in its argument is less or greater.

The arguments of all the inequalities in u then are the same as those in Π, one remarkable case only excepted, in which $A = n$; for, in this case, $A^2 - n^2 = o$; and the term having $(A\,t + B)$ for its argument, changes its form. In fact, since the constants C and C′ are arbitrary, we may change them into $C - \dfrac{a . \sin B}{A^2 - n^2}$ and $C' - \dfrac{a \;\; \cos B}{A^2 - n^2}$ respectively, in which case $C . \cos n\,t + C' . \sin n\,t$ will be changed to $C . \cos n\,t + C' . \sin n\,t - \dfrac{a}{A^2 - n^2} . \sin (A\,t + B)$. Thus u will contain the terms

$$\frac{a}{A^2 - n^2} . \sin (A\,t + B) - \frac{a}{A^2 - n^2} . \sin (n\,t + B)$$

But when $A = n$, $a \cdot \frac{\sin (A\,t + B) - \sin (n\,t + B)}{A^2 - n^2} = \frac{o}{o}$ and differentiating numerator and denominator with respect to A, it becomes simply (on making $A = n$)

$$\frac{a\,t}{2} \cdot \cos (n\,t + B) \qquad (108)$$

Similarly, if Π contained the term $\cos (n\,t + B)$, this would introduce into u the term

$$-\frac{a\,t}{2} \cdot \sin (n\,t + B) \qquad (108)$$

Terms of this kind form an exception to the law of periodicity observed by all the rest, having t disengaged from the sign sin or cos. If t represent either the time, or the mean or true longitude, of the disturbed body, they will represent inequalities, whose maxima go on continually increasing without limit. Such inequalities, if they really had an existence in our system, must end in its destruction, or at least in the total subversion of its present state; but we shall see hereafter, that when they do occur, they have their origin, not in the nature of the differential equations, but in the imperfection of our analysis, and in the inadequate representation of the perturbations, and are to be got rid of, or rather included in more general expressions, of a periodical nature, by a more refined investigation than that which led us to them. The nature of this difficulty will be easily understood from the following reasoning. Suppose that a term, such as $a \sin (A\,t + B)$ should exist in the value of u, in which A being extremely minute, the period of the inequality denoted by it would be of great length; then, whatever might be the value of the co-efficient a, the inequality would still be always confined within certain limits, and after many ages would return to its former state. Suppose now that our peculiar mode of arriving at the value of u, led us to this term, not in its real analytical form $a \cdot \sin (A\,t + B)$, but by the way of its developement in powers of t, $\alpha + \beta\,t + \gamma\,t^2 + \&c.$; and that, not at once, but piecemeal, as it were; a first approximation giving us only the term α, a second adding the term $\beta\,t$, and so on. If we stopped here, it is obvious that we should mistake the nature of this inequality, and that a really periodical function, from the effect of an imperfect approximation, would appear under the form of one not periodical.

This is what actually takes place in the theory of the problem of three bodies. These terms in the value of u, when they occur, are not superfluous; they are essential to its expression, but they lead us to erroneous conclusions as to the stability of our system and the general laws of its perturbations, unless we keep in view that they are only parts of series; the principal parts, it is true, when we confine ourselves to intervals of moderate length, but which cease to be so after the lapse of very long times, the rest of the series acquiring ultimately the preponderance, and compensating the want of periodicity of its first terms.

Section II.

General theory of the planetary perturbations depending on their mutual configurations.

Our first object in the theory of the planets is to transform the differential equations of the disturbed orbit, so as to obtain final equations in which the radius vector, and true longitude, or those parts of them arising from the action of the disturbing forces, shall be expressed in terms of the time; and to reduce them to the general form of the linear equation of the second order, whose theory we have just considered. To this end,

Let the equations (94) of m's motion be respectively multiplied by x, y, z, and added together, and we get

$$o = \frac{d\,x\,d^2\,x + d\,y\,d^2\,y + d\,z\,d^2\,z}{d\,t^2} + \mu \cdot \frac{x\,d\,x + y\,d\,y + z\,d\,z}{r^3}$$
$$+ m' \cdot \left\{ \frac{d\,\Omega}{d\,x}\,d\,x + \frac{d\,\Omega}{d\,y}\,d\,y + \frac{d\,\Omega}{d\,z}\,d\,z \right\}$$

The portion within the brackets of the last term is the differential of Ω taken on a supposition of the co-ordinates of m only varying. Let this be represented by the Roman character d, so that

$$\mathrm{d}\,\Omega = \frac{d\,\Omega}{d\,x}\,d\,x + \frac{d\,\Omega}{d\,y}\,d\,y + \frac{d\,\Omega}{d\,z}\,d\,z$$

bearing this in mind, and that $\mathrm{d}\,\Omega$ is only an abbreviated expression for this function, we have by integration

$$o = \frac{d\,x^2 + d\,y^2 + d\,z^2}{d\,t^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\,m' \int \mathrm{d}\,\Omega\,; \qquad (109)$$

in which we must be careful not to confound $\int \mathrm{d}\;\Omega$ with $\int d\;\Omega$ or Ω, $\mathrm{d}\,\Omega$ being only an incomplete differential, and the characteristic $\int$ denoting an integration relative to t, supposes the variation of the co-ordinates of the disturbing as well as the disturbed body.

Again, if we multiply the same equations (94) by x, y, z, respectively, we shall get

$$o = \frac{x\,d^2\,x + y\,d^2\,y + z\,d^2\,z}{d\,t^2} + \frac{\mu}{r} + m' \left\{ x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} \right\};$$

If we add this to the former, observing that

$$d x^2 + d y^2 + d z^2 + x\, a^2 x + y\, d^2 y + z\, d^2 z = \tfrac{1}{2}\, d^2\,(x^2 + y^2 + z^2) = \frac{d^2\,.\,r^2}{2}$$

we get

$$o = \tfrac{1}{2}\,\frac{d^2 r^2}{d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\, m' \int \mathrm{d}\,\Omega + m' \left\{ x\,\frac{d\,\Omega}{d x} + y\,\frac{d\,\Omega}{d y} + z\,\frac{d\,\Omega}{d z} \right\}$$

Now, if we put $x = \rho\,.\cos\theta$, $y = \rho\,.\sin\theta$ and $z = \rho\,.\,s$, or suppose $\rho =$ the projected radius vector r, (see fig. 11.) θ the angle it makes with the axis of the x, and $s =$ the tangent of the latitude of m, we have

$$\rho = \frac{r}{\sqrt{1 + s^2}}\,;\ r\,\frac{d\,\rho}{d\,r} = r\,.\,\frac{1}{\sqrt{1 + s^2}} = \rho$$

But

$$r\,\frac{d\,x}{d\,r} = r\,\frac{d\,x}{d\,\rho}\,.\,\frac{d\,\rho}{d\,r} = \rho\,\frac{d\,x}{d\,\rho} = \rho\,.\cos\theta = x$$

and similarly,

$$r\,\frac{d\,y}{d\,r} = y \quad \text{and}\ r\,\frac{d\,z}{d\,r} = z$$

Consequently, substituting for x, y, z, these values

$$x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} = r\left\{\frac{d\,\Omega}{d\,x}\,.\,\frac{d\,x}{d\,r} + \frac{d\,\Omega}{d\,y}\,.\,\frac{d\,y}{d\,r} + \frac{d\,\Omega}{d\,z}\,.\,\frac{d\,z}{d\,r}\right\} = r\,.\,\frac{d\,\Omega}{d\,r}$$

and it will be observed, that this property is altogether independent of the nature of the function Ω, and belongs to every possible function of the co-ordinates x, y, z, x', y', z'.

Thus we see, that

$$2\,m' \int \mathrm{d}\,\Omega + m' \left\{ x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} \right\} = m'\,.\left\{ 2 \int \mathrm{d}\,\Omega + r\,\frac{d\,\Omega}{d\,r} \right\}$$

Hence, if we put

$$\mathrm{Q} = 2 \int \mathrm{d}\,\Omega + r\,\frac{d\,\Omega}{d\,r}\,, \tag{110}$$

our differential equation becomes

$$o = \tfrac{1}{2}\,\frac{d^2\,.\,r^2}{d\,t^2} - \frac{\mu}{r} + \frac{\mu}{a} + m'\,\mathrm{Q}\,; \tag{111}$$

Let us now suppose that r represents only the elliptic value of r, and x, y, z, Ω, the elliptic values of the co-ordinates, and the value of the function Ω, which would arise from writing the elliptic values of x, y, z, x', y', z', in their expressions; and let $r + m'\,\delta\,r$, $x + m'\,\delta\,x$, $y + m'\,\delta\,y$, $z + m'\,\delta\,z$, $\Omega + m'\,\delta\,\Omega$, &c. represent the disturbed values of these quantities; then, if we neglect m'^2, we shall get by substitution

$$o = \tfrac{1}{2}\,\frac{d^2\,.\,r^2}{d\,t^2} - \frac{u}{r} + \frac{\mu}{a} + m'\,.\left\{\frac{d^2\,.\,r\,\delta\,r}{d\,t^2} + \frac{\mu\,.\,r\,\delta\,r}{r^3} + \mathrm{Q}\right\}$$

But since r represents the elliptic value of r, the first part of this equation vanishes of itself, and to determine $\delta\,r$ or $r\,\delta\,r$, we have the differential equation

$$o = \frac{d^2\,.\,r\,\delta\,r}{d\,t^2} + \frac{\mu}{r^3}\,.\,r\,\delta\,r + \mathrm{Q}\,; \tag{112}$$

This equation being linear, and of the second order, is immediately integrable by our general formula, equation (100) provided we can find the two particular integrals u' and u'' of the equation

$$\frac{d^2\,u}{d\,t^2} + \frac{\mu}{r^3}\,u = o$$

but since r, on the supposition of the term Q bearing zero, may be taken for the radius vector on the hypothesis of elliptic motion, it is obvious that the elliptic values of either x, or y, or z, will satisfy this equation, because these values are, in fact, no other than what are derived from the integration of equations precisely similar, viz.

$$\frac{d^2\,x}{d\,t^2} + \frac{\mu}{r^3}\,x = o$$

$$\frac{d^2\,y}{d\,t^2} + \frac{u}{r^3}\,y = o$$

$$\frac{d^2\,z}{d\,t^2} + \frac{u}{r^3}\,z = o$$

Consequently, we may take $u' = x$ and $u'' = y$, whence we get $u''\,\frac{d\,u'}{u''} = \frac{y\,d\,x - x\,d\,y}{y} = \frac{h\,d\,t}{y}$ because on the hypothesis of elliptic motion $y\,d\,x - x\,d\,y = h\,d\,t$, and it is of the elliptic values of x and y that we are now speaking. Similarly, $u'\,d\,\frac{u''}{u'} = \frac{x\,d\,y - y\,d\,x}{x} = -\,\frac{h\,d\,t}{x}$, and the formula (100) gives

$$\mathrm{C}'\,.\,x + \mathrm{C}''\,.\,y - x \int \frac{\mathrm{Q}\,y\,d\,t}{h} + y \int \frac{\mathrm{Q}\,x\,d\,t}{h}$$

for the complete value of u or $r\,\delta\,r$. The constants then being included under the integral sign, we have

$$r\,\delta\,r = \frac{y\int \mathrm{Q}\,x\,d\,t - x\int \mathrm{Q}\,y\,d\,t}{h}\,; \tag{113}$$

Such is the value of $r\,\delta\,r$ when we consider only the first power of the disturbing force. It would be absolutely exact, but that x and y are only particular values of u on the hypothesis of r having its elliptic value. If this supposition were not made, we should have $u' = x + m'\,\delta\,x$ and $u'' = y + m'\,\delta\,y$; and substituting these values, we should obtain terms in the expression of $m'\,\delta\,r$ depending on the square of the disturbing force, but with these we have no concern.

The perturbation in longitude ($m'\,\delta\,\theta$) is easily obtained when the value of $\delta\,r$ is found. In fact, we have

$$d\,x^2 + d\,y^2 + d\,z^2 = d\,r^2 + r^2\,d\,\theta^2$$

whence we get

$$o = \frac{r^2\,d\,\theta^2 + d\,r^2}{d\,t^2} - \frac{2\,\mu}{r} + \frac{\mu}{a} + 2\,m'\int \mathrm{d}\,\Omega \tag{114}$$

and if we subtract this from the equation (111), we find

$$o = \frac{r^2\,d\,\theta^2}{d\,t^2} - \frac{r\,d^2\,r}{d\,t^3} - \frac{\mu}{r} - m'\,.\,r\,\frac{d\,\Omega}{d\,r} \tag{115}$$

If in this we substitute $r + m'\,\delta\,r$ for r and $\theta + m'\,\delta\,\theta$ for θ we get, (after obliterating the terms which destroy each other by reason of the properties of elliptic motion, and those which contain m'^2)

$$o = \frac{2\,r^2\,d\,\theta\,d\,\delta\,\theta}{d\,t^2} + \frac{2\,r\,\delta\,r\,d\,\theta^2}{d\,t^2} - \frac{r\,d^2\,\delta\,r}{d\,t^2} - \frac{d^2\,r\,.\,\delta\,r}{d\,t^2} + \frac{\mu\,r\,\delta\,r}{r^3} - r\,\frac{d\,\Omega}{d\,r}\,; \tag{116}$$

From this, let the term multiplied by $\frac{d\,\theta^2}{d\,t^2}$ be eliminated by means of equation (115) and we get

$$\frac{2\,r^2\,d\,\theta\,d\,\delta\,\theta}{d\,t^2} = \frac{r\,d^2\,\delta\,r - \delta\,r\,.\,d^2\,r}{d\,t^2} - \frac{3\,\mu\,r\,\delta\,r}{r^3} + r\,\frac{d\,\Omega}{d\,r}\,; \tag{117}$$

And if in this we substitute for $\frac{\mu\,r\,\delta\,r}{r^3}$ its value given by the equation (112) we obtain, (restoring the value of Q)

$$d\,\delta\,\theta = \frac{d\,\{\,d\,r\,\delta\,r + 2\,r\,d\,\delta\,r\} + \left(3\int \mathrm{d}\,\Omega + 2\,r\,\frac{d\,\Omega}{d\,r}\right).\,d\,t^2}{r^2\,d\,\theta}\,; \tag{118}$$

but $r^2\,d\,\theta = h\,d\,t$, elliptic values only being considered in the second member of this equation, and consequently integrating, we have

$$h\,\delta\,\theta = \frac{d\,r}{d\,t}\,\delta\,r + 2\,r\,\frac{d\,\delta\,r}{d\,t} + \int\left(3\int \mathrm{d}\,\Omega + 2\,r\,\frac{d\,\Omega}{d\,r}\right)d\,t\,; \tag{119}$$

Now, we have $h = \sqrt{\mu\,a\,(1 - e^2)}$, and if we put $n\,t$ for the mean motion of the disturbed planet m, we have $n = \sqrt{\frac{\mu}{a^3}}$ so that $h = n\,a^2\,.\,\sqrt{1 - e^2}$; and $\frac{1}{h} = \frac{n\,a}{\mu\,\sqrt{1 - e^2}}$

Consequently we get, for the perturbation of the radius vector,

$$m'\,\delta\,r = \frac{m'\,a\left\{\cos\theta\int r\,.\,\sin\theta\,.\,\mathrm{Q}\,n\,d\,t - \sin\theta\int r\,.\,\cos\theta\,.\,\mathrm{Q}\,n\,d\,t\right\}}{\mu\,\sqrt{1 - e^2}} \tag{120}$$

and the formula expressing the perturbation in longitude, will become

$$m'\,\delta\,\theta = \frac{m'}{n\,a^2\,\sqrt{1 - e^2}}\left(\frac{d\,r}{d\,t}\,\delta\,r + 2\,r\,\frac{d\,\delta\,r}{d\,t}\right)$$

$$+ \frac{a\,m'}{\mu\,\sqrt{1 - e^2}}\left\{2\int r\,\frac{d\,\Omega}{d\,r}\,.\,n\,d\,t + 3\iint \mathrm{d}\,\Omega\,.\,n\,d\,t\right\}; \tag{121}$$

It only remains to determine the amount of the perturbation in latitude, or the value of z or of $\delta\,s$, if we put $z = r\,\delta\,s$, in which case $\delta\,s$ represents the tangent of the heliocentric latitude of m in its disturbed orbit above the plane of its elliptic motion. Now, if we treat the equation

$$o = \frac{d^2\,z}{d\,t^2} + \frac{\mu\,z}{r^3} + m'\,\frac{d\,\Omega}{d\,z}$$

in the same manner as the equation by which the value of $r\,\delta\,r$ was found, viz. regarding x and y as two particular integrals of the equation $\frac{d^2\,z}{d\,t^2} + \frac{\mu\,z}{r^3} = o$, and then completing the integration by the general formula (100)* we shall find (putting $r \,.\, \cos\theta$ and $r \,.\, \sin\theta$, for x and y)

$$m'\,\delta s = \frac{m'\,a\left\{\cos\theta\int y\,\frac{d\,\Omega}{d\,z}\,.\,n\,d\,t - \sin\theta \,:\int x\,\frac{d\,\Omega}{d\,z}\,.\,n\,d\,t\right\}}{\mu\sqrt{1-e^2}} \tag{122}$$

This is the latitude of m above its primitive orbit; and if we denote by s its undisturbed latitude above any fixed plane (as that of the ecliptic) slightly inclined to this orbit, $s + \delta s$ will be its latitude when subjected to the action of the disturbing forces.

The equation (121) gives the perturbation in longitude when that of the radius vector is known, and the latter may be computed from the expression (120) which is general; and considering the complication of the subject, as simple as can be expected. Its form enables us to compute the amount of perturbation even in the most difficult cases, as in that of a comet, by the application of the method of quadratures. Meanwhile, in the theory of the planets, where it is required to develope the value of δr in series of sines and cosines of arcs depending on the configurations of the disturbed and disturbing planet, it will be found much simpler to set out immediately from the differential equations for the disturbed radius, and proceed in the manner now to be explained.

Since the form of this equation is not precisely that of the equation $\frac{d^2 u}{d t^2} + n^2 u + \Pi = o$, the co-efficient of the second term instead of being constant, being $\frac{\mu}{r^3}$ a variable quantity, we must first endeavour to transform it by substitution into one of this form. Assuming then that u is such a quantity that its elliptic value shall satisfy the equation $\frac{d^2 u}{d t^2} + n^2 u = o$, and its disturbed value (or $u + m'\,\delta u$) the equation $\frac{d^2 (u + m'\,\delta u)}{d t^2} + n^2 (u + m'\,\delta u) + m'\,\Pi = o$, which gives $\frac{d^2\,\delta u}{d t^2} + n^2\,\delta u + \Pi = o$ we must inquire first, the relation between r and u; and secondly, the value of Π.

A satisfactory relation between r and u is easily found. In fact, since u is to satisfy $\frac{d^2 u}{d t^2} + n^2 u = o$, it must be of the form $u = \text{const.} \cos(n\,t + \text{const.})$. Now the developement of r in terms of the mean longitude gives, putting ϵ for the longitude at the epoch of the commencement of the time t, and π for the longitude of the perihelion,

$$r = a\left\{1 - e\,.\,\cos(n\,t + \epsilon - \pi) + e^2 \sin^2 (n\,t + \epsilon - \pi) + \&c.\right\}$$

So that, if we take $u = e\,.\,\cos(n\,t + \epsilon - \pi)$, we shall have

$$r = a\left\{(1 + e^2) - u\,(1 - \tfrac{3}{2} e^2) - u^2 + \&c.\right\} \tag{123}$$

This gives at once

$$\begin{aligned} \delta r &= -\,a\,\delta u\left\{1 + 2\,u + e^2 \times \&c.\right\} \\ &= -\,a\,\delta u\left\{1 + 2\,e\,.\,\cos(n\,t + \epsilon - \pi) + e^2\,.\,\&c.\right\}; \end{aligned} \tag{124}$$

by which, when δu is found, δr may be had at once.

It only remains to discover Π. Now, since our equation (111), if we put $r^2 = v$ and $m'\left\{2\int d\,\Omega + r\,\frac{d\,\Omega}{d\,r}\right\} = m'\,Q$, becomes

$$\frac{d^2 v}{d t^2} = \frac{2\,\mu}{\sqrt{v}} - \frac{2\,\mu}{a} - 2\,m'\,Q \tag{125}$$

if we multiply by $2\,d\,v$, and integrate, we shall find

$$\left(\frac{d\,v}{d\,t}\right)^2 = 8\,\mu\sqrt{v} - \frac{4\,\mu\,v}{a} - 4\,m'\int Q\,d\,v \tag{126}$$

but because u is a function of r, and therefore of r^2 or v. we have

$$\frac{d\,u}{d\,t} = \frac{d\,u}{d\,v}\,.\,\frac{d\,v}{d\,t}$$

$$\frac{d^2 u}{d t^2} = \frac{d^2 u}{d v^2}\left(\frac{d\,v}{d\,t}\right)^2 + \frac{d\,u}{d\,v}\,.\,\frac{d^2 v}{d t^2}$$

so that

$$\begin{aligned} \frac{d^2 u}{d t^2} + n^2 u = {} & \frac{d^2 u}{d v^2}\left\{8\,\mu\sqrt{v} - \frac{4\,\mu\,v}{a} - 4\,m'\int Q\,d\,v\right\} \\ & + \frac{d\,u}{d\,v}\left\{\frac{2\,\mu}{\sqrt{v}} - \frac{2\,\mu}{a} - 2\,m'\,Q\right\} + n^2 u \end{aligned}$$

Now u is a certain function of v (or of r) whose form is determined by the reversion of the series in (123) and is independent of the disturbing forces. But were these forces zero, we should have $\frac{d^2 u}{d t^2} + n^2 u = o$.

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* This formula, which perhaps is new, and which has stood us in some stead in the explanation of that chapter of the *Mécanique Céleste*, (cap. vi. liv. 2.) which I have adopted for the groundwork of this part of the present essay, may be deduced at once from the general theory of linear equations, in my paper *On various points of Analysis*.—*Phil. Trans.* 1814.

Hence the portion of the right hand member of the equation just deduced, which does not depend on the disturbing forces, must be identically zero, in virtue of the relation between u and v; and that it is so, we may assure ourselves by actual substitution. Consequently we must have, when the disturbing forces are regarded,

$$\frac{d^2 u}{d t^2} + n^2 u = \left(- 2 Q \frac{d u}{d v} - 4 \int Q\, d v \times \frac{d^2 u}{d v^2}\right) \times m'$$

that is (putting $u + m' \delta u$ for u, and disregarding terms depending on m'^2)

$$\frac{d^2 \delta u}{d t^2} + n^2 \delta u = - 2 Q \frac{d u}{d v} - 4 \frac{d^2 u}{d v^2} \int Q\, d v$$

Comparing this with the equation $\frac{d^2 \delta u}{d t^2} + n^2 \delta u + \Pi = o$ we have

$$\Pi = 2 Q \, . \frac{d u}{d v} + 4 \frac{d^2 u}{d v^2} \int Q\, d v$$

It is desirable to express this in terms of v. Now, as u is a function of v, v is reciprocally a function of u, and

$$\frac{d u}{d v} = \frac{1}{\frac{d v}{d u}} ; \quad \frac{d^2 u}{d v^2} = - \frac{\frac{d^2 v}{d u^2}}{\left(\frac{d v}{d u}\right)^3}$$

and

$$\Pi = \frac{2}{\frac{d v}{d u}} Q - 4 \frac{\frac{d^2 v}{d u^2}}{\left(\frac{d v}{d u}\right)^3} \int Q\, d v \qquad (127)$$

but since $v = r^2$, if for r we put its value in terms of u (123), we get, neglecting higher powers of u than the first, $v = a^2 (1 - 2 u + \&c.)$ and substituting this in Π, and after the differentiations writing for u its value $e \, . \cos (n t + \epsilon - \pi)$ we get

$$\Pi = - \frac{Q}{a^2}\left(1 - e \, . \cos (n t + \epsilon - \pi) \right) - \frac{1}{4} e^2 . \cos (2 n t + 2 \epsilon - 2 \pi)$$
$$- \frac{2 e}{a^2} \int Q\, n\, d t \, . \sin (n t + \epsilon - \pi) \left\{1 + e \, . \cos (n t + \epsilon - \pi)\right\} \qquad (128)$$

and this being the value of Π, we find δu from the equation

$$\frac{d^2 \delta u}{d t^2} + n^2 \delta u + \Pi = o. \qquad (129)$$

SECTION III.

Reduction of the perturbative function Q *or* $2 \int d \Omega + r \frac{d \Omega}{d r}$ *to a series of sines and cosines, and investigation of the perturbations, neglecting the eccentricities and inclinations of both orbits.*

WE have now reduced the investigation of the perturbation to the integration of the linear equation (129) and we have before seen that this is accomplished without difficulty, when Π, the last term, is reducible to sines and cosines of the independent variable and its multiples. All then that remains to be done to get their actual expressions, is to execute this reduction. This, however, is by no means a simple process; and in an essay like the present it is not possible to pursue it into all its details: we shall therefore only carry it to a certain extent necessary for our future reference, and point out the principles by which it may be, if required, carried farther.

Let us consider then the value of the function Ω. If we write in it $r \, . \cos \theta$, $r \, . \sin \theta$, $r' \, . \cos \theta'$ and $r' \, . \sin \theta'$ for x, y, x', y', and neglect zz', z'^2, $(z - z')^2$, which are either of the order of the squares of the disturbing forces, or of the products of these forces by the mutual inclination of the orbits, and put $\theta - \theta' = w$, we get

$$x x' + y y' + zz' = rr' \, . \cos w$$

$$\lambda = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \sqrt{r^2 - 2 r r' \, . \cos w + r'^2}$$

and

$$\Omega = \frac{r}{r'^2} \, . \cos w - \frac{1}{\sqrt{r^2 - 2 r r' \, . \cos w + r'^2}}$$

Let us conceive this function developed in a series of cosines of w, and its positive and negative multiples to infinity; then, since the cosines of the negative are equal to those of the positive multiples, we may represent Ω as follows:

$$\Omega = R + R' \, . \cos w + R'' \, . \cos 2 w + R''' \, . \cos 3 w + \&c.; \qquad (130)$$

where R, R′, R″, &c. are certain given, explicit functions of r and r' and of these alone, depending solely on the peculiar form of Ω, let A, A′, A″, &c. represent the same functions of a and a'. Then, if the eccentricities of the orbits were nothing, we should have

$$\left.\begin{aligned}\Omega &= \frac{a}{a'^2}\,.\cos w - \frac{1}{\sqrt{a^2 - 2\,a\,a'\,.\cos w + a'^2}}\\ &= \mathrm{A} + \mathrm{A}'\,.\cos w + \mathrm{A}''\,.\cos 2\,w + \&c.\end{aligned}\right\}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (131)$$

Were the eccentricities nothing, the orbits would be circles, and the motion in them uniform. We should therefore have $\theta = n\,t$ and $\theta' = n'\,t$, whence $w = \theta - \theta' = (n - n')\,t$, so that Ω in this case would be expressed in the very simple series

$$\Omega = \mathrm{A} + \mathrm{A}'\,.\cos\,(n - n')\,t + \mathrm{A}''\,.\cos 2\,(n - n')\,t + \&c. \qquad (132)$$

Moreover, since $\mathrm{d}\,\Omega$ represents the differential of Ω taken on the hypothesis that only the disturbed body moves, we should then have

$$\mathrm{d}\,\Omega = \frac{d\,\Omega}{d\,r}\,d\,r + \frac{d\,\Omega}{d\,\theta}\,d\,\theta = \frac{d\,\Omega}{d\,\theta}\,d\,\theta = n\,d\,t\,.\frac{d\,\Omega}{d\,\theta}$$

because, in the case of circular orbits $d\,r = o$. Thus we should have

$$\mathrm{d}\,\Omega = -\,n\,d\,t\,.\left\{\mathrm{A}'\,.\sin\,(n - n')\,t + 2\,.\,\mathrm{A}''\,.\sin\,(2\,n - 2\,n')\,t + \&c.\right\}$$

and integrating relative to t

$$\int \mathrm{d}\,\Omega = \frac{g}{a} + \frac{n}{n - n'}\left\{\mathrm{A}'\,.\cos\,(n - n')\,t + \mathrm{A}''\,.\cos 2\,(n - n')\,t + \&c.\right\}; \qquad (133)$$

$\frac{g}{a}$ being an arbitrary constant.

Again, since R, R′, R″, &c. are explicit functions of r, r', and Ω is only so far a function of r, as this symbol is contained in them, we must have

$$r\,\frac{d\,\Omega}{d\,r} = r\,\frac{d\,\mathrm{R}}{d\,r}\,. + r\,\frac{d\,\mathrm{R}'}{d\,r}\,.\cos w + \&c. \qquad (134)$$

and in the case of circular orbits, denoting by $\frac{d\,\mathrm{A}}{d\,a}$, &c. the same functions of a, a' that $\frac{d\,\mathrm{R}}{d\,r}$ denote of r, r',

$$r\,\frac{d\,\Omega}{d\,r} = a\,\frac{d\,\mathrm{A}}{d\,a} + a\,\frac{d\,\mathrm{A}'}{d\,a}\,.\cos w + \&c.$$

The values of $\frac{d\,\mathrm{A}}{d\,a}$, $\frac{d\,\mathrm{A}'}{d\,a}$, &c. are easily had in functions of a, a', when those of A, A′, &c. are found. Now, to obtain these, we may proceed as follows:

Take $c = \cos w + \sqrt{-1}\,.\sin w$. Then (by trigonometry) we shall have $\frac{1}{c} = \cos w - \sqrt{-1}\,.\sin w$; now, let us consider the function $(a^2 - 2\,a\,a'\,.\cos w + a'^2)^{-s}$ which agrees with the second term of Ω if $s = \frac{1}{2}$. This equals $a^{-2s}\left(1 - 2\,\frac{a'}{a}\,.\cos w + \left(\frac{a'}{a}\right)^2\right)^{-s}$ or $a^{-2s}\,(1 - 2\,\alpha\,.\cos w + \alpha^2)^{-s}$ putting $\alpha = \frac{a'}{a}$ But we have

$$1 - 2\,\alpha\,.\cos w + \alpha^2 = (-\,\alpha\,c)\left(1 - \alpha\,.\frac{1}{c}\right)$$

because $c + \frac{1}{c} = 2\,.\cos w$. Hence

$$(a^2 + 2\,a\,a'\,.\cos w + a'^2)^{-s} = a^{-2s}\,.\,(1 - \alpha\,c)^{-s}\,.\left(1 - \frac{\alpha}{c}\right)^{-s}$$

$$= a^{-2s} \times \left\{1 + \frac{s}{1}\,.\,\alpha\,c + \frac{s\,(s+1)}{1\,.\,2}\,\alpha^2\,c^2 + \&c.\right\} \times \left\{1 + \frac{s}{1}\,.\,\frac{\alpha}{c} + \frac{s\,(s+1)}{1\,.\,2}\,.\,\frac{\alpha^2}{c^2} + \&c.\right\}$$

$$= a^{-2s}\left\{1 + \left(\frac{s}{1}\right)^2.\,\alpha^2 + \left(\frac{s\,(s+1)}{1\,.\,2}\right)^2\alpha^4 + \&c.\right\}$$

$$+\, a^{-2s}\left\{\frac{s}{1}\,\alpha + \frac{s\,(s+1)}{1\,.\,2}\,.\,\frac{s}{1}\,\alpha^3 + \frac{s\,(s+1)\,(s+2)}{1\,.\,2\,.\,3} \times \frac{s\,(s+1)}{1\,.\,2}\,\alpha^5 + \&c.\right\}\left(c + \frac{1}{c}\right)$$

$$+\, a^{-2s}\left\{\frac{s\,(s+1)}{1\,.\,2}\,\alpha^2 + \frac{s\,(s+1)\,(s+2)}{1\,.\,2\,.\,3} \times \frac{s}{1}\,.\,\alpha^4 + \&c.\right\}\left(c^2 + \frac{1}{c^2}\right)$$

$$+ \&c.$$

but $c + \frac{1}{c} = 2\cos w$; $c^2 + \frac{1}{c^2} = 2\,.\cos 2\,w$, &c. (by trigonometry); consequently we have

$$(a^2 - 2\,a\,a'.\cos w + a'^2)^{-s} = a^{-2s}\left\{1 + \left(\frac{s}{1}\right)^2.\,a^2 + \&c.\right\} \tag{135}$$
$$+2a^{-2s}\left\{\frac{s}{1}\,a + \frac{s\,(s+1)}{1.2}\,.\,\frac{s}{1}\,a^3 + \&c.\right\}.\cos w$$
$$+2\,a^{-2s}\left\{\frac{s\,(s+1)}{1.2}\,a^2 + \&c.\right\}.\cos 2\,w + \&c.$$

In the case before us, where $s = \frac{1}{2}$, this gives at once

$$A = \frac{-1}{a}\left\{1 + \left(\frac{1}{2}\right)^2 a^2 + \left(\frac{1.3}{2.4}\right)^2 a^4 + \&c.\right\} \tag{136, 1}$$
$$A' = \frac{-2}{a}\left\{\frac{1}{2}\,a + \frac{1^2.3}{2^2.4}\,a^3 + \frac{1^2.3^2.5}{2^2.4^2.6}\,a^5 + \&c.\right\} + \frac{a}{a'^2} \tag{136, 2}$$
$$A'' = \frac{-2}{a}\left\{\frac{1.3}{2.4}\,a^2 + \frac{1^2.3.5}{2^2.4.6}\,a^4 + \&c.\right\} \tag{136, 3}$$
&c.

If a be less than unity, these series are convergent; but if greater, we have only to throw the expression into the form $a'^{-2s}\left(1 - 2\frac{a}{a'}.\cos w + \left(\frac{a}{a'}\right)^2\right)^{-s}$ previous to developement, and taking $a = \frac{a}{a'}$, instead of $\frac{a'}{a}$, a will now be less than 1, and we shall have $A = \frac{1}{a'}\left(1 + \left(\frac{1}{2}\right)^2 a^2 + \&c.\right)$ &c.

In the former case, when $\frac{a}{a'}$ is less than 1, or the orbit of the disturbed planet is *interior* to that of the disturbing, we have

$$-A = \frac{1}{a} + \left(\frac{1}{2}\right)^2.\frac{a'^2}{a^3} + \left(\frac{1.3}{2.4}\right)^2.\frac{a'^4}{a^5} + \&c.; \tag{137, 1}$$
$$\frac{d\,A}{d\,a} = +\frac{1}{a^2}\left\{1 + 3.\left(\frac{1}{2}\right)^2.\left(\frac{a'}{a}\right)^2 + \&c.\right\}; \tag{137, 2}$$
$$A' = -2\left\{\frac{1}{2}\,\frac{a'}{a^2} + \frac{1^2.3}{2^2.4}.\frac{a'^3}{a^4} + \&c.\right\} + \frac{a}{a'^2}; \tag{137, 3}$$
$$\frac{d\,A'}{d\,a} = +\frac{2}{a^2}\left\{1.\frac{a'}{a} + \frac{1^2.3}{2^2}.\left(\frac{a'}{a}\right)^3 + \&c.\right\} + \frac{1}{a'^2} \tag{137, 4}$$

and so on; and similar expressions are also readily obtained in the case when the orbit of the disturbed planet is the exterior. Thus, when the mean distances of the two planets are given, the values of A, A′, &c. and their differential co-efficients $\frac{d\,A}{d\,a}$, &c. are reducible to numerical evaluation, and may therefore be regarded as known quantities. The properties of the series on which they depend, afford many resources for facilitating their evaluation, and rules for deriving one of these quantities from another, but these we shall not stay to explain. The reader will find them with every developement in the second book of Laplace's *Mécanique Céleste*, art. 49.

These values once determined, we have Q, or $2\int d\,\Omega + r\,\frac{d\,\Omega}{d\,r}$, expressed as follows:

$$\left.\begin{aligned} Q = \frac{2g}{a} + a\,\frac{d\,A}{d\,a} + \left\{a\,\frac{d\,A'}{d\,a} + \frac{2\,n\,A'}{n-n'}\right\}\cos\,(n-n')\,t \\ + \left\{a\,\frac{d\,A''}{d\,a} + \frac{2\,n\,A''}{n-n'}\right\}\cos 2\,(n-n')\,t + \&c. \end{aligned}\right\}\ldots\ldots\ldots \tag{138, 1}$$

Such is the value of the perturbative function when the eccentricities and inclinations of the orbits are neglected. Let us, for the present, confine ourselves to this case; and, writing M, M′, M″, &c. for the successive co-efficients, we have

$$Q = M + M'.\cos\,(n-n')\,t + M''.\cos 2\,(n-n')\,t + \&c. \tag{138, 2}$$

Now the equation (128) gives, when $e = o$,

$$\Pi = -\frac{Q}{a^2} = -n^2\,a\,Q\ \left(\text{because } n^2 = \frac{1}{a^3}\right) \text{ or,}$$
$$\Pi = -n^2\,a\left\{M + M'.\cos\,(n-n')\,t + \&c.\right\}$$

so that the differential equation in δu becomes

$$o = \frac{d^2\,\delta u}{d\,t^2} + n^2\,\delta u - n^2\,a\left\{M + M'.\cos\,(n-n')\,t + \&c.\right\} \tag{139}$$

and integrating,

$$\delta u = a\,\mathrm{M} - \frac{n^2\,a\,\mathrm{M}'}{(n-n')^2 - n^2}\,.\cos (n-n')\,t - \frac{n^2\,a\,\mathrm{M}''}{4\,(n-n')^2 - n^2}\cos 2\,(n-n')\,t - \&c.$$

Consequently, since by equation (124) $\delta r = -\,a\,\delta u$ when $e = o$,

$$\delta r = -\,a^2\,.\,\mathrm{M} + \frac{n^2\,a^2\,\mathrm{M}'}{(n-n')^2 - n^2}\,.\cos\,(n-n')\,t - \frac{n^2\,a^2\,\mathrm{M}''}{4\,(n-n')^2 - n^2}\,.\cos 2\,(n-n')\,t + \&c. \qquad (140)$$

The value of δr thus obtained, $\delta \theta$ is easily got from (121); for in this case $\frac{d\,r}{d\,t} = o$,

$$2\,r\,\frac{d\,\delta\,r}{d\,t} = -\,2\,n^2\,a^3\,(n-n')\,.\,\Sigma\,\frac{i\,\mathrm{M}^i}{i^2\,(n-n')^2 - n^2}\,.\sin i\,w$$

where Σ is used to express the sum of all similar terms from $i = 1$ to $i = \infty$ inclusive, and M^i represents the i^{th} in order of the co-efficients M', M'', M''', &c. Moreover

$$\int r\,\frac{d\,\Omega}{d\,r}\,n\,d\,t = a\,\frac{d\,\mathrm{A}}{d\,a}\,.\,n\,t + \Sigma\,\frac{a\,n}{i\,(n-n')}\,\frac{d\,\mathrm{A}^i}{d\,a}\,.\sin i\,w$$

$$\int n\,d\,t\int \mathrm{d}\,\Omega = \frac{g}{a}\,.\,n\,t + \frac{n^2}{(n-n')^2}\,\Sigma\,\frac{\mathrm{A}^i}{i}\sin i\,w$$

Uniting therefore these several parts, we get

$$\delta\,\theta = \frac{1}{n\,a^2} \times 2\,r\,\frac{d\,\delta\,r}{d\,t} + 2\,a\int r\,\frac{d\,\Omega}{d\,r}\,.\,n\,d\,t + 3\,a\int n\,d\,t\int \mathrm{d}\,\Omega$$

$$= \left(3\,g + 2\,a^2\,\frac{d\,\mathrm{A}}{d\,a}\right) \times n\,t$$

$$+\,\Sigma\left\{\frac{3\,a\,n^2\,\mathrm{A}^i}{i\,(n-n')^2} + \frac{2\,a^2\,n}{i\,(n-n')}\cdot\frac{d\,\mathrm{A}^i}{d\,a} - \frac{2\,n\,a\,(n-n')\,i}{i^2\,(n-n')^2 - n^2}\,\mathrm{M}^i\right\}\sin i\,w$$

Now, first, since $n\,t$ represents the mean longitude of m as deduced from observation, the quantity n is already affected with the whole influence of the planetary perturbation, and consequently the part multiplied by $n\,t$ in this expression of the perturbation in longitude, and which, if allowed to remain, would express an additional perturbation, is superfluous. This furnishes the condition

$$g = -\,\frac{2}{3}\,a^2\,\frac{d\,\mathrm{A}}{d\,a}; \qquad (141)$$

which determines the constant g. Moreover, in the latter part of this expression, if we write for M^i its value

$$\mathrm{M}^i = a\,.\,\frac{d\,\mathrm{A}^i}{d\,a} + \frac{2\,n\,\mathrm{A}^i}{n-n'}$$

it will admit reductions, and the value of $\delta\,\theta$ will at length be found as follows:

$$\delta\,\theta = \Sigma\left\{\frac{2\,n^3\,a^2\,.\,\frac{d\,\mathrm{A}^i}{d\,a}}{i\,(n-n')\,(i^2\,(n-n')^2 - n^2)} + \frac{(3\,n^2 + i^2\,(n-n')^2)\,.\,n^2\,a\,\mathrm{A}^i}{i\,(n-n')^2\,(i^2\,(n-n')^2 - n^2)}\right\}\sin i\,w \qquad (142)$$

If these expressions of δr and $\delta \theta$ be each multiplied by m', we have the values of $m'\,\delta\,r$ and $m'\,\delta\,\theta$ the perturbations of the radius vector and the longitude, *i. e.* those parts of them which are independent of the eccentricities of the orbits. These expressions give room for some remarks. The perturbation in longitude as we observe is wholly periodical and dependent on a single angle w and its multiples. In forming then a table of the values of $m'\,\delta\,\theta$, the numerical co-efficients being computed, and the value of $m'\,\delta\,\theta$ thus reduced to the form $p\,.\sin w + q\,.\sin 2\,w + \&c.$ we may include the whole of this in one column, entered under the general argument w, instead of regarding it as consisting of an infinite number of separate inequalities.

The same remark extends to the periodical part of $m'\,\delta\,r$, its arguments are the same as in the formula for $m'\,\delta\,\theta$; but besides this, $\delta\,r$ includes, as we have seen, a constant part $-\,a^2\,.\,\mathrm{M}$ or $-\,2\,g\,a - a^3\,.\,\frac{d\,\mathrm{A}}{d\,a}$ which becomes by substituting for g its value in (141)

$$\text{const. part of } \delta\,r = \frac{1}{3}\,a^3\,.\,\frac{d\,\mathrm{A}}{d\,a},$$

and therefore

$$\delta\,r = \frac{a^3}{3}\,.\,\frac{d\,\mathrm{A}}{d\,a} + n^2\,a^2\,.\,\Sigma\,\frac{1}{i^2\,(n-n')^2 - n^2}\left(a\,\frac{d\,\mathrm{A}^i}{d\,a} + \frac{2\,n\,\mathrm{A}^i}{n-n'}\right)\cos i\,w; \qquad (143)$$

In the formation of a table of $m'\,\delta\,r$ this constant part is of course included with the variable one, but the effect is remarkable. It appears that the action of the disturbing planet alters the mean distance from the sun of the disturbed, and, of course, its mean motion and periodical time from what they would have been had the disturbing planet no existence. At the same time, it will be demonstrated in the following pages, that these alterations once produced, are permanent and unchangeable in their quantity by the subsequent actions of the bodies composing the system.

The angle w is the difference of longitudes of the two planets, or their heliocentric elongation from each other. If we call ϵ and ϵ' their epochs, or their actual longitudes at the commencement of the time t, we have

$$\begin{aligned} w &= n\,t + \epsilon - (n'\,t + \epsilon') \\ &= n\,t - n'\,t + \epsilon - \epsilon' \end{aligned}$$

and this, in fact, is the argument of the perturbations when we neglect the eccentricities and inclinations of the orbits.

Let us next examine the perturbation in latitude, we have

$$\frac{d\,\Omega}{d\,z} = \frac{z'}{r'^3} - \frac{z' - z}{\lambda^3}$$

Hence, it is evident, that if we neglect the inclinations and eccentricities, we have $\frac{d\,\Omega}{d\,z} = o$, and the plane of the disturbed orbit does not change.

We have thus determined the effect of the action of a third body on the orbit and motion of m, on the simplest supposition, and our results (to recapitulate them) amount to this.

1st. That the radius vector undergoes a permanent change in its mean value, and, of course, that the period and mean motion of m are permanently altered.

2d. That the elliptic value of the radius vector receives an accession of terms, of the form

$$p + q\,.\cos w + r\,.\cos 2\,w + s\,.\cos 3\,w + \&\text{c.}$$

and that of the true longitude, a series of terms of the form

$$q'\,.\sin w + r'\,.\sin 2\,w + s'\,.\sin 3\,w + \&\text{c.}$$

w being the difference of longitudes, or mutual elongation of the planets, from each other.

3dly. That to express the several co-efficients of these formulæ in numbers, nothing more is required than a knowledge of the mass of the disturbing planet, and the mean distances and mean motions of both.

In the cases then where the disturbing planet has satellites, as in those of Jupiter, Saturn, and Uranus, the mass is known, and the reduction of the formulæ to numbers is complete. It is fortunate that these are by far the most considerable bodies of our system, but proximity to a certain extent supplies the place of intrinsic energy; and, in the case of the perturbations of the earth, our uncertainty of the masses of Mars and Venus leaves us in some degree at a loss. But physical astronomy furnishes us in this dilemma with considerable aid. Regarding the masses of these planets as unknown quantities, we may calculate in general terms their effect, either on the places of the other planets at assigned instants, or, on the elements of their orbits themselves, which are susceptible of much more accurate determination, by bringing a long series of observations to bear on them, and comparing the variations in their values after long intervals, as computed by theory, and as given by observation, we obtain data for the determination of these delicate quantities, so much the more precise as the variations observed in the elements are greater, or, in other words, as the interval of time in which they are observed is longer. It is thus that the lapse of ages is necessary to give precision to the numerical data of our system, and that continual and patient observation must ultimately lead us to the knowledge of points which elude the direct cognizance of our senses, and defy any investigation but those in which successive generations of mankind bear a part.

In fact, if we regard the masses of all the planets as unknown quantities, but their mean distances and periodic times as known ones;—the latter afford us the means of computing the values of $\delta\,r$ and $\delta\,\theta$ in any assigned case, independent of the value of m', which does not enter into their expressions. Let us therefore represent by $m'\,\delta'\,r$ and $m'\,\delta'\,\theta$, the perturbations of the radius vector and longitude produced by the planet m'; by $m''\,\delta''\,r$ and $m''\,\delta''\,\theta$ those produced by m'', and so on. Then will the true values of these quantities at any assigned instant be

$$r + m'\,\delta'\,r + m''\,\delta''\,r + m'''\,\delta'''\,r + \&\text{c.}$$

$$\theta + m'\,\delta'\,\theta + m''\,\delta''\,\theta + m'''\,\delta'''\,\theta + \&\text{c.}$$

in which $r, \delta'\,r, \delta''\,r$, &c. and $\theta, \delta'\,\theta, \delta''\,\theta$, &c. are quantities susceptible of calculation from theory. Suppose now that we construct tables of the values of $\theta, \delta'\,\theta, \delta''\,\theta$, &c. (or, as we will for a moment write these latter quantities, θ, ϕ, ψ, &c.) then, at any assigned instant, we have only to take out of these tables the values of θ, ϕ, ψ, &c.; and the true longitude of m will be

$$\theta + m'\,\phi + m''\,\psi + \&\text{c.} = \text{L}$$

Suppose now we compare this formula with a great multitude of observations, and thus obtain a series of equations,

$$\begin{aligned} m'\,.\,\phi_1 + m''\,.\,\psi_1 + m'''\,.\,\chi_1 + \&\text{c.} &= \text{L}_1 - \theta_1 \\ m'\,.\,\phi_2 + m''\,.\,\psi_2 + m'''\,.\,\chi_2 + \&\text{c.} &= \text{L}_2 - \theta_2 \\ \&\text{c.} &= \&\text{c.} \end{aligned}$$

The only unknown quantities in these will be the masses of the disturbing planets m', m'', &c. and by resolving these, we may determine their values, and thus estimate the masses of the planets by the perturbations they produce.

In this, as in almost all such delicate inquiries, where the quantities to be determined are exceedingly small, and the observations from which they are to be discovered liable to inaccuracies, bearing a sensible proportion to the thing observed, (which in this case is $L - \theta$, or the total perturbation arising from the united action of all the planets,) we are obliged to employ a great many more observations than would be, mathematically speaking, sufficient, if each were perfect, with a view to destroy the errors of observation in the mean result. The number of disturbing planets in our system at present known does not exceed ten; and it would therefore appear that ten observations of the longitude of one disturbed planet would enable us to determine the masses of all the rest; and so they would, were the observations mathematically exact, the elements of the orbits exactly known, and the theory by which the values of ϕ, ψ, χ, &c. are computed, complete. But each of these conditions is far from being fulfilled in the present state of astronomy; and if we would use this method to determine the masses of the planets, we must accumulate many hundreds of observations made, not on one, but on all of them, especially on those subject to the greatest perturbations.

The method of treating a series of equations more numerous than the unknown quantities they contain, so as to give them all their proper influence on the result, and obtain from them a set of values which, though satisfying neither of them separately, yet when substituted in all of them, shall, on the whole, give more satisfactory results than any other, depends on the theory of probabilities and may be found in Laplace's *Theorie Analytique des Probabilités.*

If the mass of any one or more of the planets (m') for instance, be regarded as sufficiently known from other methods, we need only employ this mode for the rest, and regarding the perturbation $m' \delta' \theta$ produced by it as known, place it on the known side of the equation, which will thus become

$$m'' \delta'' \theta + m''' \delta''' \theta + \&c. = L - \theta - m' \delta' \theta$$

Thus we may determine for instance, the masses of Mars and Venus, by means of an extensive series of observations of the sun's longitude, or (which is the same thing) by employing to that end the perturbations they produce on the earth. For the masses of Jupiter, Saturn, and Uranus, being known from the periods of their satellites; and those of Mercury, and the four new planets—Ceres, Pallas, Juno, and Vesta, being so small, as to have little or no influence, we have only two unknown quantities (m', m'') to determine.

This method is laborious, certainly; but considering the perfection of modern observations, the great multitude of them which may be brought to bear upon this point, and the considerable degree of exactness which the theory of the planetary perturbations has now attained, it is not impossible that it may one day be made to render the best service in determining the masses even of those planets which have satellites. At all events, it is highly desirable that it should be applied for that purpose, as its results would lead us to judge how far the latter method can be depended on in cases like that of Jupiter and Saturn, where the great deviation from sphericity of the central body renders the application of the formula $t = \frac{2\pi \times a^{\frac{3}{2}}}{\sqrt{M + m}}$ somewhat liable to error. In fact, this formula is derived on the hypothesis of a force represented by $\frac{M + m}{(\text{dist})^2}$ but in the case of spherical bodies only does the total attraction vary precisely in that ratio*. This alone, however, will not account for the great difference which Mr. Gauss has lately found between the mass of Jupiter, as obtained from observations of its satellites, and that deduced from the perturbations of the small planets intermediate between Jupiter and Mars, so that the subject must be regarded as open to further investigation, should the calculations of the last named eminent geometer be found to coincide with a more extensive series of observations of those interesting bodies than the shortness of the time they have been known has hitherto allowed.

SECTION IV.

Of the method of taking into account the effect of the eccentricities of the orbits on the planetary perturbations, and of the origin of the secular equations of their motions.

WHEN we regard the orbits as elliptical, the whole of the foregoing investigations require modification, the value of the perturbative function, and, of course, of the perturbations themselves, receiving accessions of terms depending on the powers and products of the eccentricities. We will here endeavour to explain the manner in which these terms originate; and to a certain (though limited) extent, the course pursued by geometers in determining their form and value.

The functions Ω, $\int d\,\Omega$, $r \frac{d\,\Omega}{d\,r}$, are explicitly given in terms of r, r', and w or $\theta - \theta'$, and contain no other symbols. Hence it arose, that when r, r' were supposed constant, the only cause of the variation of these functions consisted in that of w; and θ and θ' being in this case each expressed by an arc proportional to the time, it was sufficient to develope them in cosines and sines of w, to have at once the expression of the function Π in such a form as we required for integrating our equations. When, however, the eccentricities are introduced, all these facilities are at an end; r, r', and w, are no longer constant quantities and simple functions of the time, but each of them branches out into a series of powers of the eccentricities, and sines and cosines of variable arcs.

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* Laplace *(Theorie des Satellites de Jupiter*, p. 102.) makes the deviation of the attraction of the first, compared with the fourth satellite from the law of the inverse squares of the distances, only $\frac{1}{5000}$ of the whole attraction of the former, supposing Jupiter homogeneous. In Saturn, the attraction of the ring must cause a much more considerable deviation from that law.

Our object being to reduce Ω, &c. to sines and cosines of arcs proportional to the time, or of the form $A\,t + B$, it is evident that we must substitute for r, r', and w, their values so expressed, and then develope each term of Ω to the extent we wish. At present we will confine ourselves to the first powers of the eccentricities.

Now we have
$$\Omega = R + R' . \cos w + R'' . \cos 2\,w + \&c.$$
in which R, R′, R″, &c. and w are explicit functions of r, r', &c. and $w = \theta - \theta'$ putting θ and θ' for the true longitudes of the two planets. Now, if we call ϵ, and ϵ', the longitudes at the commencement of the time t, $n\,t + \epsilon$ and $n'\,t + \epsilon'$ will be their mean longitudes after the lapse of that time, and calling π and π' the longitudes of the perihelion, the mean anomalies will be $n\,t + \epsilon - \pi$ and $n'\,t + \epsilon' - \pi'$. Hence, the *true anomalies* will be (by equation 30),
$$n\,t + \epsilon - \pi + 2\,e . \sin (n\,t + \epsilon - \pi) + e^2 \times \&c.$$
$$n'\,t + \epsilon' - \pi' + 2\,e' . \sin (n'\,t + \epsilon' - \pi') + e^2 \times \&c.$$
and the true *longitudes* of course are
$$(n\,t + \epsilon) + 2\,e . \sin (n\,t + \epsilon - \pi) + e^2 \times \&c.$$
$$(n'\,t + \epsilon') + 2\,e' . \sin (n'\,t + \epsilon' - \pi') + e^2 \times \&c.$$
Hence, if we neglect the eccentricities, we have simply $w = (n\,t - n'\,t + \epsilon - \epsilon')$ and as R, R′, &c. in this case assume their circular values A, A′, &c., the terms of Ω not depending on the eccentricities will remain as before,
$$A + A' . \cos (n\,t - n'\,t + \epsilon - \epsilon') + A'' . \cos 2\,(n\,t - n'\,t + \epsilon - \epsilon') + \&c.$$
On the other hand, the terms depending on the eccentricities have their origin,

1st. In the developement of the functions R, R′, &c.; when, instead of r, r', we put their elliptic values,
$$r = a + \Delta\,r, \text{ and } r' = a' + \Delta\,r'$$
denoting by $\Delta\,r$ and $\Delta\,r'$ the parts of r, r' arising from the eccentricities.

2d. In the substitution of $W + \Delta\,w$ for w in $\cos w$, $\cos 2\,w$, &c. W being the part of w independent of the eccentricities, or
$$W = (n - n') . t + (\epsilon - \epsilon')$$
and $\Delta\,w$ being the part depending on them, or
$$\Delta\,w = 2\,e . \sin (n\,t + \epsilon - \pi) - 2\,e' . \sin (n'\,t + \epsilon' - \pi') + e^2 \times \&c. \&c.$$

3rd. In the multiplication of these terms together.

Now, if we still continue to denote by Δ that variation in Ω r, r', &c. which arises from the eccentricities, we have
$$\Delta\,\Omega = \frac{d\,\Omega}{d\,r}\,\Delta\,r + \frac{d\,\Omega}{d\,r'}\,\Delta\,r' + \frac{d\,\Omega}{d\,w}\,\Delta\,w$$
in which the differential co-efficients $\frac{d\,\Omega}{d\,r}$, $\frac{d\,\Omega}{d\,r'}$, $\frac{d\,\Omega}{d\,w}$, are to have their *circular* values. If we would pursue the investigation further, we must add to $\Delta\,\Omega$ the terms $\frac{d^2\,\Omega}{d\,r^2} . \frac{(\Delta\,r)^2}{1 . 2}$, &c.

Now, in general, we have
$$\frac{d\,\Omega}{d\,r} = \frac{d\,R}{d\,r} + \frac{d\,R'}{d\,r} . \cos w + \frac{d\,R''}{d\,r} . \cos 2\,w + \&c.$$
$$\frac{d\,\Omega}{d\,r'} = \frac{d\,R}{d\,r'} + \frac{d\,R'}{d\,r'} . \cos w + \frac{d\,R''}{d\,r'} . \cos 2\,w + \&c.$$
the circular values of which are respectively
$$\frac{d\,A}{d\,a} + \frac{d\,A'}{d\,a} . \cos W + \frac{d\,A''}{d\,a} . \cos 2\,W + \&c. \qquad (144, 1)$$
and
$$\frac{d\,A}{d\,a'} + \frac{d\,A'}{d\,a'} . \cos W + \frac{d\,A''}{d\,a'} . \cos 2\,W + \&c. \qquad (144, 2)$$
and, in like manner, the circular value of $\frac{d\,\Omega}{d\,w}$ is
$$-\left\{ 1 . A' . \sin W + 2 . A'' . \sin 2\,W + 3 . A''' . \sin 3\,W + \&c. \right\} \qquad (144, 3)$$

The values of $\Delta\,r$ and $\Delta\,r'$ are given by equation (28) if we substitute merely for $n\,t$ the expressions $n\,t + \epsilon - \pi$, and $n'\,t + \epsilon' - \pi'$; which in this case are the mean anomalies, because the mean longitudes $n\,t$, $n'\,t$, and the constants ϵ, ϵ', π, π', are reckoned, not from the perihelion of the orbits, as in that equation, but from the line of the equinoxes. If then we put $V = n\,t + \epsilon - \pi$, $V' = n'\,t + \epsilon' - \pi'$, we have
$$\left.\begin{array}{l} \Delta\,r = -\,a\,e . \cos V + e^2 \times \&c. \\ \Delta\,r' = -\,a'\,e' . \cos V' + e'^2 \times \&c. \\ \text{and} \quad \Delta\,w = 2\,e . \sin V - 2\,e' . \sin V' + e^2 \times \&c. \end{array}\right\} ; \ldots\ldots\ldots\ldots\ldots\ldots (145)$$
Substituting therefore in $\Delta\,\Omega$ for the differential co-efficients, their circular values in (144, 1, 2, 3,) and for $\Delta\,r$, $\Delta\,r'$, and $\Delta\,w$, the values just now found, it will become

$$\begin{aligned}\Delta\,\Omega = &- a\,e\,.\cos V\left\{\frac{d\,A}{d\,a} + \frac{d\,A'}{d\,a}\,.\cos W + \frac{d\,A''}{d\,a}\,.\cos 2\,W + \&c.\right\}\\ &- a'\,e'\,.\cos V'\left\{\frac{d\,A}{d\,a'} + \frac{d\,A'}{d\,a'}\,.\cos W + \frac{d\,A''}{d\,a'}\,.\cos 2\,W + \&c.\right\}\\ &- (2\,e\,.\sin V - 2\,e'\,.\sin V')\left\{A'\,.\sin W + 2\,A''\ \sin 2\,W + \&c.\right\}\\ &+ e^2 \times \&c.\end{aligned}\qquad(146)$$

Now these series are not precisely in the form we want them; the cosines and sines of V and W being multiplied together; and to disengage them, we must employ the well known trigonometrical formula

$$\begin{aligned}\cos A\,.\cos B &= \tfrac{1}{2}\left\{\cos(A+B) + \cos(A-B)\right\}\\ -\sin A\,.\sin B &= \tfrac{1}{2}\left\{\cos(A+B) - \cos(A-B)\right\}\end{aligned}\qquad(A)$$

By the aid of this, we get

$$\begin{array}{ll}\cos V = \cos V & \cos V' = \cos V'\\ \cos V\,.\cos W = \tfrac{1}{2}\{\cos(V+W)+\cos(V-W)\}; & \cos V'\,.\cos W' = \tfrac{1}{2}\{\cos(V'+W')+\cos(V'-W')\}\\ \cos V\,.\cos 2W = \tfrac{1}{2}\{\cos(V+2W)+\cos(V-2W)\}; & \cos V'\,.\cos 2W' = \tfrac{1}{2}\{\cos(V'+2W')+\cos(V'-2W')\}\\ \&c. = \&c. & \&c. = \&c.\\ \sin V\,.\sin W = \tfrac{1}{2}\{\cos(V-W)-\cos(V+W)\}; & \sin V'\,.\sin W = \tfrac{1}{2}\{\cos(V'-W)-\cos(V'+W)\}\\ \sin V\,.\sin 2W = \tfrac{1}{2}\{\cos(V-2W)-\cos(V+2W)\}; & \sin V'\,.\sin 2W = \tfrac{1}{2}\{\cos(V'-2W)-\cos(V'+2W)\}\\ \&c. = \&c. & \&c. = \&c.\end{array}$$

Thus we see that $\Delta\,\Omega$ (if we confine ourselves to the first powers of the eccentricities) is reducible to a series of sines and cosines, whose arguments are all comprehended in the forms $V \pm i\,W$, and $V' \pm i\,W$; or, (since $\cos - A = -\cos A$) in the forms $i\,W \pm V$ and $i\,W \pm V'$. That is, (since $V = n\,t + \epsilon - \pi$, and $W = n\,t - n'\,t + \epsilon - \epsilon'$) in the forms

$$i\,(n\,t - n'\,t + \epsilon - \epsilon') \pm (n\,t + \epsilon - \pi)$$
$$i\,(n\,t - n'\,t + \epsilon - \epsilon') \pm (n'\,t + \epsilon' - \pi')$$

If we actually execute the substitutions (still, for brevity, preserving the denominations W and V) we shall obtain for the value of $\Delta\,\Omega$ the following expression—

$$\begin{array}{ll}\Delta\,\Omega = - a\,e\,.\dfrac{d\,A}{d\,a}\,.\cos V & - a'\,e'\,.\dfrac{d\,A}{d\,a'}\,.\cos V';\\ \quad - e\left\{\dfrac{a}{2}\,\dfrac{d\,A'}{d\,a} - A'\right\}.\cos(W+V) & - e'\left\{\dfrac{a'}{2}\,\dfrac{d\,A'}{d\,a'} + A'\right\}\cos(W+V')\\ \quad - e\left\{\dfrac{a}{2}\,\dfrac{d\,A'}{d\,a} + A'\right\}.\cos(W-V) & - e'\left\{\dfrac{a'}{2}\,\dfrac{d\,A'}{d\,a'} - A'\right\}\cos(W-V')\\ \quad - e\left\{\dfrac{a}{2}\,\dfrac{d\,A''}{d\,a} - 2\,A''\right\}.\cos(2\,W+V) & - e'\left\{\dfrac{a'}{2}\,\dfrac{d\,A''}{d\,a'} + 2\,A''\right\}\cos(2\,W+V')\\ \quad - e\left\{\dfrac{a}{2}\,\dfrac{d\,A''}{d\,a} + 2\,A''\right\}.\cos(2\,W-V) & - e'\left\{\dfrac{a'}{2}\,\dfrac{d\,A''}{d\,a'} - 2\,A''\right\}\cos(2\,W-V')\\ \quad - \&c. & - \&c.\end{array}\qquad(147)$$

But our object is to obtain the developement of Q the perturbative function, or $2\int d\,\Omega + r\,\frac{d\,\Omega}{d\,r}$. The part of this independent of the eccentricities is already found, and we have now only to consider that depending on them; which, in the notation above adopted, will be expressed by Δ Q, or

$$2\,\Delta\int d\,\Omega + \Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = 2\int d\,\Delta\,\Omega + \Delta\left(r\,\frac{d\,\Omega}{d\,r}\right);\qquad(148)$$

Let us first consider the value of the first part of this expression $2\int d\,\Delta\,\Omega$, and let any term of $\Delta\,\Omega$ be represented by

$$M\,.\cos(i\,W + k\,V + l\,V')$$

in which i may be any integer from o to infinity, and k and l either ± 1 or o, an expression which obviously comprehends all the terms of which $\Delta\,\Omega$ consists. This premised, it is obvious that the co-efficient M being constant, the variation of $\Delta\,\Omega$ can only arise from the variation of the angle $i\,W + k\,V + l\,V'$; and as this angle is supposed only to vary by the motion of m, we are to suppose $n\,t$ only to vary, and $n'\,t$ to remain constant. So that we have, for that part of the variation of $\Delta\,\Omega$ which arises from the term in question,

$$- M\,(i\,d\,W + k\,d\,V)\ \sin(i\,W + k\,V + l\,V')$$

because $d\,V' = o$, since $V' = n'\,t + \epsilon' - \pi'$. Now $d\,W = n\,d\,t$ and $d\,V = n\,d\,t$ also. Consequently, this

becomes
$$-(i+k)\,\mathrm{M}\,.\,n\,d\,t\,.\,\sin\,(i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}')$$
and the part of the expression $2\int \mathrm{d}\,\Delta\,\Omega$ arising from this term, will therefore be
$$\frac{2\,(i+k)\,n}{i\,(n-n')+k\,n+l\,n'}\,.\,\mathrm{M}\cos\,(i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}') \qquad (149)$$
We see therefore by the foregoing reasoning, that in order to obtain that part of the perturbative function which originates in the term $2\int d\,\Omega$, and depends on the eccentricities, we have only to take the terms of the expression for $\Delta\,\Omega$ (147) in their order, and with their proper signs, and multiply each of them respectively by that value of the fraction
$$\frac{2\,(i+k)\,.\,n}{i\,(n-n')+k\,n+l\,n'}$$
which corresponds to the values of i, k, l, in its argument, represented by $i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}'$. For instance, the term which has simply V for its argument, must be multiplied by $\frac{2\,n}{1\,.\,n}=2$, that which has V′ by o.—Again, the terms, whose respective arguments are $\mathrm{W}+\mathrm{V}$, $\mathrm{W}-\mathrm{V}$, $\mathrm{W}+\mathrm{V}'$, $\mathrm{W}-\mathrm{V}'$, are to be multiplied, according to this rule, by the respective co-efficients $\frac{4\,n}{2\,n-n'}$, o, 2, and $\frac{2\,n}{n-2\,n'}$. Similarly, the terms which have $2\,\mathrm{W}+\mathrm{V}$, $2\,\mathrm{W}-\mathrm{V}$, $2\,\mathrm{W}+\mathrm{V}'$, $2\,\mathrm{W}-\mathrm{V}'$, for their arguments, acquire the co-efficients $\frac{6\,n}{3\,n-2\,n'}$, $\frac{2\,n}{n-2\,n'}$, $\frac{4\,n}{2\,n-n'}$, and $\frac{4\,n}{2\,n-3\,n'}$; and so on.

The co-efficients thus acquired by integration depend solely on the ratios of the mean motions, or periodic times, of the disturbed and disturbing planet, and are of very great importance in the planetary theory. Were it not for these, the theory of the planetary perturbations would be very simple, as the same treatment, or nearly so, could be applied in every case, and the magnitudes of the several inequalities would go on diminishing with more or less rapidity, as the arguments contained higher multiples of the mean motions. But these co-efficients prevent this regular progression of magnitude from taking place, and render it difficult to foresee without computation whether any assigned inequality may be neglected or not, and impossible to argue from its known minuteness in the case of one pair of planets to its probable smalness in that of another. Thus an inequality, whose maximum we know to be small in the case of Venus disturbed by the earth, may have a considerable magnitude in that of Jupiter disturbed by Saturn, merely from a relation subsisting between the periodic times in the latter case which does not in the former. In fact, it is evident that if the periods should happen to be so related as to render the denominator of any of the foregoing or similar fractions very small compared to the numerator, the inequality into which it enters as a co-efficient will, on this account alone, acquire a very great preponderance. Thus, if the period of the disturbed planet were very nearly half or double that of the disturbing; the terms, multiplied by $\frac{2\,n}{2\,n-n'}$ or by $\frac{n}{n-2\,n'}$ would become very large, and the length of the period of the inequality represented by it would be proportionally increased, and in the case of exact commensurability would actually become infinite; that is to say, the disturbance would go on to such an extent, as to make a total subversion of the original conditions of the problem. The physical reason of this is equally obvious. In the case just instanced, the two planets, at every revolution of the exterior, would be placed in exactly the same circumstances—the same disturbing forces would act in the same manner for a series of ages, and their effects, instead of compensating each other by mutual opposition, would go on accumulating in the same direction, till their orbits at length became entirely changed, and the commensurability of their periods at length ceased to subsist. In fact, an equation thus limited by no period, and affecting both the longitude and radius vector of each orbit always the same way, is equivalent to an alteration of the *mean* motion and *mean* distance; and as this would take place in opposite directions on the two planets, shortening the period of one, and lengthening that of the other, their periods would continually recede from commensurability; the magnitude of the inequality in question, as well as the length of its period, would both acquire finite values, which even then would continually diminish, till reduced within limits consistent with the stability of the system. It is probably by some such gradations (if we may hazard a conjecture on a part of the planetary theory so far beyond the reach of analysis or exact reasoning,) that our system has attained, in the course of almost indefinite ages, its present admirable state of equilibrium, in which no inequality of enormous magnitude exists; and those which have any notable value, can be proved to be confined within comparatively narrow bounds.

Let us next consider the part of the developement of Q, which arises from the term $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$. Now, if we regard only the first powers of the eccentricities, and consequently neglect the squares of $\Delta\,r$, &c. we have
$$\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)=\frac{d\,\Omega}{d\,r}\,.\,\Delta\,r+r\,\Delta\,\frac{d\,\Omega}{d\,r}$$
$$=\frac{d\,\Omega}{d\,r}\,\Delta\,r+r\,.\,\frac{d^2\,\Omega}{d\,r^2}\,\Delta\,r+r\,\frac{d^2\,\Omega}{d\,r\,d\,r'}\,\Delta\,r'+r\,\frac{d^2\,\Omega}{d\,r\,d\,w}\,\Delta\,w$$

in which the differential co-efficients are to have their circular values, which we may represent by $\frac{d\,\Omega}{d\,a}$, &c. Now these are,

$$\left.\begin{aligned}
\frac{d\,\Omega}{d\,a} &= \frac{d\,\mathrm{A}}{d\,a} + \frac{d\,\mathrm{A}'}{d\,a}\cos \mathrm{W} + \frac{d\,\mathrm{A}''}{d\,a}\,.\cos 2\,\mathrm{W} + \&\text{c.}\\
a\,\frac{d^2\,\Omega}{d\,a^2} &= a\,\frac{d^2\,\mathrm{A}}{d\,a^2} + a\,\frac{d^2\,\mathrm{A}'}{d\,a^2}\,.\cos \mathrm{W} + a\,\frac{d^2\,\mathrm{A}''}{d\,a^2}\,.\cos 2\,\mathrm{W} + \&\text{c.}\\
a\,\frac{d^2\,\Omega}{d\,a\,d\,a'} &= a\,\frac{d^2\,\mathrm{A}}{d\,a\,d\,a'} + a\,\frac{d^2\,\mathrm{A}'}{d\,a\,d\,a'}\,.\cos \mathrm{W} + a\,\frac{d^2\,\mathrm{A}''}{d\,a\,d\,a'}\,.\cos 2\,\mathrm{W} + \&\text{c.}\\
a\,\frac{d^2\,\Omega}{d\,a\,d\,\mathrm{W}} &= \quad 1\,.\,a\,\frac{d\,\mathrm{A}'}{d\,a}\,.\sin \mathrm{W} + 2\,.\,a\,\frac{d\,\mathrm{A}''}{d\,a}\,.\sin 2\,\mathrm{W} + \&\text{c.}
\end{aligned}\right\}\ldots\ldots\ldots\ldots(150)$$

Let these be substituted in the above expression for $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$ and the values of $\Delta\,r, \Delta\,r'$ and $\Delta\,w$, given in (145) be put for them, and we shall have, by a process exactly similar to that gone through for $\Delta\,\Omega$,

$$\begin{aligned}
\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = &-a\,e\,.\cos \mathrm{V}\,.\left\{\left(\frac{d\,\mathrm{A}}{d\,a} + a\,\frac{d^2\,\mathrm{A}}{d\,a^2}\right) + \left(\frac{d\,\mathrm{A}'}{d\,a} + a\,\frac{d^2\,\mathrm{A}'}{d\,a^2}\right)\cos \mathrm{W} + \&\text{c.}\right\}\\
&-a'\,e'\,.\cos \mathrm{V}'\,.\left\{a\,\frac{d^2\,\mathrm{A}}{d\,a\,d\,a'} + a\,\frac{d^2\,\mathrm{A}'}{d\,a\,d\,a'}\,.\cos \mathrm{W} + \&\text{c.}\right\} \qquad (151)\\
&-(2\,e\,.\sin \mathrm{V} - 2\,e'\,.\sin \mathrm{V}')\left\{1\,a\,\frac{d\,\mathrm{A}'}{d\,a}\,.\sin \mathrm{W} + 2\,.\,\&\text{c.}\right\}
\end{aligned}$$

which, by the use of the same trigonometrical formulæ, and by properly arranging the terms, becomes

$$\begin{aligned}
\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = &-e\left\{a^2\,\frac{d^2\,\mathrm{A}}{d\,a^2} + a\,\frac{d\,\mathrm{A}}{d\,a}\right\}\cos \mathrm{V} &&- e'\,.\,a\,a'\,\frac{d^2\,\mathrm{A}}{d\,a\,d\,a'}\cos \mathrm{V}'\\
&-\frac{e}{2}\left\{a^2\,\frac{d^2\,\mathrm{A}'}{d\,a^2} - a\,\frac{d\,\mathrm{A}'}{d\,a}\right\}\cos(\mathrm{W}+\mathrm{V}) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,\mathrm{A}}{d\,a\,d\,a'} + 2\,a\,\frac{d\,\mathrm{A}'}{d\,a}\right\}\cos(\mathrm{W}+\mathrm{V}')\\
&-\frac{e}{2}\left\{a^2\,\frac{d^2\,\mathrm{A}'}{d\,a^2} + 3\,a\,\frac{d\,\mathrm{A}'}{d\,a}\right\}\cos(\mathrm{W}-\mathrm{V}) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,\mathrm{A}'}{d\,a\,d\,a'} - 2\,a\,\frac{d\,\mathrm{A}'}{d\,a}\right\}\cos(\mathrm{W}+\mathrm{V}')\\
&-\frac{e}{2}\left\{a^2\,\frac{d^2\,\mathrm{A}''}{d\,a^2} - 3\,a\,\frac{d\,\mathrm{A}''}{d\,a}\right\}\cos(2\,\mathrm{W}+\mathrm{V}) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,\mathrm{A}''}{d\,a\,d\,a'} + 4\,a\,\frac{d\,\mathrm{A}''}{d\,a}\right\}\cos(2\,\mathrm{W}+\mathrm{V}')\\
&-\frac{e}{2}\left\{a^2\,\frac{d^2\,\mathrm{A}''}{d\,a^2} + 5\,a\,\frac{d\,\mathrm{A}''}{d\,a}\right\}\cos(2\,\mathrm{W}-\mathrm{V}) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,\mathrm{A}''}{d\,a\,d\,a'} - 4\,a\,\frac{d\,\mathrm{A}''}{d\,a}\right\}\cos(2\,\mathrm{W}-\mathrm{V}')\\
&-\&\text{c.} &&-\&\text{c.}\,; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(152)
\end{aligned}$$

We are now in a condition to express the whole value of $\Delta\,\mathrm{Q}$, by combining this with the expressions (147) and (149) and it is evident that our result will be of the following form :—

$$\begin{aligned}
\Delta\,\mathrm{Q} = &(\mathrm{N}\,e\cos \mathrm{V} + \mathrm{N}'\,e'\cos \mathrm{V}') + \mathrm{O}\,e\cos(\mathrm{W}+\mathrm{V}) + \mathrm{O}'\,e'\cos(\mathrm{W}+\mathrm{V}') \qquad (153)\\
&+ \mathrm{P}\,e\cos(\mathrm{W}-\mathrm{V}) + \mathrm{P}'\,e'\cos(\mathrm{W}-\mathrm{V}') + \mathrm{Q}\,e\cos(2\,\mathrm{W}+\mathrm{V}) + \&\text{c.}
\end{aligned}$$

and the co-efficients of the several arguments will be

$$\left.\begin{aligned}
\mathrm{N} &= -\left\{a^2\,\frac{d^2\,\mathrm{A}}{d\,a^2} + 3\,a\,\frac{d\,\mathrm{A}}{d\,a}\right\}\\
\mathrm{O} &= -\frac{1}{2}\left\{a^2\,\frac{d^2\,\mathrm{A}'}{d\,a^2} + \frac{2\,n+n'}{2\,n-n'}\,a\,\frac{d\,\mathrm{A}'}{d\,a} - \frac{8\,n}{2\,n-n'}\,\mathrm{A}'\right\}\\
\mathrm{P} &= -\frac{1}{2}\left\{a^2\,\frac{d^2\,\mathrm{A}'}{d\,a^2} + 3\,a\,\frac{d\,\mathrm{A}'}{d\,a}\right\};\\
\mathrm{N}' &= -\,a\,a'\,.\,\frac{d^2\,\mathrm{A}}{d\,a\,d\,a'}\\
\mathrm{O}' &= -\frac{1}{2}\left\{a\,a'\,.\,\frac{d^2\,\mathrm{A}'}{d\,a\,d\,a'} + 2\,a\,\frac{d\,\mathrm{A}'}{d\,a} + 2\,a'\,\frac{d\,\mathrm{A}'}{d\,a'} + 4\,\mathrm{A}'\right\}\\
\mathrm{P}' &= -\frac{1}{2}\left\{a\,a'\,.\,\frac{d^2\,\mathrm{A}'}{d\,a\,d\,a'} - 2\,a\,\frac{d\,\mathrm{A}'}{d\,a} + \frac{2\,n}{n-2\,n'}\,a'\,\frac{d\,\mathrm{A}'}{d\,a'} - \frac{4\,n}{n-2\,n'}\,\mathrm{A}'\right\}
\end{aligned}\right\}\ldots\ldots\ldots(154)$$

The value of $\Delta\,\mathrm{Q}$ being thus obtained, that of Π is had by mere substitution. The part of Π, independent of the eccentricities, will remain as in the preceding section, changing only in the several arguments w into W; and if we denote this by Π, and by $\Delta\,\Pi$ the part of Π which depends on the eccentricities, the equation (128) will give

$$\Delta \Pi = \frac{e}{a^2}\left\{Q \,.\, \cos V - 2\int Q\, n\, d\, t \,.\, \sin V\right\} - \frac{\Delta\, Q}{a^2}$$

in which Q and Δ Q denote, as we have all along supposed, the parts of Q respectively independent, and dependent, on the eccentricities.

The value of Δ Π therefore consists of two parts; the latter $-\frac{\Delta\, Q}{a^2}$ is immediately obtained from the expression of Δ Q above found, and is equal to

$$-\frac{e}{a^2}\left\{N \,.\, \cos V + O \,.\, \cos (W + V) + P \,.\, \cos (W - V) + \&c.\right\}$$
$$-\frac{e'}{a^2}\left\{N' \,.\, \cos V' + O' \,.\, \cos (W + V') + P' \,.\, \cos (W - V') + \&c.\right\}$$

The former depends on Q, and must be determined by substituting for Q its value

$$Q = M + M' \,.\, \cos W + M'' \,.\, \cos 2\, W + \&c.$$

where M, M′, &c. are co-efficients, whose values are assigned in equation (138, 1). This substitution made, we find

$$Q \cos V = M \,.\, \cos V + \frac{M'}{2} \,.\, \cos (W + V) + \frac{M'}{2} \,.\, \cos (W - V) + \&c.$$

$$\int Q\, n\, d\, t \,.\, \sin V = - M \,.\, \cos V - \frac{M'\, n}{2\,(2\, n - n')} \cos (W + V) - \frac{M'\, n}{2\, n'} \cos (W - V) - \&c.$$

So that the part of Δ Π now in question becomes

$$\frac{e}{a^2}\left\{3\, M \,.\, \cos V + \frac{4\, n - n'}{4\, n - 2\, n'}\, M' \,.\, \cos (W + V) + \frac{2\, n + n'}{2\, n'}\, M' \,.\, \cos (W - V) + \&c.\right\}$$

and the whole value of Δ Π will be as follows :—

$$\Delta\, \Pi = \frac{e}{a^2}\left\{(3\, M - N) \cos V + \left(\frac{4\, n - n'}{4\, n - 2\, n'}\, M' - O\right) \cos (W + V) + \left(\frac{2 n + n'}{2\, n'} M' - P\right) \cos (W - V) + \&c.\right\}$$
$$- \frac{e'}{a^2}\left\{N' \,.\, \cos V' + O' \,.\, \cos (W + V') + P' \,.\, \cos (W - V') + \&c.\right\}; \qquad (155)$$

This found, the integral of the equation $\frac{d^2\, \delta\, u}{d\, t^2} + n^2\, u + \Pi = o$ will be obtained by the expressions 106, 107, and 108. The parts of $\delta\, u$, $\delta\, r$, and $\delta\, \theta$, independent of the eccentricities, have already been found; and calling therefore $\Delta\, \delta\, u$, $\Delta\, \delta\, r$, and $\Delta\, \delta\, \theta$, those parts of these respective quantities which depend on the eccentricities, we shall have (since $V = n\, t + \epsilon - \pi$, $W = (n - n')\, t + (\epsilon - \epsilon')$ and $V' = n'\, t + \epsilon' - \pi'$)

$$\Delta\, \delta\, u = \frac{e}{a^2}\left\{ (3\, M - N) \,.\, \frac{t}{2} \cos V + \left(\frac{4\, n - n'}{4 n - 2\, n'} M' - O\right) \frac{\cos (W + V)}{(2\, n - n')^2 - n^2} + \right.$$
$$\left. + \left(\frac{2\, n + n'}{2\, n'} M' - P\right) \frac{\cos (W - V)}{n'^2 - n^2} + \&c. \right\}$$
$$- \frac{e'}{a^2}\left\{\frac{N'}{n'^2 - n^2} \cos V' + \frac{O'}{2} \,.\, \frac{t}{2} \cos (W + V') + \frac{P'}{(n - 2\, n')^2 - n^2} \cos (W - V') + \&c.\right\} \qquad (156)$$

The perturbation of the radius vector is now easily found; for, as we have by equation 124,

$$\delta\, r = - a\, \delta\, u \left\{1 + 2\, e \,.\, \cos V + \&c.\right\}$$

this gives
$$\Delta\, \delta\, r = - a \left\{\Delta\, \delta\, u + 2\, e\, \delta\, u \,.\, \cos V\right\}$$

whence $\Delta\, \delta\, r$ is readily obtained. With regard to the perturbation in longitude, or $m'\, \delta\, \theta$, no further difficulty than the length of the substitutions remains to be encountered; for the part not depending on the eccentricities being already obtained, that which depends on them will be had by merely substituting for $\delta\, r$, $r\, \frac{d\, \Omega}{d\, r}$ and Ω, the values of $\Delta\, \delta\, r$, $\Delta \left(r \frac{d\, \Omega}{d\, r}\right)$ and $\Delta\, \Omega$ already obtained in the general expression (121).

But as this process of substitution and reduction presents no difficulties in principle, requiring only patience and exactness in performing the numerous combinations of the terms which occur in it, we shall not pursue it, but content ourselves with observing, that the part of it which depends on the first powers of the eccentricities will, on examination, be found to assume the form

$$\left.\begin{array}{l} e \,.\, a\, t \,.\, \cos V + e' \,.\, b\, t \,.\, \cos (W + V') \\ + e \left\{A \,.\, \sin (W + V) + B \,.\, \sin (W - V) + C \,.\, \sin (2\, W + V) + \&c.\right\} \\ + e' \left\{A' \,.\, \sin V' \qquad + B' \,.\, \sin (W - V') + C' \,.\, \sin (2\, W + V') + \&c.\right\} \end{array}\right\} ; \ldots\ldots\ldots (157)$$

It is here that we first encounter the secular equations of the planetary motions, in the form of two terms containing the time t disengaged from the signs sin and cos, and therefore capable of indefinite increase and diminution. They are multiplied by the eccentricities, and therefore originate from the ellipticity of the planetary orbits; and in the case of strictly circular orbits, would not exist. Similar terms occur of course in the value of δr, being introduced by the integration of the equation for δu. But as the discussion of these terms is one of the most delicate and difficult points of the planetary theory, we shall not enter upon it till we have pointed out the method of taking into account the higher powers of the eccentricities.

Section V.

Of the inequalities depending on the squares and higher powers of the eccentricities.

It is not our intention to enter into any detailed account of this very complicated part of the planetary theory. Any such attempt would lead us far beyond our proper limits; and the reader, who is desirous to follow it into its minutiæ, must consult the original memoirs of Laplace, Lagrange, &c., the *Mécanique Céleste,* and other works of a similar nature. In the foregoing sections we have however followed, as nearly as possible, the course pursued in the last named immortal work, supplying only such steps in the analysis as cannot be expected to be discovered by the ordinary student, (and they are numerous) and endeavouring throughout to place the principles of the several processes in as strong a light as possible. In the present section, the explanation of the principles on which the process of approximation is to be pursued will be almost our sole object.

Let us resume the consideration of the function Ω.

$$\Omega = \mathrm{R} + \mathrm{R}' \,.\, \cos w + \mathrm{R}'' .\, \cos 2\, w + \mathrm{R}''' \,.\, \cos 3\, w + \&\text{c.}$$

When $r + \Delta\, r$, $r' + \Delta\, r'$, and $w + \Delta\, w$, are substituted for r, r', w, in this, Ω becomes $\Omega + \Delta\, \Omega$, and we have

$$\left.\begin{aligned}\Delta\, \Omega = {} & \frac{d\, \Omega}{d\, a} \cdot \frac{\Delta\, r}{1} + \frac{d\, \Omega}{d\, a'}\, \frac{\Delta\, r'}{1} + \frac{d\, \Omega}{d\, \mathrm{W}}\, \frac{\Delta\, w}{1} \\ & + \frac{d^2\, \Omega}{d\, a^2}\, \frac{(\Delta\, r)^2}{1\,.\,2} + \frac{d^2\, \Omega}{d\, a\, d\, a'}\, \frac{\Delta\, r \,.\, \Delta\, r'}{1\,.\,1} + \frac{d^2\, \Omega}{d\, a'^2}\, \frac{(\Delta\, r')^2}{1\,.\,2} + \&\text{c.} \\ & + \frac{d^3\, \Omega}{d\, a^3}\, \frac{(\Delta\, r)^3}{1\,.\,2\,.\,3} + \&\text{c.}\end{aligned}\right\}, \ldots\ldots\ldots\ldots\ldots \quad (158)$$

The differential co-efficients of Ω are here supposed to have their circular values denoted by $\frac{d\, \Omega}{d\, a}$, $\frac{d\, \Omega}{d\, a'}$, $\frac{d\, \Omega}{d\, \mathrm{W}}$, &c.

In like manner, if we consider the value of $\Delta \left(r\, \frac{d\, \Omega}{d\, r}\right)$, or the augmentation of $r\, \frac{d\, \Omega}{d\, r}$ produced by the eccentricities, we have only to substitute for Ω in the expression (158) the circular value of the function in question, or $a\, \frac{d\, \Omega}{d\, a}$, and we get

$$\begin{aligned}\Delta \left(r\, \frac{d\, \Omega}{d\, r}\right) = {} & \frac{\Delta\, r}{1} \times \frac{d}{da}\left(a\, \frac{d\, \Omega}{d\, a}\right) + \frac{\Delta\, r'}{1} \cdot \frac{d}{da'}\left(a\, \frac{d\, \Omega}{d\, a}\right) + \frac{\Delta\, w}{1} \cdot \frac{d}{d\, \mathrm{W}}\left(a\, \frac{d\, \Omega}{d\, a}\right) \\ & + \frac{(\Delta\, r)^2}{1\,.\,2} \cdot \frac{d^2}{d\, a^2}\left(a\, \frac{d\, \Omega}{d\, a}\right) + \frac{\Delta\, r\, \Delta\, r'}{1\,.\,1} \cdot \frac{d^2}{d\, a\, d\, a'}\left(a\, \frac{d\, \Omega}{d\, a}\right) + \&\text{c.} \\ & + \&\text{c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (159)\end{aligned}$$

in which $\frac{d}{d\,a}$, $\frac{d}{da'}$, &c. denote the differentiation relative to a, a', &c. respectively of the function to which they are prefixed, and the division of the resulting differential by $d\, a$, $d\, a'$, &c. according to the very convenient system of notation explained in (Lacroix, *Differential and Integral Calculus,* 8vo. English translation, Appendix.)

Since $\Omega = \mathrm{A} + \mathrm{A}' .\, \cos \mathrm{W} + \mathrm{A}'' .\, \cos 2\, \mathrm{W} + \&\text{c.}$ we must have

$$\left.\begin{aligned}a\, \frac{d\, \Omega}{d\, a} &= a\, \frac{d\, \mathrm{A}}{d\, a} + a\, \frac{d\, \mathrm{A}'}{d\, a} .\, \cos \mathrm{W} + a\, \frac{d\, \mathrm{A}''}{d\, a} .\, \cos 2\, \mathrm{W} + \&\text{c.} \\ \frac{d}{d\, a}\left(a\, \frac{d\, \Omega}{d\, a}\right) &= \frac{d}{da}\left(a\, \frac{d\, \mathrm{A}}{d\, a}\right) + \frac{d}{da}\left(a\, \frac{d\, \mathrm{A}'}{d\, a}\right) \cos \mathrm{W} + \frac{d}{da}\left(a\, \frac{d\, \mathrm{A}''}{d\, a}\right) .\, \cos 2\, \mathrm{W} + \&\text{c.} \\ \frac{d}{d\, a'}\left(a\, \frac{d\, \Omega}{d\, a}\right) &= \frac{d}{da'}\left(a\, \frac{d\, \mathrm{A}}{d\, a}\right) + \frac{d}{da'}\left(a\, \frac{d\, \mathrm{A}'}{d\, a}\right) \cos \mathrm{W} + \frac{d}{da'}\left(a\, \frac{d\, \mathrm{A}''}{d\, a}\right) .\, \cos 2\, \mathrm{W} + \&\text{c.} \\ \frac{d}{d\, \mathrm{W}}\left(a\, \frac{d\, \Omega}{d\, a}\right) &= -\, a\, \frac{d\, \mathrm{A}'}{d\, a} .\, \sin \mathrm{W} - 2\, a\, \frac{d\, \mathrm{A}''}{d\, a} \sin 2\, \mathrm{W} - \&\text{c.} \\ \frac{d^2}{d\, a^2}\left(a\, \frac{d\, \Omega}{d\, a}\right) &= \frac{d^2}{da^2}\left(a\, \frac{d\, \mathrm{A}}{d\, a}\right) + \frac{d^2}{da^2}\left(a\, \frac{d\, \mathrm{A}'}{d\, a}\right) \cos \mathrm{W} + \frac{d^2}{da^2}\left(a\, \frac{d\, \mathrm{A}''}{d\, a}\right) \cos 2\, \mathrm{W} + \&\text{c.} \\ \frac{d^2}{d\, a\, d\, \mathrm{W}}\left(a\, \frac{d\, \Omega}{d\, a}\right) &= -\, \frac{d}{da}\left(a\, \frac{d\, \mathrm{A}'}{d\, a}\right) \sin \mathrm{W} - 2\, \frac{d}{da}\left(a\, \frac{d\, \mathrm{A}'}{d\, a}\right) \sin 2\, \mathrm{W} - \&\text{c.}\end{aligned}\right\}; \quad (160)$$

and so on. Now these series may be regarded as completely developed; for the co-efficients of their several terms cos W, sin W, &c. are quantities completely given in numbers, when a, a', A, A′, A″, &c. and their differential co-efficients are known, by the equations

$$\left.\begin{array}{l}\frac{d}{d\,a}\left(a\frac{d\,A}{d\,a}\right)=\frac{d\,A}{d\,a}+a\frac{d^2\,A}{d\,a^2};\quad \frac{d}{d\,a}\left(a\frac{d\,A'}{d\,a}\right)=\frac{d\,A'}{d\,a}+a\frac{d^2\,A'}{d\,a^2};\ \&c.\\ \frac{d}{d\,a'}\left(a\frac{d\,A}{d\,a}\right)=\frac{d^2\,A}{d\,a\,d\,a'};\quad \frac{d}{d\,a'}\left(a\frac{d\,A'}{d\,a}\right)=\frac{d^2\,A'}{d\,a\,d\,a'};\ \&c.\\ \frac{d^2}{d\,a^2}\left(a\frac{d\,A}{d\,a}\right)=2\,\frac{d^2\,A}{d\,a^2}+a\,\frac{d^3\,A}{d\,a^3};\\ \quad\&c.\end{array}\right\};\ \ldots\ldots\ldots(161)$$

If, instead of $r\,\frac{d\,\Omega}{d\,r}$, we had any other function to develope $\left(\text{such as for instance } r^2\,.\,\frac{d^3\,\Omega}{d^2 r\,d\,r'}\right)$, we might treat it exactly in the same way, and should arrive at corresponding series, in which the sines and cosines of W and its multiples would be combined with co-efficients absolutely constant, and reducible to numbers.

In the values of $\Delta\,\Omega$ and $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$ in (158) and (159) we see therefore that the differential co-efficients of Ω introduce the sines and cosines of W, and its multiples, combined only with given quantities, and not involving the eccentricities. These latter arise from the factors $\Delta\,r$, $\Delta\,r'$, $\Delta\,w$, and their powers. Let us now examine these more nearly. Supposing then, as we have done all along,

$$\mathrm{V}=n\,t+\epsilon-\pi;\quad \mathrm{V}'=n'\,t+\epsilon'-\pi';\quad \mathrm{W}=n\,t-n'\,t+\epsilon-\epsilon';$$

let us take α, β, &c. as follows:—

$$\left.\begin{array}{ll}\alpha=-\,a\,.\cos\mathrm{V} & \alpha'=-\,a'\,.\cos\mathrm{V}'\\ \beta=\frac{a}{2}\,(1-\cos 2\,\mathrm{V}) & \beta'=\frac{a'}{2}\,(1-\cos 2\,\mathrm{V}')\\ \gamma=-\,\frac{a}{8}\,(3\cos 3\,\mathrm{V}-3\cos\mathrm{V}) & \gamma'=-\,\frac{a'}{8}\,(3\cos 3\,\mathrm{V}'-3\cos\mathrm{V}')\\ \quad\&c. & \quad\&c.\\ p=2\sin\mathrm{V} & p'=2\sin\mathrm{V}'\\ q=\frac{5}{4}\sin 2\,\mathrm{V} & q'=\frac{5}{4}\sin 2\,\mathrm{V}'\\ r=-\,\frac{1}{4}\sin\mathrm{V}+\frac{13}{12}\sin 3\,\mathrm{V} & r'=-\,\frac{1}{4}\sin\mathrm{V}'+\frac{13}{12}\sin 3\,\mathrm{V}'\\ \quad\&c. & \quad\&c.\end{array}\right\};\ \ldots\ldots\ldots(162)$$

Then we shall have, by the equations (28) and (30)

$$\left.\begin{array}{l}\Delta\,r=\alpha\,e+\beta\,e^2+\gamma\,e^3+\&c.\\ \Delta\,r'=\alpha'\,e'+\beta'\,e'^2+\gamma'\,e'^3+\&c.\\ \Delta\,w=(p\,e+q\,e^2+r\,e^3+\&c.)-(p'\,e'+q'\,e'^2+r'\,e'^3+\&c.).\end{array}\right\};\ldots\ldots\ldots\ldots\ldots\ldots(163)$$

whence we obtain

$$(\Delta\,r)^2=\alpha^2\,e^2+2\,\alpha\,\beta\,e^3+(\beta^2+2\,\alpha\,\gamma)\,e^4+\&c.$$
$$\Delta\,r\,\Delta\,r'=\alpha\,\alpha'\,e\,e'+\alpha\,\beta'\,e\,e'^2+\beta\,\alpha'\,e^2\,e'+\&c.$$
$$\Delta\,r\,\Delta\,w=p\,\alpha\,e^2+(p\,\beta+q\,\alpha)\,e^3+\&c.-p'\,\alpha\,e\,e'-\&c.$$
$$(\Delta\,w^2)=p^2\,e^2+2\,p\,q\,e^3+\&c.+(p'^2\,e'^2+2\,p'\,q'\,e'^3+\&c.)-2\,p\,p'\,e\,e'-\&c.$$

and so on; and it only remains to substitute these values in the expressions for $\Delta\,\Omega$ and $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$. Confining ourselves to $\Delta\,\Omega$, since it is obvious that the process is exactly similar for the other function, we have, as before, for the part depending on the first powers of the eccentricities,

$$e\left\{a\,\frac{d\,\Omega}{d\,a}+p\,\frac{d\,\Omega}{d\,\mathrm{W}}\right\}+e'\left\{a'\,\frac{d\,\Omega}{d\,a'}-p'\,\frac{d\,\Omega}{d\,\mathrm{W}}\right\}$$

which, developed into series of sines and cosines, gives the result obtained in the last section.

The part of $\Delta\,\Omega$ depending on the squares of the eccentricities, consists of three terms multiplied respectively by e^2, $e\,e'$, and e'^2, which originate, 1st. From the terms $\beta\,e^2$, $\beta'\,e'^2$, $q\,e^2$, and $-\,q'\,e'^2$, in the simple powers of $\Delta\,r$, $\Delta\,r'$, $\Delta\,w$, and which are therefore affected with the differential co-efficients of the first order only: 2dly. With the terms $\alpha^2\,e^2$, $\alpha'^2\,e'^2$, $p^2\,e^2$, $p'^2\,e'^2$, and $-\,2\,p\,p'\,e\,e'$, in $(\Delta\,r)^2$, $(\Delta\,r')^2$, and $(\Delta\,w)^2$, and which are consequently affected with differential co-efficients of Ω of the second order.

3dly. With the terms $\alpha\, p\, e^2$, $\alpha'\, p'\, e'^2$, $\alpha\, \alpha'\, e\, \alpha'$, $\alpha\, p'\, e\, e'$, $\alpha'\, p\, e\, e'$, which arise from combinations of $\Delta\, r$ with $\Delta\, w$ and with $\Delta\, r'$.

The aggregate of all these terms, with their proper co-efficients, is

$$\left.\begin{array}{l} \dfrac{e^2}{2}\left\{\left(2\,\beta\,\dfrac{d\,\Omega}{d\,a} + 2\,q\,\dfrac{d\,\Omega}{d\,W}\right) + \left(\alpha^2\,\dfrac{d^2\,\Omega}{d\,a^2} + 2\,\alpha\,p\,\dfrac{d^2\,\Omega}{d\,a\,d\,W} + p^2\,\dfrac{d^2\,\Omega}{d\,W^2}\right)\right\} \\ + e\,e'\left\{\alpha\,\alpha'\,\dfrac{d^2\,\Omega}{d\,a\,d\,a'} - \alpha\,p'\,\dfrac{d^2\,\Omega}{d\,a\,d\,W} + \alpha'\,p\,\dfrac{d^2\,\Omega}{d\,a'\,d\,W} - p\,p'\,\dfrac{d^2\,\Omega}{d\,W^2}\right\} \\ + \dfrac{e'^2}{2}\left\{\left(2\,\beta'\,\dfrac{d\,\Omega}{d\,a'} - 2\,q'\,\dfrac{d\,\Omega}{d\,W}\right) + \left(\alpha'^2\,\dfrac{d^2\,\Omega}{d\,a'^2} - 2\,\alpha'\,p'\,\dfrac{d^2\,\Omega}{d\,a'\,d\,W} + p'^2\,\dfrac{d^2\,\Omega}{d\,W^2}\right)\right\} \end{array}\right\} ; \;\ldots\ldots\ldots\ldots (164)$$

The co-efficients of e^3, $e^2\, e'$, $e\, e'^2$, e'^3, and of the higher powers and combinations, may in like manner be easily obtained; but the number of terms of which they consist, goes on increasing so rapidly, that they at length become of extreme complexity.

Let us now consider the nature of the terms into which the expressions for Ω and $r\,\dfrac{d\,\Omega}{d\,r}$ resolve themselves by the process of developement, and the manner in which they become modified by the several processes of substitution and integration they have to undergo in obtaining the values of Q, Π, $\delta\, u$, $\delta\, r$, and $\delta\, \theta$.

It is evident, then, since α, β, γ, α', β', &c. are all composed of cosines, and p, q, p', q', &c. of sines of V, V', and their multiples, without W, that any product or combination of these letters, (such as α^2, $\alpha\, p$, $\alpha\, \alpha'$, &c.) is reducible by the trigonometrical formulæ so often employed in the foregoing pages into the simple sines or cosines of arcs, of the form $k\, V + l\, V'$. Thus, α^2 or $\alpha^2 \,.\, \cos V^2$ becomes $\alpha^2 \,.\, \dfrac{1 + \cos 2\, V}{2}$, $\alpha\, p$, or $-\, 2\, \alpha \,.\, \sin V \,.\, \cos V$ becomes $-\, \alpha \,.\, \sin 2\, V$, $\alpha\, \alpha'$ or $\alpha\, \alpha' \,.\, \cos V \,.\, \cos V'$ is reduced to $\dfrac{\alpha\, \alpha'}{2}\, (\cos\, (V + V') + \cos\, (V - V'))$, $\alpha\, p'$ into $\alpha \,.\, \sin\, (V - V') - \alpha \,.\, \sin\, (V + V')$, and so on. Moreover, it is evident, that whenever the combination in question consists only of the letters α, β, α', β', &c. or of these combined with any product of an *even* dimension, in p, q, p', q', &c., that the terms into which it is resolved will consist entirely of cosines; but when a product of an *odd* dimension in p, q, p', q', &c. occurs, then of sines. Now, the differential co-efficient of Ω combined with any such product, will, in the former case, evidently be differentiated an even, and in the latter an odd number of times relatively to W; so that in the former case it will represent a series of *cosines*, and in the latter, of *sines* of W.

Every term therefore formed by such combination, must be of one or other of the forms

$$\cos i\, W \,.\, \cos\, (k\, V + l\, V') \text{ and } \sin i\, W \,.\, \sin\, (k\, V \pm l\, V')$$

both which, being further resolved, produce terms comprehended in the form

$$\cos\, (i\, W \pm k\, V \pm l\, V')$$

It is thus demonstrated, that the co-efficients of all the powers of e, e', in the developements of Ω and $r\,\dfrac{d\,\Omega}{d\,r}$ are generally reducible to series of cosines of arguments, of the form $i\, W \pm k\, V \pm l\, V'$; but there is a connection between the multiples of V and V' contained in any argument, and the dimension of the power or product of the eccentricities to which it belongs that we must now explain. In fact, it is obvious from the process above pursued, that if we regard α, α'. p, p', as quantities of one dimension; β, β', q, q', as of two; γ, γ', r, r' of three, and so on, the dimension of every term multiplied by e, or e', will be one; that of the terms multiplied by e^2, $e\, e'$, e'^2, will be two, and so on. Now, the expressions of these quantities in V and V' involve, each, the sines or cosines of multiples of V, V', as far as the number expressing its own dimension; and when these come to be combined by multiplication, and then resolved by the usual formula, it is obvious that the resulting terms of the form $\cos\, (k\, V \pm l\, V')$ and $\sin\, (k\, V \pm l\, V')$ can only contain such multiples $k\, V$ and $l\, V'$ of V, V', as together (without regard to their signs) do not exceed the dimension of the combination from which they arose.

Moreover, since the alternate multiples of V, V' are absent in the expressions of α, β, &c. the same law will hold good in any combination of them when developed. Hence we may state it as a general law, that

The co-efficient of any power or product of the eccentricities of the dimension n, *in the developement of* Ω *or* $r\,\dfrac{d\,\Omega}{d\,r}$ *will consist of a series of cosines, the form of whose argument is*

$$i\, W \pm k\, V \pm l\, V'$$

in which i *may have every possible value from* o *to infinity, but* k *and* l *are restricted to certain particular values, viz. those which satisfy one or other of the equations*

$$k + l = n, \qquad k + l = n - 2, \qquad k + l = n - 4, \;\&c.$$

down to $\quad k + l = 1,\;$ *or* $\;k + l = o,\;$ *according as* n *is odd or even.*

Thus, as we have already seen, the parts independent of the eccentricities consist of terms of the form cos

$i\mathrm{W}$ only, and those depending on the first powers involve the arguments

$$i\mathrm{W} + \mathrm{V}, \quad i\mathrm{W} - \mathrm{V}, \quad i\mathrm{W} + \mathrm{V}', \quad i\mathrm{W} - \mathrm{V}'$$

and no other. Similarly, in the part dependent on the squares and product of the eccentricities, the arguments which can occur, are only

$i\mathrm{W}$, $i\mathrm{W} + 2\mathrm{V}$, $i\mathrm{W} - 2\mathrm{V}$, $i\mathrm{W} + \mathrm{V} + \mathrm{V}'$, $i\mathrm{W} + \mathrm{V} - \mathrm{V}'$, $i\mathrm{W} - \mathrm{V} + \mathrm{V}'$, $i\mathrm{W} - \mathrm{V} - \mathrm{V}'$, $i\mathrm{W} + 2\mathrm{V}'$, and $i\mathrm{W} - 2\mathrm{V}'$, and so on.

Here, it will be observed, we have again the argument $i\mathrm{W}$, which occurred in the part independent on the eccentricities, and it is easily seen to be a general law, that *any particular argument which first occurs combined with a power or product of the eccentricities of the dimension* n *will occur again, combined with products of the dimensions* $n + 2$, $n + 4$, *&c. to infinity, but not with* $n + 1$, $n + 3$, &c. For instance, the argument $i\mathrm{W} + 3\mathrm{V}$ cannot occur combined with any dimension of the eccentricities less than the third, and will occur again in the terms multiplied by the 5th, 7th, &c. dimensions, but not by the 4th, 6th, or any even dimensions.

Since $\mathrm{W} = nt - n't + \epsilon - \epsilon'$, $\mathrm{V} = nt + \epsilon - \pi$, and $\mathrm{V}' = n't + \epsilon' - \pi'$, the argument $i\mathrm{W} + k\mathrm{V} + l\mathrm{V}'$ is equal to

$$(i + k)\,.\,nt - (i - l)\,n't + (i + k)\,\epsilon - (i - l)\,\epsilon' - k\pi - l\pi'$$

If then we would inquire in what terms any proposed combination of n and n', such, for instance, as $(fn - gn')t$ can originate, we have only to put

$$i + k = f, \qquad i - l = g$$

which give

$$i = f - k, \qquad l = i - g = (f - g) - k$$

taking then in succession $k = 0$, $\quad k = \pm 1$, $\quad k = \pm 2$, &c. we get

$$i = f, \qquad i = f \mp 1, \qquad i = f \mp 2, \text{ \&c.}$$
$$l = f - g, \qquad l = f - g \mp 1, \qquad l = f - g \mp 2, \text{ \&c.}$$

For instance, if we would know from what terms the combination $(2n - n')\,t$ can originate, the corresponding values of i, k, l, are

1st. $i = 2$, $k = 0$, $l = -3$; 2dly. $\begin{cases} i = 1, & k = 1, \quad l = -4 \\ i = 3, & k = -1, \quad l = -2 \end{cases}$ 3dly. $\begin{cases} i = 0, & k = 2, \quad l = -5 \\ i = 4, & k = -2, \quad l = -1 \end{cases}$

So that any of the arguments comprised in the following series,

$$2\mathrm{W} - 3\mathrm{V}', \begin{cases} \mathrm{W} + \mathrm{V} - 4\mathrm{V}'; \quad 2\mathrm{V} - 5\mathrm{V}'; \\ 3\mathrm{W} - \mathrm{V} - 2\mathrm{V}'; \quad 4\mathrm{W} - 2\mathrm{V} - \mathrm{V}'; \quad 5\mathrm{W} - 3\mathrm{V}, \text{ \&c.} \end{cases}$$

will produce the combination in question. Now, the lowest sum of the co-efficients of V, V' in these arguments taken without regard to their signs, is 3: consequently, the combination $2nt - 5n't$ will first occur among the inequalities multiplied by the cubes or products of three dimensions of the eccentricities, and among them only in such terms as produce the arguments

$$2\mathrm{W} - 3\mathrm{V}', \quad 3\mathrm{W} - \mathrm{V} - 2\mathrm{V}', \quad 4\mathrm{W} - 2\mathrm{V} - \mathrm{V}', \quad 5\mathrm{W} - 3\mathrm{V}.$$

Let us now examine the co-efficients of the several arguments as they occur in the values of Q, Π, &c.

The co-efficient of any argument, such as $i\mathrm{W} \pm k\mathrm{V} \pm l\mathrm{V}'$ in the developement of Ω or $r\dfrac{d\Omega}{dr}$ will obviously consist only of combinations of a, a', A, A', A'', &c. and their differential co-efficients with a power or product of e, e', and may therefore be regarded as a given quantity, and its value, with more or less trouble, numerically computed. Taking A for the general representative of such a combination, M will be a function of a, a', e, e', and of these only, and

$$\mathrm{M}\cos\,(i\mathrm{W} \pm k\mathrm{V} \pm l\mathrm{V}')$$

will be the general form of any term of Ω or $r\dfrac{d\Omega}{dr}$.

In the value of Q, the terms of $r\dfrac{d\Omega}{dr}$ enter unchanged; but since

$$d\mathrm{W} = n\,dt, \quad d\mathrm{V} = n\,dt, \quad d\mathrm{V}' = 0$$

the term under consideration will produce in $d\Omega$ the term $-\mathrm{M}\,.\,(i \pm k)\,n\,dt\,.\sin\,(i\mathrm{W} \pm k\mathrm{V} \pm l\mathrm{V}')$ and in Q, the term

$$\frac{2\,(i \pm k)\,n}{(i \pm k)\,n \pm l\,n'}\,\mathrm{M}\cos\,(i\mathrm{W} \pm k\mathrm{V} \pm l\mathrm{V}')$$

so that Q will contain two species of terms, those whose co-efficients are of the form M, and $\mathrm{M}\dfrac{(i \pm k)\,n}{(i \pm k)\,n \pm l\,n'}$

The value of Q substituted in Π (equation 128) will produce terms comprised in one or other of the forms

$$\mathrm{M}\,.\cos\,(i\mathrm{W} \pm k\mathrm{V} \pm l\mathrm{V}')$$

$$\mathrm{M}\,.\,\frac{(i \pm k)\,n}{(i \pm k)\,n \pm l\,n'}\,.\cos\,(i\mathrm{W} \pm k'\mathrm{V} \pm l\mathrm{V}')$$

$$\mathrm{M}\,.\,\frac{n}{(i \pm k)\,n \pm l\,n'}\,.\cos\,(i\mathrm{W} \pm k\mathrm{V} \pm l\mathrm{V}')$$

$$\mathrm{M}\,.\,\frac{(i \pm k)\,n^2}{\{(i \pm k)\,n \pm l\,n'\}\,\{(i \pm k')\,n \pm l\,n'\}}\,.\cos\,(i\mathrm{W} \pm k'\mathrm{V} \pm l\mathrm{V}')$$

In the process of integration by which δu is derived from Π (or in the integration of the equation (129)) these terms again acquire factors of the form $\frac{1}{\{(i \pm k) n \pm l n'\}^2 - n^2}$ or $\frac{1}{\{(i \pm k') n \pm l n'\}^2 - n^2}$ according as k or k' occurs in the argument; and as these forms are obviously not altered in the transition from δu to δr, the terms of δr will necessarily be included in one of the forms

$$\mathrm{M} \,.\, \frac{1}{\{(i \pm k) n \pm l n'\}^2 - n^2} \cos (i \mathrm{W} \pm k \mathrm{V} \pm l \mathrm{V}')$$

$$\mathrm{M} \,.\, \frac{(i \pm k') n}{\{(i \pm k) n \pm l n'\} \{(i \pm k) n \pm l n')^2 - n^2\}} \cos (i \mathrm{W} \pm k \mathrm{V} \pm l \mathrm{V}')$$

$$\mathrm{M} \,.\, \frac{n}{\{(i \pm k) n \pm l n'\} \{(i \pm k) n \pm l n')^2 - n^2\}} \cos (i \mathrm{W} \pm k \mathrm{V} \pm l \mathrm{V}')$$

$$\mathrm{M} \,.\, \frac{(i \pm k') n^2}{\{(i \pm k') n \pm l n'\} \{(i \pm k) n \pm l n'\} \{(i \pm k) n \pm l n')^2 - n^2\}} \cos (i \mathrm{W} \pm k \mathrm{V} \pm l \mathrm{V}')$$

Let the several functions of n, n', in the co-efficients of these terms be represented indiscriminately by N, then will the general form of the terms of δr be $\mathrm{M} \,.\, \mathrm{N} \,.\, \cos (i \mathrm{W} \pm k \mathrm{V} \pm l \mathrm{V}')$.

It remains only to consider the nature of the terms of which $\delta \theta$ consists. Now these will be,

1st. Those arising from the term $\frac{d r}{d t} \delta r$, which are of the form

$$\frac{\mathrm{M} \,.\, k'' n \,.\, \mathrm{N}.}{n a^2 \sqrt{1 - e^2}} \sin (i \mathrm{W} \pm (k \pm k'') \mathrm{V} \pm l \mathrm{V}')$$

2dly. Those arising from $r \frac{d \delta r}{d t}$, whose form is

$$\frac{2 \mathrm{M} \{(i \pm k) n \pm l n'\} \mathrm{N}}{n a^2 \sqrt{1 - e^2}} \sin (i \mathrm{W} \pm (k \pm k'') \mathrm{V} \pm l \mathrm{V}')$$

3dly. Those arising from $\int r \frac{d \Omega}{d r} n \, d t$ and $\iint d \Omega \,.\, n \, d t$ whose forms are respectively

$$\frac{2 a \mathrm{M}}{\mu \sqrt{1 - e^2}} \,.\, \frac{n}{(i \pm k) n \pm l n'} \sin (i \mathrm{W} \pm k \mathrm{V} \pm l \mathrm{V}'), \text{ and } \frac{3 a \mathrm{M}}{\mu \sqrt{1 - e^2}} \,.\, \frac{(i \pm k) n^2}{\{(i \pm k) n \pm l n'\}^2} \sin (i \mathrm{W} \pm k \mathrm{V} \pm l' \mathrm{V})$$

and lastly those peculiar terms containing t out of the signs sin and cos which we have already noticed as giving rise to the secular equations.

The complete enumeration of all the possible varieties of terms which $\delta \theta$ may contain, will therefore be had by putting for N each of the four forms above assigned to it; but as those only really differ importantly in which the denominators of the fractions differ, we need only enumerate the latter quantities; which are,

$$(i \pm k) n \pm l n'; \quad \{(i \pm k) n \pm l n'\}^2;$$
$$\{(i \pm k) n \pm l n'\}^2 - n^2; \quad ((i \pm k') n \pm l n') \{((i \pm k) n \pm l n')^2 - n^2\}$$
$$\{(i \pm k') n \pm l n'\} \{(i \pm k) n \pm l n'\} \{((i \pm k) n \pm l n')^2 - n^2\}$$

These then are the various forms of the divisors with which the processes of integration, &c., affect the inequalities in longitude. They are, as we have already remarked, of the highest importance in the theory of the planets, by reason of their effect on the numerical values of the maxima of the perturbations to which they belong. Such is the immense number of terms, or rather of series of terms, branching out in all directions, of which the perturbations consist, that it is manifestly in vain to attempt to take account of them all. It is therefore of the highest consequence to have some guiding principle to direct us in our choice of the terms to be retained or neglected. Were it not for these divisors, we might safely rely on the rapid convergency of the powers and products of the eccentricities; and reject, without further examination, all in which their dimension exceeded a certain limit; but should there be an approach to commensurability in the periodic times of the two planets, (as, for instance, should five times the mean motion of the disturbed planet ($5 n' t$) be very nearly equal to twice that of the disturbing, ($2 n t$)) this circumstance will render some one of their factors ($5 n' - 2 n$) very small. In consequence, all the divisors into which this factor enters will become very small, and the inequalities affected by them will, in consequence, acquire from this cause an unnatural magnitude (if we may use such an expression) and must be retained, even though of such an order as would otherwise authorize their rejection.—The terms so affected too, will originate in a great variety of manners from the developements, and may be affected with various powers of the eccentricities; so that their number will necessarily be infinite, and for the purpose of approximation only the most prominent can be selected.

The equations of the motions of Jupiter and Saturn, known by the name of the great inequalities of these planets, were long a difficulty in the way of the theoretical astronomer, and even a stumbling block in the way of the Newtonian philosophy. It was observed, on comparing very ancient observations of the oppositions of these planets with more modern ones, that their *mean motions* had undergone an apparent alteration ; that of Saturn appearing to have been retarded, and that of Jupiter accelerated. In other words, that Saturn perpetually lagged behind, and Jupiter as constantly surpassed the places, when they ought to have been, on the hypothesis of the mean motion, or periodic time, remaining invariable.

We have seen that every inequality of very long period will appear, while on the increase, to affect the mean motion ; it is obvious it must, if the latter, as determined by observations comprised within the periods of its increase, be compared with the result of similar observations made while its value is on the diminution. Now, the length of the period of any inequality depends on the multiples of the mean motions found in its argument ; and it was not difficult for geometers to shew, that, so far as the first powers or squares of the eccentricities were concerned, no inequalities of such very long periods as the case required, could be found in the motion of either planet. The cubes and higher powers had all along been neglected without fear of error ; but Laplace having, from other considerations, ascertained that an acceleration in Jupiter's motion *being supposed*, a retardation in Saturn's *must follow* of course, and that in the very proportion observed ; and that therefore the phenomenon was not altogether inconsistent with the laws of gravity, set himself to examine the terms multiplied by the cubes of the eccentricities. Here he immediately encountered the argument $(5\,n\,t - 2\,n't + \text{const.})$; and the mean motion of Jupiter being to that of Saturn nearly in the proportion of 5 to 2, if we suppose n' to correspond to Jupiter's and n to Saturn's motion the co-efficient $5\,n - 2\,n'$ is very small, and the corresponding period is found on calculation to amount to 918 years. The resulting inequality has also $5\,n - 2\,n'$ for its divisor, and its magnitude is thus increased as well as its period lengthened. On executing the calculation, the inequalities of both planets were found to be such as would completely account for the apparent accelerations and retardations observed.

Section VI.

Of the variations of the elements of the planetary orbits, and the secular equations of their motions. Theory of the major axes, inclinations, nodes, eccentricities, and aphelia.

We have already taken occasion to observe, that the motions of the planets may be regarded as performed in ellipses, whose positions and magnitudes are continually, but very slowly, changing, by the effects of the disturbing forces. These forces are so small, that in a moderate period of time, as for instance, in a single revolution of a planet, the change is insensible ; and, if we allow for those inequalities which depend on the configurations of the disturbed and disturbing planets, the theory of which has been exposed in the foregoing sections, the motion, equated by the application of these corrections, will coincide with almost rigorous exactness with the elliptic theory. But after the expiration of many revolutions of the planet, this exact coincidence will cease to take place, even when its place is corrected for such periodical inequalities. The place so corrected is found, it is true, in the circumference of an ellipse with the sun in its focus, but it is not an ellipse of precisely the same form and position as before. Its elements have undergone a change, and this change, though imperceptible in a single revolution, becomes gradually more and more evident, till at length it is too remarkable to be overlooked.

Such slow changes are what we understand by the secular variations of the elements of the planet's orbits. But there is another point of view in which we may consider the subject, which presents peculiar facilities to the application of mathematical investigation. It consists in referring *all* the inequalities resulting from perturbation, to the variation of the elliptic elements, not merely those of long periods, but those which pass rapidly from their maxima to their minima, and depend on the configurations of the bodies. We are indebted to Lagrange for this view of the subject, and shall endeavour to give an idea in this section of the luminous analysis of that great geometer.

If, at the expiration of any instant, the disturbing forces were to cease acting, the planet would go on describing an exact ellipse, of which the infinitesimal arc described in the last instant would be an elementary portion. The plane of the ellipse would be that in which this portion and the sun's centre lie ; its eccentricity, position, and magnitude, would all be determined from the *position*, *magnitude*, and *curvature* of this element, and the laws of elliptic motion. In a word, it would be a real *ellipse of curvature* to the actual curve described by the planet, at that particular instant, subjected to the conditions of having its focus in the sun, and satisfying the other laws of elliptic motion during that moment. The elements of the planet's orbit then, at any moment, are no other than the elements of this ellipse, and are determined from three consecutive places of the planet, infinitely near each other. Thus every inequality in its motion will produce a corresponding fluctuation in the elements, which will thus be subject to as many equations, periodical or otherwise, as the planet's motion itself is affected with.

The periodical terms thus originating in the elements, will, in the course of many revolutions, compensate each other ; but if it should happen that terms not periodical should find their way into their values, these will express *secular* changes, which it becomes of the utmost importance to investigate.

To determine the ellipse of curvature at any instant, is a matter of no difficulty ; we have only to call to mind that any one of the constants in its equation may be insulated, and expressed in terms of the co-ordinates and their differential co-efficients, by the mere operations of differentiating and eliminating ; so that, supposing the co-ordinates and their differential co-efficients given at any instant, any one of the constants may have its value ascertained by simple substitution. But it is not necessary to go through these processes —we may avoid that trouble by recurring to the origin of the ellipse itself. Now all we know of it is, that it satisfies the dynamical relations of the problem, on the supposition of the disturbing forces ceasing to act at the instant $d t$. Consequently its equations, however transformed, must be such as to satisfy the differential equations

$$\frac{d^2 x}{d t^2} + \frac{\mu x}{r^3} = 0$$

$$\frac{d^2 y}{d t^2} + \frac{\mu y}{r^3} = 0$$

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} = 0$$

and its elements will be the constants introduced by the integration of these, or known functions of them.

For instance, let us consider its major semiaxis. If we pursue with these the same process of integration by which equation (109) was obtained, viz. multiply the first by $d x$, the second by $d y$, and the third by $d z$ and add, and integrate, we find

$$\frac{\mu}{a} = \frac{2 \mu}{r} - \frac{d x^2 + d y^2 + d z^2}{d t^2}$$

a being the arbitrary constant introduced by integration; and if we compare this with (33), we shall see that a is the semiaxis of the ellipse. Thus we know, that all that is necessary to obtain the semiaxis of the ellipse of curvature at any instant (let the body describe what curve it will) is merely to substitute for r, $\frac{d x}{d t}$, $\frac{d y}{d t}$, $\frac{d z}{d t}$, in the expression

$$\frac{\mu}{\frac{2 \mu}{r} - \left\{\left(\frac{d x}{d t}\right)^2 + \left(\frac{d y}{d t}\right)^2 + \left(\frac{d z}{d t}\right)^2\right\}}$$

those values which, in the case proposed, they *actually have* in virtue of the real motion of the body, such as the forces in action make it.

Now, in the case of disturbed motion, $d x$, $d y$, $d z$, are given by the equation (94); for if we integrate these after multiplying them respectively by $2 d x$, $2 d y$, $2 d z$, we find

$$\left(\frac{d x}{d t}\right)^2 = -\mu \int \frac{2 x d x}{r^3} - 2 m \int \frac{d \Omega}{d x} d x$$

$$\left(\frac{d y}{d t}\right)^2 = -\mu \int \frac{2 y d y}{r^3} - 2 m' \int \frac{d \Omega}{d y} d y$$

$$\left(\frac{d z}{d t}\right)^2 = -\mu \int \frac{2 z d z}{r^3} - 2 m' \int \frac{d \Omega}{d z} d z$$

consequently adding all together,

$$\left(\frac{d x}{d t}\right)^2 + \left(\frac{d y}{d t}\right)^2 + \left(\frac{d z}{d t}\right)^2 = -\mu \int \frac{2 x d x + 2 y d y + 2 z d z}{r^3} - 2 m' \int \left(\frac{d \Omega}{d x} d x + \frac{d \Omega}{d y} d y + \frac{d \Omega}{d z} d z\right)$$

$$= \frac{2 \mu}{r} - 2 m' \int d \Omega$$

and substituting, we get

$$a = \frac{\mu}{2 m' \int d \Omega}, \text{ or } \frac{\mu}{a} = 2 m' \int d \Omega; \qquad (165, 1)$$

The same result will be obtained as follows:—If we integrate, as in (109), and instead of adding explicitly the arbitrary quantity $\frac{\mu}{a}$ to complete the integral, regard it as included under the sign $\int$ we have

$$\frac{2 \mu}{r} - \frac{d x^2 + d y^2 + d z^2}{d t^2} = 2 m' \int d \Omega$$

but if a be the major semiaxis of the ellipse of curvature,

$$\frac{\mu}{a} = \frac{2 \mu}{r} - \frac{d x^2 + d y^2 + d z^2}{d t^2}$$

Hence we get

$$\frac{\mu}{a} = 2 m' \int d \Omega$$

If (a) be the major semiaxis at the commencement of the time t, and $\int d \Omega$ be taken, so as to vanish when

$t = o$, we have

$$\frac{\mu}{a} = \frac{\mu}{(a)} + 2\,m' \int d\,\Omega \qquad (165, 2)$$

In this instance we have had no difficulty in arriving at once at the finite expression for the varied element. But it is in other cases more commodious to express its *momentary* variation. Let us therefore denote by the characteristic δ, that peculiar variation by which the ellipse of curvature passes from the form and position it had during $d\,t$ to that which it has in the consecutive instant, then $\delta\,a$, $\delta\,e$, $\delta\,\pi$, &c. will be the momentary variations of its semiaxis, eccentricity, perihelion, &c. Moreover, $\delta\,x$, $\delta\,y$, $\delta\,z$, and $\delta\,r$, will represent the excesses of the values of x, y, z, r, in the varied ellipse; not over what they *were* in the former instant, but over what they *would have been* had that ellipse remained unaltered. But, both in the one case and the other, the point in the ellipse to which they correspond, coincides with the real place of the planet. Hence the lines x, y, z, r, are the same in the varied ellipse, in the unvaried, and in the actual curve described; so that $\delta\,x = o$, $\delta\,y = o$, $\delta\,z = o$, and $\delta\,r = o$. Again, $\delta\,\frac{d\,x}{d\,t}$, $\delta\,\frac{d\,y}{d\,t}$, $\delta\,\frac{d\,z}{d\,t}$, are the excesses of the values of $\frac{d\,x}{d\,t}$, $\frac{d\,y}{d\,t}$, $\frac{d\,z}{d\,t}$, in the varied ellipse over what they would have been had the ellipse not varied—that is, had the disturbing force not acted,—in the instant consecutive to $d\,t$. Their values therefore would vanish, had the body remained in its former ellipse, and, in general, will be obtained by subtracting from the values actually assumed by $\frac{d\,x}{d\,t}$, &c. in the consecutive instant in the curve, what *would have been* assumed by them had the body continued in the same ellipse, or had the disturbing forces ceased to act at the end of $d\,t$. Now, in the *curve* the consecutive value of $\frac{d\,x}{d\,t}$ is

$$\frac{d\,x}{d\,t} + d\,\frac{d\,x}{d\,t} = \frac{d\,x}{d\,t} - \left\{\frac{\mu\,x}{r^3} + m'\,\frac{d\,\Omega}{d\,x}\right\} d\,t$$

because $\frac{\mu\,x}{r^3} + m'\frac{d\,\Omega}{d\,x}$ being the force in the curve, we must have $d\,\frac{d\,x}{d\,t} = -\left\{\frac{\mu\,x}{r^3} + m'\,\frac{d\,\Omega}{d\,x}\right\} d\,t$. On the other hand, had the disturbing force ceased acting, we should have had simply $d\,\frac{d\,x}{d\,t} = -\frac{\mu\,x}{r^3}\,d\,t$, so that the consecutive value of $\frac{d\,x}{d\,t}$ *would have been* merely $\frac{d\,x}{d\,t} - \frac{\mu\,x}{r^3}\,d\,t$.

Hence we have

and similarly

$$\left.\begin{aligned} \delta\,\frac{d\,x}{d\,t} &= -\,m'\,\frac{d\,\Omega}{d\,x}\,d\,t \\ \delta\,\frac{d\,y}{d\,t} &= -\,m'\,\frac{d\,\Omega}{d\,y}\,d\,t \\ \delta\,\frac{d\,z}{d\,t} &= -\,m'\,\frac{d\,\Omega}{d\,z}\,d\,t \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots \quad (166)$$

To explain how the variations so obtained may be employed, let us take again the case already treated.

$$\frac{\mu}{a} = \frac{2\,\mu}{r} - \frac{d\,x^2 + d\,y^2 + d\,z^2}{d\,t^2}$$

If we differentiate this relative to the characteristic δ, we get

$$-\frac{\mu\,\delta\,a}{a^2} = -\frac{2\,\mu\,\delta\,r}{r^2} - 2\left\{\frac{d\,x}{d\,t}\,\delta\,\frac{d\,x}{d\,t} + \frac{d\,y}{d\,t}\,\delta\,\frac{d\,y}{d\,t} + \frac{d\,z}{d\,t}\,\delta\,\frac{d\,z}{d\,t}\right\}$$

in which, putting $\delta\,r = o$, and for $\delta\,\frac{d\,x}{d\,t}$ &c., their values above found, we find

$$-\frac{\mu\,\delta\,a}{a^2} = +\,2\,m'\left\{\frac{d\,\Omega}{d\,x}\,d\,x + \frac{d\,\Omega}{d\,y}\,d\,y + \frac{d\,\Omega}{d\,z}\,dz\right\} = 2\,m'\,d\,\Omega$$

$\delta\,a$ is the *momentary* variation of a in the instant $d\,t$, so that this equation may be integrated relative to t, and we get

$$\frac{\mu}{a} = 2\,m' \int d\,\Omega$$

the same result as before.

Before we proceed farther, we will stop to draw from this expression of the reciprocal axis, a most important conclusion. It is this—that all the variations to which the major axes of the planetary orbits are subjected by their mutual attraction are periodical—and that the mean distances, and consequently the mean motions of the planets are subject to no *secular* variations. In fact, when we consider only the first power of the disturbing forces, we have already proved that the developement of Ω is entirely composed of terms of the form

$$A\,.\cos\,(i\,W + k\,V + l\,V')$$

hence

$$\begin{aligned} d\,\Omega &= A\,.\left(i\,\frac{d\,W}{n\,d\,t} + k\,\frac{d\,V}{n\,d\,t}\right).\sin\,(i\,W + k\,V + l\,V') \\ &= A\,.\,(i + k)\,.\sin\left\{(i + k)\,n\,t - (i - l)\,n'\,t + (i + k)\,\epsilon - (i - l)\,\epsilon' - k\,\pi - l\,\pi'\right\} \end{aligned}$$

which is always periodic unless $i + k = o$ and $i - l = o$, when the argument becomes simply $i\,(\pi - \pi')$ a constant quantity; but in this case the whole term vanishes by the disappearance of its co-efficient. Thus $d\,\Omega$ and, of course, $\int d\,\Omega$ contains no term multiplied by t, and none but what is periodic; consequently, $\frac{\mu}{a}$ is periodic also.

This beautiful result, the demonstration of which is of almost elementary simplicity, assures us of the impossibility of any of the bodies of our system ever leaving it in consequence of the disturbances it may experience, and secures the general permanence of the whole, by keeping the mean distances and periodic times perpetually fluctuating between certain limits (very restricted ones) which they can neither exceed nor fall short of.

Let us next consider the variation in the position of the plane of the disturbed orbit. The equations (4), (5), and (6), give

$$h = y\,\frac{d\,x}{d\,t} - x\,\frac{d\,y}{d\,t}, \qquad h' = z\,\frac{d\,x}{d\,t} - x\,\frac{d\,z}{d\,t}, \qquad h'' = z\,\frac{d\,y}{d\,t} - y\,\frac{d\,z}{d\,t}$$

These quantities in the case of elliptic motion are constant, but in that of disturbed motion they will equally hold good, if h, h', h'', be regarded as variable. Now, either on the one or the other supposition, if we multiply the first by z, the second by $-y$, and the third by x, and add, we get

$$h\,z - h'\,y + h''\,x = o$$

which is the same with equation (7). But if we mutiply the first by $d\,z$, the second by $-d\,y$, and the third by $d\,x$, we shall also obtain by addition,

$$h\,d\,z - h'\,d\,y + h''\,d\,x = o$$

Consequently, even when we regard $h\;h'\;h''$ as variable, still the equation $h\,z - h'\,y + h''\,x = o$ and its differential relative to x, y, z, subsist together, just as if h, h', h'' were constant. Hence, it appears, that the body at the end of the instant $d\,t$ is still found in the plane represented by $h\,z - h'\,y + h''\,x = o$, or, that this plane is the plane in which the elementary arc described in the instant $d\,t$, lies. If therefore we call ϕ its inclination to that of the x and y taken as a fixed plane, and ω the longitude of its ascending node, we have

$$\tan\phi = \frac{\sqrt{h'^2 + h''^2}}{h}; \quad \tan\omega = \frac{h''}{h'}; \quad \mu\,a\,(1 - e^2) = h^2 + h'^2 + h''^2; \tag{167}$$

whence these elements (viz. the inclination, the longitude of the node, and the semiparameter,) are expressed in terms of h, h', h''.

That the equation $h\,d\,z - h'\,d\,y + h''\,d\,x = o$ must hold good at the same time with $h\,z - h'\,y + h''\,x = o$ is also evident from this consideration, that although the ellipse, it is true, varies from one instant to another, yet it must be regarded as invariable, while the body describes each of its elementary portions; because, by hypothesis, it is so adjusted that the body shall remain in its circumference during the whole of the instant $d\,t$, and it is not till the consecutive instant that it is necessitated to change its form, &c. to accommodate itself to the new course taken by the body. The same reasoning holds for any other finite equation of elliptic motion. Its first differential may be taken, as if the arbitrary constants it involves were rigorously such. The disturbing forces make no change on x, y, z, r, *their* consecutive values remain $x + d\,x$, $y + d\,y$, $z + d\,z$, $r + d\,r$, as before, and are common both to the curve and the ellipse, it is fo the consecutive values of $d\,x, d\,y, d\,z$, that they differ, these becoming $d\,x + d^2\,x + \delta\,d\,x$ for the curve, and $d\,x + d^2\,x$ for the ellipse.

This premised, we have only to inquire the variations of h, h', h''; and to this end, by the equations (4, 5, 6,) we have

$$\delta\,h = y\,\delta\,\frac{d\,x}{d\,t} - x\,\delta\,\frac{d\,y}{d\,t}; \quad \delta\,h' = z\,\delta\,\frac{d\,x}{d\,t} - x\,\delta\,\frac{d\,z}{d\,z}; \quad \delta\,h'' = z\,\delta\,\frac{d\,y}{d\,t} - y\,\delta\,\frac{d\,z}{d\,t}.$$

In which, substituting for $\delta\,\frac{d\,x}{d\,t}$, $\delta\,\frac{d\,y}{d\,t}$, $\delta\,\frac{d\,z}{d\,t}$, their values $-\,m'\,\frac{d\,\Omega}{d\,x}\,d\,t$, $-\,m'\,\frac{d\,\Omega}{d\,y}\,d\,t$, and $m'.\,\frac{d\,\Omega}{d\,z}\,d\,t$, we find

$$\left.\begin{aligned} \delta\,h &= m'\left\{x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x}\right\}d\,t; \\ \delta\,h' &= m'\left\{x\,\frac{d\,\Omega}{d\,z} - z\,\frac{d\,\Omega}{d\,x}\right\}d\,t; \\ \delta\,h'' &= m'\left\{y\,\frac{d\,\Omega}{d\,z} - z\,\frac{d\,\Omega}{d\,y}\right\}d\,t; \end{aligned}\right\} \dots\dots\dots\dots\dots\dots \tag{168}$$

and integrating relative to t,

$$h = (h) + m'\int\left\{x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x}\right\}d\,t; \tag{169, 1}$$

$$h' = (h') + m'\int\left\{x\,\frac{d\,\Omega}{d\,z} - z\,\frac{d\,\Omega}{d\,x}\right\}d\,t; \tag{169, 2}$$

$$h'' = (h'') + m'\int\left\{y\,\frac{d\,\Omega}{d\,z} - z\,\frac{d\,\Omega}{d\,y}\right\}d\,t; \tag{169, 3}$$

where (h), (h'), and (h''), represent the primitive values of these respective quantities at the commencement of the time t.

As these expressions are rigorous, we may derive from them all the laws which regulate the motion of the nodes and the inclinations of the planes of the orbits; but as it is only the secular variations which concern us at present, we shall not regard the periodical parts of the expressions within the brackets under the integral signs. To develope them, we must consider the orbits as inclined to the plane of the x, y; but if we take the undisturbed orbit of m for this plane, the value of z at any time t will be of the order of the disturbing forces, and z' will be a very small quantity; so that z^2, $z\,z'$, and z'^2, may be neglected. Hence $r\ (= \sqrt{x^2 + y^2 + z^2})$ and r', will represent with this degree of approximation, their projections on the plane of the x, y, and we have (calling s the tangent of m's latitude,)

$$x = r\,.\cos\theta,\quad y = r\,.\sin\theta,\quad z = r\,s;\quad x' = r'\,.\cos\theta',\ y' = r'\,.\sin\theta',\ z' = r'\,s'$$

If then we recur to the expressions for the disturbing forces $m'\dfrac{d\,\Omega}{d\,x}$, &c. in Section I. Part II. we shall find

$$y\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,y} = (y\,x' - x\,y')\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right); \tag{170, 1}$$

$$z\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,z} = (z\,x' - x\,z')\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right); \tag{170, 2}$$

$$z\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,z} = (z\,y' - y\,z')\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right); \tag{170, 3}$$

Now, since $h\,z - h'\,y + h''\,x = o$, we have $z = -\dfrac{h''}{h}\,x + \dfrac{h'}{h}\,y$

Suppose then, $\dfrac{h''}{h} = p$ and $\dfrac{h'}{h} = q$, and let the quantities corresponding to p and q in the orbit of m' be p' and q'; then, if ϕ, ϕ', be the inclinations of the two orbits to the fixed plane, and ω, ω', the longitudes of their ascending nodes, we shall have $\tan\phi = \sqrt{p^2 + q^2}$, $\tan\omega = \dfrac{p}{q}$, $\tan\phi' = \sqrt{p'^2 + q'^2}$, $\tan\omega' = \dfrac{p'}{q'}$, and thus, when p and q are determined, the inclinations and places of the nodes are easily found. We have, moreover,

$$z = q\,y - p\,x,\quad z' = q'\,y' - p'\,x'$$
$$z\,x' - x\,z' = (p' - p)\,x\,x' + q\,y\,x' - q'\,x\,y'$$
$$z\,y' - y\,z' = -\,(q' - q)\,y\,y' - p\,x\,y' + p'\,y\,x'$$

and if we therefore suppose for a moment $\mathrm{M} = \dfrac{1}{r'^3} - \dfrac{1}{\lambda^3}$,

$$\frac{d\,h}{d\,t} = m'\,\mathrm{M}\,.\,(y\,x' - x\,y')$$
$$\frac{d\,h'}{d\,t} = m'\,\mathrm{M}\,.\,\{(p' - p)\,x\,x' + q\,y\,x' - q'\,x\,y'\}$$
$$\frac{d\,h''}{d\,t} = m'\,\mathrm{M}\,.\,\{-\,(q' - q)\,y\,y' - p\,x\,y' + p'\,y\,x'\}$$

Now, our design being to eliminate h, h', h'', from the formulæ, and obtain expressions involving only p and q, from which h, h', h'', may be deduced if wanted, we differentiate the equations $p = \dfrac{h''}{h}$ and $q = \dfrac{h'}{h}$, when we find

$$\frac{d\,p}{d\,t} = -\,\frac{1}{h}\left(\frac{d\,h''}{d\,t} - p\,\frac{d\,h}{d\,t}\right)\qquad \frac{d\,q}{d\,t} = \frac{1}{h}\left(\frac{d\,h'}{d\,t} - q\,\frac{d\,h}{d\,t}\right)$$

and if we substitute in these, the values of $\dfrac{d\,h}{d\,t}$, $\dfrac{d\,h'}{d\,t}$, and $\dfrac{d\,h''}{d\,t}$, as above found, we shall get

$$\frac{d\,p}{d\,t} = m'\,\mathrm{M}\,.\,\frac{(p' - p)\,y\,x' - (q' - q)\,y\,y'}{h} = \frac{m'\mathrm{M}}{h}\,y\,\{(p' - p)\,x' - (q' - q)\,y'\}$$
$$\frac{d\,q}{d\,t} = m'\,\mathrm{M}\,.\,\frac{(p' - p)\,x\,x' - (q' - q)\,x\,y'}{h} = \frac{m'\mathrm{M}}{h}\,x\,\{(p' - p)\,x' - (q' - q)\,y'\}$$

But if we neglect the squares of the disturbing forces, and the eccentricities and inclinations of the orbits, we have $\dfrac{m'}{h} = \dfrac{m'}{\sqrt{(1 + m)\,(1 - e^2)\,.\,a}} = \dfrac{m'}{\sqrt{a}}$, so that these expressions become

$$\frac{d\,p}{d\,t} = m'\,\mathrm{M}\,.\,\frac{(p' - p)\,x' - (q' - q)\,y'}{\sqrt{a}}\,y \tag{171}$$

$$\frac{d\,q}{d\,t} = m'\,\mathrm{M}\,.\,\frac{(p'-p)\,x' - (q'-q)\,y'}{\sqrt{a}}\,x \qquad (172)$$

Let us suppose the function $(a^2 - 2\,a\,a'\,.\,\cos w + a'^2)^{-\frac{3}{2}}$ developed in a series of cosines of w, and its multiples $\mathrm{S} + \mathrm{S}'\,.\,\cos w + \mathrm{S}''\,.\,\cos 2\,w + \&c.$
then we shall have, by (135)

$$\left.\begin{aligned}
\mathrm{S} &= \frac{1}{a^3}\left\{1 + \left(\frac{3}{2}\right)^2 a^2 + \left(\frac{3\,.\,5}{2\,.\,4}\right)^2 a^4 + \left(\frac{3\,.\,5\,.\,7}{2\,.\,4\,.\,6}\right)^2 a^6 + \&c.\right\}\\
\mathrm{S}' &= \frac{2}{a^3}\left\{\frac{3}{2}\,a + \frac{3^2\,.\,5}{2^2\,.\,4}\,a^3 + \frac{3^2\,.\,5^2\,.\,7}{2^2\,.\,4^2\,.\,6}\,a^5 + \&c.\right\}\\
\mathrm{S}'' &= \frac{2}{a^3}\left\{\frac{3\,.\,5}{2\,.\,4}\,a^2 + \frac{3^2\,.\,5\,.\,7}{2^2\,.\,4\,.\,6}\,a^4 + \&c.\right\}
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots \qquad (173)$$

and if we neglect in the present research, as is allowable, the eccentricities and inclinations, or suppose the orbits circular and in one plane,

$$\mathrm{M} = \frac{1}{a'^3} - \frac{1}{\lambda^3} = \left(\frac{1}{a'^3} - \mathrm{S}\right) - \mathrm{S}'\,.\,\cos(\theta' - \theta) - \mathrm{S}''\,.\,\cos 2\,(\theta' - \theta) - \&c. \qquad (174)$$

$$r = a,\ r' = a',\ \theta = n\,t + \epsilon,\ \theta' = n'\,t + \epsilon',\ \theta - \theta' = n\,t - n'\,t + \epsilon - \epsilon' = \mathrm{W}$$

$$y\{(p'-p)\,x' - (q'-q)\,y'\} = a\,a'\,.\,\frac{q'-q}{2}\{\cos(\theta'+\theta) - \cos(\theta'-\theta)\}; \qquad (175)$$

$$+\,a\,a'\,.\,\frac{p'-p}{2}\{\sin(\theta'+\theta) - \sin(\theta'-\theta)\}$$

$$x\{(p'-p)\,x' - (q'-q)\,y'\} = a\,a'\,.\,\frac{p'-p}{2}\{\cos(\theta'+\theta) + \cos(\theta'-\theta)\}; \qquad (176)$$

$$-\,a\,a'\,.\,\frac{q'-q}{2}\{\sin(\theta'+\theta) + \sin(\theta'-\theta)\}$$

Each of these latter quantities is to be combined with M by multiplication, and in resolving each of the products of sines and cosines so originating into simple sines and cosines of sums and differences, it is obvious that constant terms will arise whenever similar terms are combined, by reason of the property $\cos \mathrm{A} \times \cos \mathrm{A} = \frac{1}{2}\cos 2\,\mathrm{A} + \frac{1}{2}$. Now the only argument common to both factors is $\theta' - \theta$; and, of course, the only terms in (174) and (175) from whose combination a constant term can originate, are $-\,\mathrm{S}'\,.\,\cos(\theta - \theta')$ and $-\,a\,a'\,.\,\frac{q'-q}{2}\,.\,\cos(\theta' - \theta)$. Consequently, if we reject all the periodical terms, and put $I = \frac{a\,a'}{4}\,\mathrm{S}'$, we have

$$\frac{d\,p}{d\,t} = \frac{m'}{\sqrt{a}}\,.\,I\,.\,(q' - q); \qquad (177,\ 1)$$

and similarly

$$-\,\frac{d\,q}{d\,t} = \frac{m'}{\sqrt{a}}\,.\,I\,(p' - p); \qquad (177,\ 2)$$

If we go through a process exactly analogous, so as to obtain differential equations for determining p' and q' relative to the orbit of m', we shall find them to be

$$\frac{d\,p'}{d\,t} = \frac{m}{\sqrt{a'}}\,I\,(q - q'); \qquad (177,\ 3)$$

$$-\,\frac{d\,q'}{d\,t} = \frac{m}{\sqrt{a'}}\,I\,(p - p'); \qquad (177,\ 4)$$

These four equations, being of the first order, and with constant co-efficients (for the secular variations of a, a', and therefore of S′ and of $\frac{a\,a'}{4} \times \mathrm{S}'$ or I, which are symmetrical functions of a, a', vanish,) are easily integrated. As they subsist simultaneously among the four variables p, p', q, q', we may integrate them all together, if we assume

$$p = \mathrm{A}\,.\,\sin(g\,t + k) \qquad p' = \mathrm{A}'\,.\,\sin(g\,t + k)$$
$$q = \mathrm{A}\,.\,\cos(g\,t + k) \qquad q' = \mathrm{A}'\,.\,\cos(g\,t + k)$$

for if we substitute these values, we find that the variable part divides off, and there remain the following equations of condition between the constants A, A′, and g,

$$g\,\mathrm{A} = \frac{m'}{\sqrt{a}}\,I\,(\mathrm{A}' \quad \mathrm{A}); \qquad (178,\ 1)$$

$$g\,A' = \frac{m}{\sqrt{a'}}\, I\,(A - A'); \qquad (178,\ 2)$$

In these one of the constants A, A′, g, remains indeterminate. Let this be A, then eliminating A′, we get for determining g,

$$\left.\begin{aligned} & g^2 + \left(\frac{m'}{\sqrt{a}} + \frac{m}{\sqrt{a'}}\right) I\, g = o \\ & g = o, \text{ or } g = -\frac{m\sqrt{a} + m'\sqrt{a'}}{\sqrt{a\,a'}}\,.\,I \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (179,\ 1,\ 2)$$

Now it is evident, that if g and g' be two values of g which satisfy the equations of condition, then A . sin $(g\,t + k)$ and B . sin $(g'\,t + k')$ will each be satisfactory values of p, and so for the rest; consequently (the equations being linear) their sum will be so, and we have

$$\begin{cases} p = A\,.\sin(g\,t + k) + B\,.\sin k' & (180,\ 1) \\ q = A\,.\cos(g\,t + k) + B\,.\cos k' & (180,\ 2) \end{cases}$$

$$\begin{cases} p' = A'.\sin(g\,t + k) + B\,.\sin k' & (180,\ 3) \\ q' = A'.\cos(g\,t + k) + B\,.\cos k' & (180,\ 4) \end{cases}$$

where A, B, k, k', are four arbitrary constants, and

$$g = -I\,.\,\frac{m\sqrt{a} + m'\sqrt{a'}}{\sqrt{a\,a'}}; \qquad A' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}\,.\,A; \qquad (181,\ 1,\ 2)$$

From these values of p and q, p' and q' it is easy to eliminate sin $(g\,t + k)$ and cos $(g\,t + k)$; for if we multiply (180, 1) by $m\sqrt{a}$, and (180, 3) by $m'\sqrt{a'}$, and add, noticing that $A\,.\,m\sqrt{a} + A'.\,m'\sqrt{a'} = o$ by reason of the relation between A and A′ (181, 2) we get

$$m\sqrt{a}\,.\,p + m'\sqrt{a'}\,.\,p' = (m\sqrt{a} + m'\sqrt{a'})\,B\,.\sin k' = \text{constant}; \qquad (182)$$

and similarly,

$$m\sqrt{a}\,.\,q + m'\sqrt{a'}\,.\,q' = (m\sqrt{a} + m'\sqrt{a'})\,B\,.\cos k' = \text{const.}; \qquad (183)$$

Moreover, we have

$$\tan\phi^2 = p^2 + q^2 = (A^2 + B^2) + 2\,AB\,.\cos(g\,t + k - k'); \qquad (184,\ 1)$$

$$\tan\phi'^2 = p'^2 + q'^2 = (A'^2 + B^2) + 2\,A'B\,.\cos(g\,t + k - k'); \qquad (184,\ 2)$$

Consequently,

$$m\sqrt{a}\,.\tan\phi^2 + m'\sqrt{a'}\,.\tan\phi'^2 = m\sqrt{a}\,(A^2 + B^2) + m'\sqrt{a'}\,(A'^2 + B^2) = \text{constant}; \qquad (185)$$

The arbitrary constants A, B, k, k', may be determined in any particular case, either by comparing the general expressions for p, p', q, q', with their actual values at any assigned instant as derived from observation, or from these last derived equations; for since $p^2 + q^2 = \tan\phi^2$ and $\frac{p}{q} = \tan\omega$, we have

$$p = \tan\phi\,.\sin\omega; \quad q = \tan\phi\,.\cos\omega; \quad p' = \tan\phi'.\sin\omega'; \quad q' = \tan\phi'.\cos\omega'; \qquad (186)$$

Now the equation (180) gives

$$(p' - p) = (A' - A)\,.\sin(g\,t + k); \quad q' - q = (A' - A)\,.\cos(g\,t + k)$$

consequently, if we take $t = o$, or if we assume for our data, the elements ϕ, ϕ', ω, ω', as they were observed at the epoch or origin of the time t, we find

$$\tan k = \frac{p' - p}{q' - q} = \frac{\tan\phi'.\sin\omega' - \tan\phi\,.\sin\omega}{\tan\phi'.\cos\omega' - \tan\phi\,.\cos\omega}; \qquad (187)$$

Hence A and A′ are found; for, the values of S, S′, &c. being known from equations (173), $I = \frac{a\,a'}{4}$ S′ is also known; and since by (181, 2)

$$A' - A = -\frac{m\sqrt{a} + m'\sqrt{a'}}{m'\sqrt{a'}}\,.\,A \text{ we get } A = -\frac{m'\sqrt{a'}\,(p' - p)}{(m\sqrt{a} + m'\sqrt{a'})\sin k} \qquad (188)$$

Again, if we divide (182) by (183), we find

$$\tan k' = \frac{m\sqrt{a}\,.\tan\phi\,.\sin\omega + m'\sqrt{a'}\,.\tan\phi'.\sin\omega'}{m\sqrt{a}\,.\tan\phi\,.\cos\omega + m'\sqrt{a'}\,.\tan\phi'.\cos\omega'}; \qquad (189)$$

$$B = \frac{m\sqrt{a}\,.\tan\phi\,.\sin\omega + m'\sqrt{a'}\,.\tan\phi'.\sin\omega'}{(m\sqrt{a} + m'\sqrt{a'})\,.\sin k'}; \qquad (190)$$

These constants once computed, the laws, periods, and limits of the motions of the planes of both orbits are known. The period in which the inequalities recur is deducible at once from the value of g. If we express the time t in Julian years, n, n' represent the mean motions of the planets m, m', in one such year, in parts of a whole circumference, and $n^2\,a^3 = 1$, that is, $\sqrt{a} = \frac{1}{n\,a}$, hence $g\left(= -I.\left(\frac{m}{\sqrt{a'}} + \frac{m'}{\sqrt{a'}}\right)\right)$ will be found as follows :—

$$g = -\,(m\,n'\,a' + m'\,n\,a)\,.\,I$$

and if we call T the whole period, $-g\,\mathrm{T} = 1$ circumference $= 1$; and

$$\mathrm{T} = -\frac{1}{g} = \frac{1}{(m\,n'\,a' + m'\,n\,a)\,.\,I}\,; \qquad (191)$$

where

$$I = \frac{a\,a'}{4}\,\mathrm{S}'$$

If n, n' be expressed in seconds, the numerator of T instead of being unity, must be $360 \times 60 \times 60'' = 2196000''$.

The limits of the variations of the inclinations are readily found; for it appears from the equations (184, 1, 2) that their maxima and minima occur when $g\,t + k - k' = o$ and 180°, and have for their corresponding values

$$\left.\begin{matrix}\mathrm{A} + \mathrm{B} \\ \mathrm{A}' + \mathrm{B}\end{matrix}\right\} \text{ and } \left.\begin{matrix}\mathrm{A} - \mathrm{B} \\ \mathrm{A}' - \mathrm{B}\end{matrix}\right\}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (192)$$

Now, it is obvious from the values of p and q, (180, 1, 2) that A and B are small quantities of the same order as p, q; so that the inclinations can never increase or diminish beyond certain very narrow limits. This follows too from the equation (185); for in the present state of our system, $\tan\phi$ and $\tan\phi'$ being extremely small, the sum $m\,\sqrt{a}\,.\tan\phi^2 + m'\,\sqrt{a'}\,.\tan\phi'^2$ is always a very minute quantity; and since $\sqrt{a}$ and $\sqrt{a'}$ must both be taken positively, (for $\sqrt{a} = \frac{1}{n\,a}$ and n is positive for both planets, the motions being both in the same direction) neither term separately can exceed the value of the constant; so that ϕ or ϕ' must remain for ever confined to a value not greatly different from what it now has, and the planes of the planetary orbits must keep for ever oscillating within very confined limits about their mean positions.

With regard to the nodes it is different. These are liable to great changes of place, and may even circulate for ever in one direction without returning. In fact, if we would determine the maxima and minima of their longitudes, we have only to put $d\,\omega = o$, $d\,\omega' = o$; the roots of which equations, if real, will indicate the stationary points; and, if imaginary, will shew that such points do not exist, or that the nodes circulate. Now we have

$$d\,\omega = \frac{d\,.\tan\omega}{1 + \tan\omega^2} = o, \text{ or } d\,.\tan\omega = o, \text{ or } \frac{d\,p}{p} = \frac{d\,q}{q}, \text{ or } q\,\frac{d\,p}{d\,t} - p\,\frac{d\,q}{d\,t} = o$$

in which equation, substituting for $\frac{d\,p}{d\,t}$ and $\frac{d\,q}{d\,t}$ their values in equations (177, 1, 2) we find

$$q\,q' - q^2 + p\,p' - p^2 = o, \quad \text{or } p\,p' + q\,q' = p^2 + q^2\,; \qquad (193)$$

in which, substituting for p, p', q, q', their values (180) we get ultimately

$$\mathrm{A} + \mathrm{B}\,.\cos\,(g\,t + k - k') = o \quad \text{which gives} \quad \cos\,(g\,t + k - k') = -\frac{\mathrm{A}}{\mathrm{B}}\,; \qquad (194)$$

Hence, if $\mathrm{B} > \mathrm{A}$ (no regard being had to the signs) this will correspond to a real value of t, and the node will then merely have a libratory motion, advancing and receding alternately; if $\mathrm{B} \angle \mathrm{A}$, they will circulate always in one direction. In the former case, if we substitute in the value of $\tan\phi^2$ (equation 184, 1) this value of $\cos\,(g\,t + k - k')$ we shall find $\tan\phi^2 = \mathrm{B}^2 - \mathrm{A}^2$; $\tan\phi = \sqrt{\mathrm{B}^2 - \mathrm{A}^2}$, which gives the inclination corresponding to the stationary points of the node. These points are attained when $\cos\,(g\,t + k - k') = -\frac{\mathrm{A}}{\mathrm{B}}$ while the maxima and minima of the inclinations happen when $\cos\,(g\,t + k - k') = \pm 1$. The stationary positions of the node therefore do not correspond either to the maxima and minima of the inclinations, or to the semi-intervals between them.

If we had considered more than two bodies the results would have been analogous, and we should have arrived at similar expressions for p and q only containing more terms, and analogous equations to those in 182, 183, and 185, viz.

$$\left.\begin{aligned} m\,\sqrt{a}\,.\,p + m'\,\sqrt{a'}\,.\,p' + m''\,\sqrt{a''}\,.\,p'' + \&\text{c.} &= \text{const.} \\ m\,\sqrt{a}\,.\,q + m'\,\sqrt{a'}\,.\,q' + m''\,\sqrt{a''}\,.\,q'' + \&\text{c.} &= \text{const.} \\ m\,\sqrt{a}\,.\tan\phi^2 + m'\,\sqrt{a'}\,.\tan\phi'^2 + m''\,\sqrt{a''}\,.\tan\phi''^2 + \&\text{c.} &= \text{const.} \end{aligned}\right\}, \ldots\ldots\ldots (195)$$

Let us apply this theory to an example, and we will take that of the orbits of Jupiter and Saturn, the two principal planets of our system. If we take for our epoch the year 1700, we have, by Halley's tables,

$$\omega = 101^\circ\ 5'\ 6''; \quad \phi = 2^\circ\ 30'\ 10''; \quad a = 9{\cdot}54007$$

$$\omega' = 97^\circ\ 34'\ 9''; \quad \phi' = 1^\circ\ 19'\ 10'; \quad a' = 5{\cdot}20098$$

Hence we find the values of p, p', q, q', for that epoch, by the equations (186) as follows :—

$$p = 0{\cdot}04078, \quad q = -\,0{\cdot}01573\,; \quad p' = 0{\cdot}02283, \quad q' = -\,0{\cdot}00303$$

whence, having computed I, and assuming $m' = \frac{1}{1067}$ and $m = \frac{1}{3358}$, we shall find A, B, k, k' and g as follows :

$B = 0{\cdot}02905$; $A = 0{\cdot}01537$; $k = 125^\circ\ 15'\ 40''$, $k' = 103^\circ\ 38'\ 40''$ and $A' = -\ 0{\cdot}00661$, and finally $g = -\ 25''{\cdot}5756$

Hence we obtain, in the case of Saturn,

$$\tan\phi = 0{\cdot}03287\ .\ \sqrt{1 + 0{\cdot}82665\ .\ \cos\left\{21^\circ\ 37' - t \times 25''{\cdot}5756\right\}}$$

and for Jupiter,

$$\tan\phi' = 0{\cdot}02980\ .\ \sqrt{1 - 0{\cdot}43290\ .\ \cos\left\{21^\circ\ 37' - t \times 25''{\cdot}5756\right\}}$$

Also we have $B + A = 0{\cdot}04442$; $B - A = 0{\cdot}01368$, so that the maxima and minima of the inclinations of Saturn's orbit are 2° 32′ 40″ and 0° 47′, and its greatest deviation from the mean state will not exceed 52′ 50″. In Jupiter's orbit the maximum is 2° 2′ 30″, and the minimum 1° 17′ 10″, and the greatest deviation from a mean state 0° 22′ 40″.

The longitude of the node, ω, has a maximum and a minimum in both orbits, because $B > A$; and the extent of its librations will be, in the case of Jupiter's orbit, 13° 9′ 40″, and in that of Saturn's, 31° 56′ 20″ on either side of its mean station, on the plane of the ecliptic supposed immoveable.

The period in which these changes take place, or the whole time in which the inclinations vary from their greatest to their least values, and the nodes from their greatest to their least longitudes, and back again, is equal to $\frac{360^\circ}{g} = \frac{360^\circ}{25''{\cdot}5756} = 50673$ Julian years; an immense period, and which may serve to give some idea of the extent to which the Newtonian theory, assisted by the refined methods of the modern analysis, enables us to carry our views of the past and future condition of our system; as this, though subject of course to some subordinate corrections, is perhaps one of the least uncertain of the results of perturbation.

Let us now consider the secular variations of the eccentricities and aphelia. Our first object, agreeable to the theory of the variation of the arbitrary constants already exposed, must be to obtain such an equation of the ellipse of curvature as shall be adapted to our purpose, by containing these elements (or convenient functions of them) in a state sufficiently disengaged from the variables x, y, z, and their differentials. Now if we differentiate the quantities $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, noticing that $r^2 = x^2 + y^2 + z^2$, we get

$$d\frac{x}{r} = \frac{r\,dx - x\,dr}{r^2} = \frac{r^2\,dx - x\,.\,r\,dr}{r^3} = \frac{r^2\,dx - \frac{1}{2}x\,d\,(r^2)}{r^3}$$

$$= \frac{(x^2 + y^2 + z^2)\,dx - (x\,dx + y\,dy + z\,dz)\,.\,x}{r^3}$$

$$= \frac{y\,(y\,dx - x\,dy) + z\,(z\,dx - x\,dz)}{r^3}$$

That is, substituting for $y\,dx - x\,dy$ and $z\,dx - x\,dz$ their values $h\,dt$, $h'\,dt$,

$$\left.\begin{aligned} d\left(\frac{x}{r}\right) &= \frac{hy + h'z}{r^3}\,dt \\ \text{similarly,}\quad d\left(\frac{y}{r}\right) &= \frac{-\,hx + h''z}{r^3}\,dt \\ d\left(\frac{z}{r}\right) &= \frac{-\,h'x - h''y}{r^3}\,dt \end{aligned}\right\}\ \text{or}\ \left\{\begin{aligned} \mu\,d\left(\frac{x}{r}\right) &= \left(h\frac{\mu y}{r^3} + h'\frac{\mu z}{r^3}\right)dt \\ \mu\,d\left(\frac{y}{r}\right) &= \left(-\,h\frac{\mu x}{r^3} + h''\frac{\mu z}{r^3}\right)dt \\ \mu\,d\left(\frac{z}{r}\right) &= \left(-h'\frac{\mu x}{r^3} - h''\frac{\mu y}{r^3}\right)dt \end{aligned}\right.$$

Now, in the case of elliptic motion, $\frac{\mu x}{r^3} = -\frac{d^2x}{dt^2}$; $\frac{\mu y}{r^3} = -\frac{d^2y}{dt^2}$ and $\frac{\mu z}{r^3} = -\frac{d^2z}{dt^2}$, consequently substituting these, we obtain

$$\mu\,d\left(\frac{x}{r}\right) = d\left\{-\,h\frac{dy}{dt} - h'\frac{dz}{dt}\right\};\quad \mu\,d\left(\frac{y}{r}\right) = d\left\{h\frac{dx}{dt} - h''\frac{dz}{dt}\right\}$$

$$\mu\,d\left(\frac{z}{r}\right) = d\left\{h'\frac{dx}{dt} + h''\frac{dy}{dt}\right\}$$

and integrating,

$$f = \mu\frac{x}{r} + \left(h\frac{dy}{dt} + h'\frac{dz}{dt}\right); \qquad (196,\ 1)$$

$$f' = \mu\frac{y}{r} + \left(-\,h\frac{dx}{dt} + h''\frac{dz}{dt}\right); \qquad (196,\ 2)$$

$$f'' = \mu\frac{z}{r} + \left(-\,h'\frac{dx}{dt} - h''\frac{dy}{dt}\right); \qquad (196,\ 3)$$

These equations will serve our purpose, as the arbitrary constants f, f', f'', are completely disengaged; but before we proceed to employ them, we must determine the values of f, f', f'', in terms of the elements, and *vice versa*; and for this purpose must first eliminate the differential co-efficients, which (from the peculiar form of the equations) is practicable, by merely multiplying the first by x, the second by y, and the third by

z; and adding the results, when we get

$$\mu r - h \frac{y\,d x - x\,d y}{d t} - h' . \frac{z\,d x - x\,d z}{d t} - h'' . \frac{z\,d y - y\,d z}{d t} = f x + f' y + f'' z$$

or, by reason of the equations (4) (5) (6),

$$\mu r = f x + f' y + f'' z + (h^2 + h'^2 + h''^2); \tag{197}$$

This equation expresses the general property of the conic sections, in virtue of which a line drawn from the focus to the circumference is always in a given ratio to a perpendicular let fall from that point on the directrix.

If we multiply (196, 1) by h'', (196, 2) by $- h'$, and (196, 3) by h, and take the sum of the results, (observing that $h'' x - h' y + h z = o$) it will be found that all the variable terms will destroy each other, leaving simply the equation of condition,

$$h'' f - h' f' + h f'' = o \tag{198}$$

Suppose X, Y, Z, the co-ordinates of the perihelion. At this point $d r = o$, or $x\,d x + y\,d y + z\,d z = o$, but if for h, h', h'', we write their values in (4) (5) (6) we have

$$h \frac{d y}{d t} + h' \frac{d z}{d t} = - x . \frac{d y^2 + d z^2}{d t^2} + d x . \frac{y\,d y + z\,d z}{d t^2} \tag{199, 1}$$

$$- h \frac{d x}{d t} + h'' \frac{d z}{d t} = - y \frac{d x^2 + d z^2}{d t^2} + d y . \frac{x\,d x + z\,d z}{d t^2} \tag{199, 2}$$

$$- h' \frac{d x}{d t} - h'' \frac{d y}{d t} = - z \frac{d x^2 + d y^2}{d t^2} + z \frac{x\,d x + y\,d y}{d t^2} \tag{199, 3}$$

At the perihelion therefore, substituting for $y\,d y + z\,d z$, $x\,d x + z\,d z$, $x\,d x + y\,d y$, their equals $- \mathrm{X}\,d\mathrm{X}$, $- \mathrm{Y}\,d\mathrm{Y}$, and $- \mathrm{Z}\,d\mathrm{Z}$, respectively, these quantities become

$$- \mathrm{X} . \frac{d \mathrm{X}^2 + d \mathrm{Y}^2 + d \mathrm{Z}^2}{d t^2}$$

$$- \mathrm{Y} . \frac{d \mathrm{X}^2 + d \mathrm{Y}^2 + d \mathrm{Z}^2}{d t^2}$$

$$- \mathrm{Z} . \frac{d \mathrm{X}^2 + d \mathrm{Y}^2 + d \mathrm{Z}^2}{d t^2}$$

Consequently we have, putting $\frac{d \mathrm{X}^2 + d \mathrm{Y}^2 + d \mathrm{Z}^2}{d t^2} = \mathrm{V}^2$, $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = \mathrm{R}^2$

$$f = \mathrm{X} \left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right); \quad f' = \mathrm{Y} . \left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right); \quad f'' = \mathrm{Z} \left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right); \tag{200}$$

Hence it is easy to obtain the following equation,

$$f^2 + f'^2 + f''^2 = \mathrm{R}^2 \left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right)^2; \tag{201}$$

Now R is the perihelion distance, $\mathrm{R} = a\,(1 - e)$ and V being the velocity at the perihelion, we have, by (33) $\mathrm{V}^2 = \mu \left(\frac{2}{\mathrm{R}} - \frac{1}{a}\right)$ Hence we get

$$f^2 + f'^2 + f''^2 = \mathrm{R}^2 \left(\frac{\mu}{a} - \frac{\mu}{\mathrm{R}}\right)^2 = \frac{\mu^2 (\mathrm{R} - a)^2}{a^2} = \mu^2 e^2$$

$$\mu e = \sqrt{f^2 + f'^2 + f''^2}; \tag{202}$$

Moreover we have

$$\frac{f'}{f} = \frac{\mathrm{Y}}{\mathrm{X}}, \quad \frac{f''}{f} = \frac{\mathrm{Z}}{\mathrm{X}}; \quad \frac{f''}{\sqrt{f^2 + f'^2}} = \frac{\mathrm{Z}}{\sqrt{\mathrm{X}^2 + \mathrm{Y}^2}}; \tag{203}$$

But $\frac{\mathrm{Y}}{\mathrm{X}}$ is the tangent of the longitude of the perihelion, or of the angle which the projection of the perihelion distance makes with the axis of the x: also, π is the longitude of the perihelion reckoned on the orbit; and if its plane is but very little inclined to that of the x, y, this angle differs from its projection on that plane only by a very small quantity of the second order; so that if we disregard, as we have hitherto done in this research, the squares of the eccentricities, inclinations, and disturbing forces, we have

$$\tan \pi = \frac{f'}{f}; \tag{204}$$

These three equations, viz. $\tan \pi = \frac{f'}{f}$, $\mu e = \sqrt{f^2 + f'^2 + f''^2}$, and $h f'' - h' f' + h'' f = o$, determine f, f', f'', in terms of the elements of the orbit. The variations of the eccentricities and longitudes of the perihelia, e and π, may be immediately determined from those of f and f', so that it is to these we shall now turn our attention. Resuming then the equations (196, 1, 2, 3) let us differentiate them on the supposition of f, f', f'', being variable quantities, their variations being such as to express the effect of the disturbing forces only. To express this we have used the characteristic δ, and we shall continue to do so, to keep the principle on which the process is founded distinctly in view. No inconvenience can arise from the confusion of two symbols, δ and d, in the same investigation, when we bear in mind that any expression such as δf denotes, strictly, the whole amount of the momentary change which the quantity f undergoes while the planet passes from *one elementary portion* of its unknown curve to *that immediately consecutive to it,* while such expressions as $d\,x, d\,y$, &c., denote the changes which x, y, &c. undergo while it passes from one end to the other of *the same elementary portion.* In both points of view, the accumulated effects during a finite time are obtained by the same rules of integration; the whole variation of f is legitimately expressed by $\int \delta f$ just as that of x is by $\int d\,x$, and when δf is expressed in terms of $d\,t$, the integration must be performed in the usual manner, as on a function of t.

We have therefore, since $\delta\,\frac{x}{r} = o$, $\delta\,\frac{y}{r} = o$, $\delta\,\frac{z}{r} = o$

$$\delta f = \frac{d\,y}{d\,t}\,\delta\,h + \frac{d\,z}{d\,t}\,\delta\,h' + h\,\delta\,\frac{d\,y}{d\,t} + h'\,\delta\,\frac{d\,z}{d\,t}$$

Substituting therefore for $\delta\,h$, $\delta\,h'$ their values in the equations (168) for $\delta\,\frac{d\,y}{d\,t}$, $\delta\,\frac{d\,z}{d\,t}$, $-\,m'\,\frac{d\,\Omega}{d\,y}$ and $-\,m'\,\frac{d\,\Omega}{d\,z}$ respectively; and for h, h', their values $\frac{y\,d\,x - x\,d\,y}{d\,t}$ and $\frac{z\,d\,x - x\,d\,z}{d\,t}$, we get

$$\delta f = -\left\{\begin{matrix} d\,y\left\{y\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,y}\right\} + d\,z\left\{z\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,z}\right\} \\ +\,(y\,d\,x - x\,d\,y)\,\frac{d\,\Omega}{d\,y} + (z\,d\,x - x\,d\,z)\,\frac{d\,\Omega}{d\,z} \end{matrix}\right\} \times m' \ldots\ldots\ldots\ldots (205,\ 1)$$

and similarly,

$$\delta f' = -\left\{\begin{matrix} d\,x\left\{x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x}\right\} + d\,z\left\{z\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,z}\right\} \\ +\,(x\,d\,y - y\,d\,x)\,\frac{d\,\Omega}{d\,x} + (z\,d\,y - y\,d\,z)\,\frac{d\,\Omega}{d\,z} \end{matrix}\right\} \times m' \ldots\ldots\ldots\ldots (205,\ 2)$$

These are the rigorous values of δf, $\delta f'$, and their integrals regarded as functions of t, express the total variations of these elements produced in that time. But, as we have done in the less complicated theory of the nodes and inclinations we shall neglect, in developing them, all their periodical terms, at least such as depend on the configurations of the planets, as well as all terms containing the squares of the disturbing forces, eccentricities, and inclinations, and take the primitive orbit of m for our fixed plane. This will simplify them greatly; for in this case z, and $\frac{d\,\Omega}{d\,z}$ are quantities of the order of the disturbing forces, and therefore, when multiplied by m', may be rejected. Moreover, on this supposition, it is indifferent whether we reckon the longitudes on the orbit or on the fixed plane, since the difference is of the order z^2; consequently, representing that of m by θ, we have

$$x = r\,.\cos\theta,\quad y = r\,.\sin\theta\,;$$

whence we find

$$x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x} = (y'\,x - x'\,y)\,.\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right)$$

$$= r\,r'\,.\sin\,(\theta' - \theta)\,.\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right) = \frac{d\,\Omega}{d\,\theta}\,; \qquad (206)$$

$$\delta f = -\,d\,y\,\frac{d\,\Omega}{d\,\theta} - h\,\frac{d\,\Omega}{d\,y}\,d\,t$$

Now we have $\frac{d\,\Omega}{d\,y} = \frac{y'}{r'^3} - \frac{y' - y}{\lambda^3} = \frac{\sin\theta'}{r'^2} - \frac{r'\,.\sin\theta' - r\,.\sin\theta}{\lambda^3}$

But since $\lambda = \sqrt{r^2 - 2r\,r'\,.\cos\,(\theta - \theta') + r'^2}$

$$\Omega = \frac{r\,.\cos\,(\theta - \theta')}{r'^2} - \frac{1}{\lambda}$$

$$\frac{d\,\Omega}{d\,r} = \frac{\cos\,(\theta - \theta')}{r'^2} + \frac{r - r'\,.\cos\,(\theta - \theta')}{\lambda^3} \qquad (207)$$

$$\frac{d\,\Omega}{d\,\theta} = -\frac{r\,.\sin\,(\theta - \theta')}{r'^2} + \frac{r\,r'\,.\sin\,(\theta - \theta')}{\lambda^3} \qquad (208)$$

Consequently

$$\frac{d\,\Omega}{d\,y} = \frac{1}{r}\left\{r\,\frac{d\,\Omega}{d\,r}\,.\sin\,\theta + \frac{d\,\Omega}{d\,t}\,.\cos\,\theta\right\} \qquad (209)$$

and as we have $d\,y = d\,r\,.\sin\,\theta + r\,d\,\theta\,.\cos\,\theta$, and $h\,d\,t = r^2\,d\,\theta$,

$$\delta f = -\left\{d\,r\,.\sin\,\theta + 2\,r\,d\,\theta\,.\cos\,\theta\right\}\frac{d\,\Omega}{d\,\theta} - r^2\,d\,\theta\,\frac{d\,\Omega}{d\,r}\,.\sin\,\theta\,; \qquad (210)$$

In like manner we should have found

$$\delta f' = +\left\{d\,r\,.\cos\,\theta - 2\,r\,d\,\theta\,.\sin\,\theta\right\}\frac{d\,\Omega}{d\,\theta} + r^2\,d\,\theta\,\frac{d\,\Omega}{d\,r}\,.\cos\,\theta\,; \qquad (211)$$

Now, if we neglect the squares of the eccentricities, we have

$$r = a\left\{1 - e\,.\cos\,(\theta - \pi)\right\},\ d\,r = a\,e\,.\sin\,(\theta - \pi)\,d\,\theta,\ r^2\,d\,\theta = n\,a^2\,d\,t,$$

whence we have

$$d\,\theta = n\,d\,t\,(1 + 2\,e\,.\cos\,(\theta - \pi) \text{ and } d\,r = n\,a\,e\,d\,t\,.\sin\,(\theta - \pi)$$

and consequently,

$$\delta f = -\,n\,a\,d\,t\,.\frac{d\,\Omega}{d\,\theta}\left\{(2 - 2\,e\,.\cos\,\overline{\theta - \pi})(1 + 2\,e\,.\cos\,\overline{\theta - \pi})\,.\cos\,\theta + e\,.\sin\,\theta\,.\sin\,\overline{\theta - \pi}\right\} - n\,a^2\,d\,t\,.\frac{d\,\Omega}{d\,r}\,;$$

which reduced, rejecting e^2, becomes

$$\delta f = -\,m'\,n\,d\,t\left\{a\,\frac{d\,\Omega}{d\,\theta}\left(2\,\cos\,\theta + \frac{3}{2}\,e\,.\cos\,\pi + \frac{e}{2}\,.\cos\,(2\,\theta - \pi)\right) + a^2\,\frac{d\,\Omega}{d\,r}\,.\sin\,\theta\right\} \qquad (212)$$

In order to find the secular part of the variation of f, we must develope this expression, retaining only such terms as are not periodical. Now, recurring to the notation of Sections 3, 4, 5, Part II. we have

$$\theta = n\,t + \epsilon + 2\,e\,.\sin\,V = V + \pi + 2\,e\,.\sin\,V$$

therefore

$$\sin\,\theta = \sin\,(V + \pi) + 2\,e\,.\sin\,V\,.\cos\,(V + \pi)$$
$$= \sin\,(V + \pi) + e\,.\sin\,(2\,V + \pi) - e\,.\sin\,\pi$$
$$2\,\cos\,\theta = 2\,\cos\,(V + \pi) - 4\,e\,.\sin\,V.\,\sin\,(V + \pi)$$
$$= 2\,\cos\,(V + \pi) + 2\,e\ \cos\,(2\,V + \pi) - 2\,e\,.\cos\,\pi$$
$$\frac{e}{2}\,.\cos\,(2\,\theta - \pi) = \frac{e}{2}\,.\cos\,(2\,V + \pi)\,;\ \frac{3\,e}{2}\,.\cos\,\pi = \frac{3\,e}{2}\,.\cos\,\pi$$

neglecting e^2 &c. Again we have

$$a^2\,\frac{d\,\Omega}{d\,r} = a^2\,\frac{d\,\Omega}{d\,a} + a^2\,\Delta\,\frac{d\,\Omega}{d\,r}$$

When this therefore is multiplied by sin θ, the term $-\,e\,.\sin\,\pi$ will combine with the constant term $a^2\,\frac{d\,A}{d\,a}$ of $a^2\,\frac{d\,\Omega}{d\,a}$, and produce $-\,a^2\,e\,.\frac{d\,A}{d\,a}\,.\sin\,\pi$, and this is the only constant term which can arise from $a^2\,\frac{d\,\Omega}{d\,a}$, because every other term contains W, and therefore both V and V′; while the value of sin θ contains only V, and, of course, V′ cannot be destroyed by their combination. Again, with respect to $\Delta\,\frac{d\,\Omega}{d\,r}$, it has been proved, that (regarding only the first powers of the eccentricities,) this is resolvable into a series of terms of the form $A\,.\cos\,(i\,W \pm V)$, $A\,.\cos\,(i\,W \pm V')$. Now, since $\Delta\,\frac{d\,\Omega}{d\,r}$ is of the order of the eccentricities, we may disregard the term of sin θ multiplied by e, and take it simply equal to sin $(V + \pi)$. If we multiply this into the series $\Sigma\,A\,.\cos\,(i\,W \pm V)$ since $W = V - V' + \pi - \pi'$, only one term can produce a constant argument, viz. that into which V′ does not enter, or in which $i = o$; that is, the term multiplied by cos V. The co-efficient in this term in $\Delta\,\frac{d\,\Omega}{d\,r}$ is $-\,a\,e\,\frac{d^2\,A}{d\,a^2}$, (as appears from (147), writing $\frac{d\,\Omega}{d\,r}$ for Ω, and of course $\frac{d\,A}{d\,a}$, &c. for A, &c.) Hence the term so originating in $a^2\,\frac{d\,\Omega}{d\,r}\,.\sin\,\theta$ will be equal to the constant part of $-\,a^3\,e\,\frac{d^2\,A}{d\,a^3}\,.\cos\,V\,.\sin\,(V + \pi)$, that is, to $-\,\frac{a^3\,e}{2}\,.\,\frac{d^2\,A}{d\,a^2}\,.\sin\,\pi$. Again the multiplication of sin $(V + \pi)$ with the series $\Sigma\,A\,.\cos\,(i\,W \pm V')$ will produce a constant term by the combination of sin $(V + \pi)$ with $\cos\,(W + V') = \cos\,(V + \pi - \pi')$. The co-efficient of cos $(W + V')$ in the developement of $\Delta\,\frac{d\,\Omega}{d\,r}$ is $-\,\frac{e'}{2}\left(a\,\frac{d^2\,A'}{d\,a\,d\,a'} + 2\,\frac{d\,A'}{d\,a}\right)$. Hence the constant part of $a^2\,\frac{d\,\Omega}{d\,r}\,.\sin\,\theta$ so originating, will be $-\frac{a^2 e'}{2}\left(a'\frac{d^2 A'}{d\,a\,d\,a'} + 2\,\frac{d\,A'}{d\,a}\right) \times$ (the constant part of $\sin(V+\pi)\,.\cos(V+\pi-\pi')$

$$= -a^2\,.\,\frac{e'}{4}\sin\pi'\left\{2\,\frac{d\,A'}{d\,a} + a'\frac{d^2 A'}{d\,a\,d\,a'}\right\}$$

Let us next consider the constant terms which can originate from the part multiplied by $a\,\frac{d\,\Omega}{d\,\theta}$. This quantity is equal to $a\,\frac{d\,\Omega}{d\,w}\,.\,\frac{d\,w}{d\,\theta} = a\,\frac{d\,\Omega}{d\,w} = a\,\frac{d\,\Omega}{d\,W} + a\,\Delta\,\frac{d\,\Omega}{d\,w}$. But we have

$$\frac{d\,\Omega}{d\,W} = -A\,.\sin W - 2\,A'\,.\sin 2\,W - \&c.$$

and as all these contain V', and this arc does not enter into any of the terms multiplying $\frac{d\,\Omega}{d\,\theta}$ in the expression of δf, it is obvious that no combination not containing V', can arise in this way, *i. e.* no constant term can occur in the part of the developement depending on $\frac{d\,\Omega}{d\,W}$. With regard to $\Delta\,\frac{d\,\Omega}{d\,w}$ being already of the order of the eccentricities, it need only be combined with the first term, $2\cos\theta$, and this term may be regarded as equal simply to $2\,.\cos(V+\pi)$. We have only then to investigate the constant part of

$$2\,a\,.\cos(V+\pi)\left\{\Delta\,\frac{d\,\Omega}{d\,w} = \frac{d^2\,\Omega}{d\,a\,d\,W}\,\Delta\,r + \frac{d^2\,\Omega}{d\,a'\,d\,W}\,\Delta\,r' + \frac{d^2\,\Omega}{d\,W^2}\,\Delta\,w\right\}$$

Now $\Delta\,r = -a\,\epsilon\,.\sin V$, $\Delta\,r' = -a'\,e'\,.\sin V'$, $\Delta\,w = 2\,e\,.\sin V - 2\,e'\,.\sin V'$

1st. The combination $-2\,a^2\,e\,.\cos V\,.\cos(V+\pi)\,.\,\Sigma - i\,\frac{d\,A^i}{d\,a'}\sin i\,W$

can contain no constant term, because the terms $\sin V$ and $\cos(V+\pi)$ cannot eliminate V' from $\sin i\,W$.

2dly. $-2\,a\,a'\,e'\,.\cos(V+\pi)\,.\cos V'\,.\,\Sigma - i\,\frac{d\,A^i}{d\,a'}\sin i\,W$

will contain a constant term, for taking $i = 1$, $\cos V' \times \sin W = \frac{1}{2}\sin(W+V') + \frac{1}{2}\sin(W-V')$, the last term of which does not contain V', being equal to $+\frac{1}{2}\sin(V+\pi-\pi')$. The combination now under consideration, will therefore contain the term

$$+\,a\,a'\,e'\,.\,1\,.\,\frac{d\,A'}{d\,a'}\,.\cos(V+\pi)\,.\sin(V+\pi-\pi')$$

which, developed, will produce a constant term,

$$-\,a\,a'\,.\,\frac{e}{2}\,.\,\frac{d\,A'}{d\,a'}\,.\sin\pi'$$

Lastly, the combination

$$4\,a\,.\cos(V+\pi)\left\{e\,.\sin V - e'\,.\sin V'\right\}.\,\Sigma\quad i^2\,.\,A^i\,.\cos i\,W$$

will produce a constant term; viz. that arising from

$$-\,4\,a\,e'\,.\cos(V+\pi)\,.\sin V' \times -\,1^2\,A'\,.\cos W$$

for $\sin V'\,.\cos W$ produces a term free from V', viz. $\frac{1}{2}\sin(V+\pi-\pi')$ which again combined with $\cos(V+\pi)$ produces $\frac{-1}{4}\sin\pi'$; so that the complete result of this combination will be

$$-\,a\,e'\,.\,A'\,.\sin\pi'$$

Assembling therefore all the constant terms so found, we get

$$\delta f = m'\,.\,a\,n\,d\,t\,.\,e\sin\pi\left\{a\,\frac{d\,A}{d\,a} + \frac{a^2}{2}\,\frac{d^2\,A}{d\,a^2}\right\} \tag{213}$$
$$+\,m'\,.\,a\,n\,d\,t\,.\,e'.\sin\pi'\left\{A' + \frac{a}{2}\,\frac{d\,A'}{d\,a} + \frac{a'}{2}\,\frac{d\,A'}{d\,a'} + \frac{a\,a'}{4}\,.\,\frac{d^2\,A'}{d\,a\,d\,a'}\right\}$$

The value of $\delta f'$ may be derived by a similar process, but without executing the whole process, we may obtain it more readily by considering the expressions (210) and (211) we then see that δf changes to $\delta f'$ simply by changing θ to $\theta - 90°$, but leaving r, $\frac{d\,\Omega}{d\,r}$ and $\frac{d\,\Omega}{d\,\theta}$ unchanged. Now Ω, $\frac{d\,\Omega}{d\,r}$ and $\frac{d\,\Omega}{d\,\theta}$ remain manifestly unaltered, whatever direction we assume as that from which we reckon the longitudes. Suppose then that we shift this direction 90°, then will each of the angles θ, θ', ϵ, ϵ', π, π', be diminished 90°, and this will produce no alteration in r, because $\theta - \pi$ will remain unchanged. Hence, the effect produced by this change on δf will transform it into $\delta f'$. The value of $\delta f'$ therefore will be had at once by changing $\sin\pi$ and $\sin\pi'$ into $-\cos\pi$, and $-\cos\pi'$ in that of δf; and this will be found to be confirmed by the direct process.

Suppose now we denote the constant co-efficients within the brackets for brevity by $-I$ and K, so as to have

$$-\ I = a\frac{d\,\mathrm{A}}{d\,a} + \frac{a^2}{2}\cdot\frac{d^2\,\mathrm{A}}{d\,a^2} \tag{214}$$

$$K = \mathrm{A}' + \frac{a}{2}\ \frac{d\,\mathrm{A}'}{d\,a} + \frac{a'}{2}\cdot\frac{d\,\mathrm{A}'}{d\,a'} + \frac{a\,a'}{4}\cdot\frac{d^2\,\mathrm{A}'}{d\,a\,d\,a'} \tag{215}$$

Then it will be observed, that K is obviously symmetrical with respect to a, a', because the co-efficient A is so; and the same will be found to hold good with I; for if we write it as follows,—

$$I = -\ \frac{1}{2}\ \frac{d}{d\,a}\left\{a^2\ \frac{d\,\mathrm{A}}{d\,a}\right\} \tag{216}$$

and for A substitute its value given in (137, 1) it will be found that the execution of the operations here indicated will lead to precisely the same result as those denoted by $\frac{d}{d\,a'}\left\{a'^2\frac{d\,\mathrm{A}}{d\,a'}\right\}$: consequently I and K are both symmetrical functions of a, and a', and do not alter when these elements are mutually transposed. With regard to I, it may be easily shewn to be identical with the function $\frac{a\,a'}{4}\ \mathrm{S}'$ which we have already denoted by the same symbol. To see this, we need only write for A and S′ their values in (136, 1) and (173, 2), and executing on the former series the operations indicated by $-\ \frac{1}{2}\cdot\frac{d}{d\,a}\left(a^2\frac{d}{d\,a}\right)$, $\left(\text{remembering that } \alpha = \frac{a'}{a},\ \frac{d\,\alpha}{d\,a} = -\ \frac{\alpha}{a};\ \frac{d^2\,\alpha}{d\,a^2} = +\ \frac{2\,\alpha}{a^2}\right)$ and multiplying the latter by $\frac{a\,a'}{4}$, the result will be identical. We have then

$$\left.\begin{array}{l}\delta f = -\ m'\,.\,n\,a\,.\,I\,d\,t\,\times\,e\,.\,\sin\pi + m'\,.\,n\,a\,.\,K\,d\,t\,\times\,e'\sin\pi \\ \delta f' = +\ m'\,.\,n\,a\,.\,I\,d\,t\,\times\,e\,\cos\pi - m'\,.\,n\,a\,.\,K\,d\,t\,\times\,e'\cos\pi'\end{array}\right\};\ldots\ldots\ldots\ldots(217)$$

Now, if we recur to the values of f, f', f'', we find

$$\frac{f'}{f} = \tan\pi;\quad \frac{f'}{\sqrt{f^2 + f'^2}} = \sin\pi,\quad f'' = \frac{h'f' - h''f}{h}$$

but h' and h'' are of the order of the disturbing forces when we take the plane of m's undisturbed orbit for our fixed plane; consequently, f'' is so, and therefore $\sqrt{f^2 + f'^2} = \sqrt{f^2 + f'^2 + f'^2} = \mu\,e$, neglecting the squares of the disturbing forces. Hence we have $f = \mu\,e\,.\,\cos\pi$ and $f' = \mu\,e\,.\,\sin\pi$.

In imitation of what we did in the theory of the nodes and inclinations, let q and p represent these quantities; then $d\,q$ and $d\,p$ being the momentary variations of q and p, will replace δf and $\delta f'$; and if in like manner we put q' and p' for $\mu'\,e'\,.\cos\pi'$ and $\mu'\,e'\,.\,\sin\pi'$, the corresponding quantities in the orbit of m', we shall have (putting $\frac{1}{\sqrt{a}}$, $\frac{1}{\sqrt{a'}}$, for $n\,a$, $n'\,a'$; and supposing $\mu = 1$, and $\mu' = 1$, which is allowable since the squares of the disturbing forces are neglected, and the quantities under consideration are all of the first order of these forces,)

$$\left.\begin{array}{l}+\ \frac{d\,p}{d\,t} = \frac{m'}{\sqrt{a}}\left\{I\,q - K\,q'\right\} \\ -\ \frac{d\,q}{d\,t} = \frac{m'}{\sqrt{a}}\left\{I\,p - K\,p'\right\} \\ +\ \frac{d\,p'}{d\,t} = \frac{m}{\sqrt{a'}}\left\{I\,q' - K\,q\right\} \\ -\ \frac{d\,q'}{d\,t} = \frac{m}{\sqrt{a'}}\left\{I\,p' - K\,p\right\}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (218,\ 1,\ 2,\ 3,\ 4)$$

Thus we see that the problem of the secular variations of the eccentricities and aphelia, depends, exactly as in the case of the nodes and inclinations, on the simultaneous integration of four differential equations of the first order with constant co-efficients. If we compare these, however, with those of (177), we shall observe that the former research is rather more complicated than the latter, by reason of the two co-efficients, I and K, which it involves, while the system of equations on which the nodes and inclinations depend, involves only the first. However, this circumstance does not render the integration more difficult; the same substitution succeeds, and the integrals have a form almost exactly similar; we have only to take

$$\left.\begin{array}{l}p\ = \mathrm{A}\,.\,\sin\,(g\,t + k) + \mathrm{B}\,.\,\sin\,(h\,t + l) \\ q\ = \mathrm{A}\,.\cos\,(g\,t + k) + \mathrm{B}\,.\cos\,(h\,t + l) \\ p' = \mathrm{A}'.\,\sin\,(g\,t + k) + \mathrm{B}'.\,\sin\,(h\,t + l) \\ q' = \mathrm{A}'.\cos\,(g\,t + k) + \mathrm{B}'.\cos\,(h\,t + l)\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(219)$$

If these be substituted, the equations will be satisfied, provided the following equations hold good between the constants,

$$\left.\begin{aligned} g\,A &= \frac{m'}{\sqrt{a'}}(IA - KA') \\ g\,A' &= \frac{m}{\sqrt{a'}}(IA' - KA) \end{aligned}\right\} \text{ and } \left\{\begin{aligned} h\,B &= \frac{m'}{\sqrt{a'}}(IB - KB') \\ h\,B' &= \frac{m}{\sqrt{a'}}(IB' - KB) \end{aligned}\right.$$

It is obvious that the elimination of A′ from the first pair, and of B′ from the second pair, will lead to equations of exactly the same form for g and h, which are therefore the two roots of the quadratic,

$$g^2 - I\,g\,.\,\frac{m\sqrt{a} + m'\sqrt{a'}}{\sqrt{a\,a'}} + \frac{m\,m'}{\sqrt{a\,a'}}(I^2 - K^2) = o \tag{220}$$

and g and h being found, and A, B, k, l, remaining arbitrary, A′ and B′ are easily found.

To adapt these values to any particular case, the general values of p, q, p', q', at any assigned epoch, must be made to coincide with the observed values of $e\,.\,\sin\pi$, $e\,.\,\cos\pi$, $e'\,.\,\sin\pi'$, and $e'.\,\cos\pi'$; which condition will furnish equations to determine all the arbitrary constants.

For example, in the case of Jupiter and Saturn, we shall find on computation $g = 21''{\cdot}9905$, $h = 3''{\cdot}5851$.
$A = 0{\cdot}04877$; $B = 0{\cdot}03532$; $A' = -\,0{\cdot}01715$; $B' = 0{\cdot}04321$; $k = 306°\,34'\,40''$; $l = 210°\,16'\,40''$
Now, since $p = e\,.\,\sin\pi$, and $q = e\,.\,\cos\pi$, we have

$$\tan\pi = \frac{p}{q}, \text{ and } e = \sqrt{p^2 + q^2} = \sqrt{A^2 + B^2 + 2\,AB\cos\{(g-h)\,t + k - l\}}$$

which gives, by substituting the numbers already found,

$$e = 0{\cdot}06021\,.\,\sqrt{1 - 0{\cdot}95009 \times \cos(83°\,42' - 18''{\cdot}4054 \times t)}$$

which is the eccentricity of Saturn's orbit; and similarly,

$$e' = 0{\cdot}04649\,.\,\sqrt{1 + 0{\cdot}68592\,.\,\cos(83°\,42' - 18''{\cdot}4054 \times t)}$$

for that of Jupiter, after any number t of years since 1700.

The longitudes of their respective aphelia, will also be known by the formulæ

$$\tan\pi = \frac{A\,.\,\sin(g\,t + k) + B\,.\,\sin(h\,t + l)}{A\,.\,\cos(g\,t + k) + B\,.\,\cos(h\,t + l)}; \tag{221}$$

$$\tan\pi' = \frac{A'.\,\sin(g\,t + k) + B'.\,\sin(h\,t + l)}{A'.\,\cos(g\,t + k) + B'.\,\cos(h\,t + l)}; \tag{222}$$

and the maxima and minima of these, or their greatest deviations from their mean places will take place, when

$$g\,A^2 + h\,B^2 + AB\,(g + h)\,.\cos\{(g - h)\,t + (k - l)\} = o$$

that is, when

$$\cos\{(g - h)\,t + (k - l)\} = -\,\frac{g\,A^2 + h\,B^2}{(g + h)\,AB}; \tag{223}$$

If this fraction be less than unity, the aphelia will librate, as in the case of the nodes about a mean position, if not, they move in one direction continually. In the case of Jupiter and Saturn now before us, $g\,A^2 + h\,B^2$ 7 $(g + h)$ AB, so that the aphelia go on for ever in one direction.

The period in which the eccentricities go through all their evolutions, and return to the same state, is represented by $\dfrac{360°}{g - h} = \dfrac{360°}{18''{\cdot}4054} = 70414$ Julian years.

The greatest and least values of the eccentricities are respectively $A \pm B$ and $A' \pm B'$, for the two planets; and in the case before us

for Saturn, ... 0·08409 and 0·01345 } the maximum of the one corresponding to the minimum of
and for Jupiter, ... 0·06036 and 0·02606 } the other planet.

Finally, we may derive relations between the eccentricities, masses, and semiaxes, similar to those obtained between the inclinations, &c. in equations (182, 183, &c.) for since $p^2 + q^2 = e^2$ it is easy to see that we must have

$$\begin{aligned} m\,.\,e^2\sqrt{a} + m'\,.\,e'^2\sqrt{a'} &= m\sqrt{a}\,(A^2 + B^2) + m'\sqrt{a'}\,(A'^2 + B'^2) \\ &\quad + 2\,(m\sqrt{a}\,.\,AB + m'\sqrt{a'}\,.\,A'B')\,.\cos\{(g - h)\,t + (k - l)\} \end{aligned} \tag{224}$$

Now we have, from the equations of condition,

$$\left.\begin{aligned} A\left(g - \frac{m'}{\sqrt{a}}\,.\,I\right) &= -\,A'\,.\,\frac{m'}{\sqrt{a}}K \\ B\left(h - \frac{m'}{\sqrt{a}}\,.\,I\right) &= -\,B'\,.\,\frac{m'}{\sqrt{a}}K \end{aligned}\right\}; \;\ldots\ldots\ldots\ldots\ldots\ldots \tag{225}$$

so that

$$AB\left(g - \frac{m'}{\sqrt{a}}I\right)\left(h \quad \frac{m'}{\sqrt{a}}I\right) = A'\,B'\,.\,\frac{m'^2}{a}K^2$$

but g and h being the roots of (220), we have

$$g\,h = \frac{m\,m'}{\sqrt{a\,a'}}\,(I^2 - K^2) \qquad g + h = I\,.\,\frac{m\,\sqrt{a} + m'\,\sqrt{a'}}{\sqrt{a\,a'}}$$

Consequently, our equation, on executing the multiplications indicated, and substituting, reduces itself to

$$m\,\sqrt{a}\,.\,\mathrm{AB} + m'\,\sqrt{a'}\,.\,\mathrm{A'B'} = o$$

so that the equation (224) becomes simply

$$m\,e^2\,.\,\sqrt{a} + m'\,e'^2\,\sqrt{a'} = m\,\sqrt{a}\,(\mathrm{A}^2 + \mathrm{B}^2) + m'\,\sqrt{a'}\,(\mathrm{A}'^2 + \mathrm{B}'^2) = \text{constant}, \tag{226}$$

because the major axes are constant; and had we considered a greater number of planets than two, we should, in like manner, have arrived at the equation

$$m\,e^2\,\sqrt{a} + m'\,e'^2\,\sqrt{a'} + m''\,e''^2\,\sqrt{a''} + \&c. = \text{constant}; \tag{227}$$

PART III.

Of the Theory of the Moon.

Section I.

Rigorous investigation of the differential equations of the moon's motion, and general expression of the disturbing forces.

The extreme minuteness of the masses of the planets, as well as their great distance, renders it allowable in the theory of their perturbations to neglect altogether the squares of the disturbing forces, and affords such facilities to the whole investigation, as to permit us to express at once the true longitude and radius vector in explicit functions of the mean longitude or the time. This is not the case in the lunar theory, in which one of the most remarkable of the inequalities depends for nearly half its value on the square of the disturbing force, and in which the whole investigation is so delicate, as to render it necessary to abandon the direct process followed in the planetary theory and adopt a route, apparently more circuitous, but possessing advantages of a peculiar kind. It consists in expressing the radius vector and the time or *mean* longitude in functions of the *true*, and thence by the reversion of series deducing the true longitude in terms of the mean. The advantage of this is, that we are thus enabled to set out from differential equations in which nothing has been neglected, and consequently have it in our power fully to appreciate the influence of all the terms we may afterwards neglect in their integration on the final result.

Our directing principle in this investigation will be to follow out, step by step, with the proper modifications, the same system of transformations, by which the differential equations 1, 2, 3, of undisturbed motion were converted into (9) and (11) expressing the radius vector and time in terms of θ.

Our first step will consist in the investigation of an equation corresponding to (9): to this end (as we have supposed $d\,t$ constant) multiply the first of the equations (94) by y, and the second by $-\,x$, and add, and we get

$$o = \frac{y\,d^2x - x\,d^2y}{d\,t^2} + m'\,.\left\{y\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,y}\right\}$$

and integrating,

$$\frac{y\,d\,x - x\,d\,y}{d\,t} = h + m'\int\left\{x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x}\right\}d\,t; \tag{228}$$

but $y\,d\,x - x\,d\,y = \rho^2\,d\,\theta$, so that this equation becomes

$$\rho^2\,\frac{d\,\theta}{d\,t} = h + m'\int T\,\rho\,d\,t \tag{229}$$

if we assume

$$T = \frac{x\,\dfrac{d\,\Omega}{d\,y} - y\,\dfrac{d\,\Omega}{d\,x}}{\rho}. \tag{230}$$

The function T is the measure, and $m'\,.\,T$ the quantity, of what, in the lunar theory, is called the tangential force, or that part of the disturbing force acting on m, which is perpendicular to the direction of the radius vector. This is evident, if we consider that $m'\,\dfrac{d\,\Omega}{d\,y}$ and $m'\,\dfrac{d\,\Omega}{d\,x}$ representing the disturbing forces in the directions QP and PM (fig. A) if we resolve these each into the direction $K\,$Q perpendicular to QM they will become respectively

Fig. A.

$$-\,m'\,\frac{x}{\rho}\,.\,\frac{d\,\Omega}{d\,y} \text{ and } +\,m'\,\frac{y}{\rho}\,.\,\frac{d\,\Omega}{d\,x}$$

so that their aggregate, or the whole *tangential force*, will have for its expression the quantity above represented by T in equation (230).

The equation (229), if m' were zero, would coincide with (9), and would express the proportionality of the areas to the times. The term $m' \int T \rho \, d t$ then expresses the momentary effect of the disturbing forces in deranging this proportionality; and since

$$\int \rho^2 \, d \theta = h \, (t + \text{const.}) + m' \int d t \int T \rho \, d t \tag{231}$$

the term $m' \int d t \int T \rho \, d t$ expresses their total effect after the lapse of the time t; and we see that this effect takes place solely in virtue of the tangential force, agreeably to what Newton has advanced in the 11th section of the *Principia.*

Our next step is to investigate an equation for the disturbed motion, corresponding to (11) in the undisturbed. The process we followed in the deduction of (11), it will be remembered, consisted

1st. In making θ the independent variable.

2dly. In eliminating t and its differentials.

Now, if in the equations (94) instead of supposing $d t$ constant, we regard it as variable, we must change $\frac{d^2 x}{d t^2}$ into $\frac{1}{d t} d \frac{d x}{d t}$ or $\frac{d^2 x}{d t^2} - \frac{d x}{d t} \frac{d^2 t}{d t^2}$, and so for $\frac{d^2 y}{d t^3}$, &c. This done, let the first of the equations (94) be multiplied by x, and the second by y; their sum will be

$$o = \left\{ (x \, d^2 x + y \, d^2 y) - (x \, d x + y \, d y) \, . \frac{d^2 t}{d t} \right\} \frac{1}{d t^2}$$

$$+ \mu \, . \frac{x^2 + y^2}{r^3} + m' . \left\{ x \frac{d \, \Omega}{d x} + y \frac{d \, \Omega}{d y} \right\}$$

but since $x = \rho \, . \cos \theta$, and $y = \rho \, . \sin \theta$, $r = \sqrt{x^2 + y^2 + z^2}$
$= \sqrt{\rho^2 + z^2}$

this equation will become (supposing $d \theta$ constant),

$$o = \left\{ \rho \, d^2 \rho - \rho^2 \, d \, \theta^2 - \rho \, d \rho \, . \frac{d^2 t}{d t} \right\} \frac{1}{d t^2}$$

$$+ \mu \, . \frac{\rho^2}{(\rho^2 + z^2)^{\frac{3}{2}}} + m' \left\{ x \frac{d \, \Omega}{d x} + y \frac{d \, \Omega}{d y} \right\} \tag{232}$$

We have only now to find the value of $\frac{d^2 t}{d t}$ on the supposition of $d \theta$ being invariable, and substitute it in this equation; and for this purpose we must employ our equation (229) which gives (multiplying it by $T \rho \, d t$)

$$T \rho^3 \, d \theta = h \, . \, T \rho \, d t + m' . \, T \rho \, d t \, . \int T \rho \, d t$$

and integrating,

$$\int T \rho^3 \, d \theta = h \int T \rho \, d t + \frac{m'}{2} \left(\int T \rho \, d t \right)^2$$

from which we get, by the solution of a quadratic equation,

$$m' . \int T \rho \, d t = - h + \sqrt{h^2 + 2 \, m' \int T \rho^3 \, d \theta}$$

and differentiating, and dividing by $m' \, T \rho$

$$d t = \frac{\rho^2 \, d \theta}{\sqrt{h^2 + 2 \, m' \int T \rho^3 \, d \theta}} ; \tag{233}$$

If we take the logarithmic differential on both sides of this, supposing $d \theta$ constant, we get

$$\frac{d^2 t}{d t} = 2 \frac{d \rho}{\rho} \quad m' . \frac{T \rho^3 \, d \theta}{h^2 + 2 \, m' \int T \rho^3 \, d \theta} ; \tag{234}$$

Substituting this then in equation (232), and for $\frac{1}{d t^2}$, writing its value $\frac{h^2}{\rho^4 \, d \, \theta^2} + 2 \, m' . \frac{\int T \rho^3 \, d \theta}{\rho^4 \, d \, \theta^2}$ given by (233), it becomes

$$o = \frac{1}{\rho \, d \, \theta^2} \left\{ \frac{d^2 \rho}{\rho^2} - 2 \frac{d \, \rho^2}{\rho^3} - \frac{d \, \theta^2}{\rho} \right\} (h^2 + 2 \, m' \int T \rho^3 \, d \theta)$$

$$+ \frac{\mu}{\rho} . \frac{1}{\left(1 + \frac{z^2}{\rho^2} \right)^{\frac{3}{2}}} + m' . \left\{ \left(x \frac{d \, \Omega}{d x} + y \frac{d \, \Omega}{d y} \right) + 7 \, . \frac{d \rho}{d \theta} \right\} , \tag{235}$$

In this equation let $\frac{z}{\rho} = s$, or $z = \rho\, s$, so that $s =$ tan heliocentric elevation of m above the plane of the x and y, $=$ tan m MQ fig. (A). Moreover, let

$$V = \frac{x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y}}{\rho}\,; \tag{236}$$

then will V be the measure, and m' . V the quantity, of that part of the disturbing force which operates to increase the gravity of m towards the central body. For if we resolve the forces $m'\,.\,\frac{d\,\Omega}{d\,x}$ and $m'\,\frac{d\,\Omega}{d\,y}$ which act in the directions of the co-ordinates x and y into the direction of the distance ρ, they will become respectively

$$+\,m'.\,\frac{x}{\rho}\,.\,\frac{d\,\Omega}{d\,x} \text{ and } +\,m'\,.\frac{y}{\rho}\,.\,\frac{d\,\Omega}{d\,y}$$

whose sum is m' . V.

These substitutions made, and ρ being put (as in the elliptic theory) equal to $\frac{1}{\mu}$, the equation will become when multiplied by $-\,\frac{\rho}{h^2}$,

$$\begin{aligned} o = {} & \frac{d^2\,u}{d\,\theta^2} + u - \frac{\mu}{h^2} \times \frac{1}{(1+s^2)^{\frac{3}{2}}} \\ & -\frac{m'}{h^2}\left\{\frac{\text{V}}{u^2} - \frac{T}{u^3}\,.\,\frac{d\,u}{d\,\theta} - 2\left(\frac{d^2\,u}{d\,\theta^2} + u\right)\int\frac{T\,d\,\theta}{u^3}\right\}; \end{aligned} \tag{237}$$

which equation is rigorous, nothing having been neglected in the previous process. But if the squares of the disturbing forces be neglected, we have $\frac{1}{(1+\,s^2)^{\frac{3}{2}}} = 1$ because s or $\frac{z}{\rho}$ is of the order of the disturbing forces; and since, in virtue of this very equation,

$$\begin{aligned} m'\left(\frac{d^2\,u}{d\,\theta^2} + u\right) & = m'\left\{\frac{\mu}{h^2}\,.\,\frac{1}{(1\,+s^2)^{\frac{3}{2}}} + m' \times \text{\&c.}\right\} \\ & = m'.\,\frac{\mu}{h^2} + m'^2 \times \text{\&c.} \end{aligned}$$

the equation becomes

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{\mu}{h^2} - \frac{m'}{h^2}\left\{\frac{\text{V}}{u^2} - \frac{T}{u^3}\,.\,\frac{d\,u}{d\,\theta} - \frac{2\,\mu}{h^2}\int\frac{T\,d\,\theta}{u^3}\right\}; \tag{238}$$

Now $\mu = 1 + m$, and if we neglect the powers and products of the disturbing forces and the mass m, the equation becomes

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{\mu}{h^2} - \frac{m'}{h^2}\left\{\frac{\text{V}}{u^2} - \frac{T}{u^3}\,.\,\frac{d\,u}{d\,\theta} - \frac{2}{h^2}\int\frac{T\,d\,\theta}{u^3}\right\}; \tag{239}$$

It is on this equation, (or the rigorous equation (237)) that the perturbation of the radius vector is made to depend in the lunar theory. It remains to derive an equation from which the perturbation in latitude can be obtained. For this purpose, all that is necessary will be to regard x and y as known, and to employ the third of our general equations (94) to obtain a value of z, or, which comes to the same, of s, since $z = \rho\, s$, and ρ is given by the previous theory.

Now, if in our equation (94) we change the independent variable from t to θ, it becomes

$$o = \frac{d^2\,z}{d\,t^2} - d\,z\,.\,\frac{d^2\,t}{d\,t^3} + \mu\,.\,\frac{z}{r^3} + m'\,.\,\frac{d\,\Omega}{d\,z}$$

but since

$$\left.\begin{aligned} \frac{1}{d\,t^2} & = \frac{h^2 + 2\,m'\int T\,\rho^3\,d\,\theta}{\rho^4\,d\,\theta^2} \\ \frac{d^2\,t}{d\,t^3} & = \frac{1}{d\,\theta}\left\{\frac{2\,d\,\rho}{\rho^5\,d\,\theta}\,(h^2 + 2\,m'\int T\,\rho^3\,d\,\theta) - m'\,.\,\frac{T}{\rho}\right\} \end{aligned}\right\}; \;\ldots\ldots\ldots\ldots\ldots \tag{240}$$

as appears from equations (233) and (234); if we substitute these, we shall get

$$\begin{aligned} o = {} & \left\{h^2 + 2\,m'\int T\,\rho^3\,d\,\theta\right\}\left(\frac{1}{\rho}\,.\,\frac{d^2\,z}{d\,\theta^2} - \frac{2\,d\,\rho}{\rho^2 d\,\theta}\,\frac{d\,z}{d\,\theta}\right) \\ & + \rho^3\left\{\frac{\mu\,z}{r^3} + m'.\,\frac{d\,\Omega}{d\,z} + m'.\,\frac{T}{\rho}\,.\,\frac{d\,z}{d\,\theta}\right\} \end{aligned}$$

Now we have $z = \rho\, s$, $d\,z = \rho\, d\,s + s\,d\,\rho$, $d^2\,z = \rho\, d^2 s + 2\,d\,\rho\, d\,s + s\,d^2\,\rho$, whence we find

$$\frac{1}{\rho} \cdot \frac{d^2 z}{d\theta^2} - \frac{2\,d\rho}{\rho^2\,d\theta} \cdot \frac{dz}{d\theta} = \frac{d^2 s}{d\theta^2} + \frac{\rho^2 s}{\rho\,d\theta^2} \left\{ \frac{d^2 \rho}{\rho^2} - \frac{2\,d\rho^2}{\rho^3} \right\}$$

But, by (235) we have

$$\left(h^2 + 2\,m' \int T \rho^3\,d\theta\right) \times \frac{1}{\rho\,d\theta^2} \left\{ \frac{d^2\rho}{\rho^2} - \frac{2\,d\rho^2}{\rho^3} \right\} =$$

$$= \frac{1}{\rho^2} \left(h^2 + 2\,m' \int T \rho^3\,d\theta\right) - \frac{\mu}{\rho} \cdot \frac{\rho^3}{r^3} - m' \left\{ \rho\,\mathrm{V} + T \cdot \frac{d\rho}{d\theta} \right\}$$

so that, by substitution we obtain,

$$o = \left(h^2 + 2\,m' \int T \rho^3\,d\theta\right) \left(\frac{d^2 s}{d\theta^2} + s \right)$$

$$+ m' \rho^3 \left\{ \frac{d\,\Omega}{dz} - s\,\mathrm{V} \right\} + m'\,T \rho^2 \left\{ \frac{d}{d\theta} - \frac{s\,d\rho}{d\theta} \right\}$$

but $dz = \rho\,ds + s\,d\rho$; so that $\frac{dz}{d\theta} - \frac{s\,d\rho}{d\theta} = \frac{\rho\,ds}{d\theta}$, and we get

$$o = \left\{ h^2 + 2\,m' \int T \rho^3\,d\theta \right\} \left(\frac{d^2 s}{d\theta^2} + s \right) + m' \rho^3 \left\{ T \frac{ds}{d\theta} - s\,\mathrm{V} + \frac{d\,\Omega}{dz} \right\}$$

this equation $\left(\text{writing } u \text{ for } \frac{1}{\rho} \right)$ becomes

$$o = \frac{d^2 s}{d\theta^2} + s + m' \cdot \frac{T \frac{ds}{d\theta} - s\,\mathrm{V} + \frac{d\,\Omega}{dz}}{u^3 \left\{ h^2 + 2\,m' \int T \frac{d\theta}{u^3} \right\}}\,; \qquad \text{(250)}$$

and when the square of the disturbing force is neglected,

$$o = \frac{d^2 s}{d\theta^2} + s + \frac{m'}{h^2\,u^3} \left\{ T \frac{ds}{d\theta} - s\,\mathrm{V} + \frac{d\,\Omega}{dz} \right\} \qquad \text{(241)}$$

Such are the fundamental equations on which the problem of the moon's motion depends. They were first investigated by Clairaut, in a piece which gained the prize of the Petersburgh Academy for 1750, and have been deduced by almost every writer since his time, as the groundwork of the lunar theory. The method we have followed shews how the expressions of the tangential and centripetal disturbing forces peculiar to this theory, originate in the general equations of the problem of these bodies; and thus connects the modern with the more ancient methods followed by Newton, Clairaut, &c.

Hitherto we have only considered the rigorous equations of the moon's motion. It remains to apply to them the methods of approximation appropriate to the case, and deduce in succession the equations of the disturbed motion. The great length to which the details of this complicated process would lead, will preclude our entering minutely into it. We must content ourselves with leading results and general principles. One great peculiarity of the lunar theory consists in this—that the mass of the disturbing body, instead of being, as in that of the planets, small in comparison with the disturbed and central bodies, exceeds them both several hundred thousand times. Its vast distance, however, makes up for this, and renders its disturbing effect small in comparison with the attraction of the central body. In the foregoing equations then the quantities m, and m', are not to be our guides in regulating the orders of the corrections. The very small quantities, whose powers determine the convergency of our approximations are T and V, or rather $m'\,T$, and $m'\,\mathrm{V}$, for m' never occurs unmultiplied by one or other of these quantities. Let us therefore, first of all, consider the nature and magnitude of these forces.

Expression of the centripetal disturbing force, $m'\,\mathrm{V}$.

$$m'\,\mathrm{V} = m' \left(\frac{x}{\rho} \frac{d\,\Omega}{dx} + \frac{y}{\rho} \frac{d\,\Omega}{dy} \right) = \frac{m'}{\rho} (x\,x' + y\,y') \left(\frac{1}{r'^3} - \frac{1}{\lambda^3} \right) + \frac{m'\,\rho}{\lambda^3}$$

$$= m'\,\rho' \cdot \cos(\theta - \theta') \left\{ \frac{1}{r'^3} - \frac{1}{\lambda^3} \right\} + \frac{m'}{\lambda^3}$$

Now, if we neglect the inclination of the moon's orbit, and take the plane of the ecliptic for our fixed plane, we have

$$z = o, \quad z' = o, \quad r = \rho, \quad r' = \rho', \quad s = o,$$

and

$$\frac{1}{\lambda} = \left\{ r'^2 \quad 2\,r\,r' \cdot \cos(\theta - \theta') + r^2 \right.^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left\{1 - \left(2\frac{r}{r'}\cos(\theta - \theta') - \left(\frac{r}{r'}\right)^2\right)\right\}^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left\{1 - \left(2\frac{r}{r'}\cos w - \left(\frac{r}{r'}\right)^2\right)\right\}^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left\{1 + \left(3\frac{r}{r'}\cos w - \frac{3}{2}\left(\frac{r}{r'}\right)^2\right) + \frac{3\,.\,5}{2\,.\,4}\left(2\frac{r}{r'}\,.\cos w - \left(\frac{r}{r'}\right)^2\right)^2 + \&c.\right\}$$

$$= \frac{1}{r'^3}\left\{1 + 3\frac{r}{r'}\cos w + \frac{9 + 15\,.\cos 2\,w}{4}\,.\left(\frac{r}{r'}\right)^2 + \&c.\right\}$$

putting $w = \theta - \theta'$, and developing in powers of $\frac{r}{r'}$. Hence, we obtain (writing r for ρ in the expression for V) after all reductions,

$$m'\,V = -\frac{m'}{r^2}\left\{\frac{1 + 3\cos 2\,w}{2}\,.\frac{r}{r'} + \frac{9\cos w + 15\cos 3\,w}{8}\left(\frac{r}{r'}\right)^2 + \&c.\right\} \tag{242}$$

Now $\frac{m'}{r'^2}$ is the accelerating force exerted by the sun on the earth, whose intensity, as we have shewn in Section 1. Part I., is represented by 2·17399 (in its mean state) on the supposition of the earth's attraction on the moon being unity. Also, $\frac{r}{r'} = \frac{60{\cdot}23799}{23405} = \frac{1}{388}$ nearly, and $\left(\frac{r}{r'}\right)^2 = \frac{1}{119920}$; so that we may safely neglect $\left(\frac{r}{r'}\right)^2$ &c. The constant, or non-periodical parts of $m'\,V$ is therefore equal to $-\frac{1}{2}\,.\frac{m'\,r}{r'^2} = -\frac{1}{2}\,m'\left(\frac{r}{r'}\right)^3 \times \frac{1}{r^2} =$ nearly $-\frac{1}{357{\cdot}5} \times \frac{1}{r^2}$.

Let a and a' be the actual mean distances of the moon and sun from the earth, and supposing

$$a = m'\left(\frac{a}{a'}\right)^3$$

a will be nearly $\frac{1}{178{\cdot}7}$, and the mean effect of the centripetal disturbing force will be $-\frac{\frac{1}{2}\,a}{r^2}$ indicating a diminution of the moon's mean gravity, amounting to $\frac{1}{357}$th of its whole quantity.

The moon's mean gravity being diminished by the action of the sun, and the velocity remaining nearly the same, the centrifugal force which would be balanced by the centripetal in a circular orbit without the sun's action, will obtain the preponderance when the sun acts and carry the moon farther out; thus the disturbing force has the effect of increasing the distance and periodic time above what they would be in the undisturbed orbit.

Let us next consider the tangential disturbing force $m'\,T$. Now we have

$$m'\,T = m'\left(\frac{x}{\rho}\,\frac{d\,\Omega}{d\,y} - \frac{y}{\rho}\,\frac{d\,\Omega}{d\,x}\right)$$

$$= m'.\frac{x\,y' - y\,x'}{\rho}\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right) = -\,m'\,r'\,.\sin\,w\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right)$$

$$= +\frac{m'}{r'^2}\left(3\frac{r}{r'}\cos w + \left(\frac{r}{r'}\right)^2.\,\&c.\right).\sin w$$

that is

$$m'\,T = \frac{3}{2}\,\frac{m'}{r'^2}\,.\,\frac{r}{r'}\,.\sin 2\,w\,; \tag{243}$$

The tangential force therefore vanishes both in syzigies and quadratures, and is a maximum in octants, or at 45° of elongation. Now we have seen, that the description of areas would be equable, were it not for this tangential force. Hence, at the former points, the equable description of areas still holds good; while, at the latter, an acceleration or retardation takes place. This, of course, produces an equation in the moon's longitude, whose period is semimenstrual, as during the first quarter of its synodic revolution, the moon is continually retarded in its orbit (supposed circular): at the first quadrature the momentary retardation ceases and changes to an acceleration—and here the effect accumulated through the whole of the quadrant has reached its maximum. In the next quadrant, the moon is gradually accelerated; and at the full moon, its motion has regained all it had lost, and is restored to its mean state, so that here the equation is nothing, and so on. This is the origin of that equation of the moon's motion to which astronomers have attached the name of the variation.

SECTION II.

Approximate integration of the equation of the moon's orbit.

In this inquiry we will commence with the equation (238), in which the square of the disturbing force is neglected. Now, the obvious mode of beginning the approximation, would be (in analogy with what we did in the theory of the planets) to substitute for u its undisturbed, or elliptic value, $\frac{1 + e \,.\, \cos \theta}{a\,(1 - e^2)}$. But, as it is obvious that the more nearly our first approximation approaches the truth, the more rapid will be the convergency of all the succeeding ones, we may take advantage of what observation has taught us respecting the form of the lunar orbit, and assume, for our first value of u, an expression representing that form, more nearly than the ellipse. Now, we learn from observation, that during a single revolution, it is true, the orbit does not deviate materially from an ellipse, but that in a very few revolutions the elliptic radius vector differs very sensibly from the real one, by reason of the rapid motion of the lunar apsides, which perform a whole circuit in little more than nine years. Instead then of representing by u the inverse radius vector of a *fixed* ellipse, we will give it such a form, in terms of θ, as shall express that of an ellipse in motion, and revolving in its own plane in the direction of the moon's motion, as observation informs us it does.

Let us then take c, such that $1 : 1 - c ::$ the moon's motion in longitude to the motion of the apsides, then will $\theta\,(1 - c)$ be the angle described by the apsis, while the moon describes θ; and, consequently, supposing (as we may) that the origin of the time t is fixed at the epoch when the apsis coincided with the axis of the x, we shall have $\theta - (1 - c)\,\theta = c\,\theta$ for the moon's true anomaly, and in the place of taking $u = \frac{1 + e \,.\, \cos \theta}{a\,(1 - e^2)}$, we may take $u = \frac{1 + e \,.\, \cos c\,\theta}{a\,(1 - e^2)}$ for our initial value of u.

Although the idea of commencing the approximation with this value of u is, it is true, taken from observation, yet, when once suggested, no matter from what source, we may look on it as a mere analytical artifice; and the truth or falsehood, convenience or inconvenience of the assumption will be tried by actual substitution; when, if we find that it reproduces the original expression with a train of small corrections multiplied by the disturbing force, or can be made to do so by a proper assumption of the constant c, we may rest assured that the assumption is (mathematically speaking) a legitimate one, and concern ourselves no farther with the way in which we arrived at it.

Now, neglecting e^2, we have

$$u = \frac{1}{a}\,(1 + e \,.\, \cos c\,\theta)$$

whence we get

$$\frac{1}{u^3} = a^3\,(1 - 3\,e \,.\, \cos c\,\theta)\,; \quad \frac{1}{u^4} = a^4\,(1 - 4\,e \,.\, \cos c\,\theta)$$

$$\frac{d\,u}{d\,\theta} = -\frac{c\,e}{a} \,.\, \sin c\,\theta$$

and, by substituting this in (238) which is equivalent to

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2} + m'\,\frac{u'^3}{h^2\,u^3} \,.\, \frac{1 + 3 \cos 2\,w}{2} - \frac{3\,m'\,u'^3}{2\,h^2\,u^4} \,.\, \frac{c\,e}{a} \sin c\,\theta \,.\, \sin 2\,w$$
$$+ \frac{3}{h^4} \int m'\,\frac{u'^3}{u^4} \sin 2w \,.\, d\,\theta$$

Now, if we neglect the eccentricity of the earth's orbit, $u' = \frac{1}{a'}$, so that $\left(\text{recollecting that } m' \,.\, \left(\frac{a}{a'}\right)^3 = a = \frac{1}{179}\right)$

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2} + \frac{a}{2\,h^2}\,(1 + 3 \cos 2\,w)\,(1 - 3\,e \,.\, \cos c\,\theta)$$
$$- \frac{3\,c\,e}{2\,h^2}\,a \,.\, \sin c\,\theta \,.\, \sin 2\,w \tag{244}$$
$$+ \frac{3\,a}{h^4} \,.\, \int (1 - 4\,e \cos c\,\theta) \sin 2\,w \,.\, d\,\theta$$

The only difficulty now in the way of integrating this, is to express $\cos 2\,w$ and $\sin 2\,w$ in terms of θ. Now we have $w = \theta - \theta'$; and since we neglect the eccentricity of the earth's orbit, θ' is proportional to the time, or $\theta' = n'\,t$; also we have θ by the equation, $h\,d\,t = r^2\,d\,\theta$, or

$$t = \frac{1}{h} \int \frac{d\,\theta}{u^2} = \frac{1}{h} \int d\,\theta\,(1 - 2\,e \,.\, \cos c\,\theta)\,a^2 = \frac{a^2}{h}\left(\theta - \frac{2\,e}{c} \,.\, \sin c\,\theta\right)$$

Hence, since $h = \sqrt{a}$ very nearly, and the motion of the apsides being slow in comparison with that of the moon itself, c is nearly unity, at least sufficiently so for a first approximation; consequently,

$$t = a^{\frac{3}{2}} (\theta - 2\,e\,.\sin c\,\theta)$$

or

$$n\,t = \theta - 2\,e\,.\sin c\,\theta$$

and eliminating t between this and the equation $\theta' = n'\,t$, we have

$$\frac{n}{n'}\,\theta' = \theta - 2\,e\,.\sin c\,\theta$$

and therefore,

$$\theta - \theta' = \theta\left(1 - \frac{n'}{n}\right) + 2\,e\,.\frac{n'}{n}\,.\sin c\,\theta$$

putting n and n' for the mean motions of the moon and earth. Hence we find, putting $\frac{n'}{n} = k$

$$\cos 2\,w = \cos 2\,(1 - k)\,\theta - \sin 2\,(1 - k)\,\theta\,.\sin\left\{4\,e\,.\frac{n'}{n}\,.\sin c\,\theta\right\}$$

$$= \cos 2\,(1 - k)\,\theta + 2\,k\,e\left\{\cos (2 - 2\,k + c)\,\theta - \cos (2 - 2\,k - c)\,\theta\right\}; \tag{245}$$

and similarly,

$$\sin 2\,w = \sin 2\,(1 - k)\,\theta + 2\,k\,e\left\{\sin (2 - 2\,k + c)\,\theta - \sin (2 - 2\,k - c)\,\theta\right\} \tag{246}$$

It only remains to substitute these values in (244) where it becomes, after reducing all the products of sines and cosines to simple multiples, and rejecting e^2,

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2}\left(1 - \frac{\alpha}{2}\right) - \frac{3\,e}{2\,h^2}\,\alpha\,.\cos c\,\theta \tag{247}$$

$$+ \alpha\left\{\mathrm{A}\,.\cos (2 - 2\,k)\,\theta + \mathrm{B}\,e\,.\cos (2 - 2\,k - c)\,\theta + \mathrm{C}\,e \cos (2 - 2\,k + c)\,\theta\right\}$$

where A, B, C, are co-efficients easily expressed in functions of k and c. If we integrate this, we get

$$u = \frac{1}{h^2}\left(1 - \frac{\alpha}{2}\right) - \frac{3\,e}{2\,h^2}\,.\frac{\alpha}{c^2 - 1}\,.\cos c\,\theta \tag{248}$$

$$+ \alpha\left\{\mathrm{A}'.\cos (2 - 2\,k)\,\theta + \mathrm{B}'\,e\,.\cos (2 - 2\,k - c)\,\theta + \mathrm{C}'.\,e\,.\cos (2 - 2\,k + c)\,\theta\right\}$$

Let us now consider the nature of this result. We set out with $u = \frac{1}{a} + \frac{e}{a}\,.\cos c\,\theta$ for a first assumed value of u; and having substituted this, have deduced another value, which ought to be a nearer approximation, and which ought to coincide with the former, if $\alpha = o$.

Now, the terms within the brackets being all multiplied by α, vanish as they should do when $\alpha = o$, and the two first terms of this expression will coincide with the uncorrected value of u, provided we take

$$\frac{1}{h^2}\left(1 - \frac{\alpha}{2}\right) = \frac{1}{a} \quad \text{or } h = \sqrt{a}\,.\sqrt{1 - \frac{\alpha}{2}} = \sqrt{a}\left(1 - \frac{\alpha}{4}\right)$$

and

$$\frac{3\,e}{2\,h^2}\,.\frac{\alpha}{1 - c^2} = \frac{e}{a}, \quad \text{or } 1 - c^2 = \frac{3}{2}\,\alpha\,.\frac{a}{h^2}$$

or

$$1 - c^2 = \frac{3}{2}\,\alpha\,.\frac{a}{h^2} = \frac{3}{2}\,\frac{\alpha}{1 - \frac{\alpha}{2}} = \frac{3}{2}\,\alpha + \frac{3}{4}\,\alpha^2$$

$$1 - c = \frac{3}{4}\,\alpha + \frac{21}{32}\,\alpha^2, \quad \text{or } 1 - c = \frac{3}{4}\,\alpha$$

neglecting α^2; that is, since $\alpha = \frac{1}{178{\cdot}7}$, $c = 0{\cdot}99580$, and $1 - c = 0{\cdot}0042$. Hence, according to this view of the subject, the progression of the apsides in one revolution of the moon is $0{\cdot}0042 \times 360^\circ$, or about $1^\circ\,31'$.

Now, it is remarkable that this progression of the lunar apsides, as determined by a first approximation, is only half the quantity actually observed; and this is the conclusion Newton had arrived at, where, after going through a process capable of being identified with that now under consideration, he remarks (in the 9th section of the *Principia*,) "Apsis lunæ est duplo velocior circiter." Every attempt to obtain a nearer coincidence, by taking into account the higher powers of the eccentricities, the inclination of the moon's orbit (which is pretty considerable, and materially modifies some of its inequalities) &c, failed; and this difficulty, which Newton evidently felt, though he had passed it in this apparently negligent manner, as being at that time beyond his reach, or deferred for farther consideration, became so great a stumbling-block in the way of succeeding geometers, as to shake their faith in the theory of gravity, till Clairaut shewed that it would vanish on pushing the process to a second approximation, and that, in fact, the value of c depends, for nearly half its quantity, on a term containing the square of the disturbing force.

To shew this, we have only to substitute the approximate value of u (248) back into the original equation

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2} + \left\{\frac{m'\,\mathrm{T}}{h^2\,u^3}\,\frac{d\,u}{d\,\theta} - \frac{m'\,\mathrm{V}}{h^2\,u^2} + 2\left(\frac{d^2\,u}{d\,\theta^2} + u\right)\int\frac{m'\,\mathrm{T}}{h^2\,u^3}\,d\,\theta\right\}$$

Supposing now we represent the first assumed value, or $\frac{1}{a}\,(1 + e \,.\, \cos c\,\theta)$ by u and by $\delta\,u$ all the terms added by the first approximation, then will the value resulting from a second approximation be had by substituting $u + \delta\,u$ for u in all the part of the above equation within the brackets, and integrating. As we only require at present to know the terms introduced by this second approximation, let us call them $\delta^2\,u$, and the value of $\delta^2\,u$ will be had at once by integrating the equation

$$o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u + \frac{m'}{h^2}\left\{\begin{aligned}&\delta\,.\,\frac{T}{u^3}\,\frac{d\,u}{d\,\theta} - \delta\,\frac{V}{u^2} + 2\left(\frac{d^2\,\delta\,u}{d\,\theta^2} + \delta\,u\right)\int\frac{T}{u^3}\,d\,\theta\\ &+ 2\left(\frac{d^2\,u}{d\,\theta^2} + u\right)\int\delta\,.\,\frac{T}{u^3}\,d\,\theta\end{aligned}\right\};\ldots\ldots\ldots(249)$$

Now, since $h^2 = a$ very nearly, and $\mathrm{a} = m'\,.\,\frac{a^3}{a'^3}$, we have

$$\frac{m'}{h^2}\,\delta\left(\frac{T}{u^3}\,\frac{d\,u}{d\,\theta}\right) = \frac{3}{2}\,\mathrm{a}\,\delta\left\{\frac{\sin 2\,w}{(a\,u)^4}\,\frac{d\,u}{d\,\theta}\right\}$$

$$= \frac{3}{2}\,\mathrm{a}\left\{\frac{2\cos 2\,w}{(a\,u)^4}\,\frac{d\,u}{d\,\theta}\,\delta\,w + \frac{\sin 2\,w}{(a\,u)^4}\,.\,\frac{d\,\delta\,u}{d\,\theta} - 4\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\frac{d\,u}{d\,\theta}\,\delta\,u\right\}$$

$$-\frac{m'}{h^2}\,\delta\,\frac{V}{u^2} = -\,\frac{\mathrm{a}}{2\,a}\,\delta\,\frac{1 + 3\cos 3\,w}{(a\,u)^3} = \frac{3\,\mathrm{a}}{2\,a}\left\{\frac{2\sin 2\,w}{(a\,u)^3}\,\delta\,w + a\,.\,\frac{1 + 3\cos 2\,w}{(a\,u)^4}\,\delta\,u\right\}$$

$$\frac{m'\,T}{h^2\,u^3} = \frac{3\,\mathrm{a}}{2}\,.\,\frac{\sin 2\,w}{(a\,u)^4}$$

$$\int\delta\,\frac{m'\,T}{h^2\,u^3}\,d\,\theta = 3\,\mathrm{a}\int\left\{\frac{\cos 2\,w}{(a\,u)^4}\,\delta\,w - 2\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\delta\,u\right\}d\,\theta$$

Let these values be substituted in the general equation (249), and it becomes

$$\begin{aligned}o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u &+ \frac{3}{2}\,\mathrm{a}\left\{\frac{2\cos 2\,w}{(a\,u)^4}\,\frac{d\,u}{d\,\theta}\,\delta\,w + \frac{\sin 2\,w}{(a\,u)^4}\,\frac{d\,\delta\,u}{d\,\theta} - 4\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\frac{d\,u}{d\,\theta}\,\delta\,u\right\}\\ &+ \frac{3}{2}\,\mathrm{a}\left\{\frac{2\sin 2\,w}{a\,(a\,u)^3}\,\delta\,w + \frac{1 + 3\cos 2\,w}{(a\,u)^4}\,\delta\,u\right\}\\ &+ 3\,\mathrm{a}\left(\frac{d^2\,\delta\,u}{d\,\theta^2} + \delta\,u\right)\int\frac{\sin 2\,w}{(a\,u)^4}\,d\,\theta\\ &+ 6\,\mathrm{a}\left(\frac{d^2\,u}{d\,\theta^2} + u\right)\int\left\{\frac{\cos 2\,w}{(a\,u)^4}\,\delta\,w - 2\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\delta\,u\right\}d\,\theta\end{aligned}\qquad(250)$$

To reduce this equation to the degree of approximation required, let us agree to neglect all terms which contain a^2 multiplied by any power of the eccentricity; or in that part of the approximation which depends on the square of the disturbing force, to regard the orbits as circular. We shall have then $u = \frac{1}{a}$ $a\,u = 1$, and supposing $\delta\,u = \frac{\mathrm{a}}{a}\,.\,\phi$, where ϕ is the sum of any number of terms of the form $A\,.\,\cos i\,\theta$, we have

$$\delta\,t = \frac{1}{h}\int\delta\,\frac{d\,\theta}{u^2} = \frac{-1}{h}\int\frac{2\,\delta\,u}{u^3}\,d\,\theta = -\,\frac{2\,a^2}{h}\,\mathrm{a}\int\phi\,d\,\theta$$

or since $n = \frac{1}{a^{\frac{3}{2}}} = \frac{h}{a^2}$

$$\delta\,.\,n\,t = -\,2\,\mathrm{a}\int\phi\,d\,\theta$$

consequently, since $w = \theta - \theta = \theta - \frac{n'}{n}\,t = \theta - k\,.\,n\,t$, we get

$$w = \theta\,(1 - k),\quad \text{and } \delta\,w = -\,k\,.\,\delta\,n\,t = 2\,k\,\mathrm{a}\int\phi\,d\,\theta$$

We have also $\frac{d\,u}{d\,\theta} = o$ and $\frac{1}{(a\,u)^n} = 1$; so that the equation (250) reduces itself to

$$\begin{aligned}o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u + \frac{3\,\mathrm{a}^2}{2\,a}\sin 2\,w\,.\,\frac{d\,\phi}{d\,\theta} &+ \frac{6\,k\,\mathrm{a}^2}{a}\sin 2\,w\,.\int\phi\,d\,\theta + \frac{3\,\mathrm{a}^2}{2\,a}\,(1 + 3\cos 2\,w)\,.\,\phi\\ &+ \frac{3\,\mathrm{a}^2}{a}\left(\frac{d^2\,\phi}{d\,\theta^2} + \phi\right)\int d\,\theta\,.\,\sin 2\,w\\ &+ \frac{12\,\mathrm{a}^2}{a}\int\left\{k\cos 2\,w\int\phi\,d\,\theta - \sin 2\,w\,.\,\phi\right\}d\,\theta\end{aligned}\qquad(251)$$

Let us now, more particularly, consider any one of the terms to which ϕ consists; as, for instance, A . cos B θ. This gives

$$\phi = \mathrm{A} \,.\, \cos \mathrm{B}\,\theta, \quad \frac{d\,\phi}{d\,\theta} = -\,\mathrm{AB} \,.\, \sin \mathrm{B}\,\theta, \quad \frac{d^2\,\phi}{d\,\theta^2} = -\,\mathrm{AB}^2 . \cos \mathrm{B}\,\theta; \int \phi\, d\,\theta = \frac{\mathrm{A}}{\mathrm{B}} \,.\, \sin \mathrm{B}\,\theta, \text{ \&c.}$$

and the equation will give, on substitution and reduction,

$$\begin{aligned} o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u &+ \left\{\frac{6\,k\,a^2}{a} \,.\, \frac{\mathrm{A}}{\mathrm{B}} - \frac{3\,a^2}{2\,a} \,.\, \mathrm{A\,B}\right\} \sin 2\,w \,.\, \sin \mathrm{B}\,\theta \\ &+ \left\{\frac{3\,a^2}{2\,a}\,(1 + 3 \,.\, \cos 2\,w) + (\mathrm{B}^2 - 1) \,.\, \frac{3\,a^2}{a} \,.\, \frac{\cos 2\,w}{2\,(1 - k)}\right\} \mathrm{A} \,.\, \cos \mathrm{B}\,\theta \\ - \frac{6\,\mathrm{A}\,a^2}{a} &\left\{\frac{k - \mathrm{B}}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)} \cos (\mathrm{B} + 2 - 2\,k)\,\theta + \frac{k + \mathrm{B}}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)} \cos (\mathrm{B} - 2 + 2\,k)\,\theta\right\} \end{aligned}$$

or resolving the products of sines and cosines by the formula (A), page 690,

$$\begin{aligned} o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2 u &+ \frac{3\,\mathrm{A}\,a^2}{2\,a}\left(\frac{2k}{\mathrm{B}} - \frac{\mathrm{B}}{2}\right)\{\cos (\mathrm{B} - 2 + 2\,k)\,\theta - \cos (\mathrm{B} + 2 - 2\,k)\,\theta\} \\ &+ \frac{3\,\mathrm{A}\,a^2}{2\,a} \,.\, \cos \mathrm{B}\,\theta \\ &+ \frac{3\,\mathrm{A}\,a^2}{4\,a}\left\{3 + \frac{\mathrm{B}^2 - 1}{1 - k}\right\}\{\cos (\mathrm{B} - 2 + 2\,k)\,\theta + \cos (\mathrm{B} + 2 - 2\,k)\,\theta\} \\ &- \frac{6\,\mathrm{A}\,a^2}{a}\left\{\frac{k + \mathrm{B}}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)} \,.\, \cos (\mathrm{B} + 2 - 2\,k)\,\theta + \frac{k + \mathrm{B}}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)} \cos (\mathrm{B} - 2 + 2\,k)\,\theta\right\} \end{aligned}$$

That is, assembling together the co-efficients of like terms,

$$\begin{aligned} o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u &+ \frac{3\,\mathrm{A}\,a^2}{a}\left\{\frac{k}{\mathrm{B}} - \frac{\mathrm{B}}{4} + \frac{3}{4} + \frac{\mathrm{B}^2 - 1}{4\,(1 - k)} - \frac{2\,(k + \mathrm{B})}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)}\right\} \cos (\mathrm{B} - 2 + 2\,k)\,\theta \\ &+ \frac{3\,\mathrm{A}\,a^2}{a}\left\{-\frac{k}{\mathrm{B}} + \frac{\mathrm{B}}{4} + \frac{3}{4} + \frac{\mathrm{B}^2 - 1}{4\,(1 - k)} - \frac{2\,(k - \mathrm{B})}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)}\right\} \cos (\mathrm{B} + 2 - 2\,k)\,\theta \\ &+ \frac{3\,\mathrm{A}\,a^2}{2\,a} \,.\, \cos \mathrm{B}\,\theta \end{aligned}$$

Which equation, integrated, gives

$$\delta^2\,u = \frac{3\,\mathrm{A}\,a^2}{2a} \,.\left\{\begin{aligned} &\frac{\cos \mathrm{B}\,\theta}{\mathrm{B}^2 - 1} + \left(\frac{2k}{\mathrm{B}} + \frac{3 - \mathrm{B}}{2} + \frac{\mathrm{B}^2 - 1}{2\,(1 - k)} - \frac{4\,(k + \mathrm{B})}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)}\right)\frac{\cos (\mathrm{B} - 2 + 2k)\,\theta}{(\mathrm{B} - 2 + 2\,k)^2 - 1} \\ &+ \left(-\frac{2k}{\mathrm{B}} + \frac{3 + \mathrm{B}}{2} + \frac{\mathrm{B}^2 - 1}{2\,(1 - k)} - \frac{4\,(k - \mathrm{B})}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)}\right)\frac{\cos (\mathrm{B} + 2 - 2k)\,\theta}{(\mathrm{B} + 2 - 2\,k)^2 - 1} \end{aligned}\right\} \quad 252)$$

Such is the value of that part of $\delta^2\,u$, which originates in the term A . cos B θ in the value of ϕ or in the term $\frac{a}{a}$. A cos B θ in u itself. A similar set of terms, having for their arguments respectively,

$$\mathrm{B}'\,\theta, \quad (\mathrm{B}' - 2 + 2\,k)\,\theta, \quad (\mathrm{B}' + 2 - 2\,k)\,\theta$$

and

$$\mathrm{B}''\,\theta, \quad (\mathrm{B}'' - 2 + 2\,k)\,\theta, \quad (\mathrm{B}'' + 2 - 2\,k)\,\theta; \text{ \&c.}$$

will arise from the other terms $\frac{a}{a}\,\mathrm{A}' \,.\, \cos \mathrm{B}'\,\theta$, $\frac{a}{a}\,\mathrm{A}'' \,.\, \cos \mathrm{B}''\,\theta$, &c. Consequently, by substituting for B, B′, B″, &c. the several values which they actually have in the first approximate value of u (equation 248), or $2 - 2\,k$, $2 - 2\,k - c$, $2 - 2\,k + c$, we shall obtain the arguments of the several terms in the second approximation, and so on. These are then as follows:—

From $c\,\theta$ arise the arguments $c\,\theta$, $2\,c\,\theta$
From $(2 - 2\,k)\,\theta$ arise the arguments $(2 - 2\,k)\,\theta$; $c\,\theta$; $(4 - 4\,k)\,\theta$
From $(2 - 2\,k - c)\,\theta$ arise $(2 - 2\,k - c)\,\theta$; $c\,\theta$; $(4 - 4\,k - c)\,\theta$
From $(2 - 2\,k + c)\,\theta$ arise $(2 - 2\,k + c)\,\theta$; $c\,\theta$; $(4 - 4\,k + c)\,\theta$

Thus we see, 1st. that each term which in the expression of u is multiplied by the first power of the disturbing force a, reproduces itself on a second approximation, multiplied by the square of a, and thus the co-efficient of every term in the general expression of u, consists in fact of an infinite series of powers of a; for what takes place at the second approximation will, of course, do so at every succeeding step; and, in fact, the very same analysis, and the same resulting formula, may be applied to deduce the terms depending on a^3, a^4, &c., in all of which the same arguments will occur.

Hence, it follows, that if we set out, as a first hypothesis, with $u = \frac{1}{a} (1 + e \,.\, \cos c\,\theta)$ the process of approximation leads us to a value of u, of the form

$$\begin{aligned} u = {} & \frac{1}{a} \left\{ \mathrm{A} + \mathrm{B}\,a + \mathrm{C}\,a^2 + \mathrm{D}\,a^3 + \&\mathrm{c.} \right\} - \frac{1}{h^2} \\ & + \cos c\,\theta \left\{ \mathrm{A}'\,a + \mathrm{B}'\,a^2 + \mathrm{C}'\,a^3 + \&\mathrm{c.} \right\} \\ & + \cos (2 - 2\,k)\,\theta \,.\, \left\{ \mathrm{A}'\,a + \mathrm{B}''\,a^2 + \&\mathrm{c.} \right\} \\ & + \cos (4 - 4\,k)\,\theta \,.\, \left\{ \mathrm{B}'''\,a^2 + \&\mathrm{c.} \right\} \\ & + \cos (2 - 2\,k + c) \left\{ \mathrm{A}^{\mathrm{iv}}\,a + \mathrm{B}^{\mathrm{iv}}\,a^2 + \&\mathrm{c.} \right\} \\ & + \cos (2 - 2\,k - c) \left\{ \mathrm{A}^{\mathrm{v}}\,a + \mathrm{B}^{\mathrm{v}}\,a^2 + \&\mathrm{c.} \right\} \\ & + \&\mathrm{c.} \end{aligned}$$

We have therefore only to take care that this shall not contradict the original hypothesis, and therefore we must have

$$1 = \mathrm{A} + \mathrm{B}\,a + \mathrm{C}\,a^2 + \&\mathrm{c.} - \frac{a}{h^2} \qquad (253)$$

$$\frac{e}{a} = \mathrm{A}'\,a + \mathrm{B}'\,a^2 + \mathrm{C}'\,a^3 + \&\mathrm{c.} \qquad (254)$$

These equations established, the two first terms of u agree with the original value, and the remaining ones are the equations due to the effect of perturbation, not capable of being expressed by the hypothesis of a revolving ellipse.

The equation (253) expresses a relation between a and h, which we shall consider presently; the other (254) gives the value of $1 - c$, or the velocity of the motion of the lunar apogee. We have already seen (248) that the value of A' is $\frac{3\,e}{2\,h^2\,(1 - c^2)}$, and if we break off the equation (254) at the first power of a we have also seen that the resulting value of $1 - c$ is $\frac{3}{4}\,a$, amounting to only half the value given by observation.

If we make B successively equal to $2 - 2\,k - c$ and $2 - 2\,k + c$ in (252), we find for the co-efficients of the argument $c\,\theta$, respectively,

$$\frac{1}{c^2 - 1} \cdot \frac{3\,a^2\,e}{2\,a}\,\mathrm{B}' \left\{ \frac{2 + 6\,k + c - 4\,k^2 - 4\,k\,c - c^2}{2\,(2 - 2\,k - c)} + \frac{(2 - 2\,k - c)^2 - 1}{2\,(1 - k)} + \frac{4\,(2 - k - c)}{c\,(2 - 2\,k - c)} \right\}$$

and

$$\frac{1}{c^2 - 1} \cdot \frac{3\,a^2\,e}{2\,a}\,\mathrm{C}' \left\{ \frac{2 + 6\,k - c - 4\,k^2 + 4\,k\,c - c^2}{2\,(2 - 2\,k + c)} + \frac{(2 - 2\,k + c)^2 - 1}{2\,(1 - k)} - \frac{4\,(2 - k + c)}{c\,(2 - 2\,k + c)} \right\}$$

when B' and C' are the co-efficients of $e \,.\, \cos\,(2 - 2\,k - c)\,\theta$ and $e \,.\, \cos\,(2 - 2\,k + c)\,\theta$ within the parentheses of equation (248).

Thus our equation (254) carried as far as a^2, will become

$$1 - c^2 = \frac{3}{2}\,a + \frac{3}{2}\,a^2 . \left\{ \begin{aligned} & \mathrm{B}' \left(\frac{2 - 6\,k + c - 4\,k^2 - 4\,k\,c - c^2}{2\,(2 - 2\,k - c)} + \frac{(2 - 2\,k - c)^2 - 1}{2\,(1 - k)} + \frac{4\,(2 - k - c)}{c\,(2 - 2\,k - c)} \right) \\ + {} & \mathrm{C}' \left(\frac{2 + 6\,k - c - 4\,k^2 + 4\,k\,c - c^2}{2\,(2 - 2\,k + c)} + \frac{(2 - 2\,k + c)^2 - 1}{2\,(1 - k)} - \frac{4\,(2 - k + c)}{c\,(2 - 2\,k + c)} \right) \end{aligned} \right\} \qquad (255)$$

This equation is to be resolved by approximation, so as to find c; and as a is a very small quantity, the approximation may be performed easily; for we have only to rescind a^2, and take $c = 1 - \frac{3}{4}\,a$ for a first approximation $= 0{\cdot}9957$, and then substitute this for c in the co-efficient of a^2, which will then become a given number (B', C', being previously computed, and k being known,) after which, if this co-efficient be called β, we have

$$1 - c^2 = \frac{3}{2}\,(a + \beta\,a^2) \quad c = \sqrt{1 - \frac{3}{2}\,(a + \beta\,a^2)} = 1 - \frac{3}{4}\,a - \frac{6\,\beta + 1}{8}\,a^2$$

and, again substituting this for c, we find a nearer approximation, and so on.

Now, it is a remarkable circumstance, that when this process is executed in numbers, the term $-\frac{6\,\beta + 1}{8}\,a^2$ thus added to the value of c is found to be very nearly equal to the term $-\frac{3}{4}\,a$ introduced by the first step of the approximation; and on pushing it still farther, the subsequent corrections are found to be inconsiderable. Now the velocity of the apogee of the lunar orbit is expressed by $1 - c$, or by $\frac{3}{4}\,a + \frac{6\,\beta + 1}{8}\,a^2 + \&\mathrm{c.}$ We have already seen that the value of this, as deduced from a first approximation (or the part $\frac{3}{4}\,a$) came out only half what observation has assigned as its real amount, and that this was regarded as a great difficulty in the way of the full admission of the Newtonian law of gravity. We now see the reason of this; the remaining half is accounted for by the term $\frac{6\beta+1}{8}\,a^2$ arising from a second approximation; and the fact, so far from being an objection against gravity, is thus converted into a most cogent argument in its favour—its effects being thus shewn to correspond not merely to general results and first approximations, but to the refinements and niceties of theory.

SECTION III.

Expression of the moon's mean longitude in terms of the true, and vice versâ; and of the variation and evection of the moon.

THE purposes of astronomy require us to know the moon's true longitude and latitude at any assigned instant, or to know the angle θ in terms of the time t. For this we must have recourse to the equation.

$$t = \int \frac{\rho^2\, d\,\theta}{\sqrt{h^2 + 2\,m' \int T\,\rho^3\, d\,\theta}} = \int \frac{d\,\theta}{u^2 \sqrt{h^2 + 2\,m' \int \frac{T}{u^3}\, d\,\theta}}$$

If we develope this, and retain only the first power of the disturbing force, we get

$$t = \frac{1}{h} \int \frac{d\,\theta}{u^2} \left\{ 1 - \frac{m'}{h^2} \int \frac{T}{u^3}\, d\,\theta \right\} = \frac{1}{h} \int \frac{d\,\theta}{u^2} \left\{ 1 - \frac{m'}{a} \int \frac{T}{u^3}\, d\,\theta \right\}$$

because $h^2 = a$, if the disturbing force and square of the eccentricity be neglected. Now, in this for u let us substitute

$$u = \frac{1}{a}\,.\,(1 + e\,.\cos c\,\theta) + a\left\{ A'.\cos(2-2\,k)\,\theta + B'\,e\,.\cos(2-2\,k-c)\,\theta + C'\,e\,.\cos(2-2\,k+c)\,\theta \right\}$$

and for $\frac{m'}{h^2} \int \frac{T}{u^3}\, d\,\theta$ its value $\frac{3\,a}{2} \int \frac{\sin 2\,w}{(a\,u)^4}\, d\,\theta$

$$= \frac{3\,a}{2} \int \sin 2\,w\,(1 - 4\,e\,.\cos c\,\theta)\, d\,\theta$$

$$= \frac{3\,a}{2} \int \left\{ \sin(2-2\,k)\,\theta - (2-2\,k)\,e \sin(2-2\,k+c)\,\theta - (2+2\,k)\,e \sin(2-2\,k-c)\,\theta \right\} d\theta$$

$$= \frac{3\,a}{2} \left\{ \frac{\cos(2-2\,k)\,\theta}{2\,(k-1)} - \frac{2\,(k-1)\,e}{2-2\,k+c} \cos(2-2\,k+c)\,\theta + \frac{2\,(k+1)\,e}{2-2\,k-c} \cos(2-2\,k-c)\,\theta \right\}$$

whence we find

$$\frac{h\,t}{a^2} = \int d\,\theta \left\{ \begin{array}{l} 1 - 2\,e\,.\cos c\,\theta - 2\,a\,a\left\{ A'.\cos(2-2\,k)\,\theta + B'\,e\,.\cos(2-2\,k-c)\,\theta + C'\,e \cos(2-2\,k+c)\,\theta \right\} \\ \qquad + 3\,A'\,a\,.\,a\,e \left\{ \cos(2-2\,k+c)\,\theta + \cos(2-2\,k-c)\,\theta \right\} \\ - \frac{3\,a}{2}(1 - 2\,e \cos c\,\theta)\left(\frac{\cos 2\,(1-k)\,\theta}{2\,(k-1)} - \frac{2\,(k-1)\,e}{2-2\,k+c} \cos(2-2\,k+c)\,\theta + \frac{2\,(k+1)\,e}{2-2\,k-c} \cos(2-2\,k-c)\,\theta \right) \end{array} \right\} \quad (256)$$

or putting $\frac{h^2}{a^2} = n$

$$n\,t = \theta - \frac{2\,e}{c}\,.\sin c\,\theta + P\,a\,.\sin(2-2\,k)\,\theta + Q\,a\,e\,.\sin(2-2\,k+c)\,\theta + R\,a\,e\,.\sin(2-2\,k-c)\,\theta \Big\} \ldots.(257)$$

A′, B′, C′, being as in (248)
where

$$\left. \begin{array}{l} P = -\,\frac{a\,A'}{1-k} + \frac{3}{8\,(1-k)^2} \\ Q = \frac{a\,(A' - 2\,C')}{2-2\,k+c} + \frac{3}{4\,(k-1)\,(2-2\,k+c)} + \frac{3\,(k-1)}{(2-2\,k+c)^2} \\ R = \frac{a\,(A' - 2\,B')}{2-2\,k-c} + \frac{3}{4\,(k-1)\,(2-2\,k-c)} - \frac{3\,(k+1)}{(2-2\,k-c)^2} \end{array} \right\} \ldots\ldots\ldots\ldots\ldots(258)$$

In order to have the value of θ in terms of $n\,t$, we must revert this series, and the simplest mode of accomplishing this will be to follow the steps by which we resolved the equation $n\,t = v - e\,.\sin v$ in the elliptic theory (vide page 656, col. 1, equation 26). Putting it therefore into the form

$$\begin{aligned} \theta = n\,t + \frac{2\,e}{c}\,.\sin c\,\theta - \mathrm{P}\,a \sin(2 - 2\,k)\,\theta & \\ - \mathrm{Q}\,a\,e\,.\sin(2 - 2\,k + c)\,\theta & \\ - \mathrm{R}\,a\,e\,.\sin(2 - 2\,k - c)\,\theta & \end{aligned} \tag{259}$$

$n\,t$ appears to be the first approximate value of θ, neglecting e and a. Let this be substituted in the whole of the second member, and we get

$$\begin{aligned} \theta = n\,t + \frac{2\,e}{c}\,.\sin c\,n\,t - \mathrm{P}\,a\,.\sin(2 - 2\,k)\,n\,t & \\ - \mathrm{Q}\,a\,e\,.\sin(2 - 2\,k + c)\,n\,t & \\ - \mathrm{R}\,a\,e\,.\sin(2 - 2\,k - c)\,n\,t & \end{aligned} \tag{260}$$

for a second approximation carried as far as the first powers of e and a. If we now substitute this value in (259) the terms multiplied by e and a only will of course remain; but in addition to those multiplied by $a\,e$, others will be introduced; and if we reject e^3 and a^2, and retain only $a\,e$, these terms will be, (putting $c = 1$)

$$\begin{aligned} & - 2\,\mathrm{P}\,a\,e\,.\sin(2 - 2\,k)\,n\,t\,.\cos c\,n\,t \\ & - 2\,\mathrm{P}\,a\,e\,(1 - k)\,.\cos(2 - 2\,k)\,n\,t\,.\sin c\,n\,t \end{aligned}$$

whose sum, after the usual reductions, is

$$- \mathrm{P}\,(2 - k)\,a\,e\,.\sin(2 - 2\,k + c)\,n\,t - \mathrm{P}\,k\,a\,e\,.\sin(2 - 2\,k - c)\,n\,t$$

Consequently, if we push the approximation only to such terms, we get

$$\begin{aligned} \theta = n\,t + \frac{2\,e}{c}\,.\sin c\,n\,t - \mathrm{P}\,a\,.\sin(2 - 2\,k)\,n\,t & \\ - \left\{\mathrm{Q} + (2 - k)\,\mathrm{P}\right\} a\,e\,.\sin(2 - 2\,k + c)\,n\,t & \\ - \left\{\mathrm{R} + k\,\mathrm{P}\right\} a\,e\,.\sin(2 - 2\,k - c)\,n\,t & \end{aligned} \tag{261}$$

The *variation* of the moon is that inequality which is represented by the term $- \mathrm{P}\,a\,.\sin(2 - 2\,k)\,n\,t$. The numerical magnitude of the co-efficient being considerable, (2146″) it, (as well as the inequality represented by the term, whose argument is $(2 - 2\,k - c)\,n\,t$, which is called the *evection*, and whose co-efficient is still greater (4830″) was long observed before its cause was known. If we observe that $n\,t$ being the mean longitude of the moon $k\,n\,t$ will be that of the sun, and if we put ☽ for the former, and ⊙ for the latter, the argument of the variation will be 2 ☽ − 2 ⊙, or simply ☽ − ⊙, and its *period*, half a synodic revolution of the moon $= 14^{d}{\cdot}765$. The argument of the evection is in like manner

$$2\,☽ - 2\,\odot - c$$

where c is the longitude of the moon's perigee, and its period is nearly that of the variation, a little longer, however, on account of the progression of the perigee during a revolution of the moon. The term depending on the argument $(2 - 2\,k + c)\,n\,t$, or 2 ☽ − 2 ⊙ + c, is found on calculation to have its co-efficient very small, and has no particular name.

In the theory we have now investigated we have been only anxious to simplify and curtail as much as possible the developements, it being far beyond our design to enter into minutiæ on so complicated a problem. It must suffice to have shewn in rather more than a general way the origin of the principal inequalities, and traced them from the differential equations to their final expression in the value of the longitude. In so doing, we have purposely neglected the inclination of the moon's orbit, and its effect both in modifying the disturbing forces which act in the plane of the orbit, and in altering the longitudes by reducing them from their actual values to their projections on the plane of the ecliptic, or by what is called in astronomy, the "Reduction to the ecliptic." The effect of the inclination, however, will be briefly touched on in the next section.

SECTION IV

Of the effect of the inclination of the plane of the moon's orbit.—Of the motion of the nodes, and the precession of the equinoxes.

In determining the effect of the inclination of the lunar orbit to the plane of the ecliptic and the motion of its nodes, we shall suppose the orbit circular, and neglect the square of the disturbing force, the cube and higher power of the inclination, and the product of the disturbing force and inclination. Thus our equation (237) will become simply,

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2\,(1 + s^2)^{\frac{3}{2}}} - \frac{m'}{h^2}\left\{\frac{\mathrm{V}}{u^2} - \frac{2}{h^2}\int\frac{T\,d\,\theta}{u^3}\right\}; \tag{262}$$

The value of the part depending on the disturbing forces $m'\,\mathrm{V}$ and $m'\,T$ in this equation remains unaltered, if we neglect $a\,s^2$; and therefore the part of u arising from perturbation is not altered when we only push the approximation to the extent now proposed. The only new terms arising in the value of u from the introduction of the inclination, are those introduced by the term $\frac{1}{h^2 (1 + s^2)^{\frac{3}{2}}}$; and to obtain them, we have only to integrate the equation

$$\frac{d^2 u}{d \theta^2} + u - \frac{1}{h^2} \left(1 - \frac{3}{2} s^2\right) = o$$

neglecting s^4. Now, in this we must put for s its undisturbed value, which is given by the equation $\frac{d^2 s}{d \theta^2} + s = o$ (making $m' = o$, in equation 240) and which is

$$s = \gamma \,.\, \sin (\theta - \text{N}) \tag{263}$$

where γ is the tangent of the inclination, and N the longitude of the ascending node. This gives $s^2 = \gamma^2 . \frac{1 - \cos 2 (\theta - \text{N})}{2}$

$$\frac{d^2 u}{d \theta^2} + u - \frac{1}{h^2} \left(1 - \frac{3 \gamma^2}{4}\right) - \frac{3 \gamma^2}{4} . \cos 2 (\theta - \text{N}) = o$$

which integrated, gives

$$u = \frac{1}{h^2} - \frac{3 \gamma^2}{4 h^3} - \frac{\gamma^2}{4 h^2} \cos 2 (\theta - \text{N}) \tag{264}$$

Thus we see that the effect of the inclination is to add to

$$u = \frac{1}{h^2} \text{ or } u = \frac{1}{a} \text{ the terms } - \frac{\gamma^2}{4 h^2} (3 + \cos 2 (\theta - \text{N}))$$

Now, in approximating to u, we assumed in the foregoing pages an expression for a first value derived not *a priori* from theory, but from the *observed* motion of the lunar apsis. It is equally a matter of observation, that the node of the lunar orbit does not remain fixed, but is continually retreating on the ecliptic, in a direction contrary to the motion of the moon in its orbit. Let us then introduce this modification into our expression for u above found, and taking for the origin of the time, the moment when the node coincided with the axis of the x, we have only to write $g\,\theta$ in place of $\theta - \text{N}$, when we get

$$u = \frac{1}{h^2} \left\{1 - \frac{3}{4} \gamma^2 - \frac{\gamma^2}{4} . \cos 2 g \theta \right\} \tag{265}$$

Let us next investigate the effect of the inclination on that part of u which arises from perturbation. Now this may be done at once, by merely taking in 250, 251,

$$\delta u = \frac{a}{a} \phi = - \frac{1}{a} \left(\frac{3}{4} \gamma^2 + \frac{1}{4} \gamma^2 \cos 2 g \theta\right)$$

or

$$\phi = - \frac{3}{4} \frac{\gamma^2}{a} . \cos o . \theta \quad \frac{1}{4} \frac{\gamma^2}{a} \cos 2 g \theta$$

that is, supposing in 252, 1st. $\text{A} = - \frac{3 \gamma^2}{4 a}$, $\text{B} = o$, and 2dly, $\text{A} = - \frac{\gamma^2}{4 a}$ and $\text{B} = 2 g$

The first supposition gives

$$\delta^2 u = \frac{9}{8 a} a \gamma^2 \left\{ 1 - \left(3 - \frac{1}{1 - k}\right) \frac{\cos (2 - 2 k) \theta}{(2 - 2 k)^2 - 1} \right\} \tag{266}$$

and the second in like manner introduces into the value of u the new arguments $\cos (2 g - 2 + 2 k) \theta$ and $\cos (2 g + 2 - 2 k) \theta$ with co-efficients affected with $a \gamma^2$.

These arguments existing in the expression of u will in like manner be introduced into the value of t or $\int \frac{d \theta}{h u^2}$ and thence, by the process of reversion before explained, will arise terms of the form $a \gamma^2 . \sin (2 g - 2 + 2 k) n t$ and $a \gamma^2 . \sin (2 g + 2 - 2 k) n t$ in the longitude, of which this mention must suffice, as their co-efficients may easily be calculated from the principles above laid down, and nothing very remarkable in the lunar theory depends on them. Other terms also will arise from the part of (250) multiplied by $\frac{d u}{d \theta}$ which in this case is not $= o$ as was supposed in (251) but is equal to $\frac{g \gamma^2}{2 h^2} \sin 2 g \theta$.

Let us now examine the manner in which the disturbing forces affect the moon's latitude. For this purpose we must take the equation (241) in which writing for s the expression $\gamma . \sin (g v - \text{N})$, for $\frac{d s}{d \theta}$ its value $g \gamma . \cos (g v - \text{N})$ for $m' T$ and $m' \text{V}$ their values $\frac{3}{2} \frac{a}{a^2} \sin 2 (1 - k) \theta$ and $- \frac{1}{2} \frac{a}{a^2} (1 + 3 \cos 2 (1 - k) \theta)$

and for $m' \frac{d\,\Omega}{d\,z}$ its value $\frac{m'\,z}{\lambda^3} = \frac{m'\,\rho\,s}{\lambda^3} = \frac{m'\,a\,s}{\lambda^3} = \frac{m'\,a\,s}{a'^3}\left(1 + 3\,\frac{a}{a'}\cos w\right)$ (see page 718, line 4,) or, neglecting the product $s \times \frac{a}{a'}$, simply

$$m'\,\frac{d\,\Omega}{d\,z} = \frac{m'\,a\,s}{a'^3} = \frac{a}{a^2}\,s$$

we shall find

$$o = \frac{d^2\,s}{d\,\theta^2} + s + \frac{3}{2}\,a\,\gamma\,.\,g \sin 2\,(1-k)\cos\theta\,(g\,\theta - \mathrm{N})$$

$$+ \frac{a}{2}\,s\,(1 + 3\cos 2\,(1-k)\,\theta) + a\,s$$

and resolving the products of sines and cosines into simple sines,

$$o = \frac{d^2\,s}{d\,\theta^2} + s + \frac{3\,a}{4}\,a\,\gamma\,.\left\{\sin(\overline{2-2\,k+g}\,.\,\theta - \mathrm{N}) + \sin(\overline{2-2\,k-g}\,.\,\theta + \mathrm{N})\right\}$$

$$+ \frac{3}{2}\,a\,.\,\gamma \sin(g\,\theta - \mathrm{N}) + \frac{3}{4}\,a\,\gamma\left\{\sin(\overline{2-2\,k+g}\,.\,\theta - \mathrm{N}) - \sin(\overline{2-2\,k-g}\,.\,\theta + \mathrm{N})\right\}$$

or

$$o = \frac{d^2\,s}{d\,\theta^2} + s + \frac{3}{2}\,a\,\gamma\,.\sin(g\,\theta - \mathrm{N}) \tag{267}$$

$$+ \frac{3}{4}\,a\,\gamma\,(1+g)\,.\sin(\overline{2-2\,k+g}\,.\,\theta - \mathrm{N})$$

$$- \frac{3}{4}\,a\,\gamma\,(1-g)\,.\sin(\overline{2-2\,k-g}\,.\,\theta + \mathrm{N})$$

The integration of this gives

$$s = \frac{3\,a\,\gamma}{2\,(g^2-1)}\sin(g\,\theta - \mathrm{N}) + a\,\gamma \sin(\overline{2-2\,k+g}\,.\,\theta - \mathrm{N}) \times \&c.$$

$$+ a\,\gamma \sin(\overline{2-2\,k-g}\,.\,\theta + \mathrm{N}) \times \&c.$$

Now the process of approximation ought necessarily to reproduce the original value $s = \gamma\,.\sin(g\,\theta - \mathrm{N})$. Hence, exactly as in the case of the apogee, we ought to have

$$\frac{3\,a\,\gamma}{2\,(g^2-1)}\,.\sin(g\,\theta - \mathrm{N}) = \gamma\,.\sin(g\,\theta - \mathrm{N})$$

and

$$g^2 = 1 + \frac{3\,a}{2}, \quad g = 1 + \frac{3}{4}\,a$$

Thus we obtain the motion of the node, for $g - 1$ or $\frac{3}{4}\,a$ expresses the ratio of the retrograde velocity of the node to that of the moon in its orbit. If we execute the calculation, we find this ratio to be that of 0·0042 : 1 (see page 720, line 15 from bottom,) and it is remarkable, that it is exactly the same as a first approximation gives for the direct motion of the apogee; but there is one remarkable difference, that in the present case, this first approximated value is very near the truth as given by observation, and is little altered by a second approximation; whereas, in the other, the repetition of the process doubles the value.

The regression of the moon's nodes is a circumstance in their theory which admits of a very easy and familiar illustration.

The part of the disturbing force which acts in the direction of the z, or which tends to draw the moon out of the plane of its orbit is $m'\,\frac{d\,\Omega}{d\,z} = m'\,.\left(\frac{z'}{r'^3} - \frac{z'-z}{\lambda^3}\right)$. If we wish to know the whole force which tends to draw the moon in a direction at right angles to the plane of its own orbit, we must assume that plane for the plane of the x, y; then will $z = o$, and

$$m'\,\frac{d\,\Omega}{d\,z} = m'\,z'\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right)$$

Now $m'\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right) = -\,\frac{3\,m'\,r}{r'^4}\cos w =$ nearly $-\,\frac{3\,a}{a^2\,a'}\cos w$, and z' being a perpendicular let fall from the sun on the plane of the moon's orbit is equal to r' or a' multiplied by the sine of the sun's angular distance from the node ($\sin(\theta' - \mathrm{N})$) and by that of the inclination or by γ. Hence, the expression of this force is

$$-\,\frac{3\,a}{a^2} \times \cos w\,.\sin(\theta' - \mathrm{N})\,.\sin I$$

w is the moon's angular distance from the sun as seen from the earth, and therefore $\cos w$ is equal to sine of

☾'s dist. from quadratures. So that we have

$$-\frac{3\,a}{a^2} \times \sin \text{☾'s dist. from quadrat.} \times \sin \text{⊙'s dist. from ☊} \times \text{sin. inclin.}; \qquad (268)$$

for the value of the force in question. Let us now examine how this force tends to produce a motion in the node. To this end, suppose the moon to set off from its ascending node, and at any point in its orbit (supposed circular) to describe an infinitely small arc which will be a portion of a great circle seen from the earth's centre. If the disturbing force at this moment ceased to act, this arc would be a portion of the undisturbed orbit, and being prolonged backwards would cut the ecliptic in the ascending node, ☊, whose position therefore would be the same at the beginning and end of this infinitesimal instant. To limit our ideas, let us conceive the moon to be within 90° from the ☊, and therefore its motion is from the plane of the ecliptic. Now, let the disturbing force act, and suppose its direction to be such as to draw the moon out of its orbit towards the plane of the ecliptic, then will the elementary arc actually described by the moon in virtue of its own inertia combined with the new impulse given by this force, be *less inclined* to the ecliptic than the last described portion; and therefore being produced backwards, will cut the ecliptic in a point *behind* the former place of the node. This point is the new or consecutive place of the node, which therefore has retreated on the plane of the ecliptic by the action of the disturbing planet.

On the other hand, had the disturbing force been directed from the plane of the ecliptic, the new path of the moon would be more inclined than in the preceding instant; and therefore, produced backwards, would cut the ecliptic in a point more *forward* than the previous place of the node, so that under these circumstances the node advances.

Thus we see that when the moon moves from the ecliptic and the force acts *to* it, the node retreats; but when the force acts from it, the node advances.

Again, let the moon be approaching the ecliptic; then, if the force act *to* that plane, it will approach it more rapidly, and will cut it in a point nearer than the node to which it is approaching. This node (and of course both) will then move to meet the moon, or in a direction contrary to its motion, *i. e.* will retreat; and *vice versâ*, if the force act from the plane, the node advances.

From this analysis of all the cases, we see that whenever the disturbing force tends to elevate the moon *from* the plane of the ecliptic, the node advances, and in every other case retreats. Now, it is easily seen, that the former condition never holds good unless the moon is between the node and the quadratures; and as the extent of the angle in which this can happen during a whole revolution of the moon in its orbit, is necessarily less than two right angles, the preponderant tendency of the node on the average of a whole revolution is always in favour of its retreat. In fact, when the node is in quadratures, it retreats at every instant of the lunation; and in the most unfavourable case, when the node is in syzigies, its retreat is barely counterbalanced by its advance, and the node only rests for an instant; the sun being then for a moment in the plane of the lunar orbit.

The general tendency of the node to recede on the ecliptic is thus clearly made out; but we may go further on these principles, and make the quantity of its recess a matter of calculation. For, let us denote by $\frac{\mathrm{K}}{a^2}$ the force expressed in (268). Then, since the lunar gravity $\left(\frac{1}{a^2}\right)$ draws the moon in the instant of time $d\,t$ through the versed sine of an arc $= a\,d\,\theta$ or through a space equal to $\frac{(a\,d\,\theta)^2}{2a}$, the force $\frac{\mathrm{K}}{a^2}$ will draw it *in the same time* through the space $\mathrm{K}\,.\,\frac{(a\,d\,\theta)^2}{2a}$. The inclination therefore of its new path to its old will be represented by the infinitely small angle

$$2\,\frac{\mathrm{K}\,.\,\frac{(a\,d\,\theta)^2}{2a}}{a\,d\,\theta} = \mathrm{K}\,d\,\theta$$

Let the new elementary portion be prolonged till it meets the ecliptic in ☊′, ☊ being the former place of the node, then we shall have ☊ ☊′ for the momentary change of the node's place. Now this is the side of a spherical triangle opposite to the infinitely small angle $\mathrm{K}\,d\,\theta$. The included side is the arc of the moon's orbit between the moon and node, or the moon's distance from its node, while the included angle is I the inclination. Hence, if we call L the longitude of the node, we shall have, by spherical trigonometry,

$$-\,d\,\mathrm{L} = \mathrm{K}\,d\theta \times \frac{\sin\,(\theta - \mathrm{L})}{\sin\,I}$$

but $K = -\,3\,a\,.\sin\,(\text{☾} - \odot)\,.\sin\,(\odot - \text{☊})\,.\sin\,I$. Hence, we have

$$d\,\mathrm{L} = 3\,a\,.\,\sin\,(\theta - k\,\theta)\,.\sin\,(k\,\theta - \mathrm{L})\,.\sin\,(\theta - \mathrm{L})\,d\,\theta$$

from which differential equation the relation between L and θ may be deduced.

If we assume L as constant during one lunation in the second member, we get by integration the whole change of L in that interval approximately, or the mean motion of the node in a lunation, which we will call $\Delta\,\mathrm{L}$

$$\triangle L = 3a \,.\int d\theta \,.\, \sin(1-k)\,\theta \,.\, \sin(k\theta - L) \,.\, \sin(\theta - L)$$

This is in principle the method followed by Newton in that part of the third book of the *Principia*, where he treats of the motion of the moon's node; the most elegant and satisfactory instance of the application of his geometry to the lunar theory.

The precession of the equinoxes is explicable on the same principles as the motion of the moon's nodes. The centrifugal force at the earth's equation throws out a portion of the matter of which it consists into the form of an oblate or flattened spheriod; and we may conceive this redundant matter, as a spheriodical shell investing an inscribed sphere. Suppose now every particle of this shell at liberty to obey any impulse, unfettered by the others, or conceive it to consist of an infinite number of infinitely small moons: each of these will describe an orbit, whose nodes have a tendency to recede on the plane of the ecliptic, and though some (those which happen to lie between their quadratures with the sine and their nodes,) will have their nodes in a state of advance; all the rest, which constitute the greater number, will have theirs in a state of recess. Conceive now the particles to cohere and form a solid ring, *unconnected* with the central globe, the motion of this ring will be a mean among all the motions of its parts, and the nodes of the ring will in consequence continually recede, with a certain velocity. Now, let the ring adhere to the sphere, then must all its motion be divided between itself and the whole mass of the earth, which *alone* has no such tendency. The redundant matter at the equator, however, bears a very small ratio to the whole mass of the earth; so that owing to this cause, the retrogradation is exceedingly diminished in rapidity; and owing to the enormous distance of the sun and the smallness of the earth (whose radius being only $\frac{1}{23405}$th part of the distance; so that here, $a = m' \,.\, \left(\frac{a}{a'}\right)^3 = \frac{m'}{(23405)^3}$ a quantity quite insensible,) is rendered too small to be distinctly perceived.

But the moon also exerts a disturbing force. That luminary is to our imaginary moons, or terrestrial molecules, what the sun is to the moon itself in the theory of the lunar perturbation. If we reduce these principles to calculation, assuming such a mass and distance of the moon as we know to be near the truth, we shall find that a retrograde motion of the earth's equator on the ecliptic of about 50″ per annum will actually result. Now this is the very phenomenon known by the name of the precession of the equinoxes; and though its strict theory is much more complicated and difficult than the general view here taken, its accordance with observation is perfect, and affords one of the most refined verifications of that admirable law which holds the frame of nature in the harmony we are now so well able to appreciate.

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FIGURE OF THE EARTH.

SINCE Astronomy first assumed the form of a Science, the inquiry into the Figure and dimensions of the Earth has always excited the interest of Philosophers. It can hardly be doubted that in the mind of a reflecting man there would always be a desire to know the nature of the Planet upon which he existed; but without Science of an exalted order, it would be impossible for him to gratify his curiosity. For the Astronomer, however, it was absolutely necessary to know something of the form of the Earth; and Astronomy alone gave the means of determining it. While the only observations of the heavenly bodies consisted in watching the rising and setting of the Sun and Stars, the form of the Earth was a matter of indifference; but when an attempt was made to reduce to a system the motions of the luminaries and Planets, it was necessary as a preliminary step to assume, and even to establish by something approaching to demonstration, a hypothesis on the Figure of the Earth.

At present the determination of the Earth's Figure possesses an interest of which formerly it was altogether destitute. The Astronomers of antiquity knew that the Earth is nearly spherical; they had also some not very correct ideas of its magnitude; and this was sufficient for their purposes. But the sphericity of the Earth, when thus established, was an isolated fact. It was proved by the impossibility of explaining certain phenomena on any other hypothesis; but it was not connected with any general theory which made this Figure a necessary consequence of the properties of matter. But by the discoveries of Newton the Figure of the Earth was shown to depend on the same theory which explains with such wonderful accuracy the motions of the Planets and their satellites. The investigations of the most profound Mathematicians have since been directed to its determination, from the Principles of Gravitation; and the labours of the most able experimenters have been employed in ascertaining it from actual observation; and the comparison of the results of theory and of observation shows that their agreement, though not perfectly exact, is sufficiently so to enable us to assert with confidence, that the Principle of Gravitation is well founded. Indeed, for one part of that Principle, (*viz.* that the attraction of a Planet is not a force directed to its centre, but is the resultant of all the forces directed to every one of its particles,) it may be considered as affording the most satisfactory proof that we can expect ever to have.

Under the head of the Figure of the Earth we propose not only to consider the theoretical form, and to give an account of the principal measures and their results, but also to explain the most remarkable phenomena depending on its Figure; namely, the variation of Gravity on its surface, and the disturbances in its own motion, and the motion of the Moon, occasioned by its deviation from the spherical form. And in connection with this subject we shall give a short account of the experiments which have been made to determine the comparative density of the superficial and the interior parts of the Earth. The whole may be considered as a collection of various classes of observations which have been made for the examination of that part of the Principle of Gravitation which asserts that "the attraction of a mass is the sum of the attractions of every one of its particles."

Section 1.—*History.*

General Principle of Meridian measures.

It was probably by observation of the form of the Earth's shadow, as seen in eclipses of the Moon, that the Figure of the Earth was first ascertained to be nearly spherical; and the necessity of an allowance for *parallax* in the Moon's declination would very soon teach Astronomers that the place of observation was not the centre about which the lunar revolutions were performed. But these remarks would in no degree enable them to form an idea of its magnitude. The method of determining the dimensions by measuring an arc of the meridian has been employed with no alteration of Principle from the time of the Alexandrian Astronomers to the present day. Let A, B, fig. 1., be two places of observation; A C, B C, lines perpendicular to the surface at those points: suppose these perpendiculars to intersect in C; let A F D, F B E, be horizontal lines; G A, H B, parallel lines, representing the directions of the visual rays from the celestial Pole, or from some very distant object, as the Sun or a Star. By observation of the angles H B E, G A D, (the altitudes of the Pole, or the Sun, &c.) their difference is known; that is, the difference of H B E and H K D, or K F B, which is equal to A C B. The length of the arc A B can be measured; knowing then the length of A B, and the angle A C B, and assuming A B to be a circular arc, the length of A C or B C can be immediately calculated. Instead of calculating the length of A C it is rather more convenient to calculate the length A M, which corresponds to an angle A C M of one degree; and this is usually called *the length of a degree on the Earth's surface.* It is plain that this process can be applied only when the vertical plane which passes through the two places, passes also through the body observed; and the observations must either be simultaneous, or must be of such a nature that a small error in the time (which in practice is inevitable) will produce no sensible error in the body's altitude. From these considerations it appears that A and B must be in the same meridian. If we assume the Earth to be spherical, we shall by this method find its radius; or if we perform the same operations in different parts of the Earth, and always find the same value for the radius, we may infer that the Earth is spherical.

This is the method pursued in the earliest measure of which we have any tradition, that of Eratosthenes. For we cannot hold with some writers that any real measure is alluded to by Aristotle, *De Cœlo*, lib. ii., when he

Estimation by Eratosthenes.

speaks of the circumference of the Earth as being 400,000 stadia, according to the opinions of the Mathematicians of his time. Eratosthenes, about 230 B. C., observed that at Syene, in Upper Egypt, the Sun was, as far as he could discover, exactly vertical at noon, on the day of the summer solstice; and that at Alexandria the Sun's zenith distance at the same season was 7° 12′. The former observation (according to Cleomedes) was made by remarking that the edges of a deep well threw no shadow on the bottom, the latter by the use of the σκάφη, a hemispherical bowl with a vertical style. These circumstances will be represented in fig. 1, if we suppose B H to be in the same straight line with C B, and G A parallel to B H, and the angle N A G = 7° 12′. He supposed Alexandria and Syene to lie in the same meridian, and reckoned their distance at 5000 stadia, which gave the Earth's circumference = 250,000 stadia. The length of the stadium here used is unknown, and it is unnecessary to point out the sources of error in this operation.

By Posidonius.

An estimation was also attempted (as the same author informs us) by Posidonius, who was contemporary with Pompey the Great. It appears to have been intended only as a rough guess, and is, in fact, much inferior in accuracy to that of Eratosthenes. He observed that the star Canopus was seen at Rhodes to just touch the horizon; at Alexandria he estimated its meridian altitude at 7½°: the distance of the places was supposed to be 5000 stadia. This gave for the circumference of the Earth 240,000 stadia. In this measure it is merely necessary to observe that refraction was neglected, not being known; and that no accurate estimation could be formed of the distance of two places separated by the Mediterranean.

By Ptolemy

In the Astronomical Works of Ptolemy (who flourished about A. D. 137) we find no notice of the Earth's dimensions. He assigns as reasons for believing the Earth to be spherical, that eclipses of the Moon, as seen at different places on the Earth, take place at different times with reference to the noon of the places of observation; and that the differences of apparent times are proportional to the distances East or West. This, we believe, is the earliest notice of the Principle of deducing terrestrial longitude from the difference of the apparent times at which an eclipse of the Moon is observed. He remarks also that on going Northward, the number of circumpolar stars is increased, appearances which could not exist if the Earth were plane, or cylindrical. But in his Geography he tacitly assumes the Earth to be spherical, and uses constantly as the length of one degree 500 stadia. This estimation seems to have been made by Marinus the Tyrian, from observations of the latitude of very distant places, and from the rough measures of the distance made by sailors and merchants.

Measure by Almamoun.

The dark Age which followed the overthrow of the Alexandrian School, put a stop to all speculations on the Figure of the Earth. It was not till the Empire of the Caliphs had extended over the greatest part of the civilized World, and the Works of the Greek Astronomers had been translated and studied by the Arabs, that another attempt was made to measure an arc of the meridian. Abulfeda relates that Abdalla Almamoun, who begun his reign at Bagdad A. D. 814, having fixed on a spot in the plains of Mesopotamia, ordered one company of Astronomers to go Northward and another Southward, measuring the distance by rods, till each should find their alteration of latitude, or alteration in the altitude of the Pole, to be one degree. One party found the distance 56 miles of 4000 cubits, the other 56⅔ miles; the latter was adopted as preferable to the former. The length of the cubit, however, is not known.

By Ferne

Seven hundred years elapsed before another estimation was attempted, and the scene was then shifted to Western Europe. Fernel, a Parisian, born in 1485, (according to Lalande, *Mém. Acad.* 1787,) published in 1528 the account of a measure made in the neighbourhood of Paris. On August 25, he observed the Sun's meridian altitude at Paris; he went Northward one degree, as nearly as he could judge, and on August 29 again observed the Sun's meridian altitude. His observations were made with a triangle, (not a quadrant,) of which one side, eight feet in length, was vertical; another side of the same length was movable round the point of connection with the first, and carried sights; the third side measured the distance between the other two, and was graduated as a line of chords to every minute of the quadrant. (This is the instrument which Ptolemy describes as proper for the observation of the Moon's parallax.) On applying to the Sun's altitudes the proper correction for change of declination, he found that his latitude had increased one degree. The distance was calculated from the number of turns made by the wheel of his carriage. The length of an arc of meridian of one degree, thus found, (56,746 toises,) agrees very well with the length given by modern observations, (about 57,060 toises.)

By Snell

The next measure (that of Snell) is distinguished by a very great improvement in the manner of measuring,—the substitution of trigonometrical operations for the actual measurement of the whole distance by the application of rods or perambulators. This method is partly described in our Treatise on TRIGONOMETRY, section 9; and will be illustrated more fully in a succeeding section of the present Essay. The account of Snell's measure was published at Leyden, in 1617, under the quaint title *Eratosthenes Batavus de Terræ ambitûs verâ quantitate, &c. a* Willebrordio Snellio *suscitatus*. He measured a base of 326.4 Rhinland perches, and two bases of verification of 348.1 and 166 perches, (each perch of 12 feet.) The length of the first base was, however, in reality concluded from a base of 87.05 perches; for the calculated length is used, though it differs a little from the measured length. His angles were observed with quadrants and semicircles. Thus he found the meridian distance between Alcmaer and Bergen-op-Zoom = 33,930.2 perches. Their latitudes he found to be 52° 40′ 30″, and 51° 29′. From this he found one degree of meridian = 28,473 perches. Between Leyden and Alcmaer he found 1° = 28,510; finally he fixed upon 28,500. As he was careful to compare his standard with the standards of other nations, this measure was readily reduced to French toises, and it gave 1° = 55,100 toises. A recalculation and reobservation of the latitudes by Muschenbroek in 1729, gave 1° = 57,033 toises. Another measure was effected (we believe in nearly the same locality) by Blaeu, or Cæsius; it was said to agree in its results with Picard's, but we are unacquainted with the details.

In 1637 was published, *The Seaman's Practice, contayning a fundamentall Probleme in Navigation experimentally verified, namely touching the Compasse of the Earth and Sea, and the Quantity of a Degree in our English Measures, &c.*; by Richard Norwood, Reader

By Norwood.

of the Mathematicks. In this Work most of the older measures are mentioned. It is stated, also, that a Mr. Edward Wright had observed the dip at Mount Edgecumbe, and had inferred the semidiameter of the Earth to be 18,312,621 feet. The author then states, that on June 11, 1633, with a sextant of more than five feet, he observed the Sun's meridian altitude near the Tower of London to be 62° 1′; and on June 6, 1635, he found the Sun's altitude at York to be 59° 33′; the difference of latitude 2° 28′. The distance was measured with a chain of 99 feet; the bearings of different parts of the road were observed with a circumferentor, and the measures reduced by a Table. Part of the distance he paced, in which he says he came through custom very near the truth. The number of chains was 9149; from which he found 1° = 61,199 fathoms.

By Riccioli.

About the same time a trigonometrical measure was attempted in Italy by Riccioli. Beginning with a base of 1094 Bolognese paces, he found the meridional distance from a point near Bologna to Modena 20,439 paces; the difference of latitudes he found to be 19′25″. The angles were measured with an instrument nearly similar to that of Fernel. From this it appeared that 1° = 63,159 paces, = 61,478 toises, or 65,521 fathoms. This measure has been universally condemned. The Geodetic part was not well conducted, and the Astronomical observations were far below the general science of the Age. It has been remarked, also, that Riccioli was persuaded of the accuracy of the Greek estimations, which (in his interpretation) were considerably greater than the later measures seemed to show. The part of his Works (*Geographia*) which comprehends this measure, contains an interesting account of all the methods of determining the Earth's dimensions which were known at that time, and of all the estimations which preceding Mathematicians had formed.

The method which he appears to have thought superior to all the others, is that of observing from each of two stations the depression of the other; and he instances some observations of his own which give nearly the same value for a degree as that which we have cited. The principle of this method will be readily understood by supposing P and Q, fig. 1, to be two elevated stations, and remarking that if the angles C P Q, C Q P, be observed, P C Q will be found by subtracting their sum from 180°; and the distance P Q being measured, the lengths C P, C Q, are easily calculated.

By Picard.

In 1669 the trigonometrical measure of Picard was commenced; and it was in every respect superior to all that had preceded it. It is true that it was not free from errors: an error of six toises was committed in the measure of the base; and his differences of latitude were vitiated by his ignorance of aberration, &c.; but, by a happy chance, these errors almost balanced each other. The extremities of his arc were Sourdon, (near Amiens,) and Malvoisine, (near Paris.) His base was 5663 toises; his base of verification 3902 toises. The difference of latitude of Malvoisine and Sourdon was found to be 1° 11′ 54″; and when the measure was extended to Amiens, the difference of latitude of Malvoisine and Amiens was found to be 1° 22′ 55″. The corresponding arcs were 68,430 and 78,850 toises; the first gave for the length of an arc of 1°, 57,064 toises, the second 57,057; the mean is 57,060 toises = 60,812 fathoms. The Astronomical observations were repeated in 1739 by Maupertuis, Clairaut, &c.: they found 1° = 57,183 toises.

Huygens's ideas of centrifugal force.

In 1673 appeared the Work of Huygens, entitled *De Horologio Oscillatorio.* In this, for the first time, were found correct notions on the subject of centrifugal force. It does not appear, however, that these were applied to the theoretical investigation of the Earth's form before the publication of Newton's *Principia.*

In 1666 Newton appears to have first entertained the idea of Gravitation. It is remarkable that, at this time, he seems to have been unacquainted with the measures of Norwood and Snell; and considering the length of a degree, according to the usual estimation, to be sixty miles, he was induced, by the disagreement of his calculations from this estimation, to lay aside his theory of Gravitation. For the Moon's parallax being known, that is the proportion of the Earth's radius to the radius of the Moon's orbit (supposed a circle) being known, the force of Gravity on the Moon was found by Newton's law; and the periodic time was also known; and from these data, and the theorems of Huygens and Newton relating to motion in a circle, it was easy to calculate the radius of the circle in which it moved, and consequently (from the proportion above mentioned) the radius of the Earth: if this were not the same as the radius given by the measures, Newton's law could not be true. The measure of Picard enabled him to establish the theory; and it is this measure that is used in the *Principia* published in 1687. In this wonderful Work a prodigious step was made towards the theory of the Earth's form.

Newton's investigation of the Earth's form.

Combining the theory of centrifugal forces with the properties of fluids, Newton showed that the Earth must be not spherical but spheroidal; that its equatorial diameter must be longer than its axis of revolution; and he actually calculated the proportion of the diameters, on the supposition that the Earth had been in the state of a homogeneous fluid, to be 229 : 230. With regard to this astonishing investigation we shall merely state, that though defective, it is not erroneous; it is one of the many instances in which Newton has obtained a correct result by means apparently quite inadequate. He showed, also, that Gravity must be less at the Equator than near the Poles; and this served to explain a very remarkable fact that had lately been observed.

Variation of Gravity on the Earth's surface.

In 1671 Richer, who had been sent by the French Government to Cayenne for the purpose of conducting a series of Astronomical observations, found that his clock, which had been regulated to mean time at Paris, lost more than two minutes every day. Similar facts were afterwards observed by Varin and Deshayes on the coasts of Africa and America; and in all cases the alteration of the clock's rate was much greater than any that could be caused by the change of temperature.

Newton's explanation of the precession of the Equinoxes.

But the most remarkable of Newton's inferences is, perhaps, his explanation of the precession of the Equinoxes. This motion of the Earth's axis was discovered by Hipparchus, (B. C. 150;) and all succeeding Astronomers had recognised its existence. Eighteen centuries after its discovery, it was shown by Newton to be a consequence of the attraction of the Sun and Moon on the Earth's equatorial protuberance. Having shown that the attraction of a distant body on a satellite would produce a retrograde motion of its nodes, he observes that if we first suppose a number of satellites to revolve in one plane, and then suppose them to be connected in such a manner as to become a solid ring revolving in its plane, we shall still have a retrograde motion of the nodes: and if the Earth be fixed within the ring, the motion of the nodes, though less than before, will still be retrograde: and the transition from this case to that of an oblate spheroid is sufficiently obvious. The calculation which Newton has attempted to found on this reasoning is in some respects erroneous; but the explanation is, perhaps, one of the strongest proofs of genius. Indeed, if at this time we might presume to select the part of the *Principia* which probably astonished, and delighted, and satisfied its readers more than any other, we should fix without hesitation on the explanation of the precession of the Equinoxes.

Huygens's investigation of the Earth's form.

In 1690 was published Huygens's Treatise, *De Causâ Gravitatis*. It contains an investigation of the figure of the Earth, supposing the attraction upon every particle to be directed towards the centre, and to be always the same at equal distances from the centre. This supposition, it will be remarked, is directly opposed to one part of the Principle of Gravitation, namely, that which states that every particle attracts every other particle. The ratio of the axes is found in this manner to be 578 : 579.

J. Cassini's Meridian measure.

In 1684 J. D. Cassini (second of that name) commenced a trigonometrical measure of an arc of meridian in France, taking Picard's base as a foundation and proceeding Southward. It was intended that this should serve as a basis for a map of France; and the more accurate measure which was afterwards executed on the same line was used for that purpose. Some interruption took place, (on the death of the Minister Colbert,) but it was at length finished in 1701. The series of triangles was then extended Northward to Dunkirk, and this part was finished in 1718. A base of 7246 toises was measured near Perpignan, and one of 5464 toises near Dunkirk. The difference of latitudes of Paris and the Southern extremity, Collioure, was found to be 6° 18′ 57″, and the distance 360,614 toises; whence 1° = 57,097. The difference of latitudes of Paris and Dunkirk was found to be 2° 12′ 9″.5, and the distance 125,468 toises; whence 1° = 56,960. It appeared, then, that the degrees shortened in going from the South to the North; and this seemed to indicate that the Earth was a prolate spheroid, the ratio of its axes being nearly 95 : 96. A conclusion in such direct opposition to Newton's theory, now extensively received, excited a great sensation among the Mathematicians of Europe. The accuracy of the measure and the certainty of the conclusion were sustained by Cassini; who, as well as his father and his son, seem to have been firmly convinced that the Figure of the Earth was elongated. To settle the point, a degree of a parallel was measured in 1734, extending from Strasburg to St. Malo. This, also, when compared with the others, seemed to lead to the same conclusion. It was alleged, however, that the difference of longitude of these stations was determined only by some ancient eclipses of Jupiter's satellites, and that no reliance could be placed on them. To escape from this state of doubt, it was determined that an arc of meridian, and if necessary, one of parallel, should be measured near the Equator, in the comparison of which with the arc measured in France, the inevitable errors of observation would produce a smaller effect; and this is the origin of the celebrated expeditions of the French Academicians.

Measure of an arc in Peru.

Every facility being afforded by Louis XV., in May, 1735, MM. Bouguer, Godin, Lacondamine, and others, sailed for South America; they then crossed the Isthmus of Darien, and sailed to Peru. The difficulty of measuring an arc of parallel, and its inutility, (as on any hypothesis it would differ little from an arc of meridian in France,) were so strongly represented by Bouguer, that it was laid aside. But the great valley between the two principal chains of the Andes was found so favourable for operations in the direction of the meridian, that they effected, with the assistance of some Spanish officers, the measure of an arc of three degrees. The Northern limit was a place called Tarqui, 2½′ North of the Equator; the Southern extremity, Colchesqui, was in South latitude 3° 4½′. A base of 6272 toises was measured in the neighbourhood of Quito, and near the Southern extremity; and a base of verification of 5259 toises near the Northern extremity. No other measure* has been conducted under such extraordinary circumstances of locality. The lowest point of their arc was at an elevation of a mile and a half above the level of the sea; and, in some instances, the heights of two neighbouring signals differed more than a mile. At some places the danger was considered by the inhabitants to be so great, that public prayers were put up in the churches for their safety. The indolence and ill-will of the natives troubled them much; they were even in danger of losing their lives in a popular tumult; and to crown the whole, the instruments by which they were to determine their difference of latitude were found to be not trustworthy, and the corrections of the places of the stars were not to be relied on. The former difficulty they overcame by an ingenious reconstruction of their instruments, the latter by simultaneous observations at the two extremities of the arc. Their final result was that the whole length of the meridian arc, reduced to the level of the lowest station, was 176,945 toises, corresponding to a difference of latitudes of 3° 7′ 1″; and, therefore, that the length of an arc of one degree was 56,767, or reduced to the level of the sea 56,748. The arc was extended about twenty minutes by Godin; but this addition is generally supposed to have been less accurately measured than the other parts. In this expedition an attempt was made by Bouguer to observe the effect of the attraction of Chimborazo; of this we shall speak hereafter. For the complete details of this very interesting measure we must refer the reader to Bouguer's and Lacondamine's accounts; the former is the more amusing and more interesting to speculative Mathematicians, but the latter will be found by practical Astronomers to be the more instructive.

Measure of an arc in Sweden.

Before the return of this party (who, in consequence of their dissensions, separated as quickly as possible, and regained Europe by different ways) another measure had been imagined and completed. Maupertuis, Clairaut, Camus, Lemonnier, and Outhier reached the Gulf of Bothnia in July 1736. They had expected to be able to make use of the islands in the Gulf for the stations of a trigonometrical measure; but they found them so low, and so near the shore, that they were obliged to give up this part of their plan. But the valley of the river Tornea, which passes by the town of the same name, seemed to offer a favourable line of country; and this was accordingly chosen. The stations (as in the Peruvian measure) were on the hills on each side; and the river gave them some facilities for the conveyance of their instruments. In the prosecution of their work great inconvenience was sustained from the cold, the birds of prey, &c., but the greatest of all from the tormenting flies and gnats. The fogs also, with which some of the hills were continually enveloped, increased the difficulties of the observations; in some instances they waited ten days on the top of a hill before they could see one of their signals. After sixty-three days of great fatigue they reached the mountain Kittis, which was fixed on for the Northern extremity. The latitudes were observed with a sector made by Graham. On their return to Tornea, the river being frozen, a base was measured on the ice nearly in the middle of the arc, but no base of verification was measured. The result being somewhat different from what they had expected, the latitudes were reobserved, and some angles of the triangles, which had before been omitted, were observed. The final conclusion was that the difference of latitudes was 57′ 29″.6, and the length of the arc 55,023 toises; whence an arc of $1^\circ = 57{,}422$. In this expedition, as well as in that to Peru, observations were made to ascertain the length of the pendulum vibrating seconds; and they all agreed in showing that Gravity increased in going from the Equator towards the Pole. For a complete account of this measure we must refer the reader to Maupertuis, *La Figure de la Terre.*

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* From this remark we ought, perhaps, to except a part of the survey of an arc of parallel made in the present century, in which the principal chain of the Alps was passed.

Remeasurement of Cassini's arc.

Before the return of Bouguer and Lacondamine, the whole of the French arc of meridian was remeasured by J. Cassini, his son Cassini de Thury, and Lacaille. The operations of this measure are the subject of the Work entitled *La Méridienne vérifiée.* On measuring new bases and making new observations of every kind, the cause of the original difficulty was soon discovered. The measure of Picard's base was erroneous by about $\frac{1}{1000}$th part of the whole, and this error had affected one part only of the arc. Some inaccuracies both Astronomical and Geodetical were found to exist in Cassini's measure. The late improvements in Astronomy, particularly the discovery of aberration and nutation, enabled the new observers to give much greater exactness to the Astronomical observations. They observed the latitudes of five points, and thus obtained, in fact, four arcs of meridian. Between Perpignan and Rodéz the length of an arc of one degree was found = 57,048.5 toises; between Rodéz and Bourges 57,040; between Bourges and Paris 57,071; and between Paris and Dunkirk 57,084. In the course of this measure an arc of parallel was measured across the mouth of the Rhone. From a station near Cette in Languedoc, and from Mont St. Victoire in Provence, the explosion of gunpowder on a church between them was observed; and from the difference of apparent times the difference of longitudes was found to be 1° 53′ 19″. The length of the arc of parallel in latitude 43° 32′ included between the two meridians was 75,599.6 toises. The comparison of this arc with the arc of meridian from Perpignan to Rodéz gave an ellipticity $\frac{1}{168}$.

Measure of an arc of parallel.

Difficulty which occurs in comparing these measures.

The comparison of any two of the three arcs in Peru, in France, considering the whole as one arc, and in Sweden, showed that the degrees increased in going towards the Pole, and that, consequently, the Earth's Figure was flattened at the Poles. But when all three were considered, a singular difficulty occurred, which we may be permitted to say is not yet wholly removed. The comparison of the Peruvian and French arcs gave an ellipticity of nearly $\frac{1}{314}$; that of the Peruvian and Swedish arcs gave $\frac{1}{213}$ for its value. If, then, the measures were accurate, it was impossible that the Earth's Figure could be elliptic: and Bouguer actually proposed, as the only hypothesis which would satisfy observations, that the increase of the lengths of degrees was proportional to the 4th powers of the sines of latitude, and that the proportion of the axes was 178 : 179. This would indicate a Figure projecting at middle latitudes above the elliptic spheroid with the same axes.

To avoid interruption in our account of several important measures, we have omitted to mention some interesting deductions from the theory of Gravitation made about the same time. These we shall now describe.

Maclaurin's investigation of the Earth's form.

The first important addition to the theory of the Earth was made by Maclaurin. The Academy of Paris proposed as a subject for the Prize Essay in the year 1740, *The Tides.* Three admirable Essays were sent, by Euler, D'Alembert, and Maclaurin: the last is generally considered as one of the most elegant Geometrical investigations that was ever made. Most of the operations can be applied, with very slight alterations, to the investigation of the Figure of the Earth, supposing it a homogeneous fluid; and the author has indicated the principal results which can be obtained by this application. In this Treatise it was proved for the first time, that the oblate spheroid is a form of equilibrium; and the equation was given by which the ellipticity can be found when the proportion of the centrifugal force at the Equator to Gravity is known. This equation is transcendental; but by an approximate solution when the centrifugal force is small, it was shown that the ellipticity (that is the proportion of the excess of the equatorial diameter above the polar diameter to the latter) is $\frac{5}{4} \times$ the ratio of the equatorial centrifugal force to the equatorial gravity; which in the case of the Earth is $\frac{5}{4} \times \frac{1}{289} = \frac{1}{230}$; and that Gravity at the Pole is greater than Gravity at the Equator, by the same proportion of the whole; and, also, that in going from the Equator to the Pole, the increase of Gravity is as the square of the sine of the latitude. These results are the same as those of the imperfect theory of Newton.

Clairaut's investigation.

In 1743 was published Clairaut's *Figure de la Terre.* This is the most valuable Work that has been written upon this subject. The same theory has since engaged the attention of the most distinguished of the modern Mathematicians, and finally of Laplace; and though the form of the investigations has been much varied, very little has been added to the results. The nature of the equilibrium of fluids is first explained. The only condition which had formerly been assumed as sufficient for the equilibrium of fluids was, that the force at the surface should be perpendicular to the surface, or that the pressure produced by the fluid in two canals of given form, terminated at one extremity in the same point, and at the other extremity in the surface, should be the same. Bouguer first showed that both of these conditions were necessary, or something equivalent to both. Clairaut has shown upon what the possibility of equilibrium depends; and he has applied these principles to the discovery of the form of equilibrium of a fluid on the

Figure of the Earth.

Clairaut's theorem.

lowing suppositions: 1st, that the Fluid is homogeneous, with a spheroidal nucleus of different density; 2dly, that the whole mass is fluid and heterogeneous. The form which makes equilibrium possible in all the variety of cases which these suppositions include, is, approximately, an elliptic spheroid; the ellipticity is different according to the law of density, &c., but in all cases the following theorems are true: 1st, the increase of the length of degrees and of Gravity, in going from the Equator to the Poles, is as the square of the sine of latitude; 2dly, the sum of the ellipticity, and of the ratio of the whole increase of Gravity to the equatorial Gravity, is $\frac{5}{2} \times$ the ratio of the centrifugal force at the Equator to the force of Gravity. The last of these theorems (one of the most important that has ever been discovered) is usually called by the name of its inventor. It is evident that, in conjunction with the former, it gives the means of determining the Earth's ellipticity from observations of the comparative force of Gravity at any two places.

Nutation discovered by observation and explained by theory.

In 1747 Bradley discovered the nutation of the Earth's axis. This had been alluded to by Newton as a consequence of his theory, but no notice seems to have been taken of his theoretical prediction. As soon, however, as the fact was established, it was treated theoretically by Mathematicians: among the best of the Treatises upon nutation may be reckoned D'Alembert's *Recherches sur la Précession, &c.*

Measure of Boscovich.

In 1750 Boscovich and Le Maire, two Jesuits, measured an arc from Rome to Rimini. Their measures were made with the pace, the measure of the Country; but it was carefully compared with the French toise, so that their results could be expressed in terms of the same standard which had already served for so many measures. A base of 6139.5 toises was measured on the Via Appia; and a base of verification of 6037.6 toises by the sea-side near Rimini. The meridian distance was found = 161,253.6 paces = 123,221.3 toises; and the difference of latitude 2° 9′ 47″, whence 1° = 56,966.3 toises. This, when some corrections were applied, was reduced to 56,979; the mean latitude 42° 59′ The whole of this measure passed over a mountainous country.

Measure of Lacaille.

In 1752 Lacaille, who had been sent to the Cape of Good Hope to make Astronomical observations, finding the circumstances of the country favourable for a trigonometrical survey, measured an arc of 1° 13′ 17″.3, which he found = 69,669.1 toises; whence 1° = 57,037. The length of his base was 6467 toises. See *Mém. de l'Acad.* 1751. This arc presents a remarkable anomaly. According to this measure, a degree in the South hemisphere, whose mean latitude is 33° 20′, is equal to a degree in the North hemisphere, whose mean latitude is about 45°. The known ability of the observer almost forbids the supposition of an error in the observations; and we have no grounds for conjecturing the cause of such a deviation from the law which seems to apply to the other arcs.

The measures described in the next three paragraphs were undertaken, we believe, at the suggestion of Boscovich.

Liesganig's measure.

History.

In 1762 Liesganig, a Jesuit, began the measure of an arc of meridian passing through the Observatory of Vienna. See the *Dimensio graduum Viennensis et Hungarici.* Every care apparently was taken to ensure the correctness of this measure. The Vienna fathom was compared with a French toise furnished by Lacondamine and Lacaille; the iron quadrant of 2½ feet, with which the angles of the triangles were observed, was repeatedly examined in all possible ways; the sector, for observation of latitude, (which we shall afterwards describe,) was on the most improved construction; the observations of latitude were sufficiently numerous; in every triangle, except one, Liesganig assures us, that the three angles were observed; and all the calculations were reexamined. The base (between Neustadt and Neunkirch) was of 6238 toises; the base of verification in Marchfeld was of 6388 toises. The concluded meridional distance of Sobieschiz and Varasdin (the North and South extremities) was 172,796 Vienna fathoms; and the difference of latitudes 2° 56′ 45″.5, whence 1° = 58,655 Vienna fathoms = 57,077 toises. Yet in spite of all these apparent securities, there appears reason for rejecting this measure. The principal objections to it may be seen in Zach's *Correspondance Astronomique, &c.* vol. vii. In the first place, it is certain that the star which Liesganig has put down as μ Draconis is not μ Draconis, but some other star; possibly 85 ι Herculis, which Zach mentions. This would occasion no error, (as the same star was observed at different stations,) except in consequence of taking erroneous reductions. In the next place, Zach (who possessed some of Liesganig's manuscripts, and who recalculated some of the observations) affirms, that many of the observations had been altered to produce greater apparent agreement. And, lastly, it appears from a repetition of part of the survey by the Austrian officers in the beginning of the present century, that in one of Liesganig's triangles it was impossible, and apparently had always been so, to observe one of his stations from another; and, in fact, when the value of one of the angles, as given by him, was compared with the sum of two angles, which ought to be equal to it, observed by the Austrian officers, a difference of 3° was found. This triangle is the last but two to the South; up to this the surveys agree within a few toises, but after this they sometimes differ in the situation which they assign to a station by more than 2500 toises. It would seem that a signal was mistaken; a similar accident happened to Snell; and there is reason to think that it has also occurred in the survey of England made under the direction of the Master-General of the Ordnance. Another arc was measured by Liesganig on the plain of the Theiss, from Kistelech in the North to Czuroch, near Petervaradin, in the South. The length was 59,990 Vienna fathoms; and the difference of latitudes 1° 1′ 34″.5; whence 1° = 58,453 = 56,881 toises. Several circumstances prevent us from attaching much value to this measure; one is, the great number of small triangles which the nature of the country compelled him to use.

Reasons for rejecting it.

Beccaria's measure.

About the same time Beccaria, a Jesuit, assisted by another of the same Order, measured a degree in the plain of Lombardy. The measure of Boscovich crossed the Apennines, and its extremities were on the sea-coasts; this of Beccaria (undertaken, as we have mentioned, at the suggestion of Boscovich) was carried over a flat country, and terminated at both extremities at the foot of lofty mountains. It was supposed that if the inequality of the country produced in the measure of a degree any sensible irregularity, the effects, in these two measures, would be of opposite kinds, and that the difference in the length of a degree would be such as to give a good idea of their magnitude. An iron toise, which had been compared by Lacondamine and Lacaille with the toise used in the measure of the arc in Peru, (commonly called the toise of Peru,) was Beccaria's standard. His instruments were in almost every respect similar to those of Boscovich. The length of the base was 6501 toises; the meridional distance from Andratæ, the Northern extremity, to Mons Regalis, (Mondovi,) the Southern extremity, was 64,890 toises. The difference of latitude was found to be 1° 7′ 44″.7; whence 1° = 57,468 toises; a quantity much greater than other measures would have led us to expect. The mean latitude is 44° 57′. An account of this measure was published under the title of *Gradus Taurinensis.* In the surveys made by the French officers during their occupation of this Country, and in the repetition of the observations for the latitudes of the stations by MM. Plana and Carlini, some discrepancies have been found, which would seem to show that the credit of this measure is rather doubtful.

Doubts of its accuracy

Measure of an arc in North America.

In 1764 Messrs. Charles Mason and Jeremiah Dixon, who had before been engaged by the Royal Society for some important Astronomical observations, were employed in settling the boundaries of Maryland and Pennsylvania in North America. The line which they traced out in the peninsula between Chesapeak Bay and Delaware Bay seemed so favourable for a meridian measure, that, on their representation, the Council of the Royal Society furnished them with standards and instructions, and procured the loan of a sector belonging to Mr. Penn. An account of the measure is given in the *Phil. Trans.* for 1768. This measure differs from all others made since the time of Norwood in this respect,—that no triangles were used, but the whole line (about 100 miles) was measured with rods. These rods were compared with a five-feet brass rod made by Bird. The whole length was found to be 538,067 feet, and the difference of latitude 1° 28′ 45″; whence 1° = 363,763 feet = $60{,}627\frac{1}{6}$ fathoms. It had been found by a comparison of standards (*Phil. Trans.* vol. xlii.) that the English fathom $= \frac{107}{114} \times$ French toise; whence one degree was inferred to be $56{,}904\frac{1}{2}$ toises. But after a new comparison of the toise and fathom, made under the inspection of Dr. Maskelyne, and after applying some small corrections, this was reduced to 60,625 fathoms, or 56,888 toises. The mean latitude was 39° 12′.

The anomalies in these measures were so considerable, that it appeared highly desirable to discover whether the attractions of mountains could have a sensible effect in altering the direction of Gravity. If in consequence of the proximity of a mountain the plumb-line was drawn from the position which it would otherwise have maintained, it is evident that the point of the heavens which corresponded to the zenith of a station, as determined by Astronomical observations, would not be the same as if the mountain did not exist. If, for instance, a mountain were at L, near the point A, in fig. 1, and drew the plumb-line towards L, the apparent direction of Gravity, instead of A C would be A *c*, and the apparent horizontal line instead of A D would be A *d*; the difference of latitudes, or the angle A C B, would be changed to A *c* B, and the length of the radius of curvature of the meridian would be changed from A C to A *c*. On account of the influence of these effects on the determination of the Figure of the Earth, as well as for the important confirmation which might be given to the Principle of Gravitation, it was to be wished that the attraction of a mountain might be ascertained by direct experiment. This had been attempted by Bouguer and Lacondamine in Peru; but the cold was so intense that it was difficult to make the observations with sufficient accuracy, and other circumstances were not favourable. Instead of observing the zenith distances of a star on opposite (North and South) sides of the mountain, they could only fix on two stations on the South side, one close to the mountain, the other at a distance of 3570 toises from the first, and 505 South of it. The effect of attraction appeared to be 7″.5; but they considered the conclusion almost unworthy of credit. In 1772 (see *Phil. Trans.* 1775) Dr. Maskelyne proposed to repeat the experiment. He pointed out Whernside in Yorkshire as a mountain on the opposite sides of which the attraction would be sensible; or he thought that the defect of matter in the valley between Helvellyn and Skiddaw, (or rather Saddleback,) in Cumberland, producing an effect of the opposite kind, might be sensible in the same kind of observations. Mr. Charles Mason was sent out to examine these, or to look for a hill proper for the purpose: these were finally rejected, and Schehallien was chosen. This is a mountain near Blair Athol, in Perthshire; it is a narrow ridge running East and West, in a comparatively flat country, and about 2000 feet higher than the general level. In 1774 the observations were made. The meridional distance of two stations on the North and South sides was found, by survey, to be 4364.4 feet, which on any estimation of the Earth's dimensions would give for the angle A C B, fig. 1, (C being considered as the centre of the Earth,) a quantity differing very little from 42″.9. The apparent difference of latitude, from Dr. Maskelyne's observations, given either by the 40 observations calculated by Maskelyne, or from the whole mass of 337 calculated by Zach, (*L'Attraction des Montagnes, &c.*) is 54″.6. The difference, or 11″.7, is the sum of the attractions which the mountain exerted in opposite directions on the plumb-line, when placed on opposite sides of the mountain. The mean density of the Earth, calculated on the theory of Gravitation from this attraction, and from an accurate survey of the mountain, was found by Dr. Hutton, (*Phil. Trans.* 1778, 1811, and 1821,) to be about 1.8 × density of Schehallien, or about 5 × density of water.

Attempt to ascertain the attraction of a mountain in Peru.

Observations on the attraction of Schehallien

The arc of meridian which had been traced so accurately through France, served for a foundation to the surveys for a very accurate map of France. Nothing of this kind had been done in England, excepting a survey of part of the Highlands, commenced in consequence of the Rebellion of 1745. The disastrous wars of the last century prevented our Government from extending the map. In the year 1785, Cassini de Thury presented to the Royal Society a memorial on the uncertainty in the difference of longitude of Greenwich and Paris; and proposed that the English and French Mathematicians in concert should determine, by Geodetic operations, the distance, measured along an arc of parallel. This was assented to, and the English survey was placed under the superintendence of General Roy, the French under that of Count Cassini, (fourth of that name,) Mechain, and Legendre. We believe it may fairly be said that in this, as in other grand experiments, though we began later than our continental neighbours, we conducted our operations with a degree of accuracy of which, till that time, no one had dared to form an idea. For the measure of the first base on Hounslow Heath, in 1784, deal rods

Junction of the observations of Greenwich and Paris.

(which had been used in all preceding measures) were rejected, and glass tubes were used. These were afterwards laid aside, and a steel chain was employed in the measure of a base of verification on Romney Marsh. In 1791 the base on Hounslow Heath was remeasured with the chain. The description of the mode of measuring the bases (*Phil. Trans.* 1785, and *Account of Trigonometrical Survey*, vol. i.) was translated into French by M. Prony. The angles of the triangles were observed with a large theodolite, made by Ramsden; and this is the first instrument with which the spherical excess of the sum of the three angles above 180° (a quantity that rarely exceeds 3″ or 4″) was observed. The French part of this survey is remarkable also, as the first instance in which the newly-invented repeating-circle was used in extensive operations. Whether this invention has contributed to the accuracy of Astronomical determinations, we may be permitted to doubt; but it can hardly be doubted that it is an excellent instrument for Geodetic operations. The chain of triangles formed by the English observers, extended from the neighbourhood of London to Dover, and across the channel to Calais, and some neighbouring stations; the French triangles joined these with the chain of triangles formerly observed in the meridian of Paris, in the neighbourhood of Dunkirk.*

English arc of parallel.

English arc of meridian.

The object originally proposed in this survey was now attained; but it will readily be imagined that advantage was taken of such an admirable commencement for an equally excellent extension. An accurate survey of the whole Kingdom was commenced; several bases of verification were measured, and in the course of the survey, an arc of parallel between Beachy Head and Dunnose was measured in 1794, (by reciprocal observations of azimuth, a method which we shall hereafter describe,) and an arc of meridian from Dunnose to Clifton in Yorkshire, in 1802. The latitudes were observed with a large zenith sector made by Ramsden. The first of these arcs gave for the length of a degree of longitude in the parallel of 50° 37′ 7″.3, 38,818 fathoms; the second gave for the length of a degree of meridian in latitude 52° 2′, 60,820. This arc of meridian was divided into two parts, nearly equal, by the station of Arbury Hill; the arc from Dunnose to Arbury gave for one degree in latitude 51° 25′, 60,864; that from Arbury to Clifton gave for one degree in latitude 52° 50′, 60,766. These partial arcs present the same anomaly as those in France; the degrees appear to diminish in going towards the Pole. (See *Account of Trigonometrical Survey*, and *Phil. Trans.* for various years.)

Origin of the new French measure.

Measure of arc from Dunkirk to Formentera.

In 1791 the National Convention of France wishing to fix on a new standard of linear measure, determined (in the true spirit of the Revolutionary Philosophy) to select one which could not be considered as belonging to one nation rather than to another, but which might claim to be esteemed a standard for all the World. The length of the pendulum vibrating seconds at a given place had been pointed out by Picard as the best practical standard; but this did not suit the expanded ideas of the French legislators. They considered that the length of the seconds' pendulum varied on varying the place of experiment; but that, assuming the Earth's surface (at least the Geometrical surface at which if channels were made communicating with the sea the water would find its level) to be an exact surface of revolution, the length of the quadrant of meridian passing through every place would be the same. The length of the quadrant of meridian passing through the Observatory of Paris was therefore to be ascertained, and one ten-millionth part of this was to be called the *mètre*. It is almost unnecessary to add, that the idea of replacing a lost standard by means of an extensive Geodetic measure is perfectly chimerical; and that the only practical method (still subject to some uncertainty) is by ascertaining the length of the seconds' pendulum, which it is presumed, in the same place, does not vary from one Age to another. The measure was undertaken by Delambre and Mechain, and is described in the *Base du Système Métrique*; a Work which cannot be too strongly recommended to the perusal of all who wish to be acquainted with the methods pursued in an extensive survey by the ablest general observers and practical Mathematicians of the Age. The difficulties with which they had to struggle were very great, arising principally from the disturbed state of the Country, and, as they approached to a termination, from the depreciation of the assignats. Fire-signals at night are preferable to all others for the operations of a survey; but these it was impossible to employ, as the people would have supposed them to be counter-revolutionary signals. In the choice of day-signals, and in procuring permission to erect and to observe them, they experienced sometimes the most serious, and sometimes the most ludicrous obstacles. It was generally necessary to receive the sanction of the popular assembly of each town through which they passed; and in many cases Delambre found himself obliged to give to the collected inhabitants a sort of Lecture on Geodesy. When a great part of their labour was completed, they were dismissed from their office for not having sufficiently distinguished themselves by their hatred of Kings. They were, however, allowed to complete the arc which they had originally intended to measure; following the course of the former measures, and using, in many instances, the same signals from Dunkirk to Barcelona. The part South of the Pyrenees was measured by the permission and with the assistance of the Spanish Government. As soon as the details of the measures could be collected, they were laid before a committee consisting of scientific deputies from several continental nations. It was there determined to proceed on the assumption, that the Earth is an exact elliptical spheroid, and to calculate its ellipticity, its dimensions, and the length of the quadrant, from a comparison of the arc newly measured with the arc measured by Bouguer and Lacondamine in Peru. The ellipticity adopted was $\frac{1}{304}$, and the mètre was fixed at 443,296 lines. The latitudes were observed at Dunkirk, Paris, Evaux, Carcassone, and Montjouy, near Barcelona. The arc was thus divided into four partial arcs, of which the middle latitudes were 49° 56′, 47° 31′, 44° 42′, and 42° 17′, and which gave respectively for the length of one degree, in toises, 57,082.7, 57,068.8, 56,977.8, and 56,946.6. This survey was afterwards extended by Biot and Arago to the Island of Formentera, near Minorca; the mean latitude of the additional arc was 40° 1′, and the length of one degree 56,956.4. The mean latitude of the whole arc, from

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* The whole of this measure has been repeated; the English part under the direction of Captain Kater, the French under that of M. Arago. See *Phil. Trans.* 1828.

Formentera to Dunkirk, is 44° 51′, and the mean length of the arc of one degree 57,006.6.

Cavendish's experiments on the attraction of leaden balls.

In 1798 a beautiful series of experiments was instituted by Mr. Cavendish, for determining the attraction of metallic balls. The observations on Schehallien had shown that the attraction of a mountain was certainly appreciable: but it required much artifice, and great experimental delicacy, to make the attraction of a sphere of lead, eight inches in diameter, distinctly sensible. This, however, was effected, and so completely that the comparison of the effects of this attraction with the effects of the attraction of the Earth, enabled him to infer the proportion of the Earth's mean density to the density of lead; and of course to the density of water. The result is very nearly the same (*Phil. Trans.* 1798 and 1821) as that given by the Schehallien experiment.

Earth's ellipticity discovered by the motion of the Moon.

Nearly at the conclusion of this century the investigations of Laplace (*Mécanique Céleste*) furnished us with a curious method of determining the ratio of the Earth's axes. He showed that, in consequence of the Earth's oblateness, the Moon's motion would not be the same as if the Earth were spherical; and that two of the resulting irregularities would rise to such a magnitude as probably to be sensible. From the accurate observations of the Moon, made at Greenwich, the existence of these inequalities has been detected, and their magnitude (about 8″ each) has been ascertained with tolerable certainty. They indicate an ellipticity of nearly $\frac{1}{300}$.

We have mentioned that the ellipticity of the Earth, (assumed to be an elliptic spheroid,) given by a comparison of the degree in Peru with that in France, was not the same as that given by the comparison of the degree in Peru with that in Sweden. The only inference to be drawn from this was, that the assumption was false, and that some such form as that stated by Bouguer must be the true one. No new measures had since been made which could be considered as decisive on this point: the measure in Peru was made with very great care: the accuracy of that in France had been confirmed by the late measure; and, as the only possible explanation, it was thought that the circumstances under which the Swedish measure was made, were not perhaps so favourable to accuracy. To set this at rest the arc was remeasured, and extended in both directions by Svanberg, in the years 1801, 1802, 1803. The details will be found in his very elegant Work entitled *Exposition des Opérations faites en Lapponie, &c.* In this survey the peculiarities of the new Astronomical School of France were pushed to an extent to which the boldest of their proposers had hardly ventured. The French standard and the French graduation alone were used; no instrument but the repeating circle was employed; the labour which English observers would have used in making good single observations was by Svanberg bestowed on the infinite repetition of angles; the imagination was taxed to discover the best manner of combining the observations in order to obtain the result which, according to the theory of chances, would be most probably correct. The methods of calculation were principally taken from Delambre. The base was nearly the same as that measured by the Academicians; the triangles, as far as the survey of the Academicians extended, were the same; those which Svanberg added were not so favourably circumstanced. The latitudes of his extreme stations, Mallorn and Pahtavara, were 65° 31′ 30″ 265, and 67° 8′ 49″.83; the distance 180,827 metres = 92,778 toises; whence 1° = 57,196 toises. (Maupertuis had found 1° = 57,422 toises.) The result of this measure seemed at once to destroy the difficulty which had so long troubled Philosophers; in fact, the arcs already mentioned agreed pretty well with the new arc in giving an ellipticity rather less than $\frac{1}{300}$. But how is the difference of the two measures to be explained? The geodetic measures, as far as they went together, agree very well; the latitude of Tornea, as determined by Maupertuis, agrees perfectly with the observations of Svanberg; the latitude of Kittis was not observed by Svanberg. It is much, very much, to be regretted that the Swedish Astronomer did not repeat the observations at the only place where an important error could be feared.

Svanberg's arc in Sweden.

Zach's observations on the attraction of Mimet.

In 1810 an attempt was made by Baron de Zach to show that the attraction of a mountain called Mimet, near Marseilles, was appreciable. (See his Work, *L'Attraction des Montagnes, &c.*) He observed the latitude of a station on the Southern face of the mountain, and that of the small island Planier: and he calculated the difference of latitudes from the distance found by survey. The attraction of the mountain appeared to produce in the latitude an error of 1″.98. It has been shown, we think satisfactorily, by Arago, (*Conn. des Temps*, 1819, *Additions*,) that the repeating circle used by Zach (as appears from other observations made with it) was not good enough for an operation of such extreme delicacy. We may add, that as the stars observed were all on the same side of the zenith, a change in the constant error of the circle would produce an error in the result without affording any means of discovering its amount.

Arc of meridian measured by Lambton in India.

In the present century an arc has been measured by Colonel Lambton in the Peninsula of India, which, for length and accuracy, must be considered, we think, as exceeding those of all the arcs yet measured. (See *Asiatic Researches* for various years, or *Phil. Trans.* 1818.) The instruments and the methods of calculation were similar to those used in the English survey. The latitude of Punnæ, the Southern station, (near Cape Comorin,) was found to be 8° 9′ 38″.39; that of Daumergidda, the Northern extremity, 18° 3′ 23″.53; the meridional distance 598,609.98 fathoms, or 680 miles. By observing the latitudes of two intermediate stations, this arc was divided into three partial arcs, whose amplitudes were 2° 50′ 10″.54, 4° 6′ 11″28, and 2° 57′ 23″.32; and whose lengths were 171,516.8, 248,188.5, and 178,904.7 fathoms; which give respectively for the length of a degree 60,472.8, 60,487.6, and 60,512.8 fathoms. A comparison of these values with those determined by the French, English, or modern Swedish measures, gives for the ellipticity a quantity rather less than $\frac{1}{300}$. An arc of parallel was also measured in the same manner as the English arc between Dunnose and Beachy Head; but the geographical situation of the place is highly unfavourable to the use of this method.

The first instance, we believe, in which instantaneous fire-signals were used to determine the difference of longitude of two places, for the purpose of comparing it with the distance measured on an arc of parallel, and

Extensive arc of parallel measured in Europe.

thus determining the radius of the parallel of a given latitude, was that in which Cassini de Thury and Lacaille measured an arc across the mouth of the Rhone, and observed the explosion of gunpowder at one intermediate point. In the years 1821, 1822, and 1823, the difference of longitude of Marennes (near Bourdeaux) and Padua was determined by this method. This extensive arc was divided into six partial arcs, and the difference of longitudes* of the extremities of each was ascertained independently of the others. The observers were MM. Brousseaud, Nicollet, Plana, and Carlini. The geographical distance of both extremities from the frontiers of Savoy and Piedmont, had been ascertained by a survey conducted by the French engineers. Considerable difficulty was found in connecting these parts of the survey, as it was necessary to select several stations on the very crest of the Alps. This was at last effected in that part of the chain which lies between the passes of Mont Cenis and Mont Genèvre; accessible stations were found, though one was more than 11,000 feet above the sea. These triangles of junction were surveyed by a mixed commission of Austrian and Piemontese officers. The result (*Conn. des Temps*, 1829, *Additions*, and *Opérations Géodésiques et Astronomiques en Piémont et Savoie*) is that the length of a degree of parallel in latitude 45° 43′ 12″, is $77{,}865^{\mathrm{m}}\cdot 75$. A comparison of this arc with the principal arcs of latitude seems to show that the ellipticity is about $\frac{1}{280}$. At the same time the arc measured by Beccaria was remeasured, and the latitudes of the stations reobserved. The result differs from Beccaria's, but not so much as to remove the anomaly in that measure.

Remeasurement of Beccaria's arc.

The arc of parallel between Beachy Head and Dunnose, measured by General Roy, was used to fix a scale of longitudes in the Ordnance map of England. It was suspected that the scale was erroneous, but it was only in 1823 that this was clearly established. Dr. Tiarks, by carrying several chronometers backwards and forwards between Dover and Falmouth, found that the difference of their longitudes was incorrect, to the amount of 4″ of time; the difference of longitudes given by the map being smaller than that given by the chronometers. It was necessary, therefore, to diminish the value of a degree of parallel, as determined by Roy's observations; and the degree so diminished agrees much better with those obtained from other measures.

Correction of the measure of parallel in England.

Besides these measures which have been undertaken expressly for the purpose of ascertaining the Earth's dimensions, others have been made in which the principal object was the mapping of the country, and from which the Earth's dimensions have been incidentally deduced, when the latitudes of extreme stations as observed, have been compared with the latitudes as calculated on assumed dimensions. We have great doubts of the accuracy, in general, of the observed latitudes; and we have not, therefore, thought these operations worthy of a particular mention. Several, undoubtedly, ought to be excepted from this remark; we may mention in particular the survey lately made, connecting Göttingen and Altona, in which we believe that the latitudes were observed with Ramsden's sector, but of which the details have not reached us.

Modern experiments with the pendulum for the variation of Gravity.

In the present century, and at the termination of the last, a great number of expeditions have been undertaken, of which one of the principal objects has been to determine the length of the seconds' pendulum at different places. We cannot here give any detailed account of them, and shall merely mention the following. Observations at places in a great extent of latitude in the Spanish expedition of 1789; (*Conn. des Temps*, 1816, *Additions*;) observations by the French at various places in France and England; (*Base du Système métrique*;) observations made in the voyages of Captains Ross and Parry; (see the accounts of those voyages, and *Phil. Trans.*;) observations by Captaine Sabine at a number of places in almost all practicable latitudes, undoubtedly the best series of observations yet made; (*Account of Experiments, &c.*;) observations made in the voyages of Freycinet and Duperrey; (accounts of the voyages, and Additions to the *Conn. des Temps*;) observations made by Captain Kater at several places in Great Britain, and by Captain Hall, Sir Thomas Brisbane, Mr. Goldingham, and others, in various parts of the world. (*Phil. Trans.* various years.) The pendulum experiments, especially those of Captain Sabine, appear to indicate a greater ellipticity than is given by the measures.

In our account of the various measures and experiments which are used to determine the Figure and constitution of the Earth, it is possible that we may have omitted some of inferior note. We believe, however, that we have included all upon which any reliance can be placed for aiding us in a most difficult and delicate inquiry.

Section 2.—Theoretical Investigation of the Form assumed by a Revolving Fluid on the Principle of Gravitation.

Physical theory.

(1.) It will be proper to commence this investigation by a repetition of the explanation given in our Treatise on Hydrodynamics, of the conditions of equilibrium of a Fluid of which different points are acted on by different forces acting in different directions.

(2.) We shall suppose that some point is taken as the origin of a system of rectangular coordinates, and that the coordinates of any point are called x, y, z; and that the force R, which acts on the Fluid at that point, is resolved into three forces parallel, respectively, to the three axes, (see Mechanics, § V.) which are called respectively X, Y, Z. These forces (or the single force of which they are parts) are of the kind called accelerating forces; they are not pressures, but, like Gravity and all kinds of attraction, they produce pressure by acting upon some mass, and the pressure so produced is proportional to that mass. We ought, in strictness, to state it thus: the force which we call R acting on the mass dm produces the pressure $R\,dm$, and this is resolved into the pressures $X\,dm$, $Y\,dm$, $Z\,dm$, parallel to the three coordinates x, y, z.

Condition of the equilibrium of a Fluid.

(3.) Now if the Fluid is at rest, we shall not disturb its rest by enclosing a part of it in tubes. Suppose then that in the fluid mass, fig. 2, we place several tubes extending from the surface to the small quantity of Fluid enclosed in a box at the point A: and now let us consider the state of the Fluid in these tubes, &c. as if the surrounding mass did not exist. The action of the forces on the different parts of Fluid in the tube C D A will produce a pressure on the Fluid at the place A; that in the tube E A, or C B A, will also produce a pressure there. Now the characteristic property of Fluids (Hydrodynamics, § II.) is, that the pressure produced by the Fluid in E A, will be transmitted by the Fluid at A, in such a manner as to cause the same pressure upwards (estimated by the pressure on a given surface) at the bottoms of the tubes C B A, C D A. If, then, the pressures produced by the Fluids in these tubes be less than that produced by the Fluid in E A, the Fluid will be forced up them; if greater, it will descend, forcing a portion of the Fluid up A E; both which conclusions are inconsistent with our supposition of rest. It is necessary, therefore, that they be equal; and thus we obtain as the condition necessary for equilibrium; the pressure produced by the Fluids in all imaginary canals from the surface to any given point of the Fluid must be equal.

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* The English reader who wishes to be acquainted with the best method of determining differences of longitude by fire-signals, is referred to an admirable paper by Mr. Herschel, *Phil. Trans.* 1826, on the determination of the difference of longitude of Paris and Greenwich.

(4.) This condition is also sufficient for equilibrium. For if we take small tubes as F G, H G, to any part of the tube E A, the Fluids in these, in consequence of this condition being satisfied, will have no tendency to disturb the Fluid in E A; from which it appears, that the equilibrium of the Fluid in E A does not depend on its being enclosed in a tube: and the same applies to any other point of the Fluid.

Expression for pressure of a Fluid.

(5.) The condition can now be reduced to a mathematical form, for which purpose we must find an expression for the pressure produced by the Fluid in a tube. Take a very small portion of the tube, as B b, and construct a parallelopiped of which B b is the diagonal. and whose sides B c, c d, d b, are parallel to the coordinates x, y, z, respectively. As these sides are the differences between the coordinates that correspond to the point B, and those that correspond to the point b, let them be called dx, dy, dz; and call B b, the increase of the length of the tube in proceeding from B to b, ds; call the section of the tube, expressed in square units of surface, K; and the density of the Fluid in that part of the tube, expressed by the weight of one cubical unit, ρ. Then the solid content of the tube B b is $K ds$; the weight of the Fluid contained in it is $K \rho ds$: the action of the force X upon this causes the pressure $K \rho X ds$ in the direction B c. Resolve this (Mechanics, Art. 29.) into two pressures, one of which is parallel to B b, the other perpendicular to it. The latter of these will only cause a pressure on the sides of the tube, and therefore is to be neglected; the former will cause a pressure in the direction of the tube, and will therefore add to the pressure of the Fluid. Its value is $K \rho X ds \times \cos b B c = K \rho X ds \times \frac{B c}{B b}$ $= K \rho X ds \times \frac{dx}{ds} = K \rho X dx$. This is the pressure which it causes on the surface K; and therefore on a surface 1 it would cause the pressure $\rho X dx$. Similarly it would appear that the pressure caused by the action of the force Y would be $\rho Y dy$; and that caused by the force Z would be $\rho Z dz$. The sum of these, or $\rho (X dx + Y dy + Z dz)$, is the quantity by which the pressure of the Fluid is increased between B and b; and therefore it is, in common language, the differential of the pressure. The whole pressure is the integral of this, and is therefore $\int \rho (X dx + Y dy + Z dz)$.

(6.) Now the condition of equilibrium requires that the pressures produced by the Fluids in the tubes C B A, C D A, should be equal, or (since this is to be the case for tubes of any form) that the pressure should be independent of the form of the tube. Now the form of the tube will be defined by two equations, one of which will express y in terms of x, and the other z in terms of x; so that, assuming at pleasure a value of x, we can find corresponding values of y and z. For every different form of the tube, y and z will be expressed in different functions of x. And since the density at any point is known from a knowledge of its situation, ρ is a function of x. On substituting these values in the quantity $\rho (X dx + Y dy + Z dz)$ it will have the shape $\phi (x) dx$, the integral of which is of the form $\psi (x)$; putting a and e for the values of x at A and C, and integrating between these limits, the pressure will be $\psi (a) - \psi (e)$. Now this expression clearly is not the same for all forms of the function ψ, that is, it is not the same for all forms of ϕ, or it is not the same for all figures of the tube; and therefore when it is necessary to integrate $\rho (X dx + Y dy + Z dz)$ in this manner, the equation of equilibrium cannot be satisfied.

Mathematical condition of equilibrium.

(7.) But there is one case in which $\rho (X dx + Y dy + Z dz)$ can be integrated without expressing y and z in terms of x; it is when $\rho (X dx + Y dy + Z dz)$ is explicitly a complete differential of some function of x, y, and z. In this case, supposing the integral to be $\chi (x, y, z)$, and putting a, b, c for the values of x, y, z, at the point A, and e, f, g for those at the point C, the pressure will be $\chi (a, b, c) - \chi (e, f, g)$. Here the expression is the same whatever be the form of the tube; and thus we get for one condition of equilibrium, that $\rho (X dx + Y dy + Z dz)$ must be explicitly a complete differential.

Form of external surface necessary for equilibrium.

(8.) Here we have compared the pressures produced by the Fluids in two tubes drawn from the same point of the surface C. If, however, we take another tube E A from another point of the surface, and call l, m, n the coordinates of E, the expression for the pressure which the fluid in it produces will be $\chi (a, b, c) - \chi (l, m, n)$. This must be the same as the former expression; hence $\chi (l, m, n)$ must be the same as $\chi (e, f, g)$. Applying this to all points of the surface we find that, l, m, n being the coordinates of any point of the surface, $\chi (l, m, n)$ must be constant; or putting x, y, z instead of l, m, n, and observing that $\chi (x, y, z)$ is the integral of $\rho (X dx + Y dy + Z dz)$, the external surface will be defined by making this differential $= 0$, or $X dx + Y dy + Z dz = 0$. And this is the second equation of equilibrium.

Relation of forces when the Fluid is homogeneous.

(9.) Suppose the Fluid to be homogeneous, or ρ, the density, to be constant and equal to k; then we must have $k (X dx + Y dy + Z dz)$ a complete differential, or $X dx + Y dy + Z dz$ a complete differential. Now it is remarkable that this condition is satisfied by the only attractive force (Gravitation) whose laws are accurately known to us, as well as by the centrifugal force of which we shall presently speak. For, let the mass of one attracting point be dm, its coordinates a, b, c, the coordinates of an attracted point x, y, z, the distance between them r, so that

Satisfied by Gravitation and centrifugal force.

$$r = \sqrt{\overline{x - a}|^2 + \overline{y - b}|^2 + \overline{z - c}|^2}.$$

Then the attraction in the direction of r, on the principle of Gravitation, will be represented by $\frac{dm}{r^2}$; that in the direction of x will be $\frac{d\,m}{r^2} \times \frac{x-a}{r} = d\,m \times \frac{x-a}{r^3}$; those in the direction of y and z, $d\,m \times \frac{y-b}{r^3}$, and $d\,m \times \frac{z-c}{r^3}$ Hence $X = d\,m . \frac{x-a}{r^3}$, $Y = d\,m . \frac{y-b}{r^3}$, $Z = d\,m . \frac{z-c}{r^3}$, and $X\,d\,x + Y\,d\,y + Z\,d\,z$

$$= d\,m . \frac{\overline{x-a}\,.\,d\,x + \overline{y-b}\,.\,d\,y + \overline{z-c}\,.\,d\,z}{(\overline{x-a}|^2 + \overline{y-b}|^2 + \overline{z-c}|^2)^{\frac{3}{2}}},$$

which is a complete differential (the integral is $\frac{-\,d\,m}{\surd\{\overline{x-a}|^2 + \overline{y-b}|^2 + \overline{z-c}|^2\}}$). The same will be true for the forces produced by the attraction of any other attracting point; and therefore for the forces produced by the sum of all the attractions of different points. The centrifugal force (as we shall show hereafter) is proportional to the distance of the point on which it acts from an axis, and acts in the direction of a line perpendicular to the axis, and passing through the point; whence it is easily found that, supposing the axis of z to be the axis of rotation, and the force to be $m \times$ distance, the force in the direction of x will be $m\,x$, and that in the direction of y will be $m\,y$, that in the direction of z, 0; therefore $X\,d\,x + Y\,d\,y + Z\,d\,z = m\,(x\,d\,x + y\,d\,y)$, which is a complete differential, its integral being $\frac{m}{2}(x^2 + y^2)$.

Form or surface of equal density when the Fluid is heterogeneous.

(10.) If ρ be not constant, and if $X\,d\,x + Y\,d\,y + Z\,d\,z$ be a complete differential, and $= d\,q$, then the quantity in Art. (7.), which must be a complete differential, is $\rho\,d\,q$. This can only be true when ρ is a function of q; and, consequently, so long as q does not vary, ρ must remain without variation. That is, if the fluid be heterogeneous, the surface which limits a stratum of any given density is defined by the equation $q = c$. Let p be the pressure; then $d\,p = \rho\,d\,q$, and $p = \int \rho\,d\,q$, which is some function of q, as $\phi\,(q)$; and if q' be the value of q at A, and q'' the value at the surface, p at $A = \phi\,(q') - \phi\,(q'')$. From this (as in Art. 8.) it appears that $\phi\,(q'')$ must be constant, or q'' must be constant, at the external surface; and thus it appears that the external surface, as well as any surface which separates strata of different density, is defined by the equation $q =$ constant. It appears, also, that $p = \phi\,(q') -$ constant, whence q' is a function of p, and ρ, which is a function of q', is also a function of p; that is, where p is the same, ρ is the same. Conversely, where ρ is the same, p is the same, provided* that the Fluid consist of indefinitely thin strata of densities varying from any one to the next in contact with it.

The same equation holds if the density depend on the pressure.

(11.) From this conclusion we derive this remarkable inference, that if we suppose the density to depend on the pressure, the equations of equilibrium will not be in any degree altered. For that supposition would only require that ρ should be the same where p is the same, which is already secured by the conclusion just mentioned.

Attraction entirely directed to centre.

(12.) *Example.* Suppose the only forces which act on each particle of a heterogeneous Fluid to be these; an attraction (not to every other particle but) to the centre, varying inversely as the square of the distance from the centre; and the centrifugal force resulting from rotation round an axis passing through the centre. (This is nearly the same as Huygens's supposition.)

Let A C B, fig. 3, be the axis, C the centre, C M, M N, N P, the three coordinates of a point P, one of which (N P or z) is parallel to the axis; join C N, C P, and draw P Q parallel to N C, which will be perpendicular to A C B. The force $\frac{f}{r^2}$ in the direction P C, represented by P C, may be resolved into P N, N M, M C, that is, the resolved parts parallel to x, y, and z are $-\frac{f\,x}{r^3}$, $-\frac{f\,y}{r^3}$, $-\frac{f\,z}{r^3}$; where r s put for P C, and the forces are considered negative because their action tends to move the particle in such a direction as to diminish x, y, and z. The centrifugal force, if T be the time of revolution, is equal (MECHANICS, § XIV.) to $\frac{4\,\pi^2}{T^2}$ Q P, or $\frac{4\,\pi^2}{T^2}$ C N; which may be resolved into $\frac{4\,\pi^2}{T^2}$ C M, and $\frac{4\,\pi^2}{T^2}$ M N, or $\frac{4\,\pi^2}{T^2}\,x$ and $\frac{4\,\pi^2}{T^2}\,y$, in the directions of x and y. These are positive, because their action tends to increase the coordinates x and y of any particle. When the rotation is taken into account by this application of centrifugal force, we may consider the conditions of equilibrium the same as if there were no rotation. Then

$$X = -\frac{f\,x}{r^3} + \frac{4\,\pi^2}{T^2}\,x,$$

$$Y = -\frac{f\,y}{r^3} + \frac{4\,\pi^2}{T^2}\,y,$$

$$Z = -\frac{f\,z}{r}.$$

From which $d\,q = -\frac{f}{r^3}(x\,d\,x + y\,d\,y + z\,d\,z) + \frac{4\,\pi^2}{T^2}(x\,d\,x + y\,d\,y) = -\frac{f\,d\,r}{r^2} + \frac{2\,\pi^2}{T^2}\,d\,(x^2 + y^2)$, and $q = \frac{f}{r} + \frac{2\,\pi^2}{T^2}(x^2 + y^2)$. The equation to the external surface will, therefore, be

$$\frac{f}{r} + \frac{2\,\pi^2}{T^2}(x^2 + y^2) = C.$$

Let a be the equatorial radius; at a point in the equator $r = a$, and $x^2 + y^2 = a^2$; consequently $C = \frac{f}{a} + \frac{2\,\pi^2}{T^2}\,a^2$, and the equation is $\frac{f}{\sqrt{x^2 + y^2 + z^2}} + \frac{2\,\pi^2}{T^2} \times (x^2 + y^2) = \frac{f}{a} + \frac{2\,\pi^2}{T^2}\,a^2$. If the centrifugal force at the equator $\left(\text{or } \frac{4\pi^2}{T^2}a\right)$ be m × whole force there $\left(\text{or } m \times \left\{\frac{f}{a^2} - \frac{4\pi^2}{T^2}a\right\}\right)$, we have $\frac{4\pi^2}{T^2} = \frac{m}{1+m} \cdot \frac{f}{a^3}$; and the equation is $\frac{f}{\sqrt{x^2+y^2+z^2}} + \frac{mf}{2(1+m)a} \times$

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* If the strata of homogeneous Fluid were thick, ρ would be the same where p was the same, yet p would not be the same where ρ was the same. For between the upper and lower surface of the stratum ρ would not be altered, but p would have had all values between the value at the upper and that at the lower surface.

$$(x^2+y^2) = \frac{f}{a} + \frac{mf}{2(1+m)a} = \frac{2+3m}{2+2m} \cdot \frac{f}{a}, \text{ or}$$

$$\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{2+3m}{2+2m} \cdot \frac{1}{a} - \frac{m}{2+2m} \cdot \frac{x^2+y^2}{a^3}.$$

The polar semiaxis, or b, will be determined by making x and $y = 0$, and $z = b$; then $\frac{1}{b} = \frac{2+3m}{2+2m} \cdot \frac{1}{a}$, or $b = \frac{2+2m}{2+3m}a$. The proportion of the diameters is, therefore, $2+3m \,.\, 2+2m$, or the flattening at the Poles is $\frac{m}{2+3m}$ × the equatorial diameter. If these were the circumstances of the Earth, m (see MECHANICS, § XIV.) would be $\frac{1}{289}$, and the flattening would be $\frac{1}{581}$.

(13.) It must be observed that the determination of the form assumed by a revolving Fluid on the Principle of Gravitation is not so simple as the solution which we have just exhibited. In fact, one of the forces which acts on each particle (the attraction of every other particle) depends on the form; that is, one of the elements necessary to the determination of the form depends itself on the form. The extreme difficulty which this introduces will prevent us from determining, analytically, the form; and will oblige us to confine ourselves to showing, synthetically, that an elliptic surface satisfies the conditions of equilibrium. For a proof that the elliptic is the only form of equilibrium, we refer the reader to a paper by Legendre, in the *Mémoires de l'Académie*, for the year 1789.

On the Figure assumed by a Homogeneous Fluid.

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(14.) We shall show that the surface formed by the revolution of an ellipse will satisfy the condition of Art. (9.) (The condition of Art. (8.), as we have explained in (10.), is already satisfied.) For this purpose we must find the attraction of a spheroid on a point at its surface.

Attraction of a pyramid on a point in its vertex.

(15.) First it is convenient to premise the expression for the attraction of a pyramid on a point at its vertex. Let V P W, fig. 4, be a pyramid of which the base V W (to which the axis is perpendicular) is very small, its area being $= A$; call the length P V, l; then the area of the section at a distance P v (which we will call s) is $A \times \frac{s^2}{l^2}$; and if another section $v' w'$ be taken at the very small distance ds from the former, the solid content of the included frustrum is $A \,.\, \frac{s^2\,ds}{l^2}$; and if k be the density of the matter, (estimated as in (5.)), the mass of the matter in the frustrum is $Ak \,.\, \frac{s^2\,ds}{l^2}$. Now by the Principle of Gravitation the force of the attraction of this matter on P will be *proportional* to this quantity divided by the square of the distance, or *proportional* to $\frac{Ak}{l^2}ds$. It is *equal* to this quantity multiplied by some constant;* but as this constant must be the same in all parts of the investigation, we will omit it in every part; and consider the attraction of the matter in $v\,w'$ on the point P, as represented by $\frac{Ak}{\quad}ds$. The integral of this, or $\frac{Aks}{l^2} + C$, is the attraction of the pyramid; making it vanish when $s = 0$, and then making $s = l$, we have for the whole attraction $\frac{Ak}{l}$.

Spheroid divided into pyramids.

(16.) Now let C M, M N, N P, fig. 5, be the coordinates x, y, z, of a point P in the surface; let the plane A P B pass through C and N P; and refer the points of the surface to three coordinates, r, s, and z, of which r is parallel to A B, s perpendicular to A B, but in the plane A B D, and z the same as before. For the point P, therefore, r is C N, s is O, z is N P. Suppose the spheroid to be divided into wedges, (one of which is represented in fig. 6,) by planes passing through P N; let the plane P Q O make with the plane A P B an angle ϕ, and P q O an angle $\phi + d\phi$. Again suppose this wedge divided into pyramids by planes, passing through P Q q, P T t; and let Q P N or q P N be called θ; T P N or t P N, $\theta + d\theta$. Let P Q, the length of this pyramid, be l. Then the area of a section of the pyramid, perpendicular to P Q, is T $t \times t\,v$. But $t\,v = \text{P}t \times d\theta = l\,d\theta$ nearly; and T t (supposing $t\,w$ perpendicular to P N) is $t\,w \times d\phi = l \,.\, \sin\theta \,.\, d\phi$. The area of the section is, therefore, $l^2 \,.\, \sin\theta \,.\, d\phi \,.\, d\theta$; and consequently, by (15.), the attraction of the pyramid is $kl\sin\theta \,.\, d\phi \,.\, d\theta$.

(17.) This is the attraction of the pyramid in the direction of its length. We may resolve this into two parts parallel to P N and N L: the former will be $kl \sin\theta \,.\, \cos\theta \,.\, d\phi \,.\, d\theta$; the latter $kl \,.\, \sin^2\theta\, d\phi \,.\, d\theta$. This latter may be again resolved into two, one parallel to C A, and one perpendicular to it; their values are $kl\sin^2\theta\cos\phi \,.\, d\phi \,.\, d\theta$, and $kl\sin^2\theta \,.\, \sin\phi \,.\, d\phi \,.\, d\theta$.

(18.) The first of these is in a direction contrary to that which tends to impress such a motion as would increase the value of z; and, therefore,

Attraction of pyramid in direction of z

$$= -kl\sin\theta \,.\, \cos\theta \,.\, d\phi \,.\, d\theta$$

In a similar manner

Attraction of pyramid in direction of r

$$= -kl\sin^2\theta \,.\, \cos\phi \,.\, d\phi \,.\, d\theta.$$

And

Attraction of pyramid in direction of s

$$= +kl\sin^2\theta \,.\, \sin\phi \,.\, d\phi \,.\, d\theta.$$

The last of these we shall neglect. For when we

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* The constant would be different according as we estimated the effects of the attraction by the *pressure* which it caused by acting on a given mass at rest, or the *acceleration* which it created in its motion.

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take the attraction of the whole spheroid, there will be a pyramid of equal dimensions, and in a similar position, on the opposite side of the plane A P B, whose attraction in the direction of s will have the same magnitude as this, but will have a different sign: and, consequently, they will destroy each other. If we took the attraction of a part only of the spheroid, this reasoning would not hold.

(19.) Now to get the attraction of the wedge in fig. 6, in the direction of z, we must integrate $k\,d\varphi\,.\int l\,.\sin\theta\,.\cos\theta\,.\,d\theta$. For this purpose we must express l in terms of θ and φ. Let r', s', z' be the coordinates of Q; a and b the major and minor semiaxes of the generating ellipse. Then (Analytical Geometry, Art. 102.) $b^2(r'^2 + s'^2) + a^2 z'^2 = a^2 b^2$. Now $r' = r - t w\,.\cos\varphi = r - l\sin\theta\,.\cos\varphi$; $s' = t w\,.\sin\phi = l\sin\theta\,.\sin\varphi$; $z' = z - \mathrm{P}w = z - l\cos\theta$. Substituting these values $b^2(r^2 - 2\,l\,r\sin\theta\cos\varphi + l^2\sin^2\theta) + a^2(z^2 - 2\,l\,z\cos\theta + l^2\cos^2\theta) = a^2 b^2$. But as P is a point in the surface, and s there is 0, $b^2 r^2 + a^2 z^2 = a^2 b^2$. Subtracting this equation from the former, $b^2 l^2 \sin^2\theta + a^2 l^2\cos^2\theta - 2\,b^2\,l\,r\sin\theta\cos\varphi - 2\,a^2\,l\,z\cos\theta = 0$; whence

$$l = \frac{2\,b^2 r\sin\theta\cos\varphi + 2\,a^2 z\cos\theta}{b^2\sin^2\theta + a^2\cos^2\theta}.$$

All the integrations must be performed between the values $\theta = 0$, and $\theta =$ that value which makes $l = 0$; that is from $\theta = 0$, to $\theta =$ the arc whose tangent is $\frac{-a^2 z}{b^2 r\cos\phi}$.

(20.) The attraction of the wedge, therefore, in the direction of z

$$= k\,d\varphi\int\frac{2\,b^2 r\sin\theta\cos\varphi + 2\,a^2 z\cos\theta}{b^2\sin^2\theta + a^2\cos^2\theta}\sin\theta\,.\cos\theta\,.\,d\theta$$

$$= d\varphi\int\left\{\frac{-2\,b^2\,r\cos\varphi.\sin^2\theta\,.\cos\theta\,.\,d\theta}{b^2\sin^2\theta + a^2\cos^2\theta} - \frac{2\,a^2 z\cos^2\theta\,.\sin\theta\,.\,d\theta}{b^2\sin^2\theta + a^2\cos^2\theta}\right\}$$

Making $b^2 = a^2(1 - e^2)$, the general integral is

$$\cos\phi\,.\,d\phi\left\{\frac{1-e^2}{e^2}\sin\theta - \frac{1-e^2}{e^3}\log\sqrt{\frac{1+e\sin\theta}{1-e\sin\theta}}\right\}$$

$$+ 2\,k\,z\,d\phi\left\{\frac{1}{e^2}\cos\theta - \frac{\sqrt{1-e^2}}{e^3}\tan^{-1*}\left(\frac{e}{\sqrt{1-e^2}}\cos\theta\right)\right\}.$$

Taking this from $\theta = 0$ to $\theta = \tan^{-1}\left(\frac{-a^2 z}{b^2 r\cos\varphi}\right) = \sin^{-1}\left(\frac{-a^2 z}{\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi}}\right) = \cos^{-1}\left(\frac{b^2 r\cos\varphi}{\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi}}\right)$,

we have the attraction of the wedge

$$= 2\,k\,r\cos\varphi\,d\varphi\left\{-\frac{1-e^2}{e^2}\,.\,\frac{a^2 z}{\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi}} - \frac{1-e^2}{e^3}\log\frac{\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi} - a^2 e z}{\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi} + a^2 e z}\right\}$$

$$+ 2\,k\,z\,d\varphi\left\{\frac{b^2 r\cos\varphi}{e^2\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi}} - \frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{b^2 e r\cos\varphi}{\sqrt{1-e^2}\,\sqrt{a^4 z^2 + b^4 r^2\cos^2\varphi}} - \frac{1}{e^2} + \frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{e}{\sqrt{1-e^2}}\right\}.$$

Now it may be observed, that all the terms, except the two last, are of such a nature, that upon increasing φ by a semi-circumference, or π, their value is the same but the sign is different, and the addition of two such expressions, one corresponding to the arc φ, and the other to the arc $\pi + \varphi$, will therefore destroy both. As we shall shortly integrate with respect to φ, from $\varphi = 0$ to $\varphi = 2\,\pi$, we shall in reality effect that addition. We may as well then reject, at once, these useless terms; and thus we find for the effective attraction of the wedge in the direction of z,

$$2\,k\,z\,d\varphi\left(-\frac{1}{e^2} + \frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{e}{\sqrt{1-e^2}}\right) \text{ or } 2\,z\,d\varphi\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1}{e^2}\right).$$

(21.) To find the attraction of the whole spheroid in the direction of z, we must integrate this with respect to φ, from $\varphi = 0$ to $\varphi = 2\,\pi$, and we get $4\,\pi\,k\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1}{e^2}\right)\,.\,z$.

(22.) The attraction of the wedge in the direction of r is

$$k\cos\varphi\,d\varphi\int\left\{\frac{-2\,b^2\,r\cos\varphi\,.\sin^3\theta\,.\,d\theta}{b^2\sin^2\theta + a^2\cos^2\theta} - \frac{2\,a^2 z\sin^2\theta\,.\cos\theta\,.\,d\theta}{b^2\sin^2\theta + a^2\cos^2\theta}\right\}.$$

The general integral is

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* We use the expression $\tan^{-1}\left(\frac{e}{\sqrt{1-e^2}}\cos\theta\right)$ to denote the arc whose tangent is $\frac{e}{\sqrt{1-e^2}}\cos\theta$; and, similarly, for other functions.

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$$2\,k\,r\cos^2\phi\,d\,\phi\left\{-\frac{1-e^2}{e^2}\cos\theta+\frac{\sqrt{1-e^2}}{e^3}\,.\,\tan^{-1}\frac{e\cos\theta}{\sqrt{1-e^2}}\right\}$$

$$+\,2\,k\,z\cos\phi\,d\,\varphi\left\{\frac{1}{e}\sin\theta-\frac{1}{e^3}\log\sqrt{\frac{1+e\sin\theta}{1-e\sin\theta}}\right\}.$$

On taking this between the limits $\theta=0$, $\theta=\tan^{-1}\,.\,\frac{-a^2z}{b^2r\cos\varphi}$, and rejecting (as in (20.)) those terms which ultimately produce no effect, we have for the effective attraction of the wedge in the direction of r

$$2\,k\,r\cos^2\varphi\,.\,d\,\varphi\left\{\frac{1-e^2}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right\}.$$

(23.) To find the attraction of the whole spheroid in the direction of r we must integrate this with respect to φ from $\varphi=0$ to $\varphi=2\,\pi$: we thus get $2\,\pi\,k\left(\frac{1-e^2}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right).\,r.$

(24.) Besides these forces of attraction there is the centrifugal force, which (as in (12.)) is $\frac{4\,\pi^2}{T^2}\,r$ in the direction of r. The whole force, therefore, in the direction of r is $2\,\pi\,k\left(\frac{1-e^2}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e+\frac{2\,\pi}{T^2\,k}\right)r.$

(25.) Resolving the force in the direction of r or C N, fig 5, into two in the directions of C M and M N, or x and y, we find that they are the same multiples of x and y which the force in the direction of r is of r.

Expressions for the forces on a particle at the surface of the spheroid.

Thus we have the following expressions:

$$X=2\,\pi\,k\left(\frac{1-e^2}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e+\frac{2\,\pi}{T^2k}\right)x=2\,\pi\,.\,k\,A\,.\,x.$$

$$Y=2\,\pi\,k\left(\frac{1-e^2}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e+\frac{2\,\pi}{T^2\,k}\right)y=2\,\pi\,k\,A\,.\,y.$$

$$Z=4\,\pi\,k\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e-\frac{1}{e^2}\right)z=4\,\pi\,.\,k\,.\,B\,z.$$

The elliptic form proper for equilibrium.

(26.) The condition in (8.) now requires that the external surface be defined by the equation $X\,d\,x+Y\,d\,y+Z\,d\,z=0$, or $2\,\pi\,k\,A\,x\,d\,x+2\,\pi\,k\,A\,y\,d\,y+4\,\pi\,k\,B\,z\,d\,z=0$, or (integrating and dividing by $\pi\,k$) $A\,(x^2+y^2)+2\,B\,z^2=C$. But the equation to the surface has already been assumed to be $b^2\,(x^2+y^2)+a^2z^2=a^2b^2$. (ANALYTICAL GEOMETRY, Art. 102.) If these equations are compatible, the elliptic form is proper for equilibrium, and a comparison of the constants will determine the degree of ellipticity. Now the form of the equations is precisely the same, and, therefore, they will be compatible if the proportion of the coefficients of x^2+y^2 and z^2 is the same (C being undetermined.) This condition requires that $\frac{A}{2\,B}=\frac{b^2}{a^2}=1-e^2$, or $A-2\,B\,(1-e^2)=0$, or

Equation for finding the ellipticity.

$$\frac{3\,(1-e^2)}{e^2}-\frac{3-2\,e^2}{e^3}\sqrt{1-e^2}\,.\,\sin^{-1}e+\frac{2\,\pi}{T^2\,k}=0,$$

an equation from which, when T^2k is known, e can be found.

(27.) If $\sin^{-1}e=f$, the equation may be put under this form, (which is more convenient for calculation,)

$$f\,.\cot f\,(1+3\cot^2 f)-3\cot^2 f-\frac{2\,\pi}{T^2\,k}=0.$$

If a curve be constructed of which the abscissa is f, and the ordinate the equation deprived of its last term, it will be found to resemble A E F B, fig. 7. If we draw a straight line, C D, parallel to A B, and at a perpendicular distance equal to $\frac{2\,\pi}{T^2\,k}$, the points in which it cuts the curve will have for abscissæ the values of f, which satisfy the equation. From this it appears that if $\frac{2\,\pi}{T^2\,k}$ does not exceed a certain limit (.22467), there are two values of f and, consequently, of e, or two forms of ellipsoid, with which equilibrium is possible; if it exceed that limit, equilibrium is not possible.

(28.) We have spoken here as if the velocity of rotation was independent of the form of the spheroid. This would not be the case, if in consequence of want of adaptation of the forces the form of the Fluid should change. Nothing, therefore, can be inferred with regard to the stability of the equilibrium. To investigate this we must make use of the following theorem of Mechanics, (known as *the principle of the conservation of areas*) If any bodies revolve round a centre, and are acted on only by their mutual attraction and by forces directed to the centre, the

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sum of the products of the mass of each by the projection (on a given plane) of the area which it describes round that centre is constant. In the case before us the particles of the Fluid are acted on only by their mutual attraction, and therefore this theorem applies. Now (without making a complete investigation) it is easily seen that the sum of the products of the mass by the area described does not depend on the length of the axis of revolution, but is proportional to the square of the major axis directly, and the time of revolution inversely. That is, $C' = \frac{a^2}{T}$, or $T = \frac{a^2}{C'}$. And since the quantity of matter is given, $C'' = a^2 b = a^3\sqrt{1-e^2}$; whence $a^3 = \frac{C''}{\overline{1-e^2}|^{\frac{1}{2}}}$,

$$a = \frac{C''^{\frac{2}{3}}}{\overline{1-e^2}|^{\frac{1}{3}}};\ \text{and}\ T = \frac{C''^{\frac{2}{3}}}{C'.\overline{1-e^2}|^{\frac{1}{3}}};\ \frac{2\pi}{T^2} = \frac{2\pi . C'^2}{C''^{\frac{4}{3}}}\overline{1-e^2}|^{\frac{2}{3}} = D.\overline{1-e^2}|^{\frac{2}{3}}$$ for abbreviation. Substituting this in the equation of (27.), and putting $\overline{\cos f}|^{\frac{4}{3}}$ for $\overline{1-e^2}|^{\frac{2}{3}}$, we have

$$f.\cot f.(1 + 3\cot^2 f) - 3\cot^3 f - \frac{D}{k}\overline{\cos f}|^{\frac{4}{3}} = 0,$$

$$\text{or}\ \frac{1}{\overline{\cos f}|^{\frac{4}{3}}}\{f\cot f.(1 + 3\cot^2 f) - 3\cot^2 f\} - \frac{D}{k} = 0.$$

If we construct a curve of which f is the abscissa, and this expression deprived of its last term the ordinate, it will be found to resemble fig. 8, the ordinate becoming infinite when $f = 90°$, or $e = 1$. And, therefore, if we draw C D parallel to A B at the distance $\frac{D}{k}$, it will cut the curve at one point and at no more. It appears, then, that the equation above admits in all cases of one solution and of no more than one; and, consequently, if a fluid mass is revolving round an axis, there is always one elliptic spheroid, and only one, in which it can remain in equilibrium. It will very easily be seen that if it be put in the form of a less flattened spheroid, its angular velocity must be increased, and therefore it will return to the form of the spheroid of equilibrium; if it be made too flat, the angular velocity will be diminished and it will also return to that form. Hence the equilibrium is always stable. We have given this proposition, perhaps, more attention than it deserves, because the preceding proposition has sometimes been stated in such a form as to imply that a mass of Fluid might have two forms of equilibrium, and might have one form of unstable equilibrium; conclusions which are quite incorrect.

A given mass, in motion, with a given velocity, has only one form of equilibrium. The equilibrium is stable.

(29.) With regard to the Earth considered as a homogeneous Fluid, we do not immediately know k, but we know the force of Gravity at the Equator, and we can calculate the centrifugal force there, and, consequently, know their ratio; let this be m. The whole Gravity at the Equator, by (23.), making $r = a$, and changing the sign, is $2\pi k.\left(-\frac{1-e^2}{e^2} + \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{2\pi}{T^2 k}\right).a$; the centrifugal force is $2\pi k.\frac{2\pi}{T^2 k}a$;

$$\text{hence}\ \frac{2\pi}{T^2 k} = m\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2} - \frac{2\pi}{T^2 k}\right),\ \text{and}\ \frac{2\pi}{T^2 k} = \frac{m}{1+m}\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right)$$

$$= \frac{m}{1+m}\left(\frac{f.\cos f}{\sin^3 f} - \cot^2 f\right).$$

The equation of (27.) is changed by this substitution into the following:

$$\frac{f\cot f(1 + 3\cot^2 f) - 3\cot^2 f}{f\cot f\,(1 + \cot^2 f) - \cot^2 f} - \frac{m}{1+m} = 0.$$

If we construct a curve whose abscissa is f, and ordinate the first term of this equation, it will have the form of fig. 9, the ordinate being $= 1$ when $f = 90°$. If the line C D be drawn parallel to A B, and at the distance $\frac{m}{1+m}$, since $\frac{m}{1+m}$ is less than 1, it will necessarily cut this curve at some point E, and only at one point. One corresponding value of f, and consequently one of e, and no more than one, may therefore be found for any given value of m; that is, the ratio of the centrifugal force at the Equator to Gravity there being known, there is but one elliptic form, which will be a form of equilibrium.

Only one form corresponding to a given ratio of centrifugal force to Gravity.

Approximate solution of the equation.

(30.) Where m is so small as in the case of the Earth $\left(\frac{1}{289}\right)$, we may be satisfied with a first approximation. Let $a = b(1 + \epsilon)$: ϵ is what is generally called the ellipticity. Then l in (19.) is

$$\frac{2r\sin\theta.\cos\phi + 2z(1 + \epsilon)^2\cos\theta}{\sin^2\theta + (1 + \epsilon)^2\cos^2\theta};$$

and if we neglect the square of ϵ, this is

$$\frac{2r\sin\theta.\cos\phi + 2z\cos\theta + 4\epsilon.z\cos\theta}{1 + 2\epsilon\cos^2\theta} = 2r\sin\theta.\cos\phi + 2z\cos\theta + \epsilon(4z\sin^2\theta\cos\theta - 4r\sin\theta.\cos^2\theta\cos\phi),$$

ure of e Earth.

Physical Theory. Earth supposed homogeneous.

The attraction of the wedge in the direction of z is, therefore,
$-k d\phi \int \{ 2 r \cos\phi . \sin^2\theta . \cos\theta . d\theta + 2 z \cos^2\theta . \sin\theta . d\theta + \epsilon (4 z . \sin^3\theta . \cos^2\theta . d\theta - 4 r \cos\phi . \sin^2\theta \cos^3\theta\, d\theta) \}$.
The general integral is

$$- k\, d\phi \left\{ \frac{2}{3} r \cos\phi . \sin^3\theta - \frac{2}{3} z \cos^3\theta + \epsilon \left(4 z . \overline{\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}} - 4 r \cos\phi . \overline{\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5}} \right) \right\};$$

taking this between the same limits as in (20.), and selecting in the same way the effective terms, we find the effective attraction of the wedge $= - k\, d\phi \left\{ \frac{2}{3} z + \frac{8}{15} \epsilon z \right\}$. Integrating this from $\phi = 0$ to $\phi = 2\pi$, we have for the attraction of the spheroid in the direction of z,

$$- 2\pi k \left\{ \frac{2}{3} z + \frac{8}{15} \epsilon z \right\} = - \frac{4\pi}{3} k \left(1 + \frac{4}{5} \epsilon \right) z.$$

(31.) The attraction of the wedge in the direction of r is
$-k \cos\phi . d\phi \int \{ 2 r \cos\phi \sin^3\theta . d\theta + 2 z \sin^2\theta . \cos\theta . d\theta + \epsilon (4 z \sin^4\theta \cos\theta\, d\theta - 4 r \cos\phi \sin^3\theta \cos^2\theta . d\theta) \}$;
The general integral is

$$- k \cos\phi . d\phi \left\{ 2 r \cos\phi \times \overline{\frac{\cos^3\theta}{3} - \cos\theta} + 2 z \frac{\sin^3\theta}{3} + \epsilon \left(4 z \frac{\sin^5\theta}{5} - 4 r \cos\phi . \overline{\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}} \right) \right\}$$

The effective part found in the same manner is

$$- k \cos\phi . d\phi \left\{ \frac{4}{3} r \cos\phi - \frac{8}{15} \epsilon r \cos\phi \right\}.$$

Integrating this from $\phi = 0$ to $\phi = 2\pi$, the attraction of the spheroid in the direction of r is

$$- \pi k \left\{ \frac{4}{3} r - \frac{8}{15} \epsilon r \right\} = - \frac{4\pi}{3} k \left(1 - \frac{2}{5} \epsilon \right) . r.$$

Expressions for the attraction of a point in the surface when the ellipticity is small.

(32.) Applying the centrifugal force $= \frac{4\pi^2}{T^2} r$, and resolving the force in the direction of r into two in the direction of x and y, we get the following expressions,

$$X = - \frac{4\pi}{3} k \left(1 - \frac{2}{5} \epsilon - \frac{3\pi}{k T^2} \right) x.$$

$$Y = - \frac{4\pi}{3} k \left(1 - \frac{2}{5} \epsilon - \frac{3\pi}{k T^2} \right) y.$$

$$Z = - \frac{4\pi}{3} k \left(1 + \frac{4}{5} \epsilon \right) z.$$

The equation to the external surface must, therefore, be $\left(\text{omitting } - \frac{4\pi}{B} k \right)$

$$\left(1 - \frac{2}{5} \epsilon - \frac{3\pi}{k T^2} \right) (x^2 + y^2) + \left(1 + \frac{4}{5} \epsilon \right) z^2 = C.$$

But it has been assumed to be $b^2 (x^2 + y^2) + b^2 . (1 + 2\epsilon) . z^2 = b^4 (1 + 2\epsilon)$. Making the proportion of the coefficients of $x^2 + y^2$ and z^2 the same,

$$1 + 2\epsilon = \frac{1 + \frac{4}{5}\epsilon}{1 - \frac{2}{5}\epsilon - \frac{3\pi}{k T^2}}, \text{ or } 1 + \frac{8}{5}\epsilon - \frac{3\pi}{k T^2} = 1 + \frac{4}{5}\epsilon; \text{ whence } \epsilon = \frac{15\pi}{k T^2}.$$

(33.) Using m to signify the same thing as in (29.), $m = \dfrac{\frac{4\pi^2}{T^2}(1 + \epsilon)}{\frac{4\pi}{3} k \left(1 - \frac{2}{5}\epsilon \right) (1 + \epsilon)\, b} = \dfrac{12\pi}{k T^2}$ nearly; whence $\frac{\pi}{k T^2} = \frac{m}{12}$, and $\epsilon = \frac{5}{4} m$. If the Earth were a homogeneous fluid, m being $\frac{1}{289}$, ϵ would be $\frac{5}{4} \times \frac{1}{289} = \frac{1}{230}$: and the ratio of the axes would be $1 : 1 + \epsilon$ or $230 : 231$.

Expression for Gravity.

(34.) We have nothing remaining but to investigate the magnitude of Gravity (including the effects of attraction and centrifugal force) at any point of the surface. Since the forces X, Y, Z are at right angles to each other, the force compounded of them all is $\sqrt{X^2 + Y^2 + Z^2}$, (MECHANICS, § VI.) which, taking the expressions in (25.) $= 2\pi k \sqrt{A^2 (x^2 + y^2) + 4 B^2 z^2}$. But by (26.) $\frac{A}{2B} = \frac{b^2}{a^2}$: whence the force

Figure of the Earth.

Physical Theory. Earth supposed homogeneous.

$= 2\pi k\sqrt{\frac{b^4}{a^4}\,4\,B^2(x^2+y^2)+4\,B^2z^2} = 4\pi k\,B\sqrt{\frac{b^4}{a^4}r^2+z^2}$. Let fig. 10 represent the section of the spheroid through the point P and the axis; draw P R the normal at P. Then (Conic Sections, Art. 34.) $RN = \frac{b^2}{a^2}CN = \frac{b^2}{a^2}r$; and $PN = z$; therefore $PR = \sqrt{\frac{b^4}{a^4}r^2+z^2}$, and, consequently, the force is proportional to P R.

Gravity expressed in terms of the latitude.

(35.) An expression in terms of the angle P R N (or the latitude of P) may be thus obtained. Let $PRN = \lambda$; $PR = n$; P N or $z = n\sin\lambda$; C N or $r = \frac{a^2}{b^2}RN = \frac{a^2}{b^2}n\cos\lambda$; and substituting these in the equation to the ellipse, $r^2b^2 + a^2z^2 = a^2b^2$, it becomes $n^2\left(\frac{a^4}{b^2}\cos^2\lambda + a^2\sin^2\lambda\right) = a^2b^2$, whence $n = \frac{b^2}{\sqrt{a^2\cos^2\lambda + b^2\sin^2\lambda}}$. If $a = b\,(1+\epsilon)$ and powers of ϵ higher than the first be neglected, this becomes $n = \frac{b}{\sqrt{1+2\epsilon\cos^2\lambda}} = b\,(1-\epsilon\cos^2\lambda) = b\,.\,\overline{1-\epsilon}\,.\,(1+\epsilon\sin^2\lambda)$: and the force is, therefore, proportional to $1+\epsilon\sin^2\lambda$, or $1+\frac{5}{4}m\,.\,\sin^2\lambda$.

On the Figure assumed by a Heterogeneous Fluid.

(36.) The difficulties attending this investigation are so considerable that it would be useless to attempt a complete solution. We shall confine ourselves to an approximation including all terms depending on the first powers of the ellipticities: the reader who may desire to see an approximation including the terms depending on the squares of the ellipticities, and obtained by a different method, is referred to the *Phil. Trans.* for the year 1826.

Method of proceeding in the investigation for a heterogeneous Fluid.

(37.) We shall assume that the form of each homogeneous stratum is elliptical, and that the ellipticity is different for the different strata; so that if we consider the extent of any homogeneous stratum to be sensible, it will be bounded by surfaces of different ellipticities. We shall then investigate the forces produced by each stratum, and by the whole, upon a point in any position: adding the centrifugal force, we shall form the expressions for X, Y, Z; and as we know (Art. 9.) that $X\,dx + Y\,dy + Z\,dz$ is a complete differential, we shall only assume that $X\,dx + Y\,dy + Z\,dz = 0$ is the equation to the surface limiting every homogeneous stratum. We shall then show that this equation agrees with that assumed in the first instance, provided a certain condition be satisfied; from which we shall infer at once the most important deductions relating to the Figure of the Earth.

(38.) Our first object then must be, to investigate the attraction of a shell, bounded by two spheroidal surfaces of different ellipticities, on a point within it or without it. As this is the same as the difference of attractions of two spheroids whose axes and ellipticities are different, we shall begin with investigating the attraction of a spheroid; and first on a point within it.

(39.) It is convenient to premise, that if two pyramids of different lengths have equal solid angles, (that is, if their sections at a given distance from the vertex be equal,) their attractions are proportional to their lengths. For the attraction of one such pyramid (15.) is $\frac{A\,k}{l}$; that of another $\frac{A'\,k}{l'}$; their proportion is $\frac{A\,l'}{A'\,l}$; but if their solid angles be equal, $\frac{A}{A'} = \frac{l^2}{l'^2}$; therefore the proportion of the attractions is $\frac{l}{l'}$.

Attraction of a homogeneous spheroid on a point within it.

(40.) Now let fig. 11 be constructed in the same manner as fig. 5, except that P, instead of being at the surface of the spheroid, is in the interior; and produce the planes which are the sides of the pyramid Q P to meet the other surface about the point Q. The attraction of the pyramids P Q, P Q′, are in opposite directions, and proportional to their lengths, and, consequently, their combined attraction is that of a pyramid whose length is P Q — P Q′, and whose solid angle is the same. Now, using the same notation as in (19.), and forming the same equation, we have

$$l^2(b^2\sin^2\theta + a^2\cos^2\theta) - 2\,l\,(a^2z\cos\theta + b^2r\sin\theta\,.\,\cos\phi) = a^2b^2 - a^2z^2 - b^2r^2;$$

whence

$$l = \frac{a^2z\cos\theta + b^2r\sin\theta\cos\phi}{b^2\sin^2\theta + a^2\cos^2\theta} \pm \frac{1}{b^2\sin^2\theta + a^2\cos^2\theta}\ \{(a^2b^2 - a^2z^2 - b^2r^2)(b^2\sin^2\theta + a^2\cos^2\theta)$$
$$+ (a^2z\cos\theta + b^2r\sin\theta\cos\phi)^2\}^{\frac{1}{2}}$$
$$= \frac{a^2z\cos\theta + b^2r\sin\theta\cos\phi}{b^2\sin^2\theta + a^2\cos^2\theta} \pm \sqrt{R}\ \text{suppose.}$$

One of these roots represents P Q; the other represents P Q′ estimated in the direction of P Q, that is, if its sign be changed it represents the length of P Q′. Thus

$$PQ = \sqrt{R} + \frac{a^2z\cos\theta + b^2r\sin\theta\cos\phi}{b^2\sin^2\theta + a^2\cos^2\theta},$$

Physical Theory. Earth supposed heterogeneous.

$$\mathrm{PQ}' = \sqrt{\mathrm{R}} - \frac{a^2 z \cos\theta + b^2 r \sin\theta . \cos\phi}{b^2 \sin^2\theta + a^2 \cos^2\theta},$$

whence

$$\mathrm{PQ} - \mathrm{PQ}' = \frac{2 a^2 z \cos\theta + 2 b^2 r \sin\theta \cos\phi}{b^2 \sin^2\theta + a^2 \cos^2\theta}$$

These two pyramids are now to be treated as if they were a single pyramid of length $\mathrm{PQ} - \mathrm{PQ}'$; and hence we shall have (calling the density ρ)

$$\text{force in direction of } z = -\rho\,(\mathrm{PQ} - \mathrm{PQ}') \sin\theta . \cos\theta . d\phi . d\theta,$$

$$\text{force in direction of } r = -\rho\,(\mathrm{PQ} - \mathrm{PQ}') \sin^2\theta . \cos\phi . d\phi . d\theta.$$

The force in the direction of s may be neglected as before.

(41.) Let $a = b\,(1 + e)$; and neglect e^2, e^3, &c. Then

$$\mathrm{PQ} - \mathrm{PQ}' = 2 r \cos\phi \sin\theta + 2 z \cos\theta + e\,(4 z \sin^2\theta \cos\theta - 4 r \cos\phi . \sin\theta . \cos^2\theta).$$

The force in the direction of z

$$= -\rho\, d\phi \{ 2 r \cos\phi . \sin^2\theta . \cos\theta . d\theta + 2 z \cos^2\theta . \sin\theta\, d\theta + e\,(4 z \sin^3\theta . \cos^2\theta . d\theta - 4 r . \cos\phi \sin^2\theta . \cos^3\theta . d\theta \}.$$

Integrating this from $\theta = 0$ to $\theta = \pi$, we have for the attraction of the whole double wedge L Q O Q′ L,

$$-\rho\, d\phi \left\{ \frac{4 z}{3} + \frac{16}{15} e z \right\}.$$

Integrating this from $\phi = 0$ to $\phi = \pi$, we have, attraction of whole spheroid in the direction of z

$$= -\frac{4\pi}{3} \rho \left(1 + \frac{4}{5} e\right) z.$$

(42.) For the force in the direction of r we must in the same way integrate

$$-\rho \cos\phi . d\phi . \{ 2 r \cos\phi . \sin^3\theta\, d\theta + 2 z \sin^2\theta . \cos\theta\, d\theta + e\,(4 z \sin^4\theta . \cos\theta\, d\theta - 4 r \cos\phi . \sin^3\theta . \cos^2\theta . d\theta) \},$$

and we find that the attraction of the whole spheroid in the direction of r

$$= -\frac{4\pi}{3} \rho \left(1 - \frac{2}{5} e\right) r.$$

These expressions are independent of the dimensions of the spheroid, provided that it contain the point, and that the ellipticity have the given value e. If, then, two spheroidal surfaces of equal ellipticity but of different dimensions surround the point, the attraction of the matter included between them, upon that point, is nothing. But this will not apply to our present purpose.

Attraction of a homogeneous spheroidal shell on a point within it.

(43.) Now let the spheroid in fig. 11 be surrounded by another spheroid indefinitely near, and of which the ellipticity is $e + de$. Then if the matter whose density is ρ extend to the first spheroid, its attraction on P in the direction of z is $-\frac{4\pi}{3}\rho z - \frac{16\pi}{15}\rho e z$. If it extend to the second spheroid, its attraction is $-\frac{4\pi}{3}\rho z - \frac{16\pi}{15}\rho e z - \frac{16\pi}{15}\rho z . de$. The difference, or the attraction of the stratum included between them, is $-\frac{16\pi}{15} z \rho\, de$. In the same manner, the attraction of the same stratum in the direction of $r = \frac{\quad}{15} r . \rho\, de$.

Attraction of a series of spheroidal shells of different densities on a point within them.

(44.) Now, if we suppose a great number of such thin strata to rest one upon another, as in fig. 12, it will be evident that their attraction in the directions of z and r is to be found by integrating the expressions in the last article. As ρ and e will both be known from a knowledge of the polar semiaxis of the spheroid to which they belong, that is as ρ and e are functions of b, we can put the attractions in this form:

$$\text{Attraction of assemblage of strata in direction of } z = -\frac{16\pi}{15} z \int \rho \frac{de}{db} . db.$$

$$\text{Attraction in direction of } r = \frac{8\pi}{15} r \int \rho \frac{de}{db} . db.$$

The function $\int \rho \frac{de}{db} db$ we shall denote by $\chi(b)$; and if b' be the axis of the inner stratum (which, however, must include or contain P), and b'' that of the exterior stratum, the value of the integral $\int \rho \frac{de}{db} db$ in the present instance will be $\chi(b'') - \chi(b')$; and then the attractions of the assemblage of strata in the directions of z and r are respectively $-\frac{16\pi}{15} z\,(\chi(b'') - \chi(b'))$, and $\frac{8\pi}{15} r\,(\chi(b'') - \chi(b'))$.

(45.) The attraction of a spheroid on a point without it is not found so directly. It will be shown to bear a known proportion to the attraction of another spheroid on a point in a different situation within it; from which, with the assistance of the expressions obtained in (41.) and (42.), its value can be found. For this purpose we must investigate the attraction of a small prism in the direction of its length on a point without it. Let A B

Attraction of a prism on a point without it.

fig. 13, be a prism, the area of whose section is very small and $= \text{A}$; upon the axis produced draw P C, a perpendicular; let $\text{CA} = a$, $\text{CB} = b$, $\text{PC} = c$, $\text{CQ} = x$; at the distances x and $x + dx$, let sections be made; the solid content intercepted is $\text{A} . dx$; and if ρ be the density, the quantity of matter is $\rho . \text{A} . dx$; the distance from P is $\sqrt{c^2 + x^2}$, and therefore the attraction is $\frac{\rho . \text{A} . dx}{c^2 + x^2}$ in the direction P Q, the resolved part of this in the direction C Q is $\frac{\rho . \text{A} . dx}{c^2 + x^2} \times \frac{x}{\sqrt{c^2 + x^2}} = \rho . \text{A} . \frac{x\,dx}{(c^2 + x^2)^{\frac{3}{2}}}$; by integration we get the attraction of the prism $=$ constant $- \frac{\rho . \text{A} .}{\sqrt{c^2 + x^2}}$; and taking this between the limits $x = a$, $x = b$, it is

$$\rho . \text{A} . \left(\frac{1}{\sqrt{c^2 + a^2}} \frac{1}{\sqrt{c^2 + b^2}}\right) = \rho . \text{A} . \left(\frac{1}{\text{PA}} - \frac{1}{\text{PB}}\right)$$

(46.) Now, in fig. 14, let A B be an ellipsoid, of which the three semiaxes are a, b, c; and let the coordinates of the point P be r, s, z. Parallel to the axis of b and c take two sections whose distances from C are f and $f + df$; and divide the included space into prisms by sections parallel to the axes of a and c, of which two are at the distances g and $g + dg$ from C B; let TT′, the length of that prism, be $2h$. Then the area of its section is $df . dg$; also $\text{PT} = \sqrt{\overline{r - f}|^2 + \overline{s - g}|^2 + \overline{z - h}|^2}$, and $\text{PT}' = \sqrt{\overline{r - f}|^2 + \overline{s - g}|^2 + \overline{z + h}|^2}$, whence (by the last article) the attraction of the prism on P in the direction PN′ is

$$\rho . df . dg \left\{ \frac{1}{\sqrt{\overline{r - f}|^2 + \overline{s - g}|^2 + \overline{z - h}|^2}} - \frac{1}{\sqrt{\overline{r - f}|^2 + \overline{s - g}|^2 + \overline{z + h}|^2}} \right\}.$$

This is to be integrated, first with respect to g from $g = -\text{RV}$ to $g = \text{RW}$, which will give the attraction of the slice, and then with respect to f from $f = -a$ to $f = +a$, which will give the attraction of the whole ellipsoid on P in the direction PN′.

Expression to be integrated for the attraction of a homogeneous spheroid on a point without it.

(47.) Let $f = ma$, $g = nb$, $h = pc$. The general equation to the ellipsoid is $\frac{f^2}{a^2} + \frac{g^2}{b^2} + \frac{h^2}{c^2} = 1$, or $m^2 + n^2 + p^2 = 1$; and at the points V and W, $h = 0$, and $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$, or $m^2 + n^2 = 1$, whence the extreme values of n are $\pm \sqrt{1 - m^2}$. Also $df = a\,dm$, $dg = b\,dn$. Hence the attraction of the ellipsoid is

$$\rho\, a\, b \int dm \int dn \left\{ \frac{1}{\sqrt{\overline{r - ma}|^2 + \overline{s - nb}|^2 + \overline{z - pc}|^2}} - \frac{1}{\sqrt{\overline{r - ma}|^2 + \overline{s - nb}|^2 + \overline{z + pc}|^2}} \right\},$$

where the first integration is to be made from $n = -\sqrt{1 - m^2}$ to $n = +\sqrt{1 - m^2}$, and the second from $m = -1$ to $m = +1$.

(48.) In the same manner the attraction of an ellipsoid whose semiaxes are a', b', c' in the same direction on a point whose coordinates are r', s', z', is

$$\rho . a' . b' . \int dm \int dn \left\{ \frac{1}{\sqrt{\overline{r' - ma'}|^2 + \overline{s' - nb'}|^2 + \overline{z' - pc'}|^2}} - \frac{1}{\sqrt{\overline{r' - ma'}|^2 + \overline{sn - b'}|^2 + \overline{z + pc'}|^2}} \right\},$$

where the integrals are to be taken between the same limits as before.

(49.) Now the terms within the brackets will be precisely equal in the two expressions, if the following equations hold,

$$a' r' = a r, \quad \text{or} \quad \frac{r'}{a} = \frac{r}{a'} : \text{ let this } = \mu.$$

$$b' s' = b s, \quad \text{or} \quad \frac{s'}{b} = \frac{s}{b'} \qquad = \nu.$$

$$c' z' = c z, \quad \text{or} \quad \frac{z'}{c} = \frac{z}{c'} \qquad = \pi.$$

Relation between the attraction of a spheroid on a point without it, and the attraction of another spheroid on a point within it.

Also, $m^2 a'^2 + n^2 b'^2 + p^2 c'^2 + r'^2 + s'^2 + z'^2 = m^2 a^2 + n^2 b^2 + p^2 c^2 + r^2 + s^2 + z^2$; which equation, putting $1 - m^2 - n^2$ for p^2 and $a - \mu^2 - \nu^2$ for π^2, will become

$$c'^2 + m^2 (a'^2 - c'^2) + n^2 (b'^2 - c'^2) + a c^2 + \mu^2 (a^2 - c^2) + \nu^2 (b^2 - c^2)$$
$$= c^2 + m^2 (a^2 - c^2) + n^2 (b^2 - c^2) + a c'^2 + \mu^2 (a'^2 - c'^2) + \nu^2 (b'^2 - c'^2).$$

And as m and n are variable, this can only be satisfied by taking

$$a^2 - c^2 = a'^2 - c'^2$$
$$b^2 - c^2 = b'^2 - c'^2$$
$$a = 1.$$

The last gives $\mu^2 + \nu^2 + \pi^2 = 1$; that is $\frac{r^2}{a'^2} + \frac{s^2}{b'^2} + \frac{z^2}{c'^2} = 1.$

When all these equations are satisfied, the terms within the brackets will be the same, and as the integrals are to be taken between the same limits, the attraction of the first spheroid on the first point perpendicular to a principal section, is to the attraction of the second spheroid on the second point perpendicular to the corresponding section as $a\,b$ to $a'\,b'$, that is as the area of the section of the first to the area of the section in the second.

(50.) The utility of this proposition arises from this circumstance, that if the first point be *without* the first ellipsoid, the second point will be *within* the second ellipsoid, the attraction of which has already been found. For $a^2 - c^2 = a'^2 - c'^2$, or $a^2 - a'^2 = c^2 - c'^2$; similarly $b^2 - b'^2 = c^2 - c'^2$; hence the spheroids will not cut one another (if concentric and similarly situated.) Also since $\mu^2 + \nu^2 + \pi^2 = 1$, or $\frac{r'^2}{a^2} + \frac{s'^2}{b^2} + \frac{z'^2}{c^2} = 1$, the second point is on the surface of the first spheroid; and similarly the first point is on the surface of the second spheroid; therefore, the surface of the second spheroid is without that of the first, and, therefore, the second point is completely surrounded by the second surface.

(51.) This proposition we have demonstrated in its most general form, because the demonstration is thus made somewhat easier. We shall now return to the case which more immediately concerns us, where the semi-axes of the attracting spheroid are b, $b\,(1 + e)$, and $b\,(1 + e)$, and the coordinates of the attracted point r and z, s being 0.

(52.) Let b', $b'\,(1 + e')$, r', and z', be the corresponding quantities in the imaginary spheroid and attracted point; then we must have $b'^2\,(1 + e')^2 - b'^2 = b^2\,(1 + e)^2 - b^2$; or neglecting e^2, e'^2, $b'^2\,e' = b^2\,e$, and $e' = \frac{b^2}{b'^2}\,e$. Also $\frac{z^2}{b'^2} + \frac{r^2}{b'^2\,(1 + e')^2} = 1$, or $\frac{z^2}{b'^2} + \frac{r^2}{b'^2 + 2\,b^2\,e} = 1$, or $\frac{z^2 + r^2}{b'^2} - \frac{2\,b^2\,r^2}{b'^4}\,e = 1$; or (taking the reciprocal) $\frac{b'^2}{z^2 + r^2} + \frac{2\,b^2\,r^2}{(z^2 + r^2)^2}\,e = 1$, whence $b'^2 = z^2 + r^2 - \frac{2\,b^2\,r^2}{z^2 + r^2}\,e$, and $b' = \sqrt{z^2 + r^2} - \frac{b^2\,r^2}{(z^2 + r^2)^{\frac{3}{2}}}\,e.$

Then the attraction of the fictitious spheroid on the fictitious point in the direction of z, by (41.), is $-\frac{4\,\pi}{3}\,.\,\rho \times \left(1 + \frac{4}{5}\,e'\right).\,z' = -\frac{4\,\pi}{3}\,\rho\left(1 + \frac{4}{5}\,e'\right)\frac{b}{b'}\,.\,z$; and, therefore, by (49.), the attraction of the real spheroid on the real point is $\frac{b^2\,(1 + e)^2}{b'^2\,(1 + e')^2} \times -\frac{4\,\pi}{3}\,\rho\left(1 + \frac{4}{5}\,e'\right)\frac{b}{b'}\,z, = -\frac{4\,\pi}{3}\,.\,\rho\,.\,z\,.\,\frac{b^3}{b'^3}\left(1 + 2\,e - \frac{6}{5}\,e'\right)$. And the attraction of the fictitious spheroid on the fictitious point in the direction of r by (42.) is $-\frac{4\,\pi}{3}\,\rho\left(1 - \frac{2}{5}\,e'\right)r' = -\frac{4\,\pi}{3}\,\rho\,.\left(1 - \frac{2}{5}\,e'\right).\,\frac{b\,(1 + e)}{b'\,(1 + e')}\,r$; therefore the attraction of the real spheroid on the real point is

$$\frac{b^2\,(1 + e)}{b'^2\,(1 + e')} \times -\frac{4\,\pi}{3}\,\rho\,.\left(1 - \frac{2}{5}\,e'\right).\,\frac{b\,(1 + e)}{b'\,(1 + e')}\,.\,r = -\frac{4\,\pi}{3}\,\rho\,.\,r\,.\,\frac{b^3}{b'^3}\,.\left(1 + 2\,e - \frac{12}{5}\,e'\right).$$

(53.) Putting for $\frac{b^3}{b'^3}$ its value $\frac{b^3}{\overline{z^2 + r^2}|^{\frac{3}{2}}}\left(1 + \frac{3\,b^2\,r^2}{(z^2 + r^2)^2}\,e\right)$, and for e' its value $\frac{b^2}{z^2 + r^2}\,e$, we find

Expressions for the attraction of a homogeneous spheroid on a point without it.

attraction of spheroid in the direction of z

$$= -\frac{4\,\pi}{3}\,\rho\left\{\frac{z}{(z^2 + r^2)^{\frac{3}{2}}}\,.\,b^3\,(1 + 2\,e) + \frac{9\,r^2\,z - 6\,z^3}{5\,(z^2 + r^2)^{\frac{7}{2}}}\,.\,b^5\,.\,e\right\};$$

attraction of spheroid in the direction of r

$$= -\frac{4\,\pi}{3}\,\rho\left\{\frac{r}{(z^2 + r^2)^{\frac{3}{2}}}\,.\,b^3\,(1 + 2\,e) + \frac{3\,r^3 - 12\,r\,z^2}{5\,(z^2 + r^2)^{\frac{7}{2}}}\,.\,b^5\,e\right\}$$

(54.) From this we find the attraction of a thin stratum bounded by two spheroidal surfaces in the same manner as in (44.). If we suppose the spheroid in fig. 14 to be surrounded by a concentric spheroid, whose axis and ellipticity are $b + d\,b$ and $e + d\,e$, we shall have, in the direction of z,

Attraction of a spheroidal homogeneous shell on a point without it

attraction of large spheroid

$$= -\frac{4\,\pi}{3}\,\rho\left\{\frac{z}{(z^2 + r^2)^{\frac{3}{2}}}\,\{\,b^3\,(1 + 2\,e) + d\,.\,\overline{b^3\,(1 + 2\,e)}\,\} + \frac{9\,r^2\,z - 6\,z^3}{5\,(z^2 + r^2)^{\frac{7}{2}}}\,(b^5\,e + d\,.\,b^5\,e)\right\},$$

attraction of small spheroid

$$= -\frac{4\,\pi}{3}\,\rho\left\{\frac{z}{(z^2 + r^2)^{\frac{3}{2}}}\,.\,b^3\,(1 + 2\,e) + \frac{9\,r^2\,z - 6\,z^3}{5\,(z^2 + r^2)^{\frac{7}{2}}}\,b^5\,e\right\}.$$

The difference, or the attraction of the included stratum,

$$= -\frac{4\pi}{3}\rho \cdot \left\{\frac{z}{(z^2+r^2)^{\frac{3}{2}}} \cdot d \cdot \overline{b^3(1+2e)} + \frac{9r^2z - 6z^3}{5(z^2+r^2)^{\frac{7}{2}}} d \cdot b^5 e\right\}$$

(55.) From this, by reasoning precisely similar to that in (44.), we find that the attraction of an assemblage of strata of different densities, each of which is bounded by spheroidal surfaces, concentric and similarly situated, as in fig. 16, on a point without it, in the direction of z, is

$$-\frac{4\pi}{3}\left\{\frac{z}{(z^2+r^2)^{\frac{3}{2}}} \int \rho \cdot d \cdot \overline{b^3(1+2e)} + \frac{9r^2z - 6z^3}{5(z^2+r^2)^{\frac{7}{2}}} \int \rho \cdot d \cdot b^5 e\right\}.$$

The integrals in this expression must in general be taken from $b = 0$ to $b =$ some assigned value. Let $\int \rho \cdot d \cdot \overline{b^3(1+2e)}$, corrected so as to vanish when $b = 0$, be called $\phi(b)$; let $\int \rho \cdot d \cdot b^5 e$ be called $\psi(b)$; then the attraction in the direction of z

Attraction of a series of spheroidal shells of different densities on a point without them.

$$= -\frac{4\pi}{3}\left\{\frac{z}{(z^2+r^2)^{\frac{3}{2}}} \phi(b) + \frac{9r^2z - 6z^3}{5(z^2+r^2)^{\frac{7}{2}}} \cdot \psi(b)\right\}.$$

Similarly the attraction in the direction of r

$$= -\frac{4\pi}{3}\left\{\frac{r}{(z^2+r^2)^{\frac{3}{2}}} \phi(b) + \frac{3r^3 - 12rz^2}{5(z^2+r^2)^{\frac{7}{2}}} \psi(b)\right\}.$$

(56.) We can now express the force which acts upon any point P, in any one of the strata in such a system as we have described, fig. 17. We must use the expressions that we have just found, taking the integrals from $b = 0$ to $b = $ C D, D P being the spheroidal surface of equal density that passes through P, and the expressions of (44.), taking the integrals from $b =$ C D to $b =$ C A.

(57.) Let C D $= \beta$, C A $=$ b; let the corresponding ellipticities be ϵ and e. Then the whole forces acting on P (taking into account the centrifugal force) are

force in the direction of r

$$= -\frac{4\pi}{3}\left\{\frac{r}{(z^2+r^2)^{\frac{3}{2}}} \phi(\beta) + \frac{3r^3 - 12rz^2}{5(z^2+r^2)^{\frac{7}{2}}} \psi(\beta)\right\} + \frac{8\pi}{15} r \cdot \{\lambda(\mathrm{b}) - \chi(\beta)\} + \frac{4\pi^2 r}{\mathrm{T}^2},$$

force in the direction of z

$$= -\frac{4\pi}{3} \cdot \left\{\frac{z}{(z^2+r^2)^{\frac{3}{2}}} \phi(\beta) + \frac{9r^2z - 6z^3}{5(z^2+r^2)^{\frac{7}{2}}} \psi(\beta)\right\} - \frac{16\pi}{15} \cdot z\{\chi(\mathrm{b}) - \chi(\beta)\}.$$

And if we resolve the former into forces in the direction of x and y, and observe that $r^2 = x^2 + y^2$, we shall finally obtain

Attraction of a heterogeneous spheroid on one of its particles.

$$\mathrm{X} = -\frac{4\pi}{3}\left\{\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \phi(\beta) + \frac{3x(x^2+y^2) - 12xz^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}} \psi(\beta) - \frac{2}{5}x\{\chi(\mathrm{b}) - \chi(\beta)\} - \frac{3\pi}{\mathrm{T}^2}x\right\}.$$

$$\mathrm{Y} = -\frac{4\pi}{3}\left\{\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \phi(\beta) + \frac{3y(x^2+y^2) - 12yz^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}} \psi(\beta) - \frac{2}{5}y\{\chi(\mathrm{b}) - \chi(\beta)\} - \frac{3\pi}{\mathrm{T}^2}y\right\}.$$

$$\mathrm{Z} = -\frac{4\pi}{3}\left\{\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \phi(\beta) + \frac{9z(x^2+y^2) - 6z^3}{5(x^2+y^2+z^2)^{\frac{7}{2}}} \psi(\beta) + \frac{4}{5}z\{\chi(\mathrm{b}) - \chi(\beta)\}\right\}.$$

(58.) Now we have explained in (10.) and (37.) that the equation to the surface bounding a stratum of equal density, must be $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0$; and that this condition is sufficient. But our whole investigation has gone on the supposition that the surface bounding a stratum of equal density is a spheroid, whose semi-axis and ellipticity are β and ϵ, and which is, therefore, defined by the equation $\frac{x^2+y^2}{\beta^2(1+\epsilon)^2} + \frac{z^2}{\beta^2} = 1$, or $\frac{x}{1+2\epsilon}dx + \frac{y}{1+2\epsilon}dy + z\,dz = 0$. The single question then now remains:—is this consistent with the equation $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0$? or can the coefficients have the same proportions? In this inquiry we are not to take into account the squares or higher powers of ϵ, as they have been uniformly neglected.

(59.) Assuming the coefficients in the two differential equations to have the same proportion, we get this equation,

$$0 = \frac{2\epsilon \cdot \phi(\beta)}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{6}{5}\frac{\psi(\beta)}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{6}{5}\{\chi(\mathrm{b}) - \chi(\beta)\} - \frac{3}{\mathrm{T}^2},$$

Physical Theory. Earth supposed heterogeneous.

in which we have omitted the terms $\epsilon \,.\, \psi(\beta)$, $\epsilon \,.\, \{\chi(\mathrm{b}) - \chi(\beta)\}$, and $\epsilon \,.\, \frac{3\pi}{\mathrm{T}^2}$, because an inspection of the quantities by whose integration the functions are produced will show that though $\phi(\beta)$ is not small, yet $\psi(\beta)$ and $\chi(\beta)$ depend entirely upon e for their value, and, therefore, $\epsilon \,.\, \psi(\beta)$ and $\epsilon \,.\, \chi(\beta)$ are in fact of the second order with regard to ϵ. And $\frac{3\pi}{\mathrm{T}^2}$, which is a multiple of the centrifugal force, is to be estimated of the same order as ϵ, because the ellipticity is immediately occasioned by it, and, therefore, $\epsilon \,.\, \frac{3\pi}{\mathrm{T}^2}$ is also of the second order.

(60.) Again, the quantity $x^2 + y^2 + z^2$, which is in the denominator of the first two terms, differs from β^2 only by $\frac{(x^2 + y^2)(2\epsilon + \epsilon^2)}{\beta^2 (1 + \epsilon)^2}$, that is by a quantity of the same order as ϵ, and, therefore, it may be considered in this equation as equal to β^2. The equation now becomes

$$0 = \frac{2\epsilon \,.\, \phi(\beta)}{\beta^3} - \frac{6}{5} \,.\, \frac{\psi(\beta)}{\beta^5} - \frac{6}{5}\{\chi(\mathrm{b}) - \chi(\beta)\} - \frac{3\pi}{\mathrm{T}^2};$$

or putting b for β,

$$\frac{2e \,.\, \phi(b)}{b^3} - \frac{6}{5}\frac{\psi(b)}{b^5} - \frac{6}{5}\{\chi(\mathrm{b}) - \chi(b)\} - \frac{3\pi}{\mathrm{T}^2} = 0:$$

and if this equation is possible, the equilibrium is possible.

Equilibrium is possible, with the form assumed.

(61.) Differentiating this equation twice, so as to eliminate $\psi(b)$ and $\chi(b)$, both which contain e, and observing that as $\phi(b)$ is multiplied by e, we may consider it as $= \int \rho \,.\, d(b^3)$, instead of $= \int \rho \,.\, d(b^3 \,.\, \overline{1 + 2e})$, we find

$$b^3 \phi(b) \,.\, \frac{de}{db} + 3\psi(b) - 3e \,.\, b^2 \phi(b) = 0,$$

$$\frac{d^2 e}{db^2} + \frac{2\rho b^2}{\int \rho b^2 db} \,.\, \frac{de}{db} + \left(\frac{2\rho b}{\int \rho b^2 db} - \frac{6}{b^2}\right) e = 0.$$

Upon expressing ρ in terms of b it will be possible to solve this equation, and to obtain e in terms of b. The solution will contain two arbitrary constants, whose values may be determined by making the expression satisfy the original equation. Hence equilibrium is possible.

(62.) Using the letter m (as in the investigation of the form assumed by a homogeneous fluid) to denote the ratio of the centrifugal force at the Equator to the force of Gravity there, and observing that at the surface $\beta = \mathrm{b}$, $x = \mathrm{b}(1 + \mathrm{e})$, $y = 0$, $z = 0$, we have

$$m = \frac{\frac{3\pi}{\mathrm{T}^2}\mathrm{b}(1 + \mathrm{e})}{\frac{\phi(\mathrm{b})}{\mathrm{b}^2(1 + \mathrm{e})^2} + \frac{3}{5}\frac{\psi(\mathrm{b})}{\mathrm{b}^4(1 + \mathrm{e})^4} - \frac{3\pi}{\mathrm{T}^2}\mathrm{b}(1 + \mathrm{e})}:$$

and neglecting the terms of the second order, $m = \frac{3\pi}{\mathrm{T}^2} \,.\, \frac{\mathrm{b}^3}{\phi(\mathrm{b})}$; whence $\frac{3\pi}{\mathrm{T}^2} = \frac{m\phi(\mathrm{b})}{\mathrm{b}^3}$. The equation of (60.), in the form in which we are compelled to use it for investigation of the Earth's constitution, is, therefore,

Equation to be used for investigation of the Earth's form.

$$\frac{2e \,.\, \phi(b)}{b^3} - \frac{6}{5} \,.\, \frac{\psi(b)}{b^5} - \frac{6}{5}\{\chi(\mathrm{b}) - \chi(b)\} - \frac{m\phi(\mathrm{b})}{\mathrm{b}^3} = 0.$$

We have already remarked in (11.) that this equation also holds if we suppose the density of the fluid to depend on the pressure. We may further remark that if we suppose a solid nucleus to exist, similar in general constitution to what we have assumed to be the state of the Earth, but having for its different strata any arbitrary densities and ellipticities, and then if we suppose fluids of different densities to be suffered to arrange themselves freely round this nucleus, the same expressions will hold for the forces, as if the whole were fluid, (the integrals being taken as well for the solid as for the fluid part,) and the equation of this article will apply in the same manner with regard to the fluid parts. And, therefore, the equations respecting the surface (which are in fact the only important ones, the others being only useful in conducting us to them) are equally true whether the whole be supposed to be fluid, or a fluid heterogeneous mass be supposed to surround such a nucleus.

(63.) At the surface $b = \mathrm{b}$; and the equation of the last article becomes $(2\mathrm{e} - m)\frac{\phi(\mathrm{b})}{\mathrm{b}^3} - \frac{6}{5} \,.\, \frac{\psi(\mathrm{b})}{\mathrm{b}^5} = 0$.

Equation simplified when applied to the surface.

(64.) One of the most remarkable applications of this equation is to be found in the expression for Gravity at any part of the Earth's surface. If in (57.) we put b for b, and $\frac{m\phi(\mathrm{b})}{\mathrm{b}^3}$ for $\frac{3\pi}{\mathrm{T}^2}$, we have for the forces acting on a point at the Earth's surface

$$X = -\frac{4\pi}{3}\left\{\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\phi(\mathrm{b}) + \frac{3x(x^2+y^2)-12xz^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}}\psi(\mathrm{b}) - x\frac{m\phi(\mathrm{b})}{\mathrm{b}^3}\right\}.$$

$$Y = -\frac{4\pi}{3}\left\{\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\phi(\mathrm{b}) + \frac{3y(x^2+y^2)-12yz^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}}\psi(\mathrm{b}) - y\frac{m\phi(\mathrm{b})}{\mathrm{b}^3}\right\}.$$

$$Z = -\frac{4\pi}{3}\left\{\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\phi(\mathrm{b}) + \frac{9z(x^2+y^2)-6z^3}{5(x^2+y^2+z^2)^{\frac{7}{2}}}\psi(\mathrm{b})\right\}.$$

Expression for Gravity at the Earth's surface.

The whole force or $\sqrt{X^2+Y^2+Z^2}$, putting r^2 for x^2+y^2, and neglecting terms of the second order, is

$$\frac{4\pi}{3}\left\{\frac{\phi(\mathrm{b})}{r^2+z^2} + \frac{3}{5}\cdot\frac{r^2-2z^2}{(r^2+z^2)^3}\psi(\mathrm{b}) - \frac{r^2}{(r^2+z^2)^{\frac{1}{2}}}\cdot\frac{m\phi(\mathrm{b})}{\mathrm{b}^3}\right\}.$$

Now the equation to the external surface is $\frac{r^2}{\mathrm{b}^2(1+\mathrm{e})^2}+\frac{z^2}{\mathrm{b}^2}=1$, whence $r^2+z^2=\mathrm{b}^2+2\,\mathrm{e}\,r^2$; or if λ be the latitude, we may assume (nearly enough to the truth for substitution in the small terms) $r^2=\mathrm{b}^2.\cos^2\lambda$; whence $r^2+z^2=\mathrm{b}^2(1+2\,\mathrm{e}\,.\cos^2\lambda)=\mathrm{b}^2(1+2\,\mathrm{e}-2\,\mathrm{e}\sin^2\lambda)$. Hence $r^2-2z^2=\mathrm{b}^2(1-3\sin^2\lambda)$, omitting e, as this expression is used only to multiply a small term; and $\frac{r^2-2z^2}{(r^2+z^2)^3}=\frac{1-3\sin^2\lambda}{\mathrm{b}^4}$. Also $\frac{r^2}{(r^2+z^2)^{\frac{1}{2}}}=\mathrm{b}(1-\sin^2\lambda)$. Substituting, we get for the whole force

$$\frac{4\pi}{3}\left\{\frac{\phi(\mathrm{b})}{\mathrm{b}^2}(1-2\,\mathrm{e}+2\,\mathrm{e}\sin^2\lambda)+\frac{3}{5}\cdot\frac{\psi(\mathrm{b})}{\mathrm{b}^4}(1-3\sin^2\lambda)-\frac{m\phi(\mathrm{b})}{\mathrm{b}^2}(1-\sin^2\lambda)\right\}$$

$$=\frac{4\pi}{3}\cdot\frac{\phi(\mathrm{b})}{\mathrm{b}^2}\cdot\left(1-2\,\mathrm{e}-m+\frac{3}{5}\cdot\frac{\psi(\mathrm{b})}{\mathrm{b}^2\phi(\mathrm{b})}\right)\cdot\left\{1+\overline{2\,\mathrm{e}-\frac{9}{5}\,\frac{\psi(\mathrm{b})}{\mathrm{b}^2.\phi(\mathrm{b})}+m}\,.\sin^2\lambda\right\}.$$

Let

$$\frac{4\pi}{3}\cdot\frac{\phi(\mathrm{b})}{\mathrm{b}^2}\cdot\left(1-2\,\mathrm{e}-m+\frac{3}{5}\cdot\frac{\psi(\mathrm{b})}{b^2\phi(\mathrm{b})}\right)=\mathrm{E};$$

Gravity expressed in terms of the latitude.

then the expression for Gravity is

$$\mathrm{E}\left\{1+\left(2\,\mathrm{e}-\frac{9}{5}\cdot\frac{\psi(\mathrm{b})}{\mathrm{b}^2.\phi(\mathrm{b})}+m\right)\sin^2\lambda\right\};$$

from which it appears that, in going from the Equator to the Poles, Gravity increases as the square of the sine of latitude.

(65.) The coefficient of the square of the sine of latitude is $2\,\mathrm{e}-\frac{9}{5}\cdot\frac{\psi(\mathrm{b})}{\mathrm{b}^2.\phi(\mathrm{b})}+m$. Now from (63.) we get $\frac{9}{5}\cdot\frac{\psi(\mathrm{b})}{\mathrm{b}^2.\phi(\mathrm{b})}=3\,\mathrm{e}-\frac{3m}{2}$. Substituting, the coefficient of $\sin^2\lambda$ is $\frac{5m}{2}-\mathrm{e}$; and Gravity is now expressed by $\mathrm{E}\left(1+\overline{\frac{5m}{2}-\mathrm{e}}\,.\sin^2\lambda\right)$, a remarkably simple expression, all functions, and, in fact, every thing depending on the internal constitution of the spheroid having disappeared. It appears from this, that if by observations of the force of Gravity at different latitudes we can express it by a formula of this kind $\mathrm{E}(1+\mathrm{F}\sin^2\lambda)$, then e, the ellipticity of the Earth's surface, will $=\frac{5m}{2}-\mathrm{F}$. This is the celebrated theorem known by the name of *Clairaut's Theorem.*

Clairaut's theorem.

(66.) Another remarkable application of the equation of (63.) occurs in the expression for the attraction of a point exterior to the spheroid. The forces in the directions of r and z will be found by putting b for b in the expressions of (55.); resolving these into forces in the directions of x, y, and z, and calling them X', Y', Z', we have

Attraction of heterogeneous spheroid on a point without it.

$$X' = -\frac{4\pi}{3}\cdot\left\{\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\phi(\mathrm{b}) + x\frac{3(x^2+y^2)-12z^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}}\psi(\mathrm{b})\right\}.$$

$$Y' = -\frac{4\pi}{3}\cdot\left\{\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\phi(\mathrm{b}) + y\frac{3(x^2+y^2)-12z^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}}\psi(\mathrm{b})\right\}.$$

$$Z' = -\frac{4\pi}{3}\cdot\left\{\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\phi(\mathrm{b}) + z\frac{9(x^2+y^2)-6z^2}{5(x^2+y^2+z^2)^{\frac{7}{2}}}\psi(\mathrm{b})\right\}.$$

Earth supposed heterogeneous.

Substituting for $\frac{3 \,.\, \psi(\mathrm{b})}{5}$ the value $\left(\mathrm{e} - \frac{m}{2}\right) . \mathrm{b}^2 . \phi(\mathrm{b})$ found in (63.), and making $\frac{4\,\pi}{3} . \phi(\mathrm{b}) = \mathrm{M}$, we have

$$\mathrm{X}' = -\,\mathrm{M}\left\{\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} + x\,\frac{\mathrm{b}^2(x^2+y^2-4\,z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}}\left(\mathrm{e} - \frac{m}{2}\right)\right\}.$$

$$\mathrm{Y}' = -\,\mathrm{M}\left\{\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} + y\,\frac{\mathrm{b}^2(x^2+y^2-4\,z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}}\left(\mathrm{e} - \frac{m}{2}\right)\right\}.$$

$$\mathrm{Z}' = -\,\mathrm{M}\left\{\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} + z\,\frac{\mathrm{b}^2(3\,x^2+3\,y^2-2\,z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}}\left(\mathrm{e} - \frac{m}{2}\right)\right\}.$$

It appears, then, that the attraction of a spheroid upon a body without it is not in the direction of a line joining the body with the centre of the spheroid; (for if it were, X′, Y′, Z′ would be proportional to x, y, z respectively;) and, in consequence, the motion of the satellite of a spheroidal body will not be the same as if the body were spherical. The irregularity occasioned by this difference is sensible in the motion of our own Moon, and in those of Jupiter's satellites; and as it depends on $\mathrm{e} - \frac{m}{2}$, it gives us the means of ascertaining the value of e without any knowledge of the internal constitution of the Earth.

Resulting inequality in the Moon's motion.

(67.) We cannot here go through the whole process of calculating the inequality of the Moon produced by the Earth's ellipticity, but we can indicate the principal steps. Let the axis of x be supposed to be directed to the first point of Aries, that of y to the point in the Equator whose right ascension is 90°, and that of z to the pole of the Equator. Now take a new system of coordinates, that of x being the same as before, that of y' in the plane of the Ecliptic directed to the first point of Cancer, and that of z' directed to the pole of the Ecliptic. Resolve the three forces X′, Y′, Z′ in these three directions. From the Moon let fall a perpendicular on the plane of the Ecliptic, and join the Earth's centre with the foot of the perpendicular, and resolve the forces just found into one in the direction of the joining line, one in the direction of the perpendicular, and one at right angles to both these. Let θ be the Moon's longitude, ω the obliquity of the Ecliptic. Then it will be found that the last-mentioned forces are $\mathrm{X}' \cos\theta + (\mathrm{Y}'\cos\omega + \mathrm{Z}'\sin\omega)\sin\theta$, $\mathrm{Z}'\cos\omega - \mathrm{X}'\sin\omega$, and $\mathrm{X}'\sin\theta - (\mathrm{Y}'\cos\omega + \mathrm{Z}'\sin\omega)\cos\theta$. Adopting the notation in the Treatise on Physical Astronomy, Part III., and substituting for x, y, z the values $\frac{1}{u}\cos\theta$, $\frac{1}{u}(\sin\theta . \cos\omega - s\sin\omega)$, $\frac{1}{u}(\sin\theta . \sin\omega + s\cos\omega)$, we find the terms depending on the Earth's ellipticity which are to be added to $m'\mathrm{V}$, $m'\mathrm{T}$, and $m'\frac{d\,\Omega}{d\,z}$. (See Physical Astronomy.) They are rather complicated, but the only terms which produce any sensible effect are in $m'\mathrm{T}$, $-\,\mathrm{M}\left(\mathrm{e} - \frac{m}{2}\right) . 2\,\mathrm{b}^2 u^4 . \sin\omega . \cos\omega . \cos\theta . s$; and in $m'\frac{d\,\Omega}{d\,z}$, $\mathrm{M}\left(\mathrm{e} - \frac{m}{2}\right) \times 2\,\mathrm{b}^2 u^4 \sin\omega . \cos\omega . \sin\theta$. If we treat these terms like the others in the Treatise alluded to, it will easily be found that the resulting inequality in the Moon's latitude, expressed in seconds, (considering the sum of the masses of the Earth and Moon = 1) is $-\frac{4 \,.\, \overline{60}|^3}{\pi} . \overline{\mathrm{e} - \frac{m}{2}} . \left(\frac{\text{Earth's period}}{\text{Moon's period}}\right)^2 \left(\frac{\text{Earth's radius}}{\text{Moon's distance}}\right)^2 . \sin\omega . \cos\omega . \sin\theta = -\,\overline{e - \frac{m}{2}} . 4891'' . \sin\theta$ nearly. The perturbation in longitude is not so easily calculated.

Law of density which makes the equation for the ellipticity of the different strata integrable.

(68.) We shall conclude this investigation with remarking that the differential equation of (61.) may be put in a simpler form by assuming $\int \rho\, b^2\, d\, b = p$, and $p\, e = v$; substituting these values,

$$\frac{d^2 v}{d\,b^2} - \frac{6\,v}{b^2} - v\,\frac{b}{p} . \frac{d\,p}{d\,b} = 0.$$

A form of ρ which makes this easily integrable, (first pointed out, we believe, by Legendre,) and which probably represents pretty well the law of density in the interior of the Earth, (giving a density gradually increasing to the centre,) is $\rho = \mathrm{A}\,\frac{\sin q\,b}{b}$, A and q being constant. The general value of v is found to be $\mathrm{C}\left(\sin \overline{q\,b + \mathrm{C}'} + \frac{3}{q\,b}\cos\overline{q\,b + \mathrm{C}'} - \frac{3}{q^2\,b^2}\sin\overline{q\,b + \mathrm{C}'}\right)$, and determining the constants by substitution in the equations from which this differential equation is derived, it is found that e, or $\frac{v}{p}$, =

$$\frac{5\,m}{2} . \frac{\left(1 - \frac{q\,\mathrm{b}}{\tan q\,\mathrm{b}}\right)^2}{2 - q^2\mathrm{b}^2 - \frac{q\,\mathrm{b}}{\tan q\,\mathrm{b}} - \frac{q^2\,\mathrm{b}^2}{\tan^2 q\,\mathrm{b}}} . \frac{1 - \frac{3}{q^2 b^2} + \frac{3}{q\,b \,.\, \tan q\,b}}{1 - \frac{q\,b}{\tan q\,b}}.$$

Earth supposed heterogeneous.

To get the ellipticity of the surface we must put b for b, and if $f = 1 - \frac{q\,\mathrm{b}}{\tan q\,\mathrm{b}}$, it is found that $e = \frac{5\,m}{2} \cdot \frac{1 - \frac{3f}{q^2\,\mathrm{b}^2}}{3 - f - \frac{q^2\,\mathrm{b}}{f}}$. The value $q = \frac{5}{6} \cdot \frac{\pi}{\mathrm{b}}$ gives a law of density that probably represents nearly the Earth's density.

The probable error, arising from neglect of the higher powers of the ellipticity, may be inferred from those in the extreme cases.

(69.) We shall now consider the probable magnitude of the error which we have incurred by neglecting the square of the ellipticity. We have investigated, accurately, two cases which may be considered as extremes in the series of laws of density, within which the law of the Earth's density is included. One of these is, the case of a homogeneous Fluid; the other is that of (12.), in which the assumption of attraction directed only to the centre, amounts to the same as (on the principle of gravitation) the assumption that the matter near the centre is infinitely more dense than that near the surface. We shall examine, then, how far, in these two cases, the two principal results of our investigation (the ellipticity of form, and Clairaut's theorem) are true; and if we find them nearly true in these cases, we shall conclude that they are nearly true in the case of a density gradually increasing to the centre.

If the Fluid be homogeneous, the form is accurately elliptic

(70.) First as to the ellipticity of figure. When the Fluid is homogeneous, it appears by the investigation extending from (14.) to 26.), that the figure is *accurately* that of an elliptic spheroid. When the attraction of the particles is neglected in comparison with the attraction to the centre, it appears from (12.) that the equation to the external surface, using r for the distance of any point from the centre, is

$$\frac{1}{r} = \frac{2 + 3\,m}{2 + 2\,m} \cdot \frac{1}{a} - \frac{m}{2 + 2\,m} \cdot \frac{x^2 + y^2}{a^3}, \text{ or } \frac{1}{r} = \frac{1}{b} - \left(\frac{a}{b} - 1\right) \cdot \frac{x^2 + y^2}{a^3} = \frac{1}{b} - \frac{e}{b^3\,(1 + e)^3}\,(x^2 + y^2).$$

Let r' be the corresponding line in an elliptic spheroid with the same axes for the same values of x and y; then proceeding from the equation $\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$, it will be found that

$$r' = b\sqrt{\left\{1 + \frac{a^2 - b^2}{a^2\,b^2} \cdot (x^2 + y^2)\right\}} = b\sqrt{\left\{1 + \frac{2\,e + e^2}{1 + 2\,e + e^2} \cdot \frac{x^2 + y^2}{b^2}\right\}}.$$

Expanding the expressions for r and r' as far as terms of the order e^2, it is found that $r' - r = \frac{3}{2\,b^3}\,e^2 \cdot (x^2 + y^2) \cdot (b^2 - \overline{x^2 + y^2})$. This is greatest when $x^2 + y^2 = \frac{b^2}{2}$, or at latitude 45° nearly; its value is then $\frac{3\,e^2}{8}\,b$. If $b = 4000$ miles, and $e = \frac{1}{580}$, (the ellipticity of the Earth that would correspond to this supposition,) this would be 24 feet, a quantity quite insensible. We may conclude, then, that the theoretical figure of a heterogeneous Fluid, on the principle of Gravitation, would (since r is less than r') be somewhat flatter at latitude 45° than the elliptic spheroid, but that if the circumstances were those of the Earth, the difference would be insensible in measures of degrees.

If the Fluid be infinitely dense at the centre, the form does not sensibly differ from an elliptic spheroid.

(71.) Secondly, as to the law of Gravity at the surface. In the homogeneous ellipsoid it has been found (35.) that, putting λ for the latitude of any point, the Gravity there is proportional to $\frac{1}{\sqrt{\sin^2\lambda + \overline{1 + e}|^2\cos^2\lambda}}$. Expanding this so as to include the second power of e, it may be put under the form $(1 - e + e^2)\,\{1 + e\sin^2\lambda \quad \frac{3\,e^2}{2}\sin^2\lambda\cos^2\lambda\}$, or Gravity is proportional to $\{1 + e\sin^2\lambda - \frac{3\,e^2}{2}\sin^2\lambda \cdot \cos^2\lambda\}$. Now as Clairaut's theorem is commonly understood, Gravity is proportional to $\{1 + \overline{\frac{5\,m}{2} - e} \cdot \sin^2\lambda\}$. We must then investigate accurately the relation between e and m. In the equation of (29.) make $f = \frac{1}{\cot f} - \frac{1}{3} \frac{1}{\cot^3 f} + \frac{1}{5} \cdot \frac{1}{\cot^5 f} - \frac{1}{7} \cdot \frac{1}{\cot^7 f}$, and we find at length $\frac{m}{1 + m} = \frac{4}{5}\,e\left(1 - \frac{57}{70}\,e\right)$, whence $\frac{5\,m}{2} = 2\,e - \frac{1}{35}\,e^2$, and $\frac{5\,m}{2} - e = e - \frac{1}{35}\,e^2$. Consequently, according to Clairaut's law, Gravity should be expressed by $1 + \left(e - \frac{e^2}{35}\right)\sin^2\lambda$; it actually is expressed by $1 + e\sin^2\lambda - \frac{3\,e^2}{2}\sin^2\lambda\cos^2\lambda$. We ought then to increase Gravity in the homogeneous spheroid by equatorial Gravity $\times\, e^2\sin^2\lambda\left(\frac{3}{2}\cos^2\lambda - \frac{1}{35}\right)$ in order to make it follow Clairaut's law. For circumstances similar to those of the Earth, e must be taken $= \frac{1}{230}$, and the quantity to be added is about

If the Fluid be homogeneous, Clairaut's law is nearly true

Physical Theory. Earth supposed heterogeneous

$\frac{\text{equatorial Gravity}}{35300} \times \sin^2\lambda\left(\cos^2\lambda - \frac{2}{105}\right)$. This is nearly a maximum when $\lambda = 45°$, and it then is nearly $= \frac{\text{equatorial Gravity}}{140000}$, a quantity barely sensible in the most accurate pendulum experiments.

When the attractive force is wholly directed to the centre, it appears (see 12.) that the resolved parts of Gravity are

$$f\left\{\frac{x}{r^3} - (2e - 6e^3)\frac{x}{b^3}\right\}, \quad f\left\{\frac{y}{r^3} - (2e - 6e^3)\frac{y}{b^3}\right\}, \quad f\frac{z}{r^3}$$

Adding the squares of these, observing that $\frac{1}{r} = \frac{1}{b}\left\{1 - e \,.\, \frac{x^2+y^2}{b^2} + 3e^2\frac{x^2+y^2}{b^2}\right\}$, and extracting the square root, it is found that Gravity is proportional to

$$1 - 4e\frac{x^2+y^2}{b^2} + 14e^2 \,.\, \frac{x^2+y^2}{b^2} + e^2\left(\frac{x^2+y^2}{b^2}\right).$$

Now from the consideration that $\sin\lambda = \frac{\text{force in direction of } z}{\text{Gravity}}$, we find $\frac{x^2+y^2}{b^2} = (1+4e)\cos^2\lambda - 2e\cos^4\lambda$; and thus we obtain at length that Gravity is proportional to

$$1 + \overline{4e + 9e^2}\sin^2\lambda - 9e^2\sin^2\lambda\cos^2\lambda.$$

But from (12.) $\frac{a}{b} = \frac{2+3m}{2+2m}$, whence $e = \frac{m}{2+2m}$, and $m = 2e + 4e^2$, $\frac{5m}{2} = 5e + 10e^2$, $\frac{5m}{2} - e = 4e + 10e^2$, and therefore Gravity, according to Clairaut's formula, ought to be expressed by

$$1 + \overline{4e + 10e^2} \,.\, \sin^2\lambda.$$

If the density be infinitely great at the centre, Clairaut's law is nearly true.

We ought, therefore, to increase the actual Gravity by equatorial Gravity $\times\, e^2\sin^2\lambda\,(9\cos^2\lambda + 1)$, in order to bring it under Clairaut's law. The maximum value of this is nearly $\frac{9e^2}{4} \times$ equatorial Gravity; which, if $e = \frac{1}{580}$, is about $\frac{\text{equatorial Gravity}}{150000}$; a quantity as small as that found before.

Conclusions with regard to heterogeneous spheroid.

(72.) We may conclude, then, that the following theoretical results may be considered as sufficiently accurate for a heterogeneous fluid mass in the circumstances of the Earth.

The external surface has the form of an elliptic spheroid.

At the latitude λ Gravity is proportional to $1 + \left(\frac{5m}{2} - e\right)\sin^2\lambda$, (where m is the proportion of centrifugal force at the Equator to Gravity at the Equator, and $1 : 1 + e$ is the proportion of the axes.)

Disturbance produced by a small mass near the Earth's surface.

(73.) If a mass of different density were placed any where in the Fluid it is plain that it would disturb all the laws that we have found. For a calculation of its effects we shall refer to a paper by Dr. Young, in the *Phil. Trans.* for 1819, from which we extract the following results.

Prop. A. If a plumb-line be drawn aside by a small mass not far below the Earth's surface, the distance of the points where the greatest deviations are produced, is to the depth of the mass, as $\sqrt{2}$ to 1.

B. The sine of the greatest deviation $= ,385 \times \frac{a}{c^2}$ $\left(\text{where } a = \frac{\text{disturbing mass}}{\text{Earth's mass}}, \text{ and } c = \frac{\text{depth}}{\text{Earth's radius}}\right)$.

C. The greatest elevation of the general surface above the sphere $= \frac{a}{c}$.

D. Gravity at the highest point will be increased by $\frac{a}{c^2} - \frac{2a}{c}$.

E. At the place where the greatest deviation of the plumb-line is produced, the proportional increase of Gravity is to the deviation (estimated as an arc to radius 1) as $\sqrt{2}$ to 1.

We have conducted the theoretical investigation of the Earth's form by an analysis which, though far from elegant, is sufficiently powerful to master this subject, and sufficiently simple to be understood after a common acquaintance with the Differential Calculus. The analysis of Laplace we have thought it desirable to exclude, where it is possible to succeed without its assistance. The difficulty of understanding the fundamental points of that calculus, and the obscurity which, to those who have given most attention to it, has always appeared to hang upon many of its applications, (see *Phil. Trans.* 1812, &c. and *Cambridge Transactions*, vol. iii.) will serve for our apology. We have thus lost some of Laplace's results, (especially those relating to the form of a Fluid spread over an irregular solid;) but we have thought it better to give them up at once, than to compose a treatise which, to the greater part of our readers, would be (and that without necessity) unintelligible. The reader who wishes to be acquainted with a calculus the most singular in its nature, and the most powerful in its applications that has ever appeared, is referred for explanation and example to our Essay Electricity, or to the Vth Book of the *Mécanique Céleste.*

Section 3.—On the Manner of determining the Figure of the Earth by Geodetic Measures.

The investigations which we have just given seem to show that if the Earth had been originally fluid, and if the different liquids of which it consisted had been allowed to settle into a state of rest, and if it had then become solidified, and had afterwards been subjected to no alteration of shape, its external form would be nearly that of an oblate spheroid. The ellipticity of the spheroid, however, is not determined; it has only appeared that if the Earth were supposed to have been a homogeneous Fluid, (a thing in the highest degree improbable,) and the attraction of each molecule proportional to its mass, the proportion of the equatorial axis to the polar axis would be 231 : 230 ; if the whole attraction were directed to one attractive point at the Earth's centre, or if, on the same theory of universal attraction, the density at the centre were infinitely greater than that at any other point, the proportion would be 581 : 580 ; and if (as seems *à priori* most probable) the density increased gradually to the centre, and if every particle attracted with a force proportional to its mass, the proportion of the axes would be between these two.

Now it is plain that none of these suppositions correspond exactly to the state of the Earth's surface. The irregularities in its external form are considerable. The height of the mountains in some parts is perhaps equal to $\frac{1}{7}$ of the difference between the polar and equatorial semiaxes. The depth of the sea in many places is possibly much greater. Still the examination of the Earth's surface will serve to verify the correctness of the general principle of Gravitation. We may expect that there will be a considerable approximation to the spheroidal form, subject to irregularities of which, in some cases, the causes cannot be discovered, and in other cases a probable explanation can be given by reference to the form of the neighbouring mountains or the surrounding country. If (as some Philosophers have supposed) the interior of the Earth be yet fluid within a small distance of the surface, it is impossible that the general form of the Earth can differ very much from that of a fluid mass.

The extent of the watery covering of the Earth is a circumstance highly favourable to the agreement of the calculated and observed forms of the Earth. We may consider the sea as a Fluid, acted on by the attraction of the solid part of the Earth, (whose form does not much differ from that of equilibrium,) and by the centrifugal force, and the reciprocal attraction of its own particles. The irregularities in the Earth's attraction will be very much smaller than those of the Earth's form ; and, consequently, the irregularity which they produce in the form of the sea will be very much smaller than the irregularities in the form of the land. If, then, we refer our measures, which must necessarily be made on land, to the surface of the sea, we may expect an agreement of the theoretical form and the ascertained measures to very considerable exactness.

The theory will apply, approximately, to the Earth's form.

It is natural, therefore, that in making the comparison we should begin by discussing the properties of the spheroid. By combining with them a consideration of the practical exactness which it is possible to attain in the different measures, we shall be able to choose the kind of measure which is best adapted to our purpose. And by calculating from different combinations of measures the length of the Earth's axes and the value of its ellipticity, we can infer, from the agreement or discordance of the results, the correctness or incorrectness of our original assumptions.

We have explained in the first section the general principle upon which the Earth's form is ascertained by measures. If the distance between two points can be found, and if the angle can be found which is included between the verticals drawn at these points, the distance of the intersection of those verticals can be found ; and this is the same as the radius of curvature of the curve joining those points. The lengths of the extensive arcs which have been used for this purpose, have always (with one exception) been ascertained by geodetic operations, for a general idea of which we refer the reader to the Essay on Trigonometry, § IX., and which we shall explain with greater minuteness in the following section. For the mere measure, the general direction of the chain of triangles, or the direction of the line joining the extreme points, is a matter of perfect indifference. But there are practical considerations which require us to fix on one of two directions. The latitudes of the places P and Q, fig. 18, whether on the same meridian or not, are the complements of the angles $p\,\mathrm{P}\,s$, $q\,\mathrm{Q}\,s$, respectively, which are included by the verticals at the places and the lines drawn to the celestial Pole. And if S be any star which can be observed at both places the angle $s\,\mathrm{P}\,p = s\,\mathrm{P}\,\mathrm{S} + \mathrm{S}\,\mathrm{P}\,p$, and $s\,\mathrm{Q}\,q = s\,\mathrm{Q}\,\mathrm{S} + \mathrm{S}\,\mathrm{Q}\,q$; considering, therefore, the angles $s\,\mathrm{Q}\,s$, $s\,\mathrm{P}\,s$, as equal, the difference of latitudes is the same as the difference of $\mathrm{S}\,\mathrm{P}\,p$, $\mathrm{S}\,\mathrm{Q}\,q$. That is, it is the same as the difference of the zenith distances of the same star at the two places, and can, therefore, easily be found. Now if the places P and Q be on the same meridian, their verticals will intersect in some point D, and the difference of latitudes, which is the difference of $s\,\mathrm{Q}\,q$ and $s\,\mathrm{P}\,p$, or ($\mathrm{P}\,r$ being parallel to $\mathrm{Q}\,q$) the difference of $s\,\mathrm{P}\,r$ and $s\,\mathrm{P}\,p$, is equal to $r\,\mathrm{P}\,p$, or Q D P, the angle contained by the verticals. The length P Q being known from measures, and the angle P D Q, or the difference of latitude, being found by observations of the zenith distances of a star, the length of P D or Q D, or the radius of curvature, is found.

Differences of latitude found astronomically.

Again, if T and V, fig. 19, be two places on different meridians, and if planes be drawn through these places, and through the axis A C of the Earth, the angle made by these planes (or the difference of longitude) may be determined astronomically. For in consequence of the inclination of the planes, the place T will be brought by the Earth's motion under the celestial meridian passing through some known star sooner than the place V;

Determination by Geodetic Measures.

Difference of longitude found astronomically.

and as the Earth's rotatory motion is uniform, the difference of time will serve to measure the angle.* If, then, we compare by any means the clocks at the two places, and observe the times indicated by these clocks at the passage of the same star over the meridians, the angle between the planes can be ascertained. Now if instead of T we have a place t, whose latitude is the same as that of V, and if we draw V W, t W, perpendicular to the axis, the angle between the planes will be the same as the angle V W t. The distance V t being measured, and the angle V W t, or the difference of longitude being found, the length of V W, or t W, or the radius of a parallel, is found.

Either of these measures then will give the length of a line that will assist materially in determining the Earth's form and dimensions. But they cannot easily be combined. Both have been extensively used; but the difference of latitude can be ascertained with so much greater accuracy than the difference of longitude, that measures of the former kind have generally been relied on.

Now in practice it is found extremely difficult to determine with accuracy the direction of any one line with respect to the meridian, or the azimuth of any station with respect to any other station. We must, therefore, always consider that we are liable to a small uncertainty in the direction of a side of one of the triangles, and, consequently, in the general direction of the chain of triangles, which depends entirely on this line. And we must choose the direction so that a small error may produce the smallest possible effect in the line to be measured.

The general direction of the chain of triangles ought to be either in the meridian or perpendicular to it.

Now, in fig. 20, let P Q be the meridian, P the first extremity of the chain of triangles, X the last; let X, x, Y, y, be in the circumference of a circle whose radius is P. Draw X Q, $x\,q$, Y R, $y\,r$, arcs of parallels. From the uncertainty in the direction of the triangles, it is uncertain perhaps whether X or x is the true situation of the last station. It is uncertain, therefore, whether the point in the meridian P Q, whose latitude is the same as that of the last station, is Q or q. But it is plain that this uncertainty is much less than it would be if Y or y were the situation of the last station; and that by bringing the position as near as possible to the meridian P Q, the uncertainty Q q may be made as small as we please. The same, it will readily be seen, is true with respect to an arc measured nearly in the direction of a parallel.

It appears, therefore, that we must determine on measuring an arc either in the direction of a meridian, or in that of a parallel; and that when we have chosen between these, we must make the general direction of the chain of triangles coincide as nearly as possible with that of the line that we have chosen.

We shall now, from the properties of the spheroid, express the length of the lines above described in terms of the axes and ellipticity of the spheroid.

Radius of parallel in a spheroid, expressed in terms of the latitude.

Let fig. 21 be the generating ellipse of the spheroid, in the position in which its plane passes through the place P; P M the normal; P N, P W, perpendiculars to the axes. Let B C $= a$, A C $= b$. Then (Conic Sections, Art. 34.) N M $= \frac{b^2}{a^2}$ C N. Also P s being parallel to the axis, or in N P produced, and M P being produced to p, the angle p P s or N P M is the colatitude, and P M N the latitude of P; call this L. Then N P $= \text{M N} \tan \text{L} = \frac{b^2}{a^2}\,\text{C N} \,.\, \tan \text{L}$. Now (Conic Sections, Art. 26.) $\text{P N}^2 = \frac{b^2}{a^2}(a^2 - \text{C N}^2)$ or $\frac{b^4}{a^4}\,\text{C N}^2 \,.\, \tan^2 \text{L} = \frac{b^2}{a^2}(a^2 - \text{C N}^2)$, or $b^4 \,.\, \text{C N}^2 \,.\, \tan^2 \text{L} = a^4 b^2 - a^2 b^2 \,.\, \text{C N}^2$, whence

$$\text{C N}^2 = \frac{a^4}{b^2 \tan^2 \text{L} + a^2} = \frac{a^4 \cos^2 \text{L}}{b^2 \sin^2 \text{L} + a^2 \cos^2 \text{L}} = \frac{a^4 \cos^2 \text{L}}{b^2 + (a^2 - b^2) \cos^2 \text{L}}, \text{ and C N} = \frac{a^2 \cos \text{L}}{\sqrt{\{ b^2 + (a^2 - b^2) \cos^2 \text{L} \}}}.$$

Let $a = b\,(1 + e)$; $\text{C N} = b\, \frac{(1 + 2\,e + e^2) \cos \text{L}}{\sqrt{1 + (2\,e + e^2) \cos^2 \text{L}}}$. If we suppose e so small that its square may be neglected,

$\text{C N} = \frac{b\,(1 + 2\,e) \cos \text{L}}{\sqrt{1 + 2\,e \cos^2 \text{L}}} = b \cos \text{L}\,(1 + 2\,e - e \cos^2 \text{L})$; or $b \cos \text{L}\,(1 + e + e \sin^2 \text{L})$. This is the value of P W, the radius of the parallel passing through P.

Degree of parallel in terms of the latitude.

If then D be the difference of longitude expressed in seconds, of two places on the same parallel, and L their latitude, the length of the arc between them is $b \,.\, \cos \text{L}\,(1 + e + e \sin^2 \text{L})\, \text{D} \sin 1''$; D sin 1″ being the length of the arc, which with radius 1 subtends the angle D″. The length of the arc corresponding to a difference of longitude of 1° is $b \cos \text{L}\,(1 + e + e \sin^2 \text{L}) \,.\, 3600 \,.\, \sin 1''$; this is called *a degree of parallel.*

Radius of curvature of meridian in a spheroid, expressed in terms of the latitude.

The radius of curvature of the meridian is found by the expression $-\frac{\left\{1 + \left(\frac{d\,y}{d\,x}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2\,y}{d\,x^2}}$, where $x = \text{C N}$, $y = \text{N P}$.

But $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, or $b^2 x^2 + a^2 y^2 = a^2 b^2$; whence $\frac{d\,y}{d\,x} = -\frac{b^2\,x}{a^2\,y}$, and $\frac{d^2\,y}{d\,x^2} = -\frac{b^4}{a^2\,y^3}$; and the radius of curvature $= \frac{1}{b^4\,a^4} \times (a^4\,y^2 + b^4\,x^2)^{\frac{3}{2}}$. But we have just found $x^2 = \text{C N}^2$ or $\text{P W}^2 = \frac{a^4 \cos^2 \text{L}}{b^2 \sin^2 \text{L} + a^2 \cos^2 \text{L}}$; and

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* This angle may also be ascertained in other ways, of which we shall speak hereafter.

$y^2 = \mathrm{P\,N}^2 = \mathrm{N\,M}^2 \tan^2 \mathrm{L} = \frac{b^4}{a^4}\, \mathrm{C\,N}^2 . \tan^2 \mathrm{L} = \frac{b^4 \sin^2 \mathrm{L}}{b^2 \sin^2 \mathrm{L} + a^2 \cos^2 \mathrm{L}}$; whence the radius of curvature

$$= \frac{1}{b^4 a^4} \times \left(\frac{a^4 b^4}{b^2 \sin^2 \mathrm{L} + a^2 \cos^2 \mathrm{L}}\right)^{\frac{3}{2}} = \frac{a^2 b^2}{(b^2 \sin^2 \mathrm{L} + a^2 \cos^2 \mathrm{L})^{\frac{3}{2}}}.$$

If $a = b\,(1 + e)$, the radius of curvature $= \frac{b\,(1 + 2\,e + e^2)}{(1 + \overline{2\,e + e^2}\,.\, \cos^2 \mathrm{L})^{\frac{3}{2}}}$; or if e is so small that e^2 may be neglected, the expression becomes $b\,(1 + 2\,e - 3\,e \cos^2 \mathrm{L})$, or $b\,(1 - e + 3\,e \sin^2 \mathrm{L}.)$

If then λ and λ' are the latitudes of two stations on the same meridian, and if the difference between λ and λ' be not great, the included arc may be considered as a circular arc whose radius is the radius of curvature, corresponding to the middle point between them, or $b\left(1 - e + 3\,e \sin^2 \frac{\lambda + \lambda'}{2}\right)$. And if $\lambda' - \lambda$ be expressed in seconds, the length of the arc, subtending the angle $\lambda' - \lambda$, in a circle whose radius is 1, will be $(\lambda' - \lambda) \sin 1''$, and, therefore, the arc of meridian or the circular arc equal to it, will be $b\left(1 - e + 3\,e \sin^2 \frac{\lambda + \lambda'}{2}\right) . (\lambda' - \lambda) \sin 1''$. The length of the arc corresponding to a difference of latitudes of 1° is $b\left(1 - e + 3\,e \sin \frac{\lambda + \lambda'}{2}\right) . 3600 \sin 1''$; this is called *a degree of latitude.*

Degree of meridian in terms of the latitude.

If, however, the arc be considerable with respect to the radius, the following process must be used. If two normals be drawn at points whose latitudes are L and L + d L, they will intersect at the distance $b\,(1 - e + 3\,e \sin^2 \mathrm{L})$. Let s be the arc of meridian, $d\,s$ that part included betwen the latitudes L and d L, then $d\,s = d\,\mathrm{L} \times b\,(1 - e + 3\,e \sin^2 \mathrm{L}) = b\,d\,\mathrm{L} \times \left(1 + \frac{e}{2} - \frac{3\,e}{2} \cos 2\,\mathrm{L}\right)$, and $s = b \times \left(\overline{1 + \frac{e}{2}}\,\mathrm{L} - \frac{3\,e}{4} \sin 2\,\mathrm{L}\right)$. This expression gives the length of the arc from the Equator to the point whose latitude is L; if then we put λ and λ' successively in place of L, and take the difference of the expressions, we shall have for the length of the arc included between the latitudes λ and λ',

$$b\left(\overline{1 + \frac{e}{2}} . \overline{\lambda' - \lambda} - \frac{3\,e}{4} . \overline{\sin 2\,\lambda' - \sin 2\,\lambda}\right) = b\left(\overline{1 + \frac{e}{2}} . \overline{\lambda' - \lambda} - \frac{3\,e}{2} \cos \overline{\lambda' + \lambda} \sin \overline{\lambda' - \lambda}\right)$$

$$= b\left(\overline{1 + \frac{e}{2}} - \frac{3\,e}{2} . \cos \overline{\lambda' + \lambda} . \frac{\sin \overline{\lambda' - \lambda}}{\lambda' - \lambda}\right) . (\lambda' - \lambda).$$

Here (as in all expressions given by integration) it is supposed that $\lambda' - \lambda$ is measured by its proportion to the radius; if expressed in seconds, we ought to put $(\lambda' - \lambda) \sin 1''$, and thus the length

$$= b\left\{\overline{1 + \frac{e}{2}} . - \frac{3\,e}{2} . \cos \overline{\lambda' + \lambda} . \frac{\sin \overline{\lambda' - \lambda}}{\lambda' - \lambda}\right\} \times (\lambda' - \lambda) \sin 1''.$$

From the former expression it will be seen that in an oblate spheroid a degree of meridian (found by making $l' - l = 3600$) near the Pole, is greater than one near the Equator, and that, generally, on increasing the latitude the degrees increase. The general fact of the oblateness of the Earth will, therefore, be proved by the augmentation of degrees in approaching the Pole; a diminution would show that the Earth's form was prolate.

The length of the axis, and the oblateness, may thus be found from two measures. Suppose λ and λ' the latitudes of the extreme points of one arc, whose length, as determined by measures, = A; suppose μ, μ', and B, the extreme latitudes and measured length of another arc. Then

Ellipticity determined from two measures of meridian arcs.

$$\frac{\mathrm{A}}{(\lambda' - \lambda) \sin 1''} = b\left(1 - e + 3\,e \sin^2 \frac{\lambda' + \lambda}{2}\right),$$

$$\frac{\mathrm{B}}{(\mu' - \mu) \sin 1''} = \left(1 - e + 3\,e \sin^2 \frac{\mu' + \mu}{2}\right).$$

$$\therefore \frac{\mathrm{A}}{(\lambda' - \lambda) \sin 1''} - \frac{\mathrm{B}}{(\mu' - \mu) \sin 1''} = 3\,e\,b\left(\sin^2 \frac{\lambda' + \lambda}{2} - \sin^2 \frac{\mu' + \mu}{2}\right).$$

The value of b will be found nearly enough by neglecting e in either equation; thus $b = \frac{\mathrm{A}}{(\lambda' - \lambda) \sin 1''}$ or $= \frac{\mathrm{B}}{(\mu' - \mu) \sin 1''}$. Considering this then as known, e

$$= \frac{\frac{\mathrm{A}}{(\lambda' - \lambda) \sin 1''} - \frac{\mathrm{B}}{(\mu' - \mu) \sin 1'}}{3\,b\left(\sin^2 \frac{\lambda' + \lambda}{2} - \sin^2 \frac{\mu' + \mu}{2}\right)}.$$

Now, if an error should be committed in measuring either A or B, the influence of this error on the value of e would

Determination by Geodetic Measures.

depend entirely on the magnitude of the divisor $3\,b\left(\sin^2\frac{\lambda'+\lambda}{2}-\sin^2\frac{\mu'+\mu}{2}\right)$ It would be greatest when the divisor was least; and *vice versâ*. The most favourable combination, or that which would make $\sin^2\frac{\lambda'+\lambda}{2}-\sin^2\frac{\mu'+\mu}{2}$ greatest, would be one arc of which the middle was under the Equator, and another of which the middle was at the Pole. The latitudes of the middle points, or $\frac{\lambda'+\lambda}{2}$ and $\frac{\mu'+\mu}{2}$, being then, respectively, 90°, and 0, the quantity $\sin^2\frac{\lambda'+\lambda}{2}-\sin^2\frac{\mu'+\mu}{2}=1$. If the middles of the two arcs have nearly the same latitude, the divisor will be so small, and the consequent influence of errors so great, that no dependence can be placed on the value of e so determined. In general, the arcs should be separated by as wide an interval as possible. No advantage, however, is gained by measuring an arc beyond the Equator; as, on any supposition upon which investigations can be made, the degrees of meridian in South latitude ought to be equal to the degrees of meridian in the same North latitude. It is scarcely necessary to mention that if the arcs are long, and measured with accuracy, two equations should be made of the form

$$\frac{\mathrm{A}}{(\lambda'-\lambda)\sin 1''}=b\left\{\overline{1+\frac{e}{2}}\,.\,-\frac{3\,e}{2}\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}\right\},$$

and should be rigorously solved for the determination of b and e.

An arc of meridian, and one of parallel, may also be combined for the same purpose. For from a measured arc of meridian we shall have the equation

$$\mathrm{A}=b\left(1-e+3\,e\sin^2\frac{\lambda'+\lambda}{2}\right)(\lambda'-\lambda)\sin 1''.$$

Ellipticity determined from a measure of a meridian arc and a measure of an arc of parallel.

And if D be the difference of longitude (in seconds) between two places on the same parallel, whose latitude = L, and if the measured distance = C, we shall have the equation

$$\mathrm{C}=b\cos\mathrm{L}\,(1+e+e\sin^2\mathrm{L})\,\mathrm{D}\sin 1''.$$

Whence

$$\frac{\mathrm{A}}{(\lambda'-\lambda)\sin 1''}-\frac{\mathrm{C}}{\mathrm{D}\cos\mathrm{L}\sin 1''}=b\,e\,\left(3\sin^2\frac{\lambda'+\lambda}{2}-\sin^2\mathrm{L}-2\right),$$

and

$$e=\frac{\frac{\mathrm{A}}{(\lambda'-\lambda)\sin 1''}-\frac{\mathrm{C}}{\mathrm{D}\cos\mathrm{L}\sin 1''}}{b\,\left(3\sin^2\frac{\lambda'+\lambda}{2}-\sin^2\mathrm{L}-2\right)}=\frac{\frac{\mathrm{C}}{\mathrm{D}\cos\mathrm{L}\sin 1''}-\frac{\mathrm{A}}{(\lambda'-\lambda)\sin 1''}}{b\left(2+\sin^2\mathrm{L}-3\sin^2\frac{\lambda'+\lambda}{2}\right)}.$$

As before, b may be taken $=\frac{\mathrm{A}}{(\lambda'-\lambda)\sin 1''}$, or $=\frac{\mathrm{C}}{\mathrm{D}\cos\mathrm{L}\sin 1''}$. The denominator will be greatest when $\sin\mathrm{L}$ is nearly $=1$, and $\sin\overline{\frac{\lambda'+\lambda}{2}}=0$, or when the arc of parallel is near the Pole, and that of meridian near the Equator. It will be least when $\sin^2\mathrm{L}$ is nearly equal to $3\sin^2.\frac{\lambda'+\lambda}{2}-2$; such a combination ought therefore to be avoided. If both arcs be measured in nearly the same latitude, the divisor is $2-2\sin^2\mathrm{L}$, or $2\cos^2\mathrm{L}$. For such measures a place near the Equator should be chosen.

In the comparison of two arcs of meridian, it appears that the greatest divisor is $3\,b$; in the comparison of one arc of meridian and one of parallel, the greatest divisor is also $3\,b$. It would, at first sight, appear that these combinations were equally advantageous. But, in fact, the difference of longitudes cannot be ascertained with so great exactness as the difference of latitudes, and larger errors may be expected in the numerator when the arc of parallel is used, than when only arcs of meridian are employed. On this account the principal reliance is placed on the comparison of meridian arcs. We believe that two arcs of parallel have never been compared; but it is plain that if they could be measured with sufficient exactness, two arcs of parallel might be used to determine the axis and ellipticity.

Elements of Earth's form may be corrected from new measures.

Instead, however, of using two measures in the way which we have described, every new measure, when the elements of the Earth's form have been approximately determined, may be employed for the purpose of correcting previous determinations. This method is now extensively used, as it bears great analogy with the process commonly employed in correcting other astronomical elements; we shall therefore describe it here. By methods which we shall discuss in the next section, when two points are connected by a chain of triangles, and the latitude of one is known, the latitude of the other can be calculated with assumed dimensions of the Earth. But this latitude may also be observed; if the observed latitude differs from that calculated, it shows that the assumed dimensions are erroneous. Suppose, now, b and e to be the assumed axis and ellipticity; $b+\delta b$, and $e+\delta e$, the real axis and eccentricity; λ' the calculated latitude of the second point, $\lambda'+\delta\lambda'$ the observed latitude. Then the first calculation being made for a spheroid, we have

$$\text{Measured arc} = b\left(1+\frac{e}{2}-\frac{3\,e}{2}\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}\right)\times(\lambda'-\lambda)\sin 1''.$$

And as the same arc corresponds to the observed latitudes on the spheroid really existing we have (omitting $\delta\lambda'$ in $\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}$, in which it makes no sensible difference)

$$\text{Measured arc} = (b+\delta b)\left\{1+\frac{e+\delta e}{2}\left(1-3\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\lambda'-\lambda}{\lambda'-\lambda}\right)\right\}(\lambda'-\lambda+\delta\lambda')\sin 1''.$$

Taking the difference, and neglecting the products of δb, δe, $\delta\lambda'$,

$$0=\{(\lambda'-\lambda)\,\delta b+b\,\delta\lambda'\}\,.\left(1+\frac{e}{2}-\frac{3\,e}{2}\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}\right)\sin 1''$$
$$+\,b\,(\lambda'-\lambda)\left(\frac{1}{2}-\frac{3}{2}\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}\right).\,\delta e\,.\sin 1''.$$

Whence

$$0=\frac{d\lambda'}{\lambda'-\lambda}+\frac{d\,b}{b}+\frac{1-3\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}}{2+e-3\,e\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}}.$$

Form of equations of condition.

Or (as e is small)

$$0=\frac{\delta\lambda'}{\lambda'-\lambda}+\frac{\delta b}{b}+\left(\frac{1}{2}-\frac{3}{2}\,.\cos\overline{\lambda'+\lambda}\,.\,\frac{\sin\overline{\lambda'-\lambda}}{\lambda'-\lambda}\right)\delta e.$$

Every measure gives an *equation of condition* of this form; these are to be used in the same manner as similar equations in other astronomical investigations. A similar process can be extended to arcs of longitude.

Deviations from the elliptic form.

Hitherto, in the whole of this section, we have supposed the Earth's form to be a perfect spheroid. We shall now consider the indications which, using the same observations and the same measures, would show the existence of some deviation from that form.

Dissimilarity of Northern and Southern hemispheres.

If the Northern and Southern hemispheres of the Earth were dissimilar, the radii of curvature in corresponding North and South latitudes would be unequal, and therefore the lengths of degrees in corresponding North and South latitudes would not be the same. Conversely, if it is found that the length of a degree in any North latitude is not the same as that in equal South latitude, we must conclude that the Northern and Southern hemispheres are not similar.

Earth's form not a solid of revolution.

If the Earth were not a Solid of revolution, the different meridians would be different curves, and, therefore, at the same latitude, but in different longitudes, the degrees of meridian would be unequal. The parallels, also, would not be circles, and, consequently, different degrees of longitude on the same parallel would have different lengths. An inequality then either in the degrees of latitude or in those of longitude, depending on the difference of longitude, will show that the Earth's form is not one of revolution.

Form generated by the revolution of a curve differing from the ellipse.

But the Earth's form, even though perfectly symmetrical, may be formed by the revolution of some figure differing from the ellipse. The generating curve for instance, as A Q D, fig. 21, may project between A and D above the ellipse which has the same axes. This deviation of form would be thus discovered. It is plain that at A and D the curvature is diminished, or the radius of curvature increased; and at Q the curvature is increased, and the radius of curvature diminished. Consequently, the length of degrees near A, and near D, is greater, and near Q it is less, than in the elliptic spheroid. In the spheroid the degrees at Q are longer than those at D, and shorter than those at A. Consequently, the difference of the length of degrees at Q and D is diminished; and the difference of those at Q and A is increased. The ellipticity (as is easily seen in the formulæ above) is found by dividing the difference of degrees in different latitudes by a coefficient depending only on the latitudes. This process is properly applicable only to an elliptic spheroid; but if we use it for such a curve as A Q D, it is clear that an ellipticity greater than the true value will be found where the difference of degrees is greater than in an ellipse, and less than the true value where the difference is less. Consequently, by comparing two arcs measured at D and Q, and using the elliptic formula, we should get an ellipticity smaller than the truth; by comparing those measured at Q and A we should get a value greater than the truth. Conversely, if the comparison of arcs near the Equator, and arcs in middle latitudes, give a smaller value of the ellipticity than the comparison of arcs in middle latitudes, and arcs near the Pole, we shall be entitled to conclude that the Earth's form projects at middle latitudes beyond the elliptic spheroid which has the same axes. If the difference be of the contrary kind, we may conclude the Earth's form to be more flattened at middle latitudes than the ellipse which has the same axes.

Indicated by the comparison of three meridian arcs.

Or by the comparison of two meridian arcs with an arc of parallel.

The same deviation from the elliptic form might also be discovered thus. By a projection at Q above the elliptic form, the degree of parallel passing through Q, would be made greater than in the ellipse at the same latitude. Suppose the latitude of Q to be 45°, and suppose that two meridian arcs were measured of which the middle points were in latitudes 0°, and 45°, respectively. For $\frac{A}{(\lambda'-\lambda)\sin 1''}$ in the equatorial arc put f, and in the other f'; and for $\frac{C}{D\cos L\sin 1''}$ put g. Then the values of the ellipticity found by comparing each of the meridian arcs with the arc of longitude would be as follows:

$$\text{by the Equatorial arc } \frac{2}{5} \cdot \frac{g-f}{b},$$

$$\text{by the other } 2\ \frac{g-f'}{b}.$$

Now g and f are greater than in the ellipse, and f' is less. The ellipticity therefore given by comparing the arc of parallel with the Equatorial arc of meridian, might, or might not, be too great; but that given by comparing it with the other arc would certainly be much too great. That is, a larger value of ellipticity would be found by comparing the arc of parallel with the arc of meridian in latitude 45°, than by comparing it with one measured across the Equator. If then we find, on comparing an arc of parallel nearly in latitude 45° with arcs of meridian near the Equator, and near latitude 45°, and applying the elliptic formulæ, that the first comparison gives a smaller value for the ellipticity than the second, we may conclude that the Earth's surface at latitude 45° is further from the centre than if the form were that of an elliptic spheroid.

Having thus explained the mode of using measures of meridians and parallels for the determination of the Earth's Figure, we shall now proceed to give an account of the principal measures which have been made.

Section 4.—Measures of Arcs of Meridian.

It will not be expected that we should lay before our readers a complete account of all the measures that have been made. Our limits will not allow this; nor is it by any means necessary. It is our object to give such information as will enable any one to understand the calculations of the several arcs, to appreciate their exactness, and to estimate their importance as conducing to our knowledge of the Earth's form. We shall touch lightly on those measures, which though on a large scale, have not been conducted with the extreme care requisite in such delicate operations, but we shall describe fully the methods, and explain accurately the theories of those which have been adopted, almost by general consent, as the only ones proper to determine the Figure of the Earth. For details we must refer to the original accounts.

Of the rough estimations of the Greek Astronomers, and the somewhat less rough measure of the Arabian Caliph, we have given in the first section as full an account as they deserve. The still more accurate measures of Fernel and Norwood have received sufficient attention. As specimens of attempts made before good instruments were invented, or refined theories formed, they are curious; and their results are not so inaccurate as might, at first sight, be expected. But the great principle of triangulation was then unknown; and little reliance can be placed on the measures in which this was not adopted. We shall commence with a general account of the methods followed in those measures in which triangulation has been used.

The first part in order of calculation (though not always the first in order of time) is the measure of a base. The method which has generally been pursued is something like this. The extremities being chosen, (the requisites of which are, a sufficient distance, as four or five miles on a pretty level plain, and the power of seeing the same two signals, or at least one, at each extremity,) a telescope, mounted like a transit instrument, is placed at one extremity, and an observer, by means of it, is enabled to direct an assistant to fix stakes in a straight horizontal line, or rather in a vertical plane, on the whole length of the base line. These serve as guides in the actual measure. The base is sometimes measured roughly with a common chain, to prevent any serious error of omission in the subsequent measure, and to point out the places at which it is necessary (in some methods) to fix posts for the accurate measure.

Preparations for measure of base.

For this purpose fir-rods were universally employed till the commencement of the English survey. From some experiments made by General Roy, it appeared that they were subject to considerable alterations of length from moisture; they were laid aside, and glass tubes (and ultimately a steel chain) employed. But the reader must not suppose that these alterations had never been observed before, or that the base-measures anterior to this time were affected by the errors which they would occasion. It was usual in the former measures to keep an iron rod of a toise (a little more than a fathom) in length, as the real standard, and to compare it several times a day with the wooden rods with which the work was actually performed. As the length of the rods in general was several times that of the iron toise, brass pins were fixed at distances as nearly as possible equal to the length of the standard, upon which fine dots were made. The comparison was made by means of *beam-compasses*, (a long rod having at one end a fixed point perpendicular to its length, and at the other a point attached to a frame movable along the rod by means of a micrometer-screw.) The beam-compasses being so adjusted by the standard, that the distance between their points was exactly a known length, the fixed point was applied to one of the dots on the wooden rod, and if the movable point would not touch another it was moved by the micrometer-screw till it came in contact, and thus the difference of length was ascertained with great accuracy. This continual comparison was very troublesome, but we apprehend that little has been gained in point of accuracy by the use of the subsequent methods. When this was done, it is plain that the measured length was the same as if it had been effected with the metallic standard; and, consequently, the same corrections for the expansion or contraction of the metal from change of temperature were necessary.

Rods used for base-measure.

Manner of placing the rods.

Several rods being provided, they were sometimes laid on the ground, but more frequently were supported on trestles. The end of the first was adjusted to the extremity of the base by a plumb-line; the others were in some instances brought in contact; but more frequently they were placed so as to leave a small interval, which was measured by a small graduated tongue or slider attached to one, and which was moved till it touched the next. Sometimes all were placed horizontal by a carpenter's square, or a spirit-level; then, if the ground was not level, it was necessary to suspend a plumb-line from the extremity of one, and to make it pass over a given point of the next. Sometimes they were inclined so as always to bring the ends in contact; then the inclination was measured, and allowance made for it afterwards. One principle was adopted in every instance, (except that of the English chain-measure,) that at least two rods should be in their places when a third was applied. Of the different methods, we cannot say that any one is preferable to the others.

Reduction for elevation above the sea.

The arc being thus measured, and the reduction for inclination applied, (if the rods were inclined,) and the correction for temperature being also applied, we have the sum of a series of lines similar to the short lines in fig. 22, measured with the standard bar at the standard temperature. Each of these lengths ought now to be reduced to the projection on some surface (the level of the sea has always been chosen) by lines drawn in the direction of Gravity. If r be the Earth's radius or the radius of the surface of the sea, (which is known nearly enough,) h the elevation, the measured lengths must be multiplied by the fraction $\frac{r}{r+h}$, or $1-\frac{h}{r}$, or they must be diminished by the part $\frac{h}{r}$ of the whole. If the surface slopes uniformly, the mean height may be taken; if it is very irregular, it may be divided into several parts. The reduced length $a\,b$, fig. 22, of the base A B, is now found; if the length of the chord be required, it is easily found by subtracting $\frac{\mathrm{A\,B}^3}{24\,r^2}$.

Signals.

The signals at the principal stations have generally been flagstaffs, poles supported by piles of stones, pyramids of wood, broad plates of wood or metal with a hole through which the sky was visible, or Bengal lights; placed for the purpose on the same spot as the instrument with which the angles were observed. Sometimes advantage has been taken of church spires, windmills, &c., near which the quadrant or theodolite could be placed; a reduction is then necessary, to make the angle such as it would have been if the instrument had been placed exactly on the signal. Thus, in fig. 23, suppose E and F signal stations, from which the signal G has been observed; the instrument cannot be placed at G. but is placed at g; the length Gg, and the angle GgE, GgF, are observed. The real angle E G F is greater than the observed angle E g F by G F g − G E g. And $\sin \mathrm{GF}g = \frac{\mathrm{G}g}{\mathrm{GF}} \sin \mathrm{G}g\mathrm{F}$, which is easily calculated, as an approximate knowledge of G F is sufficient. Similarly G E g is calculated. Thus the angle is found which would have been observed at G.

Measure of angles between signals.

The instruments which have been used for measuring the angles are quadrants, theodolites, and repeating-circles. When we consider the various errors to which quadrants are liable, it seems strange that the use of whole circles should not have been sooner introduced. But these evils (like most others) can be evaded if sufficient labour is bestowed upon them. The principal errors to which they are liable are, inequality in the parts of the graduation; and eccentricity, by which a further apparent inequality is created, and the proportion which the whole arc bears to the circumference is vitiated. The last defect was discovered and measured by placing the quadrant in a situation where the horizon was clear, and observing the apparent angular distance from an object A to another B, from B to C, and so on to A again, when the sum of all ought to equal 360°. The first was found by measuring the same angle on different parts of the quadrant, or by some equivalent method. When these points were attended to, we doubt whether the defects peculiar to the quadrant produce errors of any importance. The English theodolite and the French repeating-circle are undoubtedly superior, chiefly because the art of making instruments is improved. In respect of accuracy we should be inclined to prefer the theodolite; but the portability of the repeating-circle is a most valuable quality, especially in the survey of a mountainous country.

Verification quadrant.

Angles reduced to horizontal angles.

With the quadrant and the repeating-circle, the angle contained between two signals was observed; with the theodolite, the horizontal angle was observed, and not the angle actually subtended by the signals, except both were in the horizon. In some of the earlier surveys, the true angles, unreduced, were used, and the length of the opposite sides was calculated, and this process was continued through the whole chain of triangles. For the subsequent calculations, however, it was necessary to reduce them to horizontal angles. Here it is plain that the sides of the triangles were the actual lengths from one station to another, measured in straight lines, which would be in general inclined to the horizon. But in most instances the observed angles have been reduced to horizontal angles by the approximate method given in Trigonometry, Art. 181.; sometimes by rigorous calculations of the triangle, whose angles are at the zenith, and the two signals. Then the triangles were generally calculated as plane ones. The accuracy of the English survey made the difference between plane and spherical triangles sensible; and the English reduced the spherical triangle to a plane one, whose sides were the chords joining the stations, or rather the projections of the stations on the spheroid, whose surface is the level of the sea, by the formula in Trigonometry, Art. 182. The French calculated the triangles by Legendre's formula, given in Trigonometry, Art. 183. The first of these methods has been used in the whole of the English survey, and in that of India; the second has commonly been employed on the Continent since the year 1780. The spherical excess, it must be observed, is the same on a spheroid as on a sphere, where the latitudes and difference of longitudes of the stations are the same; of this we shall give a demonstration when we treat of the determination of differences of longitude.

Calculation of triangles.

Determination of azimuth of one signal.

The next step is, to determine the direction of one of the sides with regard to the meridian. The general principle is this. The error of a clock or chronometer with respect either to solar or to sidereal time, may be found by means of a transit-instrument, or by observations of equal altitudes, or by absolute altitudes if the latitude of the place be well known. It is easy to calculate the azimuth of a star or the Sun at any given time; and, consequently, (the clock-error being known,) at any given clock-time. And if the star or sun be used when near the horizon, the angle made with another object will not be much affected by the uncertainty of refraction. At a signal station, therefore, the angle made by another signal with some celestial body near the horizon is observed, and the time noted; by applying the clock-error the true time is found; the azimuth at the instant is calculated; and by adding or subtracting the observed angle the azimuth of the station is found. The Sun, at rising or setting, or some circumpolar star, as Capella, passing the meridian below the Pole, has generally been used. The azimuth of Polaris, at its extreme digressions, has been observed with a theodolite; or the distance of Polaris from a signal, when apparently nearest to it or furthest from it with a repeating-circle. The former method requires extremely good adjustment of the theodolite; the latter, a very precise knowledge of the latitude and the refraction. The following method seems preferable where it is practicable. A temporary mark is fixed as nearly as possible in the meridian. A transit-instrument is adjusted upon it, and the transits of stars at different Polar distances are observed. The deviation of the transit-instrument, or the azimuth of the mark, can be determined thus with extreme accuracy. By means of the theodolite or repeating-circle, the angle between this mark and one of the signals can be observed, and thus its azimuth can be found. The Baron de Zach has suggested the observation of what he calls circum-meridian observations of the Sun. When the Sun is near the meridian his motion in azimuth is nearly uniform, and can easily be calculated. Repeated observations can be made; and the time being noted, his azimuth can be calculated more easily than when he is in any other part of his course. It is necessary that the clock-error be determined by transits of the Sun; for the Solar Tables (Delambre's or Carlini's) are not sufficiently exact to give his azimuth at a time determined by transits of stars, and the observation of equal altitudes is not sufficiently accurate. If night-signals can be used, the observation of a circumpolar star under the Pole would possess the same advantages without the same liabilities to error.

Calculation of arc of meridian.

We are now able to calculate the distance of the parallels which pass through the extreme stations of the chain of triangles. This, in the light in which we have considered these measures, is the sole object of the survey; but in point of fact nearly all the measures of arcs have been parts of surveys for mapping the country. Perhaps it is owing to this circumstance that the method of parallels and perpendiculars has been so much used. In the first extensive survey of France (of which Picard's arc was the beginning, and the arc measured and remeasured by J. Cassini and Cassini de Thury a continuation) an imaginary meridian line was drawn through Paris, and lines perpendicular to it were drawn across the kingdom at intervals of 60,000 toises, and chains of triangles were carried as nearly as possible along them; which were the foundation of the grand survey of France. In the English survey, meridian lines were traced through eight of the principal stations; and though the course of the triangles was not particularly directed by them, the places of all the stations were referred to them. It is plain that no method of determining the situation of a place could be more convenient than to give the distance from the fundamental point along a given line to the foot of the perpendicular passing through the point in question, and to give the length of that perpendicular; it is in fact the same as giving the rectangular coordinates x and y of the point. It is also convenient for the meridian measure. The only point of importance to be attended to is, that the perpendiculars be not so long that the difference between the spherical length of the sides, and their length when projected on the tangent plane, (or rather the tangent cylinder, which touches the sphere in the line to which the points are referred) become sensible. This was secured in the mapping surveys by using several lines of reference instead of one; and in the chain of triangles used for a meridian measure there is no fear that the perpendiculars will be too long.

Method of parallels and perpendiculars.

Now let fig. 24 represent such a chain of triangles with the perpendiculars drawn from the stations upon the meridian passing through A one extremity. The length of A B, and its azimuth, (or the angle B A b,) are known, and therefore, A b and b B can be found; and similarly A c and c C. Through B draw g B h parallel to b A. The angle A B h is equal to B A b, and C B A and C B D have been observed; hence D B g = A B h + C B A + C B D − 180° is known. Also D B is known; and hence D g and g B can be calculated. Adding D g to g d or B b, which has been calculated, we have D d; and adding g B to b A we have A d. These can also be determined from the position of C by the length and direction of the line C D; and if the two results differed it would be proper to take the mean. Then the meridian distance and perpendicular for F, would be found from those at C and D; and so on to the end. It must be observed that g B h is not the meridian passing through B, and therefore D B g is not the azimuth of D as seen from B; it differs from it by an angle termed *the convergence of meridians*, of which we shall speak hereafter. We may also remark that there is no inconsistency in taking into account the spherical excess, while we neglect here the difference between the lengths of lines on a sphere or spheroid, and the corresponding lines on a cylinder; the former varies, *cæteris paribus*, as the square of the sides, and the latter as the cube, and therefore, when the triangles are very small in comparison with the Earth's radius, the latter is smaller in proportion to the quantities retained than the former.

Distance of feet of perpendiculars calculated.

Now let F be the last of the stations. The distance A f is not the distance between the parallels passing through A and F. For produce A f to P the Pole of the Earth, and describe a small circle, F f', of which P is the centre. Then A f' is the real distance between the parallels, and ff' must be subtracted from the ascertained length A f. This quantity may be calculated in the following manner. F f is part of a great circle, (since it is produced in a vertical plane,) F f' is part of a parallel. In fig. 25, we have represented the chain of triangles on the surface of the sphere, or spheroid; with the plane of the great circle F f G M, and that of the small circle F f' G N, intersecting in the straight line F m G. Fig. 26 represents the plane of the circle P A; m is the point at which the radius of the great circle and the small circle intersect. Now $ff' = mf \,.\, \tan fmf' = mf \,.\, \tan f\mathrm{K}e = mf \,.\,$ tan latitude of f; or $= mf$. tan latitude of F (the difference being very small.) But mf is evidently the versed sine of the arc of F f to radius K f, and therefore

Difference between perpendicular and small circle.

$$= \frac{\overline{\mathrm{F}f}|^2}{\text{Earth's diameter}};$$

consequently,

$$ff' = \frac{\overline{\mathrm{F}f}|^2 \,.\, \text{tan lat. of F}}{\text{Earth's diameter}}.$$

We believe that this would be the easiest way of making the calculation. But it has generally been done thus. From the approximate knowledge of the Earth's dimensions find the arc of a great circle in degrees and minutes corresponding to the length F f. Consider P F f as a right-angled spherical triangle, of which the side P f will be determined by this formula,

$$\cos \mathrm{P}f = \frac{\cos \mathrm{PF}}{\cos \mathrm{F}f} \text{ (TRIGONOMETRY, Art. 105.)}$$

Take the difference between P F and P f in degrees, and from the assumed radius of the Earth convert it into fathoms. This method appears troublesome, it requires logarithms of 7 places; that above would not require logarithms of more than 4 places.

Convergence of meridians.

We may now easily see what is meant by the convergence of meridians. In fig 24, or 25, join P F by an arc of great circle. Then P F is not perpendicular to F f, and therefore in fig. 24 it is not parallel to P f. The angle is easily found by solution of the same spherical triangle, P F f, thus, $\cos \mathrm{PF}f = \frac{\tan \mathrm{F}f}{\tan \mathrm{PF}}$. (Trigonometry, Art. 109.) If now (as frequently happens) the azimuth of B had been observed at A, and that of E at F, and it was desired to know how nearly one coincided with the result calculated from the other, the process to be used would be this. Since fF E is known, (by the antecedent calculations,) and A F f is known, (by the solution of the spherical triangle just given,) their sum P F E is known; and this is the angle to be compared with the observed azimuth.

Delambre's method of calculating arcs of meridian.

The method invented by Delambre, and used in the measure of the French arc commenced in 1792, and in most of the continental surveys since that time, is certainly more elegant, and, perhaps, in some respects, more convenient than that which we have described. The triangles were calculated as spherical triangles, and therefore the spherical excess was computed for no other purpose than to discover the sum of the errors in the observation of the angles. When the side a, and the two adjacent angles B, C, of a triangle are known, and its area roughly calculated, from which the spherical excess x is found, then A = 180 + x − B − C, and

$$\mathrm{Sin}\, b = \frac{\mathrm{Sin}\, a \,.\, \sin \mathrm{B}}{\sin \mathrm{A}}, \quad \mathrm{Sin}\, c = \frac{\mathrm{Sin}\, a \,.\, \sin \mathrm{B}}{\sin \mathrm{A}},$$

where by Sin a we mean the sine in the circle whose radius = R = Earth's radius. Now $\mathrm{Sin}\, a = a \frac{\mathrm{Sin}\, a}{a}$; and a is a quantity whose logarithm is found easily from the common tables, and $\log \frac{\mathrm{Sin}\, a}{a}$ is a quantity that varies very slowly, and can be taken from a very small table constructed for the purpose. Now the general problem is this. Given the Earth's dimensions to tolerable accuracy, the length of A B, fig. 27, the angle P A B, and the latitude of A, to find the difference of latitudes of A and B, the distance of the parallels, the difference of longitudes, and the angle P B A. Draw A Q the normal at A, B S that at B. As the Earth's form is not spherical, these normals will not (in general) meet the axis P S in the same point. Join B Q, A S; and suppose that a sphere described from the centre Q with radius 1, cuts the lines Q P, Q A, Q B, in p, a, b. The angle at P, or the difference of longitudes will be the same as the angle at p; that observed at A will be the same as that at a, since each is

Azimuth at second station may be calculated as on a sphere.

the inclination of the planes A Q P, A Q B; but the angle observed at B will not be the same as the angle $p\,b\,a$. For suppose the angle measured with a theodolite; its axis must be vertical; that is, it must be in the direction B S; consequently the angle which it measures is the angle made by the planes B S A, B S P; whereas $p\,b\,a$ is the angle made by the planes B Q A, B Q P. We have now to inquire, Are the planes B S A, B Q A, equally inclined to the plane B P Q S? In general they are not; but we can investigate their difference thus. Draw A a' perpendicular from A on the meridian plane P B; and from a' draw $a's'$, $a'q'$, perpendicular to B S, B Q. This is more clearly represented in fig. 28. Then the plane A $a's'$ is perpendicular to B S, and therefore the tangent of the angle observed at B is $-\frac{\mathrm{A}\,a'}{a'\,s'}$. Similarly the tangent of the angle $p\,b\,a$ is $-\frac{\mathrm{A}\,a'}{a'\,q'}$. Hence tan $p\,b\,a$ = tan angle at B $\times \frac{a'\,s'}{a'\,q'}$. These will be absolutely equal when $a'\,q' = a'\,s'$; or when the elevation of A actually observed at B is equal to the depression which would be observed if the axis of the theodolite were in the line B Q; that is, if A be very near the horizon of B. The error of the assumption, that $p\,b\,a =$ the angle observed at B, depends therefore entirely on this circumstance, that a perpendicular at A, when observed at B, does not appear perpendicular. This, however, is not discoverable by the senses when assisted by the best instrument; and, therefore, we may at once assume that $p\,b\,a$ is equal to the azimuth of A as observed at B. Now, in the spherical triangle $b\,a\,p$, we have given $p\,a =$ colatitude of A; $b\,a\,p =$ observed azimuth; $a\,b = \frac{\mathrm{A\,B}}{\mathrm{A\,Q}}$, whose determination therefore requires a knowledge of A Q as well as of A B (if the form be assumed to be elliptical A Q = P V, in fig. 10,

$$= \frac{\mathrm{C\,N}}{\mathrm{R\,N}} \,.\, \mathrm{P\,R} = \frac{a^2}{b^2}\ \mathrm{P\,R}$$

$$= \frac{a^2}{\sqrt{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}} = b\,(1 + 2\,e - e \cos^2 \lambda),$$

if we neglect powers of the ellipticity) and a small table of the value of log A Q at different latitudes must be made: it varies very slowly. Then solving the triangle as a spherical triangle, for which the method in Trigonometry, Art. 186. was generally used by Delambre, and is shorter than the accurate computation, we have the azimuth P B A = $p\,b\,a$; and the distance $a\,b''$ of the parallels passing through a and b; whence the distance A b' = A Q × $a\,b''$ is found. For a survey of meridian lines these two determinations (the azimuth of A as seen from B, and the distance in linear measure between their parallels) are all that are necessary; but as it is always desirable, and in mapping necessary, to determine the latitudes and longitudes of the stations, on assumed dimensions of the Earth, the following must also be calculated. The difference of longitudes B P A, which is the same as $b\,p\,a$; and the difference of latitudes, which is obtained by dividing the distance between the parallels by the radius of curvature of the meridian at the middle point, or (in the ellipse)

Distance of parallels calculated.

Difference of longitudes and of latitudes calculated.

by

$$\frac{a^2\,b^2}{\left(b^2 \sin^2 \frac{\lambda + \lambda'}{2} + a^2 \cos^2 \frac{\lambda + \lambda'}{2}\right)}$$

or

$$b\left(1 - e + 3\,e \sin^2 \frac{\lambda + \lambda'}{2}\right).$$

We will remark here, once for all, that the theory of the shortest lines that can be traced on an ellipsoid between two points has no application to the sides of Geodetic triangles, as some writers have supposed. The lines which determine the azimuths, &c. are the lines which represent the rays of light passing from one signal to another; and light does not move in the shortest curves that can be traced on ellipsoids, but in straight lines. In consequence of this, (as may be observed in the theory just given,) the truth or falsehood of some of our propositions will depend on the relative elevation of the signals. And in all the more delicate parts of the theory, we must consider the signals as being elevated above the curved surface. Thus Delambre treated the subject; and thus it had been treated long before in some masterly Papers by Dalby, in the *Phil. Trans.* for 1791 and 1795, which are transcribed in the first volume of the *Account of the Trigonometrical Survey.* If we chose to take into account the terrestrial refraction, and to suppose that refraction always bends the light towards such a *shortest line,* we ought to correct our results by about $\frac{1}{12}$th of the difference between them and those obtained by the theory that we have rejected; but such a refinement would be ridiculous.

Theory of shortest lines does not apply in Geodetic theorems.

Determination of difference of latitudes by observation.

We have here given every thing necessary for determining the distance between the parallels passing through the extreme stations of the chain of triangles; and the Geodetic part is completed. We now come to the Astronomical determination of the latitudes at the extremities; a determination on the accuracy of which the exactness of the whole operation depends, and whose difficulties far exceed all the difficulties of the other determinations

A small error in the absolute latitudes is of no consequence.

From the description of the general principle which we have given in the Ist section, and from the formulæ in the IIId section, it will appear that if the difference of latitudes of the extreme stations be accurately determined, the radius of curvature at the middle point will be known; and, if a small error be committed in fixing on the absolute latitudes, (from which the latitude of the middle point is found,) it will have no sensible effect on the resulting dimensions of the Earth. For if we take the expression which gives the ellipticity by comparison of two measured arcs,

$$e = \frac{\frac{A}{(\lambda' - \lambda)\sin 1''} - \frac{B}{(\mu' - \mu)\sin 1''}}{3\,b\left(\sin^2\frac{\lambda' + \lambda}{2} - \sin^2\frac{\mu' + \mu}{2}\right)},$$

in which the numerator is certainly small, and the denominator is not generally small, it is evident that the value of the denominator will not be much altered by a small alteration in $\frac{\lambda' + \lambda}{2}$; for instance, if

$$\frac{\mu' + \mu}{2} = 0, \text{ and } \frac{\lambda' + \lambda}{2} = 45°,$$

an error of 1 in $\frac{l' + l}{2}$ would only alter the denominator from $3\,b\sin^2 45°$ to $3\,b\sin^2 44° 59'$. But an error in $\lambda' - \lambda$ will produce a very serious effect in the numerator; as if $\lambda' - \lambda = 1°$, an error of $1'$ will alter $\frac{A}{(\lambda' - \lambda)\sin 1''}$ from $\frac{A}{3600 \sin 1''}$ to $\frac{A}{3540 \sin 1''}$, or will increase it by $\frac{1}{60}$th part; and if $\frac{A}{(\lambda' - \lambda)\sin 1''}$ exceeds $\frac{B}{(\mu' - \mu)\sin 1''}$ by only $\frac{1}{360}$th part, it will from the effects of the error exceed it by $\frac{7}{360}$th part, or the resulting ellipticity will be seven times as great as before. In estimating, then, the accuracy of the observations for determining the latitudes, we have no need to inquire into the exactness of the determination of the absolute latitudes, but only into that of the difference of latitudes.

An error in the difference of latitudes is very important.

The instrument with which the difference was commonly determined in the surveys previous to 1787 is the zenith-sector. With this instrument the zenith-distance of stars that pass near the zenith can be ascertained with great accuracy. At the two extremities of an arc the same stars were observed at their meridian passage; and, without any knowledge of their declination, the difference of their zenith-distances was found; which is equal to the difference of the Astronomical latitudes. As much of the credit of several of the measures depends on the capability of this instrument, we shall give a general description of it, and of the method of using it. In fig. 29, A B is a bar of iron with a cross-piece C D, the whole in one piece. The top, A, is formed in such a way that the instrument can be turned half round in azimuth when suspended at the top, and that the bottom can be moved freely in the directions D C, or C D. It might be made as represented in the figure; or it might move on a hinge for its ordinary motion, and in a swivel for its azimuthal motion. The bracket, or other support, E, on which it rests, is attached to some firm part of the building. To the bar, A B, is firmly attached a telescope, F G. At a point, *a*, near A, is attached a plumb-line, *a* H; sometimes it is fastened at a point of attachment which is movable, in order that, by moving the point of suspension, the plumb-line may be made to pass over a fine dot at *a*. The limb, CD, is graduated, sometimes on a circular arc, of which *a* is the centre, and sometimes on a straight line. L D is a screw passing through a block strongly connected with the floor, and acting with its point against the end D of the piece C D; and M N O is a string and weight pressing the sector against the point of the screw.* The method of using it is this. By transits, or equal altitudes, a clock is regulated to solar or sidereal time; and then, the time at which celestial bodies pass the meridian being known, the meridian line is easily found with tolerable correctness. A cord is stretched across the room, and this serves well enough (as great accuracy is not necessary) to direct the astronomer in fixing guides which will compel the sector to move in the direction of the meridian. When a star is to be observed, the screw L D is turned till the plumb-line falls exactly on some point, K, of the graduation, such that the telescope is very nearly directed to the star; and the number of

General description of the zenith-sector.

Method of using the zenith-sector

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* It will readily be understood that this description, and the figure belonging to it, are intended to illustrate only the general principle of this instrument, and that most of the details, which can be supplied without difficulty by those conversant with instruments, are wholly omitted. For instance, the screw generally acts in a movable piece which is clamped in any convenient position, so as to diminish very much the range of the screw. The construction represented in the figure is probably the worst form in which the instrument has ever been used.

turns and fractions of a turn (as shown by a scale and divided head, not represented in the figure) are read. When the star enters the field of view, the screw is again turned, till it appears to glide exactly along the wire fixed in the focus of the eye-glass, and the turns and parts of a turn again read. The motion of the screw being thus ascertained, its value in degrees and minutes is to be added to the value of the graduation, K, and thus the apparent zenith-distance of the star is found.

Reversion necessary.

Now, it is evident that this is not the true zenith-distance, except the place of the division 0, be quite correct. It is impossible to insure this; but it is easy to see, that if the instrument be turned half round in azimuth, and the same observation made, the apparent zenith-distance will be just as much greater than the true, as in the former observation it was less. The mean of the two determinations will be the true zenith-distance.

For accuracy it is indispensably necessary, that the telescope F G preserve the same position with regard to the frame in the observations before and after reversion. In the best sectors this has been secured by omitting the bar, A B, entirely, and attaching the limb, C D, to the tube of the telescope. In several, instead of turning the screw, L D, to bring the wire upon the star, the wire is moved by a micrometer-screw within the telescope. And in some, a graduated scale slides in a groove on the arm, C D, and when the instrument has been moved till the star glides along the middle wire, the scale is moved by a micrometer-screw till a division is brought under the plumb-line, and the reading of the micrometer-head is then registered.

Advantages of this instrument.

We request the reader to notice the construction of this instrument, because all the discordances between ancient and modern measures on the same line, appear to be owing to the difference of determinations with the zenith-sector and the repeating-circle. We confess that to us it appears impossible, when the instrument is made with any reasonable care, that the error in its results can be at all serious. The stars observed are in a part of the heavens where the tremor and dancing, which generally affect stars in other positions, are seldom seen; there is no uncertainty about the effects of refraction; the telescope and the whole instrument are not subject to flexure; the variation of temperature produces no sensible effect. We may be excused for reminding the reader, that not only the existence but the magnitude of aberration and nutation were discovered by means of a zenith-sector, and that subsequent observations have added little to the accuracy of the determination.

History of the repeating-circle.

The repeating-circle was invented late in the last century, and was first introduced to the notice of the scientific world by its employment in the French part of the survey for connecting the Observatories of Greenwich and Paris. It was again used in the measure of the arc from Dunkirk to Barcelona, conducted by Delambre and Mechain. From this time, it attained among the French a degree of popularity which can hardly be imagined. The minds of Philosophers, not less than those of Politicians, seem at that period to have been distracted with the idea of obtaining every thing desirable by the application of one sweeping principle. They imagined that the effects of errors of division and errors of observation might be entirely destroyed by repetition, and that no new error would be introduced in their place.* And from merely reading the French Works on Astronomy, or almost any other Science, published early in the present century, one might almost imagine that it was impossible to make an Astronomical or Optical observation without a repeating-circle. This instrument, however, was never much used by English observers. This has been remarked by M. Littrow, (in Zach's *Correspondance Astronomique*,) with the commentary, "the English are a practical nation;" which we believe to be the true explanation.

Discordance of latitudes of Barcelona and Montjouy

One of the first observations which shook the credit of the repeating-circle occurred in the course of the French survey of 1792. In consequence of the breaking out of a war between France and Spain, Mechain, who undertook the survey of the Southern part of the chain, was detained in Spain. He had before observed the latitude of Fort Montjouy, about a mile from Barcelona, and he now wished to repeat the observations; but as he was not allowed to enter the fort, he observed the latitude of Barcelona, and connected it by a small triangulation with Montjouy. The latitude of Montjouy, thus obtained, differed more than 3″ from that found by observation at the place. The chagrin which he felt after attempting in vain to reconcile these observations, undoubtedly contributed to accelerate his death. It was not till the nature of the instrument was better understood that Astronomers had it in their power to explain this apparent inconsistency.

Constant error of repeating-circles.

It is now universally acknowledged, that all repeating-circles are liable to an error which cannot be removed by any number of repetitions, and which, on that account, is called the constant error: "*constante toutefois jusqu'à un certain point: car il suffit souvent de transporter l'instrument d'un lieu dans un autre, ou de le démonter, pour produire un changement dans le rapport matériel des pièces, et par suite une variation dans la quantité de l'erreur.*" We cite this passage from an interesting Memoir by M. Nicollet on the discordances above mentioned, in the Additions to the *Conn. des Temps*, for 1831. It contains nearly all that is known upon the subject;† for the causes of the error are still in obscurity. We beg to refer the reader who wishes to inquire more accurately into the nature of the repeating-circle, to a Paper by Mr. Troughton in the Ist volume of the *Memoirs of the Astronomical Society.* The only mode of observation which appears to be a probable, but not certain, method of removing the error is, if the latitude of a place is the result sought, to observe the zenith-distances of stars passing the meridian at nearly equal distances on both sides of the zenith; the mean of the results will probably be near the truth.

Probable method of obviating its effects.

To make this more intelligible we shall give a short

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* It is said to have been the opinion of a celebrated French Mathematician engaged in the survey of 1787, that it was no longer necessary to bestow any care on making the triangles well-conditioned, as all possibility of error in the angles would be removed by the use of the repeating-circle.

† In the *Exposé des travaux relatifs à la reconnaissance hydrographique des côtes occidentales de France*, p. 86, M. Daussy thinks, that the constant error in observing horizontal angles is proportional to the angle. In some cases, however, he admits that the observed angles could not be corrected by this hypothesis. The Baron de Zach (*Correspondance*, vol. ii.) mentions a repeating-circle with which the observed angles decreased continually; and facetiously insinuates, that if he had continued the repetition of observations of a right angle for several years he should have reduced it to nothing.

Method of observation with the repeating-circle.

account of the mode of observing with this instrument. Figs. 30 and 31 represent it as used for finding the angular distance of two objects, A and B, in any positions, (the stand, &c., being omitted for clearness.) Both telescopes are movable on the circle, which is graduated on one side only; and the telescope on that side carries one, or (more generally) four verniers. Suppose now in fig. 30, C, the telescope on the graduated side, is directed to the object B, and D, the other telescope, is pointed to A; in this state the vernier or verniers of C are read off. Then the whole circle is turned in its plane till the telescope D (which remains clamped to it) is directed on B; and the telescope C is then released and turned, while the circle remains fixed, till it is pointed to A. It is plain that C has been turned through an angle equal to twice the angle sought. Now the whole circle must be turned again till C points to B, and D must be turned till it points to A. Then the circle must be turned again till D points to B, and C must be turned till it points to A: when C will have been turned through four times the arc; and by a repetition of this process the arc can be multiplied any even number of times. The general rule is, to move the circle only in one direction, and the telescopes only in the opposite direction. Two observers are generally necessary. It is necessary to read the verniers only at the beginning and the end, except it is wished to ascertain whether the angles are generally equal.

Observation of zenith-distances with the repeating-circle.

For observing zenith-distances of stars, advantage is taken of a spirit-level attached to the telescope D, and the telescope C only is used. The verniers of C are read, and then the whole circle is turned in its plane till C is pointed to the star S; and D is moved on the circle till the bubble of the level E is either in the middle, or is so near it, that by the divisions of the level-scales the error can be accurately estimated. Then the instrument is turned half round on its vertical axis, so that, in fig. 33, the face presented to the reader is the ungraduated face; and (if necessary) the circle is moved till the level-bubble is in the same position as before, or else the error is noted Then leaving the circle and D fixed, C is released and again pointed on the star; after which it is necessary to look to the adjustment of the level. C has now described an angle equal to twice the zenith-distance. This (as before) can be repeated any number of times; the motion of the star in the mean time must be calculated from theory. The correctness of the whole depends on the goodness of the level. If the bubble is kept always at the middle, two observers are necessary, one to look through the telescope, and the other to attend to the level. If the level-scales are read every time, a single observer is sufficient. We prefer this mode, not only as requiring fewer persons, but also as being more accurate.

Stars near the zenith cannot be observed with it.

One circumstance which limits the use of the repeating-circle is this. It is impossible to place the plane of the circle perfectly vertical. The error of verticality in most of the repeating-circles which we have seen[*] cannot be easily ascertained, (except by observing a star, alternately by direct vision and by reflection from mercury, and marking the time and azimuth.) Suppose, now, in fig. 34, Z is the zenith, S the star, T S the great circle representing the plane in which the circle and its telescope move when the telescope is pointed to S. On reversing the instrument, T′ S is the position of the plane. From Z draw Z T, Z T′, perpendicular to T S, T′ S. T and T′ are the highest points of the circle in the two positions; and, consequently, T′, or rather the point corresponding to it in the circle, is the same as T in the other position. The telescope, therefore, has been moved through S T + T′ S, or 2 S T; and half of this, or S T, is the estimated zenith-distance. To find how much this differs from the true zenith-distance, since $\cos TS = \dfrac{\cos ZS}{\cos ZT}$, we have

$$\frac{\cos TS - \cos ZS}{\cos TS + \cos ZS} = \frac{1 - \cos ZT}{1 + \cos ZT},$$

or

$$\tan \frac{TS + ZS}{2} \cdot \tan \frac{ZS - TS}{2} = \tan^2 \frac{ZT}{2};$$

or

$$\tan \frac{ZS - TS}{2} = \cot ZS \cdot \tan^2 \frac{ZT}{2}$$

nearly. If Z S is small, this expression may become considerable. It is necessary, therefore, to avoid observing stars near the zenith, that is in the part of the heavens most favourable for observation.

The telescopes of the repeating-circle very small.

The smallness of the telescopes attached to the repeating-circles diminishes very much the confidence which we might else be disposed to place in them. The largest that we have seen is only 24 inches long. We apprehend that the long telescopes of the zenith-sectors made before the achromatic object-glass was invented, would give a better image of a star. The telescopes attached to the repeating-circle used by Delambre[*] would not separate the double star ζ Ursæ Majoris; these two stars are distant 14″ or 15″. With such a telescope, the *minimum visibile* is of course not very small. "However an instrument may be constructed, or in whatever manner it may be used, I have no faith that it can give results nearer the truth than a quantity that is visible in the telescope." This opinion we quote from Mr. Troughton's Paper before cited; and we subscribe to it most heartily.

Magnitude of the constant error.

We know not how far[†] the flexure of the telescopes and the defects of the level are found to introduce errors in the observations of zenith-distances from which observations of horizontal angles are free; but we should expect that such additional errors exist, and that their magnitude is sensible. Now to give an idea of the amount of errors in observations of horizontal angles, we take the first triangle in the *Opérations Géodésiques et Astronomiques en Piémont et Savoie*, of which the angles were observed by a company of Piemontese officers, and by another company of Austrian officers.

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[*] In some of Reichenbach's repeating-circles provision is made for the application and reversion of a cross-level.

[*] We copy this statement from the Memoir of M. Nicollet, *Conn. des Temps*, 1831, *Additions*. M. Nicollet has also attempted to prove that the telescope attached to Mechain's circle was unable to separate this star. This circle is now preserved in the Observatory at Milan, where (through the kindness of MM. Cesaris and Carlini) we lately had an opportunity of examining it, and of trying the telescope upon the star in question. It effects the separation perfectly well. We have since found in an anonymous Memoir of great ability inserted in the *Philosophical Magazine* for March 1829, that Mechain himself had with this telescope observed the star to be double, and had published this observation in the *Monatliche Correspondenz* of Zach. This circle (made by Lenoir) appeared to us to be of very inferior workmanship.

[†] It is the practice of the ablest modern observers with the repeating-circle to determine the flexure of the telescope by observing the angular distance (through the zenith) between two opposite marks in the same straight line with the centre of the instrument, or between the wires of two collimating telescopes, which have been adjusted one on the other. The observed distance will be less than 180° by twice the effect of flexure. It is found in general that the flexure produces an error of several seconds.

Stations.	Number of Series.	Angles found by the Piemontese.	Number of Series.	Angles found by the Austrians.
Trélod	6	81° 51′ 57″.64	2	81° 51′ 52″.13
Granier ...	8	55 26 42 .22	4	55 26 51 .42
Colombier..	6	42 41 22 .70	3	42 41 20 .99

Here between two values of the angle observed on Mont Granier. one of which is the mean of 80 observations, and the other the mean of 48, we find a difference of more than 9″. A serious error, therefore, exists, which repetition has not been able to remove. But the fact is that the repetition was useless; for the results of the different series agree extremely well. The smallest of the Piemontese results is 55° 26′ 40″.0, and the greatest 55° 26′ 44″.5; the smallest of the Austrian is 55° 26′ 50″.0, and the greatest 55° 26′ 52″. What kind of *constant error* existed in these instruments we can hardly imagine, since in the other angles the difference is of the opposite kind. It is true that this is (as far as we have examined) the worst of the triangles; it is true also that these instruments were not repeating-circles of the usual construction, but repeating-theodolites of only eight inches diameter. But the points which we wish to impress on the reader remain, that repeating instruments are subject to an error which cannot be removed by any number of repetitions, and that the amount of this error is such as to make their results unfit to be put in competition with those of a good zenith-sector. Judging from the agreement of the observations with the same instrument in the instance given above, we apprehend that the smallness of the instruments has had no effect on the discordance of the results.

We have treated at length on the relative value of the zenith-sector and the repeating-circle, because in some instances we shall find different amplitudes assigned to the same arc, according as it has been observed with one or other of these instruments, and because some arcs which have been absolutely rejected by writers on the Figure of the Earth will appear, when the nature of the instruments is examined, to be nearly as trustworthy as those which they have retained. It has been so much the fashion for some years (chiefly among French writers) to extol the repeating-circle, that we know no Work in which the relative merits of the two instruments are stated, to which we could refer the reader.

We shall now mention individually some of the principal measures.

Methods pursued in J. Cassini's measure.

In J. Cassini's measure, the bases were measured with wooden rods of 4 toises each. The angles were observed with a quadrant of 39 inches, and an octant of 36 inches radius. For the azimuth, the clock-errors were determined by equal altitudes, and transits of Capella below the Pole were observed. Some of the azimuths, however, were determined by observation of the Sun at rising and setting. The zenith-distances of stars were observed with a sector of much greater arc than that which we have described above. We think it unnecessary to allude further to this measure, as it gave no result which is of any use at the present day.

Methods used in the Peruvian measure.

In the Peruvian arc, the base was measured with greater care than had before been bestowed on that operation, but with less than would now satisfy Astronomers. The standard measure of France was an iron toise preserved in the Châtelet; but it was so rough that a new standard was necessary; and in fact that used in this measure, and called from that circumstance *the toise of Peru* has always been considered as the real standard. Wooden rods were used for the actual measure, and these were compared with the iron toise every day. Two at least were always in their places while the third was moved; they were always kept in a horizontal position. It does not appear that the thermometer was observed during the measure, but the observers compared the temperature by their bodily sensations with that at other seasons, which had been ascertained by the thermometer; and on this rough estimation, the allowance for the expansion was founded. The temperature to which the measure was reduced was 13° of Reaumur, or 61¼° of Fahrenheit. The expansion of the iron toise for 1° of temperature was found in the following elegant manner. The toise was mounted on knife edges so as to vibrate like a pendulum, and another bar was similarly mounted so as to be synchronous with it at a given temperature. The air in the place containing the toise was heated, in consequence of which the toise lengthened, and, therefore, vibrated more slowly than the other pendulum; by observing the number of vibrations which elapsed before its vibrations coincided with those of the pendulum, the alteration of time of oscillation, and consequently the alteration of length, was found: it was ·0117 line for 1° of Reaumur. The length of the base was 6274 toises, 3 inches, and 2 lines; this was obtained by two measures which differed only 3 inches, one party beginning at one end, and at the same time another party at the other end. The South end of the base was 126 toises higher than the North; the base, and in fact the whole measure, was corrected so as to refer it to the level of the lowest extremity of the base. The number of principal triangles was 33. The signals were heaps of stones, pyramids of wood, or tents; the angles were observed with quadrants of three feet radius. The triangles were first calculated without any reduction of the observed angles to horizontal angles, and thus the length of the base of verification appeared to be 5260.03 toises; its measured length was 5258.95. The angles were then reduced to horizontal angles; as the apparent elevation of one signal from another was sometimes 6°, this was done not by an approximate formula, but by accurate computation of the spherical triangle. Then the triangles were again calculated, and the positions of the projections of the signals on the level of the lowest end of the base found by the method of parallels and perpendiculars. The azimuths were determined by observations of the Sun near the horizon; there were in all about 20 observations. The general direction of the chain of triangles made an angle of about 14° with the meridian, extending from North-East to South-West. It seems tha. Bouguer calculated the series considering the meridians through the different points (on account of the proximity to the Equator) as perfectly parallel, while La

Expansion of standard ascertained.

Condamine made allowance for the convergence of meridians. La Condamine also thought it proper to diminish the calculated length half as much as the measured length of the second base was less than the calculated length. Finally the distance of parallels of the Observatories erected near the extremities of the arc was found according to Bouguer, 176,940 toises, according to La Condamine 176,950.

Construction of the zenith-sectors used in Peru.

The difference of latitudes was determined by observations of α Aquarii and θ Aquilæ, but more particularly of ϵ Orionis. The sector which they carried out with them was of 12 feet radius, and had an arc of 30°, but they soon found the convenience and accuracy of reducing the extent of arc. The new arc they redivided in a very ingenious manner; and afterwards constructed another sector on the same principle. It was as follows. The zenith-distance of the star used principally at the Southern observatory was about 1° 41′. If the reader will consider the method of observing with the zenith-sector by reversion, as described above, he will find that it is only necessary to have two points on the limb settled with great accuracy (one on each side of the centre) whose distance is nearly $2 \times 1° 41'$, or 3° 22′. Now the chord of the arc of 3° 22′ 15″ is $\frac{1}{18}$th of the radius. To divide an assumed radius into 18 equal parts is not easy: but it is very easy, assuming the length of the chord, to multiply it 18 times, and thus form the radius. This they accordingly did; then describing an arc of a circle, and taking on it two points at the assumed distance, the graduation was completed. The observation was made by moving the instrument till the plumb-line fell on the point, and a wire in the field of view of the telescope was then moved by a micrometer-screw, till the star was bisected. For the Northern observatory, the zenith-distance of the stars being less, they used in the same manner the arc, whose chord $= \frac{1}{20}$th of the radius.

In the first sector the telescope was connected with the principal bar of the sector by supports of several inches in length. The connection appeared to be not sufficiently firm, and they reduced the length of the supports. After numerous observations it still appeared that the telescope bent or deviated in some manner, and that no confidence could be placed in the instrument. At last Bouguer rivetted all the screws, and wrapped the bar and telescope round and round with iron wire. From this time the results were perfectly consistent.

Another difficulty now presented itself. If the same instrument were used to determine the latitudes of both extremities, as the apparent places of the stars would change between the observations, a perfect acquaintance with their apparent motions was necessary. But the laws and magnitudes of the newly discovered corrections for nutation and aberration were not then sufficiently known to enable them to calculate accurately the apparent places. But having constructed another sector, they determined on observing the same star simultaneously at both extremities. Bouguer accordingly went to Cotchesqui the Northern extremity, and La Condamine to Tarqui the Southern, and at length a satisfactory conclusion was obtained. More than three years, however, had been spent in determining the difference of latitudes. They fixed on 3° 7′ 1″.

One reduction remained. The length of the projection of the arc on the level of the lowest station was found, but it was necessary to find the length of the projection on the level of the sea. To ascertain the height of the lowest station above the sea was difficult, as the mountains between it and the sea were not visible at the coast on account of the constant fogs hanging on the forests below the mountains. At last they found that two of the peaks were visible from an island in the Emerald River; and they finally succeeded in ascertaining the height of the lowest station to be 1226 toises. The length of a degree at the level of the lowest station according to Lacondamine was 56,770 toises, and therefore at the level of the sea 56,749. Bouguer, however, made it 56,746; but he observed that this ought to be increased by seven toises on account of the expansion of the iron toise when compared with the wooden measuring-rods. Delambre recalculated this degree; he found the difference of latitudes to be 3° 7′ 3″; and the length of the degree 56,737 toises.

Value of this measure.

We have the advantage of possessing different accounts of this survey written by persons, each of whom had made nearly enough of independent observations to obtain a result. They had calculated the observations, each in his own way; and their personal friendship was not so ardent as to prevent either of them from pointing out any thing unfair in the proceedings of the other. When to this we add that it was one part of their system to make no important observation, if possible, without witnesses, it will appear that this arc is in all respects worthy of credit.

Methods pursued in the first Swedish measure.

In the Swedish measure conducted by Maupertuis, Clairaut, &c. the signals were commonly cones of barked trees, and the angles were observed with a two feet quadrant. The observed angles were reduced to horizontal angles. The azimuth was determined at Kittis (the Northern station) by observing the Sun's transit over the vertical of the next signal, the clock being regulated by equal altitudes. At Tornea the azimuth was found by observing the Sun near the horizon. The arc was calculated by parallels and perpendiculars, and allowance was made for the difference between a small circle and a great one. The base was then measured on the frozen river; eight rods were made each of four toises in length, and were compared with an iron toise (since called the *toise of the North*, and which had been compared with that used in Peru) at the temperature of 15° of Reaumur or 65$\frac{3}{4}$° of Fahrenheit. The rods were laid on the snow; and two troops made independent measures which differed but four inches. The length was 7406 toises 5 feet; whence the distance between the parallels of the observatories was found to be 55,023.4 toises. The sector for determining the difference of latitudes was made by Graham. A detailed description of it may be found in the *Dégré du Méridien entre Paris et Amiens;* it resembled, in most parts, the sector used by Bradley, now preserved in the Observatory of Greenwich. The telescope tube (which was large and strong, and nine feet in length) was itself the radius of the sector; it carried a limb of small extent, the graduation of which had been carefully verified. The whole was moved by a micrometer-screw acting in opposition to a weight, and care was taken that the screw should always act on the same point. The sector was not reversed, (according to the method of using this instrument described in our general account of it,) and thus the absolute zenith-distances of the stars observed were not determined. But as the same part of the limb was used in the observations of the same star at the two stations, the difference of zenith-distances, and consequently the difference of latitudes, was found, subject only to the error which might arise from disturbance in the relative positions of the telescope and the limb. The firmness of the instrument, (which consisted of but one piece,) the care used in transporting it, and the agreement of the results obtained after two transportations, seem fully to justify the Academicians in this use of the sector. Before the measure of the base δ Draconis was observed at Kittis and Tornea, five times at each; the difference of latitude (neglecting refraction) appeared to be 57′ 26″.93. After the measure of the base α Draconis was observed three times at each place; the difference of latitude was 57′ 30″.42,* the mean is 57′ 28″.7. In this estimation, refraction was not taken into account; it ought therefore to be increased by about 0″.7, which makes the difference of latitude 57′ 29″.4. Consequently the length of a degree is 57,425.5. We have stated in the first section that the later terrestrial measures in the same place, as far as they go, give results that agree perfectly with these. In order to make this measure and that of Svanberg agree in all points, it is necessary to suppose an error of 12″ or more in the French observations of latitude at Kittis. From the excellence of the instrument, the reputation of its maker, the care and fidelity of the observers, as shown in the points that have been examined, and the circumstance of their having repeated the observations under the fear that some error had crept into the first set, we have no hesitation in expressing our opinion that this is *impossible.* And we are glad to find that M. Rosenberger, after a careful examination of all the observations, (see the *Astronomische Nachrichten*, No. 121, 122.) has come to the same conclusion. From the difference of the altitudes of α and δ Draconis, compared with their known difference of declination, he has shown that the line of collimation was in the same state before the journey from and after the return to Tornea, as well as before the journey from and after the return to Kittis. And from observations made in France with the same instrument, (see the *Dégré du Méridien entre Paris et Amiens,*) it appears that after repeated voyages the line of collimation was not sensibly changed. The length of the arc which M. Rosenberger is inclined to adopt, is 55,020.16 toises, and the difference of latitude 57′ 30″.44; whence 1° = 57,405.02. The mean latitude is 66° 19′ 37″.

Reasons for retaining this measure.

Methods followed in the verification of the arc of J. Cassini.

In the measure which forms the principal subject of the work entitled *La Méridienne Vérifiée*, a base of 6224 toises was measured near Dunkirk, one of 5094 toises near Amiens, one of 5749 toises near Paris, one of 7492 near Bourges, one of 4422 near Rodéz, and one of 7929 toises near Perpignan. The Dunkirk base crossed the entrance of the harbour. This part was not measured, but its length was ascertained by drawing a line perpendicular to the base, and finding by trial the point at which the rays coming from the signal made with this perpendicular an angle of 45°. The base measured near Bourges was one of the sides of a principal triangle. The base near Paris (on the road to Fontainbleau) was measured with four iron rods of 15 feet each; but all the others were measured with fir rods, well painted, and armed at the ends with iron caps. These wooden rods were 18 or 24 feet long, 3 or 4 inches broad, and 2 inches thick; they were compared three times each day with four iron rods of 3 feet each, which had been compared at the temperature of 14° of Reaumur, (63° of Fahrenheit,) with a line of 10 toises traced in the *salle de la Méridienne* of the Observatory at Paris. The mean of the difference was applied as a correction to the length of the wooden rods; it would appear also, (though we have not found any distinct statement,) that a correction was applied for the temperature of the iron rods at the time of comparison. In all cases the rods were applied end to end. The bases were measured at least twice: that near Paris five times. In the bases of Bourges and Rodéz, the angles made by the rods with the horizon were observed, and allowance made for the inclination.

The signals were generally church-towers, windmills, &c.; in a few instances it was found necessary to erect posts or other signals where no conspicuous object could be found.

The angles were measured with a quadrant of two feet radius, and were reduced to the centre of the station. The observed angles were altered to make the calculated lengths of the bases of verification and the calculated azimuth agree with those observed; but in no case did the alteration exceed 5″.

The azimuth of one side of a triangle was determined at each of the principal stations, (Dunkirk, Paris, Bourges, Rodéz, and Perpignan,) by observations of the Sun near the horizon.

South of Bourges, the observed angles were reduced to horizontal angles. The calculation of the terrestrial arc of meridian was made by the method of parallels and perpendiculars.

The sector for celestial observations was of 6 feet radius, and had an arc of 50°, divided by means of its index. The general accuracy of the arc was verified by taking angles all round the horizon, and observing whether their sum amounted to 360°; and the equality of the divisions, by determining the zero point from observations of different stars. The telescope had an internal micrometer. The only stars on which their determinations rest, are α Lyræ, α Cygni, α Persei, Capella, γ Draconis, and η Ursæ Majoris. Each star was observed eight or ten times at each station; and the sector was several times reversed in the course of each series. The final result was, that the length of the arc between Dunkirk and Paris was 125,431 toises, and its amplitude (or the difference of latitudes of the extremities) 2° 11′ 50″.28; between Paris and Bourges, 99,990 and 1° 45′ 7″.33; between Bourges and Rodéz, 155,767 and 2° 43′ 51″.5; between Rodéz and Perpignan, 94,308 and 1° 39′ 11″.2. The latitude of Paris was supposed to be 48° 50′ 10″, or 48° 50′ 12″. Of the excellence of this measure we conceive there can be no doubt.

Methods used by Boscovich.

In the survey conducted by Boscovich, the bases were measured with wooden rods of 27 palms (about 18 French feet) each. These were compared continually (by means of beam-compasses, as described in our general account) with an iron rod of 9 palms, that had been measured by a French iron toise,* furnished by Mairan. Corrections for temperature were applied: the

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[*] About 1″ of the difference of these results depended on the inequality of the degrees on the limb, as ascertained in the verification.

[*] It is stated in a note to the French translation of Boscovich's Work, that this toise was shorter than the *toise of Peru* by $\frac{8}{75}$ of a line, and shorter than the *toise of the North* by $\frac{1}{20}$ of a line. The latter is the same as the toise used in the French arc last described. We know not on what authority the difference between the *toise of Peru* and the *toise of the North* is given: in every other statement that we have seen they are asserted to be equal. See the *Base du Système Métrique*, tom. iii. p. 413; where the comparison of Mairan's toise with the others will be found.

standard temperature was the same, we presume, as that adopted in the Peruvian measure, or 13° of Reaumur's scale. The base near Rimini consisted of two parts, making an angle of 170° 52′ 15″; a part of it crossing a river was measured by a triangle. The rods were supported on trestles; they were not placed in contact, but the distance between their extremities was measured. In uneven ground they were placed in a horizontal position; and the end of one rod was made to touch a plumb-line suspended from the end of the other. Allowance was made for the curvature of the rods when it appeared sensible. The signals were a sort of pyramidal huts, made of large branches of trees. The angles were measured with a quadrant of three feet radius, made by a priest. The triangles were calculated in the same manner as those of Peru. The azimuths were determined by three observations of the setting Sun at Rome, and three at Rimini. The length of the meridian arc was found by parallels and perpendiculars, calculating from the base of Rimini, to be 123,221.3 toises. The sector for celestial observations (made by the same priest) was of nine feet radius; it was similar to that described in our general account, with this difference, that the object-glass as well as the eye-piece of the telescope were attached to the bar, and independent of the intermediate tube; and that the graduations were upon a straight piece, sliding in a groove with a micrometer-screw, forming, in fact, a line of tangents divided into equal parts. The bar, &c. were of iron, and the slider of brass, and an apparatus was made for observing whether the difference of expansion would produce any sensible effect: Boscovich states that he never found it sensible. The stars observed were α Cygni and μ Ursæ Majoris: the number of observations 43. Between the first observations at Rome and the observations at Rimini, the line of collimation had altered its position by more than 2′; and on repeating the observations at Rome, its position was again changed. But as the sector was reversed several times during each series of observations, and the results of the different observations of each series agreed very well, there seems no reason for doubting the accuracy of the ultimate result. The mean amplitude is 2° 9′ 47″, whence 1° = 56,966.3 toises. But as the base near Rome was greater by the measure than by calculation, and some of the sides are lengthened by a different calculation, (Boscovich does not specify what,) and as some of the celestial observations appeared preferable to the rest, Boscovich adopted as the length of 1°, 56,979 toises. As Mairan's toise was shorter by $\frac{1}{29}$ of a line than the toise of Peru, these numbers ought to be diminished by $\frac{1}{25000}$ of the whole. The mean latitude is 42° 59′.

We have given every thing which seems necessary to enable the reader to judge of the value of this measure. Objections have been made to some parts of it by the Baron de Zach. (*Correspondance Astronomique*, vol. vi.) The azimuth observed at Rimini was reobserved by Zach, and the results differ more than 1′. From whatever source it rises, this is of no consequence, as the chain of triangles follows very nearly the direction of the meridian. The same Astronomer has calculated from the observations of Boscovich the latitude of Rimini, and finds that it differs nearly 4″ from that which he has determined by 110 observations of Polaris and α Aquilæ with a repeating-circle. But there is a fallacy in his mode of observation. The advantage of observing, with a repeating-circle, stars on both sides of the zenith, consists in this: that if the declinations of these stars have been determined by an instrument which has no constant error, the constant error of the repeating-circle will be eliminated from the result. But the declination of α Aquilæ used by Zach, was determined by the same repeating-circle, and differs from that commonly used; consequently, his results are affected with the constant error which affects the observations of Polaris. And as the sector of Boscovich was reversed three times during the observations of α Cygni at Rimini, and five times during those of μ Ursæ Majoris, we cannot allow the result to be set aside for any observations made with a repeating-circle.[*] From the appearance of the observations, and the character of the observer, we consider that great reliance may be placed on the results of this measure.

Methods followed in Lacaille's measure at the Cape of Good Hope.

The standard used by Lacaille was an iron toise, constructed by the same artist who made the toise of Peru and the toise of the North. From this was made a wooden toise, terminated with brass plates, and exactly of the same length. By means of these, the measuring rods (four fir-rods, each 18 feet long, 3 inches broad, 2 inches thick, and well painted) were verified four times each day. Nothing is mentioned of a correction for temperature; Lacaille merely observes, that the sky was clouded and the South-West wind strong, and that nothing was to be feared on the score of temperature. It would seem that the rods were laid on the ground: Lacaille himself made the contact. Only four triangles were used. The signals were fires, rocks, &c. The angles were measured with a three-feet quadrant; and reduced to the centre of the station and to the horizontal angle. The azimuth was determined by 17 observations of the rising Sun at the Cape Observatory, which was the Southern extremity of the arc. The meridian arc was computed by parallels and perpendiculars. The amplitude was determined from observations made with the same sector which was used in the verification of the French meridian. Sixteen stars were observed at the Northern station (Klipfonteyn) for six days, the sector being three days in one position, and three days in the reversed position. The number of observations at the Cape was more considerable; they are to be found in Lacaille's *Fundamenta Astronomiæ*. The statements of Lacaille respecting the error of collimation (*Astron. Fund.* p. 158.) appear quite satisfactory. The deduced amplitude with Lacaille's last corrections (*Fundamenta*, p. 184.) was 1° 13′ 17″.5; the meridian arc 69,669.1 toises; whence 1° = 57,034.4 toises. The mean latitude 33° 18½′.

It is proper to remark, that Lacaille was so much surprised at this result, that he repeated the measure of the base and all the calculations. The observations of terrestrial angles and celestial zenith-distances did not seem subject to any doubt. Whatever be the consequence, we conceive that this measure must be received as equally certain (excepting, perhaps, some doubt about the trifling correction for the temperature in the measure of the base) with any of the former, and almost any of the subsequent measures.

The localities have since been examined by Captain Everest:[†] (*Astronomical Transactions*, vol. i.) it appears that there are large mountain masses to the North-West of the Northern extremity, and to the South-West of the Southern extremity, the effect of which would be to diminish the apparent difference of latitudes.

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[*] We have just received (through the kindness of the author) Mr. Rumker's observations made at Paramatta. The difference of the latitudes as found by the mean of many repetitions North of the zenith, and the mean of many South of the zenith, is 15″. The instrument is made by Reichenbach.

[†] We believe that if this distinguished officer had not been pre-

Methods used by Liesganig.

In Liesganig's arc, the standard was an iron toise, compared, at the temperature 13° of Reaumur, ($61\frac{1}{4}°$ Fahrenheit,) with the *toise of Peru*. The measuring rods were each six Vienna fathoms long, spliced like the mast of a ship; they were laid on the ground, and placed in contact. The angles were observed with a quadrant of $2\frac{1}{2}$ feet, that had been carefully verified; they were reduced to horizontal angles. The azimuth was determined by observations of the Sun near the horizon. The meridian arc was calculated by parallels and perpendiculars. The sector of 10 feet radius was similar in all respects to that of Boscovich; its divisions were verified. The stars observed were α and β Aurigæ, α and δ Cygni, γ Draconis, and ι Herculis, (set down as μ Draconis;) each star was observed three or four times in each position of the sector, which was only once reversed. The concluded amplitude was 2° 56′ 45.″5; and the concluded arc of meridian 172,796 Vienna fathoms = 168,149 toises.

We have already stated the reasons for rejecting this measure. If we adopt for the Southern part of the arc the numbers given by the later survey, we ought to increase the meridian arc by 264 toises. But it appears from Zach's statements, (who, however, is no friend to the Jesuits,) that the astronomical observations were falsified. It will be better upon the whole to reject any conclusion derived from this arc of meridian.

Methods pursued in Beccaria's measure.

In the survey made by Beccaria, the base was measured with wooden rods laid on trestles. Two small parts were added by triangulation. The toise (compared at 13° of Reaumur with the toise of Peru) was furnished by Lacondamine: and the correction used for the temperature was Lacondamine's. The signals in general were posts. The angles (observed with a quadrant like that of Boscovich) were reduced to horizontal angles, and these to the angles formed by the chords. The azimuth was observed, and the calculations made, as in the last-described surveys. The sector* was nearly similar in construction to that of Boscovich. The stars observed were α and δ Cygni, and β Aurigæ: the sector was reversed twice or three times at each place. The details of the micrometer readings, &c. are not given. The concluded distance and amplitude were 64,889.6, and 1° 7′ 44″.71, the extreme difference of the results from the three stars amounting only to 2″. It is unnecessary to enter into greater details, as the same line has since been surveyed by MM. Plana and Carlini, whose conclusions (which we shall shortly give) are undoubtedly more accurate than those of Beccaria.

Methods used in the North-American arc.

In the measure of the line between Maryland and Pennsylvania, by Mason and Dixon, the whole was measured with two "rectangular levels, or measuring frames." As we do not perfectly understand the description of them given in the *Phil. Trans.* 1768, we shall copy it *verbatim*. "The levels used in this work were each 20 feet in length, and 4 feet in height. They were made of pine, an inch thick, and in form of a rectangle; the breadth of the bottom board was $7\frac{1}{2}$ inches, that of the top 3 inches, of the ends $4\frac{1}{2}$ inches, and the bottom and top were strengthened with boards firmly fixed to them at right angles. The joints were secured with plates of iron, and the ends were plated with brass. The plumb-lines used in setting them level were 3 feet 2 inches in length, and hung in the middle of the levels, being secured in a tube from the wind, in the manner of carpenters' levels; wherefore we called these by the same name." The levels were frequently compared with a brass standard of 5 feet, and the temperature was noted. The allowance for expansion was taken from Smeaton's experiments, and the length was reduced to what it would have been if the brass rod had been used at the temperature 62° of Fahrenheit. By fixing in the ground a stake with a movable piece of iron, which was brought to a certain mark on each level, it was easy to observe whether one level moved when the other was placed in front of it. The line was carried over a few rivers by a triangle, as in the former surveys. For the azimuth, a clock was regulated by equal altitudes, and the clock time of stars' culminations being found, a transit-instrument was directed to them at that instant, and by means of it a meridian mark was fixed; thus the azimuth of a given line could be found, or the direction of a meridian could be fixed. The lines measured were the following. N P, fig. 35 = 78,290.72 feet, in the direction of the meridian: P a = 2,991 miles; this line is in such a direction that if continued to the length of 10′ of a great circle, it would meet the parallel of latitude passing through P: a C = 45.5 feet, in the meridian. C D = 26,608.06 feet, in the direction of the meridian. (These two parts were measured separately from a mistake in fixing the point C, which ought to have been in the parallel of P.) D B = 1491 feet; and the angle C D B = 93° 28′. B A = 434,011.64 feet, making an angle of 3° 43′ 30″ with the meridian. Hence, the distance of the parallels of N and P is 78,290.7 feet; that of P and a, − 14.1; that of a and C, 45.5; that of C and D, 26,608.1; that of D and B, 89.8; that of B and A (computed as described in our

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vented, by his absence from Europe, from consulting Lacaille's account of the survey, he would have found that some of his objections were without foundation. Lacaille states that the ground was cleared and levelled for the base measure; he states that the instruments were not placed at the centre of the stations, (which was rarely done before the English survey,) and gives the numbers necessary for the reductions. The magnitude of fires visible at the distance of 45 miles need not, perhaps, be so great as to throw any great uncertainty on the terrestrial measure. The instability of the observatory, though always injurious, is of comparatively little consequence, we conceive, in observations with the zenith-sector, and, with reasonable care, its effects can be easily detected.

* Through the politeness of the Members of the Turin Academy we have been allowed to examine this sector, which is now preserved at their apartments. We are at a loss to conceive how such errors can have been committed as those attributed to Beccaria's observations. The tube containing the object-glass appears to be firmly attached to the bar; and the object-glass, though not quite tight in its cell, is tight enough to prevent accidental derangement. This tube is unconnected with the rest of the tube. The eye-piece also appears to be firmly connected with the bar. The object-glass-tube projects about two inches beyond the last of its two supports, and the eye-piece about four inches beyond the last of its supports. Can the weight of the telescope tube (weakly attached to the axis) have twisted the eye-piece? Or are the divisions of the tangent-scale, or the length of the radius, wrongly estimated? No other explanation suggests itself to us. It must be remarked that the wires in the eye-piece, and the plumb-line, are wanting: the pin for the plumb-line suspension is as good as those commonly used. We have no hesitation in saying that we should place more confidence in the observations made by any careful person with this instrument, than in those made with Mechain's circle. The character of the observer, as it appears to us, can alone explain the errors of the observations.

general account) 433,078.8; whence the distance of parallels of N and A is 538,098.8 feet. As the five-feet brass rod, on comparison with the Royal Society's standard, was found to be too short by $\frac{13}{360000}$ of the whole, these numbers ought to be diminished by 19.4; and the distance of the parallels is 538,079.4 feet. By a comparison of the standard just mentioned with the toise of Peru, instituted on this occasion, it was found that the toise = 76,734 inches; whence the distance of the parallels = 84,147.2 toises.

The observations of stars were made with a six feet sector, constructed by Bird, in which the plumb-line was made to bisect a dot at the centre of the instrument. The stars observed were α Lyræ, γ Andromedæ, β Persei, δ Persei, β Aurigæ, Castor; Capella also was observed, but the result disagreeing a few seconds from that given by the others, was rejected. Each star was observed at N and P, three or four times in each position of the sector. The sector was reversed only once at each station. Finally, the difference of latitude = 1° 28′ 44″.99, and the mean latitude 39° 12′; whence 1° = 363,772 feet = 60,628.7 fathoms = 56,888.3 toises. The Royal Society's standard yard has been found by Capt. Kater (*Phil. Trans.* 1821) to be longer than Shuckburgh's by .001365 of an inch; whence the distance of parallels is 538,099.8 feet, and 1° = 363,785 feet = 60,630.9 fathoms, by Shuckburgh's standard.

The only chance of error in this operation consists (we conceive) in the possibility of losing in the measure some whole number of level-lengths. This was provided against with tolerable certainty by previously measuring the whole line with a chain, and fixing posts at certain distances. Of the goodness of the astronomical observations we think there can be no doubt. The results of this measure must, we think, be received as equal in authority to those of any other measure.

Junction of Greenwich and Paris.

The survey of 1787, connecting the meridians of Greenwich and Paris, has sometimes been used in the estimation of the degree deduced from the French measures. The meridian arc, of which the Southern extremity is in France, has sometimes been a little extended by taking Greenwich for its Northern termination. We mention this only to record our protest against such an extension. The chain of triangles connecting the meridians runs East-South-East, and West-North-West, and is of considerable length, and consequently the smallest error in the azimuth (which it is impossible to avoid) will produce a considerable error on the length of the meridian arc.

Methods used in the survey of the French meridian.

The bases on which the operations of Delambre and Mechain were founded, (at Melun and Perpignan,) were measured with rods of platinum, 2 toises long, 6 lines broad, and 1 line thick. To each was attached, at one end, a rod of brass; the proportion of the expansions of brass and platinum being known, the expansion of the platinum rod was inferred from the observed difference of expansion of the two rods. This apparatus was enclosed in a wooden box. Four of these compound rods were used: they were placed on trestles, not in contact, the distance being measured by a small slider. The lengths of the bases were expressed in numbers of the length of the toise of Peru, taken at the temperature of 13° of Reaumur, or 61¼° of Fahrenheit.* Allowance was made for the inclination of the bars. The bases consisted each of two parts, making a small angle. The length of the base of Melun was 6075.9 toises: that of Perpignan 6006.25. The length of the latter calculated from the former, through the chain of triangles, differed only 11 inches from the measured length. The angles were observed with repeating-circles. The triangles were calculated as spherical triangles, by Delambre's method. (TRIGONOMETRY, Art. 182.) The azimuths were determined by observations of the Sun near the horizon, and by measures of the angular distance of a signal from the Pole-star. (TRIGONOMETRY, Art. 185.*) The arc of meridian was then calculated by the method which (in our general account of methods) we have described as Delambre's. In the prolongation of this arc to Formentera, the same methods were used, except that the azimuth was ascertained by transits as described in TRIGONOMETRY, Art. 185. The latitudes were found almost entirely by observations of the Pole-star, and β Ursæ Minoris, made with Lenoir's repeating-circle. To give an idea of the extent to which this instrument was used in these determinations, it will be sufficient to state that the latitude of Formentera was fixed by 2500 observations of Polaris, and 1400 of β Ursæ Minoris.

We have already mentioned the discordances in the observations of Mechain at Barcelona and Montjouy, and the doubt thrown by them on the results of observations with the repeating-circle. They have been considered by M. Nicollet in the Memoir before referred to, (*Conn. des Temps.* 1831, *Additions:*) and his conclusions (omitting every thing relating to the inability of the telescope to separate ζ Ursæ Majoris) appear to us perfectly correct. Mechain's observations at these places were not confined to the Pole-star and β Ursæ Minoris, but included several stars passing to the North, and several passing to the South of the zenith. Now M. Nicollet shows that if we use Piazzi's declinations, and find the latitude of each place, first by a mean of the observations of stars passing North of the zenith, and next by a mean of those of stars passing South of the zenith, and take the mean of these results, the difference of latitudes is almost precisely what we should have expected from the known distance of parallels; and thus the difficulty disappears. It has been answered, (*Philosophical Magazine*, March 1829,) that if we compare the observations on the same stars made at the two stations, as with zenith-sector observations, the original difficulty remains. But we think that the author of this answer has not sufficiently considered the nature of the defect attributed to repeating-circles. Of the stars observed at both stations, only one passes South of the zenith. Now if the *constant error* of the circle had undergone any alteration (as is probable after the interval of a year) *all* the stars North of the zenith would give the same error, and with the same sign in the difference of the latitudes. But the stars South of the zenith would give the same error with a contrary sign, and in fact the only South star observed at both places, gives a discordance of an opposite kind. We think, therefore, that M. Nicollet is quite right in fixing the latitude of Montjouy at 41° 21′ 44″.535.

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* As there has been a little confusion respecting these reductions for temperature, the reader is referred to the *Phil. Trans.* 1826, p. 568, for an abstract of all the passages in the *Base du Système Métrique* relating to this point, as well as to the comparison of the new standard with the ancient and foreign standards.

* In the Essay on TRIGONOMETRY to which we have referred, it is said that the azimuth is determined by a right-angled spherical triangle. This is only true when the signal is exactly in the horizon; when it is elevated or depressed, it is necessary to calculate the zenith angle of the triangle, whose angles are at the zenith, the Pole, and the signal.

The latitude of Mountjouy is, however, of no essential importance. It is much to be wished that the latitudes of Dunkirk and Formentera were determined with greater certainty. The latter rests on an enormous number of observations, but entirely on stars North of the zenith, and with a circle whose level was attached to the axis; it is probably subject to a much greater error than if it had been fixed by ten observations with Graham's or Ramsden's sector. In the determination of the latitude of Dunkirk, many observations were rejected; and in that of Paris, Mechain rejected observations by thousands, apparently preserving only those whose results presented the greatest agreement. The latitudes of the other intermediate points are still more doubtful. In our present uncertainty we shall adopt from the *Base du Systême Métrique* the following numbers: latitude of Dunkirk 51° 2′ 8″.7; latitude of Formentera 38° 39′ 56″.1; distance of parallels 705,189.4 toises; mean length of degree 57,007 toises.

Svanberg's methods.

The base measured by Svanberg was nearly in the same line as that measured by Maupertuis and Clairaut. The measuring rods were iron bars of six mètres each. The observed length was corrected for the flexure of the bars, their horizontal inclination with the direction of the base, &c., but the correction for temperature remains uncertain from a cause which deserves a distinct explanation.

On the manner of using standard measures.

By a toise, or a yard, or a mètre, we mean a certain extent of space in one direction, without any reference to wooden rods or metallic bars. But as it is impossible to measure a line without some material standard, we are compelled to adopt as the practical definition of a toise or a yard, &c. the length of a certain bar of metal, called a standard. If this length be invariable, the measure is fixed without any further condition. But if from change of temperature, or any other change of circumstances, the length of the bar (as compared with the length of others not subjected to the same trial) be found to change, then we must specify the degree of temperature, and the other circumstances under which this certain bar must be placed in order to present the exact length required. If then the length of a base, or the length of a pendulum, has been measured with the standard, at a higher temperature than that specified, and if we know that at this higher temperature the bar was longer than at the specified temperature, it is plain that the length of the bar has not been contained so many times in the line measured as it would have been if used at the specified temperature, and, consequently, the resulting number (which expresses how often the length of the bar is contained in the line) must be increased. Similarly if measured at a lower temperature the number must be diminished.

Reduction for temperature.

It was not till geodetic measures had acquired considerable accuracy, that this refinement was thought of. And when necessity compelled Philosophers to fix on some temperature for the measure of the standard bar, convenience directed them to choose one which could be commanded at all times with little trouble, and which was as likely to occur in ordinary measures as any other. Thus the French toise was the measure of the iron bar called the *toise of Peru* at the temperature of 13° Reaumur, or 61¼° Fahrenheit, or 16¼° Centigrade; the English yard was the measure of a brass bar, (or rather some one of several brass bars,*) at the temperature of 62° Fahrenheit.

Temperature for the old standards

Different temperature for the new French standard.

But the new French measure was adopted at a time when all other considerations gave way to theory. The mètre is *defined* by 443.296 lines of the toise of Peru, at the temperature 13° of Reaumur. But it is *represented* by the length of a standard platinum bar, at the temperature of freezing water, or 0° of Reaumur's and the Centigrade scale. This change of temperature for the measure of the standard, has introduced no little confusion into the subject of measures generally.† In the instance now before us, additional confusion is introduced by the change of the metal used for the construction of the standard.

It is uncertain whether the iron bar used by Svanberg was compared with the platinum standard preserved at Paris, at the temperature of 0°, or 16¼° Centigrade. If the expansions of the two metals, with the same change of temperature, were the same, this would be of no consequence. But their expansions being different, if the two bars had the same length at the temperature 0°, they would not have the same length at the temperature 16¼°. If they were compared at the temperature 0°, it would be right to apply to the measured length of Svanberg's base, the correction corresponding to the expansion of iron between 0° and the observed temperature. If compared at the temperature 16¼°, the length must be corrected for the expansion of iron, between 16¼° and the observed temperature, and for the expansion of platinum between 0° and 16¼°. It is most probable that they were compared at the temperature 0°.

The signals for Svanberg's triangles were so constructed as to allow the light of the sky to be seen through them. The angles were observed with repeating-circles. The azimuths were found by observing the transits of the Pole-star and other stars over the same vertical; from equal altitudes of stars; and from observations of the Sun. The triangles and arc of meridian were calculated by Delambre's method. The latitudes were found by a great number of observations on the Pole-star, made with Lenoir's repeating-circle. The final results are, that the distance between the parallels of Mallorn and Pahtavara is 180,827.68 mètres = 92,777.98 toises, (or 180,794.06 mètres = 92,760.73 toises, if the standards were compared at the temperature 16¼° Centigrade;) the latitude of Mallorn = $72^g.8056372$ = 65° 31′ 30″.26; the latitude of Pahtavara = $74^g.6079723$ = 67° 8′ 49″.83; whence the difference of latitudes = 1° 37′ 19″.57. This ought to be diminished by 0″.28, because in the calculation of nutation, the true place of the Moon's node has been used instead of the mean place. Hence 1° = 57,198.9 toises.

There is no doubt of the general excellence of this measure. The astronomical determinations, we conceive, are not equal to those in the measure of the French

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* For a comparison of Shuckburgh's, Bird's, Roy's, Ramsden's, and the Royal Society's standard, the reader is referred to a paper by Capt. Kater, in the *Philosophical Transactions*, 1821.

† The reader, who may think this explanation unnecessary, is referred for a proof of the confusion on this subject to the *Phil. Trans.* 1818, where Col. Lambton has applied an unnecessary correction to the length of the French arc; to the *Phil. Trans.* 1822, where Mr. Goldingham has applied a correction with the wrong sign to the length of the seconds' pendulum at London; and to Zach's *Correspondance*, vol. viii. p. 150, where Zach has fallen into an error in the comparison of Beccaria's toise with the French mètre.

Academicians in the same Country, but they are as good as could be made with the instrument employed. The anxiety of Svanberg and his companions to omit no reduction, whose value could be sensible, has led them into many refinements that are useless, and some that are erroneous,* but none to such an amount as to hurt the credit of the result.

Methods used in the English survey.

The rods with which the first base was measured in England were thin glass tubes, 20 feet long, and about an inch in diameter, enclosed in wooden boxes, (except at the ends,) and supported on trestles. Each rod had at one end a small slider driven outwards by a weak spring; and the succeeding rod was pressed against it till the slider was pushed up to a certain point. But the apparatus ultimately adopted was a steel chain of 100 feet in length, consisting of forty links. One chain only was used in the actual measure, but another was carried for the purpose of occasional comparison. The mode of measuring finally employed was this. The chain was supported in several troughs resting on trestles, and extended by a weight of 28 pounds. In any position of the chain, two posts were necessary at each end. One post at the preceding end carried a pully, over which passed the rope sustaining the weight which stretched the chain. To one post at the following end was fixed an apparatus of screws, by which the chain was supported in opposition to the action of the weight, and by which it could be moved slowly. The other post at the following end of the chain carried a scale, to one division of which a mark on the end of the chain was brought by the screw motion; and the other post at the preceding end carried a scale, of which one division was brought by a screw motion to a mark at that end of the chain. When the chain was moved to the next position, the mark on the following end of the chain was brought to that division of the scale with which the mark on the preceding end had coincided. It will easily be seen that the correctness of the operation depends on the immobility of the scale-posts, (which are subject to no strain,) and that in fact three posts at each end were necessary. The expansion of the glass tubes and of the chain were determined by experiments expressly made for the measure. Corrections were applied for the temperature, the inclination, and the height above the sea. The length of the base on Hounslow Heath was 27,404 feet; of that on Salisbury Plain 36,575; and of that on Misterton Carr, (near the Northern extremity of the arc of meridian,) 26,342 feet. Several other bases were measured in different parts of the Kingdom.

The signals were usually flag-staffs, or white lights. The angles were observed with a large theodolite, made by Ramsden, three feet in diameter; an instrument undoubtedly superior to any other that had been used in similar observations. As with this instrument the horizontal angle was immediately observed, no reduction for the height of the signals was necessary. The horizontal angles were reduced to chord angles. The first azimuth was obtained from the meridian mark of the transit-instrument at Greenwich Observatory; afterwards the azimuths were found by observing with the theodolite the Pole-star at its greatest azimuth. The meridian arc was computed by parallels and perpendiculars. The distance of the parallels of Dunnose (in the Isle of Wight) and Arbury Hill (near Daventry) was found to be 586,319.5 feet; that between the parallels of Dunnose and Clifton (near Doncaster) 1,036,337 feet: and that between the parallels of Dunnose and Burleigh Moor 1,442,852.5 feet. The chains were measured with Ramsden's brass yard, which it appears from Captain Kater's comparison (*Phil. Trans.* 1821) exceeded Shuckburgh's, now adopted as the standard, by .002505 parts of an inch, or $\frac{1}{14370}$ of the whole. Thus the meridional distances are found to be 586,360.3, 1,036,409.1, and 1,442,952.9 feet, when measured by Shuckburgh's standard.

The difference of latitudes was determined by observations with a zenith-sector made by Ramsden, apparently the best which has ever been constructed. For a description of this superb instrument we must refer to the *Phil. Trans.* 1803, to the *Account of the Trigonometrical Survey*, vol. ii., or to Pearson's *Practical Astronomy*, vol. ii. It is sufficient here to state that the telescope was eight feet long, and carried the arc; that it was moved by an external micrometer; that the plumb-line bisected a dot at the centre; and that from a very severe examination it does not appear that the error of collimation was sensibly altered during the observations. For the difference of latitude of Dunnose and Burleigh Moor, eight stars were observed; those of the other stations were found by a greater number. In general each star was observed about eight times, the sector being reversed after every day's observation. The amplitudes were thus found to be, between Dunnose and Arbury Hill 1° 36′ 19″.98; between Dunnose and Clifton 2° 50′ 23″.38; between Dunnose and Burleigh Moor 3° 57′ 13″.1. The mean latitude of the last arc is 52° 35′ 45″; the length of $1° = 364,968.3$ feet. The middle latitude between Dunnose and Arbury Hill is 51° 25′ 18″; the length of $1° = 365,208.4$; the middle latitude between Arbury Hill and Clifton is 52° 50′ 30″; the length of 1°, 364,625.1; the middle latitude between Clifton and Burleigh Moor 54° 0′ 56″; the length of 1°, 365,002.5. By a comparison of the French mètre with Shuckburgh's standard, Captain Kater (*Phil. Trans.* 1818) has found the mètre = 39.37079 inches; and as the mètre = 443.296 lines of the toise of Peru, it follows that the toise = 6.394596 feet. Hence the distance between the parallels of Dunnose and Burleigh Moor = 225,651.9 toises. We believe the results of this measure to be far more exact than those of any of the measures hitherto described.

Methods used in the East-Indian survey.

The methods adopted by Colonel Lambton were in almost all respects the same as those used in the English survey. The base near Madras was measured with a steel chain which was occasionally compared with a standard chain that had been measured by Ramsden's brass standard when the temperature was 50° of Fahrenheit; the length of the base was therefore corrected for the expansion of the steel to this temperature, and for the contraction of the brass to the temperature 62°. The expansion of the chain was determined by independent experiments. Corrections were applied for the inclination, the height above the sea, &c. The same course was pursued in the measure of the base near Bangalore, of that in the Coimbetoor, of that near Palamcottah, of that near Gooty, and of that near Daumergidda: their lengths were respectively 40,006, 39,794, 32,301, 30,507, 32,609, and 30,806 feet. On comparing the chains previous to the measure of the base near Gooty, there appeared to be some reason for supposing that the standard chain had altered its length; and it was then compared by means of beam-compasses with a brass bar.

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* Among these we may reckon Svanberg's method of taking the mean of observations made with the repeating-circle. Instead of taking the difference between the first and last readings, and dividing by the number of multiplications, he takes every possible combination of one, two, three, &c. multiples, and gives to each a vote proportional to its number. But it seems plain that every observation of a double angle is perfectly independent, (as the telescopes are pointed anew for every observation,) and, therefore, we have no right to take two or three double angles together, as we should have if it were possible to observe at once four or six times the angle. Every observation being independent, and therefore all being *à priori* equal in accuracy, the probably best determination is the mean of all, that is the difference of the first and last readings divided by the number of multiplications. In the instance that he has mentioned, (p. 33.) the difference of determinations, by Svanberg's, and by the common method, is $1''.4$.

The signals were flag-staffs and white lights. The angles were observed with a large theodolite. The observed horizontal angles were reduced to chord-angles, and the triangles calculated as plane triangles. The azimuths were determined by observations of Polaris at his greatest azimuth, made with the theodolite. The meridian arc was calculated by parallels and perpendiculars; but the principal stations were so nearly in the same meridian that no allowance was made for the difference between the perpendicular and the small circle passing through a station. Two different meridians were, in fact, measured; in the first of these (near the Coromandel coast) the meridional distance between Trivandeporum and Paudree was found = 574,328 feet; and in the second, the distance from Punnæ (near Cape Comorin) to Putchapolliam = 1,029,101; that from Putchapolliam to Namthabad = 1,489,131; and that from Namthabad to Daumergidda = 1,073,428 feet. To reduce these to Shuckburgh's standard, it must be observed (*Phil. Trans.* 1823) that the triangles extending over the shorter meridian arc, and the Southern part and 105,996 feet of the middle part of the longer arc, were calculated from the measure by Ramsden's chain; and those on the rest of the middle part and the Northern part from the measure made with the brass bar. The former is too long by .00250 inch, and the latter too short by .00064 inch in every yard. The lengths thus corrected are 574,368, 1,029,173, 1,489,198, and 1,073,409 feet.

The latitudes were found by observations with a zenith-sector of five feet radius, constructed by Ramsden, and similar to that used in the English survey. It appears that its error of collimation changed by a very small quantity during the operations, but was steady enough at each station. For the determination of some of the amplitudes, seventeen stars were used, each of which had been observed eight or ten times, the sector being reversed after every day's observation. The absolute latitudes depended on a smaller number of principal stars. The final results were as follows: latitude of Trivandeporum 11° 44′ 52″.6; of Paudree 13° 19′ 49″.0; latitude of Punnæ 8° 9′ 38″.4; of Putchapolliam 10° 59′ 48″.9; of Namthabad 15° 6′ 0″.2; and of Daumergidda 18° 3′ 23″.5. The latitudes of some intermediate points were observed, and in one of them it appeared that the disturbance of local attraction was sensible. The length of 1° is, in the arc from Trivandeporum to Paudree 362,988 feet, in the Southern part of the long arc 362,864, in the middle part 362,941, and in the Northern part 363,071 feet.

Considering the extent of this arc,[*] as well as the exactness of every part of the operations, we do not hesitate to state our opinion that it is superior to every other that we have described.

Methods used in the remeasurement of Beccaria's arc.

Some triangles observed by the Baron de Zach, for the estimation of the accuracy of Beccaria's measure, were founded on a base of little more than 600 yards in length. An erroneous reduction was applied for temperature, and the base was measured but once; MM. Plana and Carlini, therefore, measured a new base of nearly the same length, and in nearly the same situation. The measuring rods were of wood, 12 mètres in length, in the form of hollow parallelopipeds. To the preceding end of each was attached a metallic frame in the form of a hollow parallelogram and in the same plane as one side of the parallelopiped. A fine thread was stretched across this frame, so that when the rod was horizontal the thread was vertical. In measuring, every rod was placed horizontal upon trestles; the second rod was not made to touch the first, but was so placed that a fine dot near its following end was bisected by the thread carried by the frame on the preceding end of the first rod. By this simple contrivance the use of the plumb-line was entirely avoided, and there was no fear of disturbing one rod by the contact of another. The rods were compared after each of the two measures with a standard mètre, by means of a microscopic apparatus. Some triangles, observed with the repeating-circle, connected this with one of Beccaria's principal sides; and the rest of the triangles were observed by the surveyors of the French Government. A chain of triangles had also been observed, connecting Beccaria's with a base measured near the Ticino; and others connecting it with the triangles crossing the Alps, and ultimately with the bases measured in France. The results from the different bases agreed very nearly. The azimuth was determined at the Observatory of Turin, from the meridian mark of the transit-instrument. The parts of meridian were calculated by Delambre's formulæ. The length of Beccaria's arc, deduced from the base on the Ticino, was 126,385.25 mètres, or 64,845 toises.

The latitudes were determined by observations made with a repeating-circle. At each extremity, Polaris, and two stars South of the zenith, as well as the Sun, were observed; several series of observations were made on each, the mean of the latitudes given by Polaris was taken as one determination, subject to the constant error of the circle, and the mean of the latitudes given by the other stars and the Sun, as another determination, subject to the constant error with the contrary sign. The mean of these was free from the constant error. Thus the latitude of Beccaria's station at Mondovi was found to be 44° 23′ 44″.37; that of Andrate 45° 31′ 15″.44, the amplitude 1° 7′ 31″.07, less by 13″ than that given by Beccaria.

The whole of this measure, and in particular the determination of the latitudes, seem to have been made with so much care, that we must allow its excellence. It is, in fact, almost the only one in which proper caution has been used in the determination of the latitudes with the repeating-circle.

Methods used in the Hanoverian measure.

The apparatus used in the Hanoverian[*] base-measure is described in a pamphlet, entitled *Schreiben an den Herrn Olbers von H. C. Schumacher, enthaltend eine Nachricht über den Apparat, dessen er sich zur Messung*

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[*] This arc has been extended by Captain Everest to latitude 24° 7′. The details are not yet published.

[*] We had not received the account of this and the following measure when the first section of this Treatise was sent to press.

der Basis bei Braack im Jahre 1820 bedient hat. Three bars of hammered iron were used, each 12 French feet in length, and $1\frac{1}{3}$ inch broad and deep; the ends were armed with steel plates, at one end flat, and at the other spherical. They were carried in wooden boxes completely closed, the ends only projecting; to prevent flexure they were supported by counterpoises applied at different parts. The boxes were placed on trestles of variable heights; the rods were placed horizontal, and at the same height, as long as the nature of the ground allowed. A small interval was left between the flat end of one rod and the spherical end of the next, and this was measured by dropping between them a glass wedge whose depth was forty-eight times the thickness of its back. When it became necessary to alter the height of the bars, a cylinder of bell-metal was employed, which by means of a level was placed very exactly vertical, and the same glass wedge was used to measure the distance between the top of the cylinder and the upper bar, as well as between the bottom of the cylinder and the lower bar. Of the length of the base, and the details of the survey, we find no mention; and we believe that no further account has been published.

The difference of latitudes of the observatories at Göttingen and Altona, was ascertained by observations made with Ramsden's zenith-sector, (the same which was used in the English survey.) Forty-two stars were observed at both places; each about six times in each position of the sector. In general, the sector was reversed after each night's observation. The details will be found in the *Bestimmung des Breitenunterscheides zwischen den Sternwärten von Göttingen und Altona, &c. von* C. F. Gauss. The results were, latitude of the observatory at Göttingen, 51° 31′ 47″.85; of the observatory at Altona, 53° 32′ 45″.27; amplitude of the arc, 2° 0′ 57″.42. The last is, undoubtedly, one of the most accurate determinations ever made.

It is stated by M. Gauss that the observatory of Altona is 115,163.7 toises North, and 7.2 toises West, of that at Göttingen. This estimation is founded on the base mentioned above; the measuring bars have not been definitely compared with the French toise, but M. Gauss states, that it is impossible that there can be any serious error in the estimation. The resulting length of one degree is 57,127.2 toises.

Methods used in the measure of the Russian arc.

The arc measured by M. Struve, and extending from Jacobstadt, on the Düna, to the island Hochland, in the Gulf of Finland, depends on a base of 2315 toises. This line was but once measured with iron bars of two toises in length, each of which carried at one end a lever of great sensibility, (*Fühlhebel*,*) by which the contact with the next was made. By means of a microscopic apparatus, they were compared with a standard made by Fortin. To detect any error in the base measure, it was divided into two parts, and these were connected by a triangulation. The signals were poles supported by wooden pyramids. The angles were measured with Reichenbach's universal instrument, (a sort of theodolite,) in which the horizontal circle was 14 inches in diameter. For the azimuths, the Pole star was observed with the same instrument, at Jacobstadt, Dorpat, and Hochland; at Dorpat, also, another determination was obtained from the meridian mark of the transit-instrument. The triangles were calculated by Legendre's theorem, and the distance of parallels by a method explained by Bessel in the *Astronomische Nachrichten*, No. 3. It consists in supposing arcs of great circles, A D, A F, &c., drawn from the station at one extremity A of the chain, fig. 36, to each of the other stations, and calculating each of the triangles A B D, A D F, &c., either as spherical triangles, or with the assistance of Legendre's theorem as plane triangles. Thus knowing A B, B D, and the angle A B D, the side A D and the angles B A D, B D A are found. Subtracting B D A from the sum of B D C, C D E, E D F, the angle A D F is found; and A D, D F, are known: hence A F, F A D, A F D, are found. This process is continued to the last station, as F; then all the angles at A being known, and the azimuth of A B being known, that of A F is known; also the length of A F is known; and then the distance of the parallels can easily be found by either of the methods already explained. The amplitude was found from observations of γ, ζ, and η, Ursæ Majoris, with an eight-feet transit-instrument adjusted to move in the prime vertical. It was carefully levelled, and was reversed after each night's observation. The clock (upon the goodness of which the exactness of this operation entirely depends) was made by Repsold. The latitudes were also determined by observations of Polaris and β Ursæ Minoris, with 18-inch circles made by Reichenbach: allowance was made for the flexure of the telescopes. By a mean, the latitude of Jacobstadt was found to be 56° 30′ 4″.7; that of Dorpat, 58° 22′ 47″.4; that of Hochland, 60° 5′ 9″.9. The terrestrial distances of the parallels were, from Jacobstadt to Dorpat, 107,281.6 toises; from Dorpat to Hochland, 97,538.5 toises. The lengths of 1° given by these parts are 57,108.8, and 57,165.5 toises. M. Struve remarks that the general flatness of the country, and the apparent absence of disturbing causes, would not have led him to expect so great a difference. An abstract of these operations will be found in the *Astronomische Nachrichten*, No. 164.

Bessel's method of calculating the distance of parallels.

The method of determining the latitude by transits (upon which it appears that M. Struve principally relied) has been so little used that we are unable to form a decided opinion on its merits. But the well-known character of the observer, and the caution evidently employed in every part of the operation, leaves no doubt of the general accuracy of these results.

Continuation of the Indian arc.

Since the last sheet was sent to press, the account of the continuation of the Indian arc by Captain Everest, mentioned in the note to p. 212, has been published. The new bases at Takal Khera (near Ellichpoor) and near Kulliampoor, (in the vicinity of Seronj,) were measured with the same chain and referred to the same brass bar which Colonel Lambton had used. Their lengths were 37,913 and 38,412 feet. Night-signals were generally preferred, as it was found that in the healthy season of the year the nights were very favourable, and the magnitude of terrestrial refraction at night such as to make stations visible which were not visible in the day. The large theodolite with which the greater part of the triangulation was performed had sustained much injury from an accidental blow while used by Colonel Lambton for the purposes of general geography, and the limb was completely distorted. Colonel Lambton, however, by the use of wedges, screws, pullies, and wooden hammers, restored the circle so nearly to its original shape, that the difference between the readings on any point of the limb, and on the point 90° from it, did not exceed 26″. By altering the zero point nine times for each observed angle it was presumed that the effect of this distortion was almost annihilated. Great skill is shown in the apportionment of the errors of observation. In the computation of the triangles, Legendre's method was used; as it appeared to be more exact than the method of referring to the chord triangles (used generally by the English geodists.) The azimuths were determined by observations of Polaris, β Ursæ Minoris, and β Cephei, at their greatest digressions. The meridian arc was calculated by computing separately the intervals between the points where the sides of the triangles (produced if necessary) cut the meridian of Takal Khera. The meridional distance from Daumergidda to Takal Khera, as deduced from the Beder base, was 1,105,618 feet: that deduced from the Takal Khera base 1,105,381. The difference is not explained; Captain Everest has adopted the mean, or 1,105,499 feet. The distance from Takal Khera to Kulliampoor is 1,097,320 feet; and the results from the two bases at its extremities agree almost exactly. The amplitude was found by numerous observations of 24 stars with the zenith-sector used by Colonel Lambton. The extent of the Indian arc has now become so great that it is impossible to observe the same stars at both extremities with the sector, and therefore the absolute latitudes must be calculated from the catalogued places of the stars. Colonel Lambton's observations were therefore recomputed with the improved catalogue, and the united results are: latitude of Punnæ, 8° 9′ 31″.685; of Daumergidda, 18° 3′ 16″.075; of Takal Khera, 21° 5′ 51″.940; and of Kulliampoor, 24° 7′ 11″.851. For details we must refer to Captain Everest's interesting *Account of the Measurement of an Arc of the Meridian between the Parallels of* 18° 3′ *and* 24° 7′

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* We are not acquainted with the construction of this instrument.

Section 5.—*Measures of Arcs of Parallel.*

Method used in the measure of the arc of parallel across the mouth of the Rhone.

In the verification of the French meridian by Cassini de Thury and Lacaille, the arc of parallel extending from Mont St. Victoire, near Aix, to the meridian of a station near Cette was thus measured. In fig. 37, let P be the pole of the Earth; V, Mont St. Victoire; S, the station near Cette; V*s*, an arc of parallel; C K, H *u*, perpendiculars on P H, P V. By a triangulation founded on a base measured near Arles, and by observation of the azimuth of H at V, with the usual calculations by parallels and perpendiculars, H *u* = 24,710.7 toises, *u* V = 10,422.7; whence the distance of the parallels of H and V = 10,328.3, which, considering 1° = 57,048, corresponds to a difference of latitudes 10′ 52″. The latitude of V being found = 43° 31′ 50″, that of H is found. And the length of H *u* will not sensibly differ from the length of the arc of parallel passing through H; increasing this in the proportion of cos lat. H : cos lat. V, V *k* is found = 24,785.2. Now investigating the convergence of meridians, and applying it to the angle P V H, V H *k* is found; and V H C having been determined by the triangulation, C H *k* and C H P are found. From this angle, and the length of C H, *c k* is found in the same manner as V *k*: it is 33,149.5. By a similar process *c s* = 20,728.9. The sum or V*s* = 78,663.6.

Measures of Arcs of Parallel.

It has been explained in Section 3, that the difference of longitude of two places can be found if we have any means of comparing clocks regulated to the solar or sidereal time at those places. The most convenient method is, to observe from both places some instantaneous signal, and to note the time indicated by both clocks at the time of observation. In the present instance, the difference of longitudes was determined by observing from V and S the explosion of ten pounds of gunpowder at a point between them. The duration of the flash was estimated at less than half a second. The clocks were regulated by equal altitudes of the Sun and stars. By a mean of four explosions, on December 14 and 15, 1739, and January 4 and 5, 1740, the difference of longitudes was found to be $7^m\,33^s\tfrac{1}{4}$ of time, or 1° 53′ 19″ of arc; whence 1° of parallel in latitude 43° 31′ 50″ = 41,618.

Of the observations for determining the clock errors, (upon which the difference of the longitude wholly depends,) no details are given. The extreme difference of the four determinations, which are quite independent, is 1½ second of time. The result, we imagine, is not subject to any great error, though the omission abovementioned makes it difficult to assert this positively.

Method used in the measure of the English arc of parallel.

In the course of the English survey, several arcs of parallel have been measured. The only one which, from its length, appears worthy of confidence, is that between Beachy Head and Dunnose. The difference of longitudes was thus ascertained *geometrically*. Proceeding from the bases measured on Hounslow Heath and Salisbury Plain, by the method of parallels and perpendiculars, and considering the latitude of Greenwich = 51° 28′ 40″, and the length of 1° = 60,851 fathoms, the latitude of Beachy Head was found = 50° 44′ 23″.71. Similarly the latitude of Dunnose was found = 50° 37′ 7″.31. At Beachy Head the azimuth of the signal on Dunnose was found, by observing the angle between it and a staff at a convenient distance, and by observing with the great theodolite, the difference of azimuth of this staff, and the Pole-star at its greatest azimuthal digression. By nine observations the angle between Dunnose and the North meridian was found to be 96° 55′ 58″. Similarly, by seven observations at Dunnose, the angle between Beachy Head and the North meridian was found to be 81° 56′ 53″. The problem now to be solved is this: Given the astronomical latitudes of two stations on a surface (not necessarily spheroïdal) differing little from a sphere, and their reciprocal azimuths, to find the difference of longitude, or the inclination of their meridian planes. The following beautiful solution is given by Dalby, and is the foundation of the method used in the English and Indian surveys; the considerations connected with it are almost sufficient for a complete theory of spheroidal triangles.

Difference of longitude may be found by reciprocal observations of azimuth.

In fig. 38, let B and O be two stations sufficiently elevated to be reciprocally visible; O B the straight line described by the rays of light; B R, O S normals, drawn in the direction of gravity; let P R S be that line parallel to the Earth's axis of rotation which cuts both these normals. (There is but one such line, as will easily be seen by supposing the whole figure projected on an equatorial plane; and whether it coincides or not with the axis of rotation is of no consequence.) The azimuth of O as seen at B is the inclination of the planes O B R, P B R; similarly, the azimuth of B as seen from O is the inclination of B O S, P O S. With radius C *p* = 1 construct a sphere, and take two points b, o, whose astronomical latitudes are the same as those of B and O, and whose meridian planes coincide with their meridian planes; that is, let C b and C o be parallel to R B and S O. Now we are to find the relation between the azimuths of o seen from b, and b seen from o, and those of O seen from B, and B seen from O.

Sum of azimuthal angles on spheroid equal to the sum of azimuthal angles at corresponding points of sphere.

From R draw R G parallel to S O, or C o; and from S draw S Q parallel to R B, or C b. Then each of the planes G B R, O Q S, is parallel to $o\,b$ C; and P B R, $p\,b$ C, are the same plane; also P O S, $p\,o$ C, are in the same plane. Consequently, the inclination of the planes P O S, Q O S, is equal to the angle o; and the inclination of P B R, G B R, is equal to the angle at b. Hence the azimuthal angle at O is *less* than that at o by the inclination of the planes B O S, Q O S; and the azimuthal angle at B is *greater* than that at b by the inclination of the planes O B R, G B R. If then the inclination of B O S, Q O S, be equal to the inclination of O B R, G B R, the *sum* of the azimuthal angles at B and O will be equal to the *sum* of those at b and o. We must now discover in what cases those inclinations are equal.

First, they are equal when R coincides with S, as both inclinations then vanish. This happens when the figure is spherical, with any latitudes, or with equal latitudes when the figure is any solid of revolution.

Secondly, from B draw B H perpendicular on O S, and B L perpendicular on the plane Q O S; and from O draw O K perpendicular on B R, and O M perpendicular on the plane G B R. Then the tangent of the inclination of O B R, G B R, is $\frac{\mathrm{O\,M}}{\mathrm{K\,M}}$; and the tangent of the inclination of B O S, Q O S, is $\frac{\mathrm{B\,L}}{\mathrm{H\,L}}$. The numerators are equal, (as they are the perpendicular distances of two parallel planes at different points;) are the denominators also equal? To discover this, project the figure on the plane G B R, (fig. 39,) then the lines K M, H L will be equal, if O N, B N are equal. This will happen when B O N = O B N, which holds when the reciprocal depressions are equal. Further than this we cannot proceed; but it will easily be seen that O N, B N are very nearly equal in all cases where O and B are equally elevated above a surface whose curvature does not alter very rapidly. If unequally elevated, the error in the assertion (that the inclinations of O B R, G B R, and of B O S, Q O S, are equal) arises only from this circumstance, that a perpendicular to the horizon at O does not appear perpendicular when seen from B; the effect of which, as we mentioned before, is quite insensible. We may therefore assert, that, for all practical purposes, these inclinations are equal; and, consequently, that the sum of the azimuthal angles at B and O is equal to the sum of those at b and o.

Spherical excess in spheroidal triangle equal to that in spherical triangle.

The first conclusion that we shall derive from this theorem is, that the spherical excess in a spheroidal triangle is the same as in a spherical triangle, whose vertices have the same astronomical latitudes and the same difference of longitude. In fig. 40, let B O C be a spheroidal triangle, and $b\,o\,c$ the corresponding spherical triangle. Adding the equations

$$\mathrm{POB} + \mathrm{PBO} = pob + pbo,$$

$$\mathrm{PCB} + \mathrm{PBC} = pcb + pbc,$$

we have

$$\mathrm{POB} + \mathrm{PCB} + \mathrm{OBC} = pob + pcb + obc.$$

Subtracting

$$\mathrm{POC} + \mathrm{PCO} = poc + pco,$$

there remains

$$\mathrm{COB} + \mathrm{OCB} + \mathrm{OBC} = cob + ocb + obc,$$

and, consequently,

$$\mathrm{COB} + \mathrm{OCB} + \mathrm{OBC} - 180^\circ = cob + ocb + obc - 180^\circ.$$

Theorem for finding the difference of longitude.

We shall next observe, (and this, in fact, is the application for which the theorem was invented,) that any equation relating to the spherical triangle $p\,o\,b$, fig. 38, in which only $p\,o$, $p\,b$, $o\,p\,b$, and $o + b$ are concerned, is equally true if we substitute for them colat. of O, colat. of B, difference of longitudes of O and B, and sum of azimuthal angles at O and B, (these being respectively equal to the former.) Now (Trigonometry, Art. 117.) $\tan\frac{op\,b}{2} = \frac{\cos\frac{p\,b - p\,o}{2}}{\cos\frac{p\,b + p\,o}{2}} \cdot \cot\frac{b + o}{2}$, and

therefore $\tan\frac{1}{2}\,\text{diff. longitudes} = \frac{\cos\frac{1}{2}\,\text{diff. latitudes}}{\sin\frac{1}{2}\,\text{sum of latitudes}} \times$ $\cot\frac{1}{2}$ sum of azimuthal angles. This is the theorem to be used for finding the difference of longitudes. An examination of it by the method given in Trigonometry, Art. 161, will make it evident that, in general, a small error in the latitudes produces no sensible error in the determination, while an error in the azimuths is of great importance; but that when the latitudes are small, the errors receive large multipliers in the ultimate result. This method ought not, therefore, to be used for stations near the Equator; and we shall therefore take no notice of the arc of parallel measured by Col. Lambton in India.

Ought not to be used in low latitudes.

From the data given above, the difference of longitude of Beachy Head and Dunnose is found to be 1° 26′ 47″.87; which will not be sensibly altered by the minute alterations since made in the latitudes.

The terrestrial arc of parallel was found by a method equivalent to the following. In fig. 41 let B L, D E be two parallels through B and D; B W and D R perpendiculars to the meridians P B, P D. In the triangles W B D, R B D, the necessary angles being known, and the chord of B D being found by triangulation = 339,397.6 feet, the arc B W was found = 336,119.1, and D R = 336,983.5. Now the latitudes of B and W differ so little that we may suppose their normals to intersect the axis in the same point T, fig. 42; construct a sphere, with centre T, passing through B; then W and L will be nearly in its surface. Then it is easily seen that tan W T B = cos lat. B × tan diff. long.; and $\mathrm{B\,W} = \mathrm{T\,B} \times \mathrm{W\,T\,B} = \mathrm{T\,B} \times \frac{\mathrm{W\,T\,B}}{\tan \mathrm{W\,T\,B}} \times \cos \text{lat.}\,\mathrm{B}$ × tan diff. long. Also the arc of small circle B L = T B × cos lat. B × diff. long.; consequently $\mathrm{B\,L} = \mathrm{B\,W} \times \frac{\text{diff. long.}}{\tan \text{diff. long.}} \times \frac{\tan \mathrm{W\,T\,B}}{\mathrm{W\,T\,B}}$. Now diff. long. = 1° 26′ 47″.87, and W T B = 54′ 56″.21; whence B L is found = 336,076.2. Similarly, D E is found = 336,940.5. These are arcs of parallel corresponding to a difference of longitude of 1° 26′ 47″.87, in latitudes 50° 44′ 23″.7 and 50° 37′ 7″.3; the corresponding lengths of one degree are 232,316.5 and 232,914.0.

It is stated by Captain Kater, (*Phil. Trans.* 1828,) that the theodolite used for determining the azimuth angles at Beachy Head and Dunnose (upon the accuracy of which the value of the result wholly depends) did not admit of adjustment sufficiently exact to give these angles with the necessary precision. Be this as it may, it appears certain (from the description of the mode of observation) that every care was taken to insure all the accuracy that it was possible to obtain. The result of these observations was intended to fix a scale of longitude for the great Map of England; and the observers knowing fully the importance of the determination, appear to have been well satisfied with its accuracy. Whether the error of collimation was corrected does not appear; but in the description of the instrument it is expressly remarked, that the telescope admitted of being reversed for this purpose. The result, however it may disagree with that which we shall next present, must, we imagine, be considered as equally valuable in proportion to the extent of the arc.

Correction of the difference of longitude of Dover and Falmouth by transportation of chronometers.

From the value of an arc of longitude nearly coinciding with those which we have given, (the lengths of 1° of parallel given in the *Trigonometrical Survey* arc 38,718 and 38,818 fathoms,) the difference of longitude between Dover and Falmouth was calculated to be $6^\circ\ 20'\ 52''.5$, or $25^{m}\ 23^{s}.5$ of time. To determine this independently, Dr. Tiarks was sent by sea with twenty-four chronometers to compare the apparent times at Dover and Falmouth. Three trips were made; the apparent time was determined by equal altitudes; this was compared with the time shown by the chronometers; and thus, the error of the chronometers with regard to each of these apparent times being found, the difference of the Dover and Falmouth time at the same instant was found. The result was that the difference of longitude is $25^{m}\ 28^{s}.4$; and this probably cannot be subject to any great error. This amounts to the same as saying, that an arc in the parallel of Beachy Head (lat. $50^\circ\ 44'\ 23''.7$) whose length is 1,474,672 feet, embraces a difference of longitude of $25^{m}\ 28^{s}.4$ of time, or $6^\circ\ 22'\ 6''$; or that 1° of parallel $=$ 231,563 feet.

Measure of the arc of parallel from Padua to Marennes.

In the determination of the difference of longitude of two places at a very great distance from each other, and where the transportation of chronometers easily and rapidly cannot be accomplished, the following methods may be used. The difference of longitude may be ascertained by astronomical observations carried on simultaneously at the two extremities; as observations of the Moon's right ascension at the time of her transit; observations of eclipses of Jupiter's satellites, (which are a sort of signals seen at both places at the same absolute time;) and observations of occultations of stars by the Moon, or of solar eclipses, (which differ from the last only in this respect, that the absolute time of the phenomenon is not exactly the same at both places, but that the difference admits of calculation.) The last is the most accurate of all known methods. Or the difference of longitude may be found by dividing the arc into several partial arcs, such that a point comprised in each may be visible at both its extremities, and establishing at each of those extremities a temporary observatory, and then determining the difference of longitude of these extremities by observation of artificial signals as in the French arc; and thus making the whole determination the sum of the several partial and independent determinations. Or instead of making all the determinations independent of each other, and dependent on the correctness of the apparent time used at each of the temporary observatories, the signals may be made at the signal posts, and observed at the temporary observatories, along the whole line in the same evening: then if there be any error in the assumed apparent time at one of the intermediate observatories, it will have the effect of increasing the determined extent in longitude of the arc on one side, and diminishing it for that on the other side, so that their sum will be unaltered. It is only necessary that the rates of the clocks be known pretty accurately. (See Mr. Herschel's Paper, *Phil. Trans.* 1826.) It is found in practice extremely difficult to establish the system of cooperation necessary for this method. In the arc which we are now considering, the second method was generally used, but the principle of the last was introduced in one part of it.

Difference of longitude of Padua and Milan.

In August 1824, signals were made by the explosion of gunpowder on four evenings on the summit of Monte Baldo, a mountain on the East bank of the Lago di Garda. Ten signals were made each evening; the quantity of powder used for each signal was three-quarters of a Viennese pound. They were observed by several astronomers at the observatories of Padua and Milan, (as well as at Bologna, Modena, and Verona.) Their difference of longitude was found to be $10^{m}\ 43^{s}.27$ in time. All the observations are given in the Milan *Ephemeris* for 1826: they appear to have been made with all possible care, and the extreme discordance of the results is little more than a second of time. The difference of longitude of the church of Santa Giustina at Padua, and the cathedral at Milan, was inferred to be $10^{m}\ 45^{s}.38$.

Difference of longitude of Milan and Mont Cenis.

In September 1821, ten signals were given on each of three evenings, on the mountain Roche Melon. They were observed at the observatory of Milan, and at a temporary observatory on the plain of Mont Cenis. The error of a chronometer (by Earnshaw) on mean solar time was found from observations with a transit-instrument erected near the hospice, and the chronometer was then carried to the place at which Roche Melon is visible. The difference of longitude was found $= 9^{m}\ 0^{s}.20$; whence the difference of longitude of the cathedral at Milan and the hospice at Mont Cenis $= 9^{m}\ 0^{s}.81$. The details are given in the *Opérations Géodésiques, &c.* It is easily seen that the difficulty of fixing firmly the instruments and other causes render this determination less certain than the last; but it appears to be satisfactory.

Difference of longitude of Mont Cenis and Geneva.

In September 1822, observations were made for the purpose of finding the difference of longitude of Mont Cenis and Mont Colombier, a hill on the right bank of the Rhone below Geneva. At Mont Cenis it was found necessary to carry Earnshaw's chronometer to a point 2000 feet above the observatory. The chronometer was compared, before going and after returning, with the transit clock; it was also compared when at the place of observation by secondary signals seen there and at the observatory. The results were different: the mean was generally taken. The tempestuous state of the weather made the determination of the absolute time at Mont Colombier very uncertain, and the following arrangement was therefore made. On some of the evenings on which signals made on Mont Tabor were observed at Mont Cenis and the observatory on Mont Colombier, signals were also made on another part of Mont Colombier, and observed at the observatory of Mont Colombier and that of Geneva. Consequently (as we have already explained) the difference of longitude between Mont Cenis and Geneva was found, unaffected by any error in the absolute time at Mont Colombier. The mean of the results from twelve signals (six on September 5, and six on September 7) was $3^{m}\ 8^{s}.76$. As this is affected by the errors attending four observations of signals, and as the number of signals is small, this determination cannot be considered equal to the last.

Difference of longitude of Geneva and Puy d'Isson.

In precisely the same manner, and at the same time, the difference of longitude of Geneva and Solignat was found. Signals made on Pierre-sur-autre were observed at Solignat and Colombier, and signals on another part of Colombier were observed on Colombier and at Geneva. The time at Solignat was determined by absolute zenith distances of stars observed with repeating-circles. The mean of the results of twelve signals on September 6 and 7, was $11^{m}\ 53^{s}.97$. See *Conn. des Temps*, 1828. This is liable to the same objections as the last; but if combined with the last, the result will be almost free from any error in the absolute time at Geneva. The difference of longitude of Geneva and the station on the Puy d'Isson (near Solignat) was inferred to be $11^{m}\ 57^{s}.82$.

If the signals on September 7 were thought sufficiently numerous, there would be no necessity for using the observations made at Geneva. On that evening the difference of longitude of Mont Cenis and Colombier was found to be $4^{m}\ 44^{s}.28$, and that of Colombier and Solignat $10^{m}\ 18^{s}.81$, whence that of Mont Cenis and Solignat $= 15^{m}\ 3^{s}.09$, and that of Mont Cenis and Puy d'Isson $15^{m}\ 6^{s}.94$, unaffected by the error of absolute time on Colombier.

Difference of longitude of Puy d'Isson and Sauvagnac.

In August 1823, the difference of longitude of Solignat and La Jonchère was determined by twenty signals on the Mont d'Or. The mean was $6^{m}\ 49^{s}.99$; from which the difference of longitude of the Puy d'Isson and Sauvagnac (near La Jonchère) $= 6^{m}\ 51^{s}.39$. Signals were, in fact, made on three evenings; but the results given by the first evening's observations differ steadily two seconds from those of the other evenings. No explanation of this anomaly could be given, and the results of the first evening's observations were rejected. This circumstance seems to throw some doubt upon the conclusion adopted. The absolute times were determined from zenith distances of stars.

Difference of longitude of Sauvagnac and St. Preuil.

In September 1823, the difference of longitude of La Jonchere and St. Preuil, was found in the same way from ten signals on September 20, given on Puy-Cogneux. Though signals were given on eight evenings it was only possible (from some atmospheric cause) to see them on one. M. Nicollet considers, however, that the favourable circumstances under which these observations were made leave no doubt of the accuracy of the result. The difference of longitude of La Jonchère and St. Preuil appeared to be $6^{m}\ 28^{s}.34$; that of Sauvagnac and St. Preuil, $6^{m}\ 23^{s}.09$.

Difference of longitude of St. Preuil and Marennes.

In October 1823, similar observations were made at St. Preuil and Marennes. The mean of forty-six signals gave for the difference of longitude $3^{m}\ 49^{s}.01$; and between St. Preuil and the steeple of Marennes, $3^{m}\ 48^{s}.99$.

Adding together these differences, we find for the difference of longitude between the steeple of St. Giustina at Padua, and the steeple of Marennes, $51^{m}\ 56^{s}.25$, or $12^{\circ}\ 59'\ 3''.75$. The terrestrial length of the corresponding arc of parallel in latitude $45^{\circ}\ 43'\ 12''$, as found from the French and Piedmontese survey, (we know not by what method of calculation,) is 1,010,996 mètres.

Determination of Figure of Earth from Geodetic Measures.

Value of this determination.

It is difficult to estimate the value of this determination. We are not, ourselves, inclined to rate it very high. Of the accuracy of the determinations on the Italian side of the Alps there can be little doubt. The credit of the Mont Cenis observations rests entirely on the steadiness of rate of a single chronometer, which was every day carried on a mountain path to a considerable elevation. Of the French determinations it seems probable that one (if not two) is liable to considerable errors. Those who know practically the difficulty of determining the time at a fixed observatory, and with the best instruments, to an accuracy of one-tenth of a second, will probably allow that there may have been at any of the temporary stations an error of a quarter of a second of time in the correction of the clock. When to this are added the errors of chronometers, and of observations of signals, and the doubt which is thrown on some parts by large and unexplained discrepancies, and when it is considered that the determinations of the six partial arcs are absolutely independent; we cannot flatter ourselves that the difference of longitude of the extreme points is determined within one second of time. And the principal observers were themselves so sensible of this uncertainty that they had (we believe) concerted a plan for the immediate determination of the difference of longitude of Marennes and Padua, by occultations or other astronomical phenomena observed at both places. The respective governments (as we are informed) refused to defray the expenses, on the ground that the arc might in a short time be extended further to the East. It is much to be regretted that the accurate determination of the extensive arc already surveyed should in the mean time be withheld from the scientific world.

Section 6.—Determination of the Figure of the Earth, from the Geodetic Measures.

Abstract of elements of all the arcs.

The following Table contains an abstract of the elements of the arcs above described. The foreign measures are reduced by supposing the toise = 6.394596 English feet,* (according to Shuckburgh's standard,) and the mètre = 3.280899 English feet.

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* This number is deduced from Captain Kater's measure of the mètre, (*Phil. Trans.* 1818,) supposing the mètre = 443,296 lines of the toise of Peru. The comparison mentioned in the *Phil. Trans.* 1768, allowing for the difference between the Royal Society's standard and Shuckburgh's standard as determined by Captain Kater, (*Phil. Trans.* 1821,) gives for the length of the toise 6.394743 feet of Shuckburgh's standard. Boscovich's arc has been corrected for the quantity (.03413 line) by which Mairan's toise was found to be shorter than the toise of Peru. (*Base du Système Métrique*, tom. iii. p. 414.) The English arcs of parallel, as well as the other English arcs, are reduced to Shuckburgh's standard.

Figure of the Earth.

Determination of Figure of Earth from Geodetic Measures.

Arcs of Meridian.

No.	Description.	Latitude of Middle Point.	Amplitude.	Length in English feet.	Remarks.
1	Peruvian arc as calculated by Delambre	−1°31′ 0″	3° 7′ 3″.1	1131057	Mountainous country. The amplitude probably well determined.
2	Maupertuis's Swedish arc.	66 19 37	0 57 30.4	351832	Mountainous country. A very little doubt about the amplitude.
3	French arc by Lacaille and Cassini de Thury......	46 52 2	8 20 0.3	3040605	Apparently very good.
4	Roman arc by Boscovich .	42 59	2 9 47	787919	Mountains in the middle of the arc, and sea at both extremities. The determination appears a good one.
5	Lacaille's arc near the Cape of Good Hope	−33 18 30	1 13 17.5	445506	Mountains in the neighbourhood. The determination appears good.
6	American arc by Mason and Dixon	39 12	1 28 45.0	538100	The country favourable; and apparently no doubt about the result.
7	French arc from Formentera to Dunkirk.......	44 51 2	12 22 12.6	4509402	The latitudes depend on inadequate observations with repeating-circles.
8	Svanberg's Swedish arc...	66 20 10	1 37 19.3	593278	Mountainous country. The latitudes doubtful, as in the last arc.
9	English arc from Dunnose to Burleigh Moor......	52 35 45	3 57 13.1	1442953	Excellent.
10	Lambton's first Indian arc.	12 32 21	1 34 56.4	574368	Excellent.
11	Lambton's second Indian arc, as extended by Everest.................	16 8 22	15 57 40.2	5794599	Excellent.
12	Piedmontese arc by Plana and Carlini...........	44 57 30	1 7 31.1	414657	The determination excellent. Mountains at both extremities.
13	Hanoverian arc by Gauss .	52 32 17	2 0 57.4	736426	Excellent, subject only to a small doubt about the standard.
14	Russian arc by Struve	58 17 37	3 35 5.2	1309742	Excellent.

Arcs of Parallel.

No.	Description.	Latitude.	Extent in Longitude.	Length in English feet.	Remarks.
15	Arc across the mouth of the Rhone, by Lacaille and Cassini de Thury	43°31′50″	1°53′1.9″	503022	Pretty good.
16	Roy's arc between Beachy Head and Dunnose	50 44 24	1 26 47.9	336099	Apparently very good.
17	Arc from Dover to Falmouth................	50 44 24	6 22 6	1474775	Apparently good.
18	Arc from Padua to Marennes	45 43 12	12 59 3.8	3316976	Subject to the accumulated errors of six independent determinations in difficult circumstances.

Equations of the arcs.

Forming for each of the meridional arcs the equation of p. 194, line 22, and for each of the arcs of parallel the expression of p. 193, line 42, we obtain the following Table.

No.	Equations.
1	$b \times .0544111 - be \times .05426 = 1131057$
2	$b \times .0167280 + be \times .02536 = 851832$
3	$b \times .1454455 + be \times .08688 = 3040605$
4	$b \times .0377524 + be \times .01489 = 787919$
5	$b \times .0213197 - be \times .00203 = 445506$
6	$b \times .0258163 + be \times .00512 = 538100$
7	$b \times .2159001 + be \times .10627 = 4509402$
8	$b \times .0283097 + be \times .04293 = 593278$
9	$b \times .0690040 + be \times .06160 = 1442953$
10	$b \times .0276169 - be \times .02371 = 574368$
11	$b \times .2785749 - be \times .20944 = 5794599$
12	$b \times .0196403 + be \times .00978 = 414657$
13	$b \times .0351849 + be \times .03132 = 736426$
14	$b \times .0625662 + be \times .07326 = 1309742$
15	$b \times .0238980 + be \times .03523 = 503022$
16	$b \times .0159783 + be \times .02556 = 336099$
17	$b \times .0703392 + be \times .11251 = 1474775$
18	$b \times .1582185 + be \times .23931 = 3316976$

We shall now proceed to discuss these equations.

Earth's dimensions found by different comparisons of meridian arcs.

1. Comparing the longest arcs, Nos. 7 and 11, we find $b = 20852460$, $e = .003318$.

2. Comparing No. 7 and No. 2, $b = 20816330$, and $e = .006850$.

3. Comparing No. 7 and No. 8, $b = 20851450$, and $e = .003418$.

4. As there is some doubt about both the Swedish arcs, (Nos. 2 and 8,) take the sum of the equations corresponding to them, and compare this with the equation of No. 7. Thus we find $b = 20838100$, and $e = .004722$.

5. The comparison of No. 1 and No. 3 gives $b = 20861200$, and $e = .003553$. The comparison of No. 1 and No. 7, gives $b = 20853720$, and $e = .003196$. This is nearly the same comparison as that from which the French determined the length of their mètre.

6. As the Roman and Piedmontese arcs (Nos. 4 and 12) are in circumstances of locality opposite in character, we may perhaps take the sum of their equations to represent nearly the equation which would be given by an arc in circumstances unaffected by these

peculiarities. Comparing this sum with No. 11 we have $b = 20897940$, $e = .006180$.

7. Comparing No. 11 with the sum of Nos. 9, 13, and 14, we find $b = 20853650$, $e = .003367$.

8. The comparison of the two American arcs (No. 1 and No. 6) gives $b = 20834100$, $e = .002283$.

9. The comparison of Nos. 5 and 11 gives $b = 20910300$, $e = .006961$.

Remarks on the discordances.

The discordance of these results, especially in the values which they give for the ellipticity, seems to show that several of the measures are affected with some considerable cause of error. This supposition at least is necessary if we suppose the Earth's Figure (at the level of the sea) to be one of revolution, and the generating curve to be at all similar to an ellipse. The most probable disturbing cause is the attraction of mountains; and it will be seen above, that the greatest discordances exist in those comparisons in which Nos. 2, 4, 5, and 12 are used; all of which are in mountainous Countries. Let us then exclude all which are in that predicament, namely, Nos. 1, 2, 4, 5, 8, and 12, and with the others endeavour to ascertain the Earth's dimensions.

10. Comparing the sum* of Nos. 3, 6, 7, 9, 13, 14 with the sum of Nos. 10, 11, we find $b = 20854270$, $e = .0033808$.

11. If we confine ourselves to the modern arcs, we may compare the sum of Nos. 7, 9, 13, 14 with the sum of Nos. 10, 11, and we get $b = 20853355$, $e = .0033232$.

The mean of the two last may, perhaps, be considered a good determination. (This amounts to the same as giving the modern arcs double the credit, *cæteris paribus*, of the ancient arcs.) Thus we find $b = 20853810$ feet, $e = .0033520$.

Comparison of observed and calculated arcs of meridian.

Substituting these values in each of the equations of meridian arcs, we get the following apparent errors. These may be produced by errors in the geodetic measures, or by errors in the astronomical determination of amplitude.

No.	Latitude of Middle Point.	Calculated Length.	Measured Length.	Error in Measure.	Corresponding Error in Amplitude.
1	−1°31′ 0″	1130886	1131057	+171	−1″.7
2	66 19 37	350615	351832	+1217	−12.0
3	46 52 2	3039166	3040605	+1439	−14.2
4	42 59	788322	787919	−403	+4.0
5	−33 18 30	444455	445506	+1051	−10.4
6	39 12	538726	538100	−626	+6.2
7	44 51 2	4509768	4509402	−366	+3.6
8	66 20 10	593366	593278	−88	+0.9
9	52 35 45	1443302	1442953	−349	+3.4
10	12 32 21	574260	574368	+108	−1.1
11	16 8 22	5794708	5794599	−109	+1.1
12	44 57 30	410259	414657	+4398	−43.4
13	52 32 17	735929	736426	+497	−4.9
14	58 17 37	1309865	1309742	−123	+1.2

Determination of Figure of Earth from Geodetic Measures.

The reader is now able to judge whether the dimensions which we have used, or any others, are likely to represent with tolerable accuracy the measured lengths of meridian arcs on the Earth's surface. We have sometimes thought that the measured arcs appeared too small, or the amplitudes too great, in middle latitudes; or that the comparison of equatorial arcs with those in middle latitudes gave a smaller ellipticity than that of arcs in middle latitudes with Northern arcs. These indications would show (see Section 3) that the Earth's Figure is protuberant at middle latitudes above the elliptic spheroid. But after careful examination we believe that there is no unequivocal appearance of this And we do not perceive, that the difference between the calculated and the measured arcs would be materially diminished (at least for the best arcs) by adopting a different value for the ellipticity. And observing that Nos. 1 and 6 are nearly in the same longitude, that Nos. 2, 3, 4, 5, 7, 8, 9, 12, 13, 14, are nearly in the same longitude 90° from the former, and that Nos. 10 and 11 are in a third longitude 60° from the last, we think ourselves entitled to conclude that there is no important difference between the different meridians of the Earth.

The assumed dimensions satisfy the arcs of meridian.

On the whole we are inclined to think, that as far as meridian measures go, the Earth's form may be very well represented by that of an ellipsoid of revolution with the dimensions that we have assigned. The greatest differences are at the places and in the directions that we should *à priori* have expected. No. 12 is measured across the great basin of Piedmont and absolutely abuts on the feet of very lofty mountains at both extremities; No. 3 is terminated by the Pyrenees;* No. 5 is in circumstances partly similar to those of No. 12; and No. 2 is in a mountainous country. The other differences are not greater than such as might be produced by a small lump of unequal density below the Earth's surface, or (as Captain Everest has shown in his *Account of the Measurement, &c.*) by a mountain range at a considerable distance.

Discussion of arcs of parallel.

We shall now proceed to consider the arcs of parallel.

12. Comparing No. 17 with No. 9, $b = 20841200$, $e = .003763$. These then are the dimensions of the ellipsoid that corresponds most exactly to the curvature of England.

13. Comparing No. 18 with No. 11, $b = 20855220$, $e = .003466$.

14. Comparing No. 18 with No. 7, $b = 20848880$, $e = .003667$.

Here we may observe that upon comparing an arc of parallel nearly in latitude 45°, with two arcs of meridian of which one is in a low latitude, and the other

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* In determining elements from a number of observations or measures, the method of *minimum squares* has frequently been used. We have rejected the use of this method, for the following reason. It is perfectly certain, that the elements determined by this method, if substituted in the equations of condition, will generally give the greatest apparent errors of linear measure in the smallest arcs, and *vice versâ*. A consequence so directly opposed to common sense cannot, we think, be supported by any symbolical reasoning. The doctrine of chances (from which this method is deduced) is more liable than any other to errors of omission in the preliminary considerations for the solution of any problem; and we prefer resting in the belief that there is some such error in the proof of this method, to receiving the consequence abovementioned. We have, therefore, thought best to use the method commonly employed in Astronomy, *viz.* to take the sum of groups of the equations of condition, and to consider each sum as one equation: the groups being selected so as to make the coefficient of e large and positive in one sum, and large and negative in the other.

* A part of this error is undoubtedly to be attributed to the errors of bases, &c. mentioned by Delambre, *Base du Système Métrique*, tom. iii. p. 163. Perhaps a small part of his differences is to be assigned to the later measure.

Figure of the Earth.

nearly in latitude 45°, the first comparison gives a smaller ellipticity than the second. This, as we have mentioned in page 197, would indicate that the Earth's Figure projected at middle latitudes above the spheroidal form. But we place little reliance on this conclusion, because the determination of the extent in longitude of the arc of parallel does not appear to us sufficiently certain.

Comparison of calculated and measured arcs of parallel.

If we substitute the values $b = 20853810$, $e = 0033520$, in the equations of arcs of parallel, we get the following apparent errors.

No.	Latitude.	Calculated Length.	Measured Length.	Error in Measure.	Error in Amplitude.		Lateral Deviation of Plumb-line.
					In Space.	In Time.	
15	43°31′50″	500827	503022	+2195	−30″.0	$-2^{s}.00$	21″.7
16	50 44 24	334995	336099	+1104	−17.2	−1.15	10.9
17	50 44 24	1474705	1474775	+70	−1.1	−0.07	0.7
18	45 43 12	3316187	3316976	+789	−11.1	−0.74	7.8

The assumed dimensions satisfy the arcs of parallel.

The magnitude of these errors, and the fact of their all having the same sign, appear at first sight rather alarming. It must, however, be remarked, that the two first arcs are measured across low land or sea, and terminated at both extremities by hills, and the effect of the attraction of the hills would be to make the amplitude smaller than if no such inequality existed. The extremities of the third and fourth arcs are more favourably situated; and in these the errors are within the limits of errors of observation. On the whole we are pretty well satisfied with the agreement between the computed and measured arcs.

Conclusion deduced from the measures of arcs.

The following are our conclusions from the discussion of the measures.

1. The measured arcs may be represented nearly enough *on the whole* by supposing the Earth's surface (at the level of the sea, or at the level at which water communicating freely with the sea would stand) to be an ellipsoid of revolution, whose polar semiaxis is 20,853,810 English feet, or 3949.585 miles, and whose equatorial radius is 20,923,713 feet, or 3962.824 miles. The ratio of the axes is 298.33 : 299.33; and the ellipticity (measured by the quotient of the difference of the axes by the smaller) is $\frac{1}{298.33}$, or .003352. The meridional quadrant is 32,811,980 feet.

2. In order, however, to conciliate the various measures with this assumed form, it is necessary to suppose that in some observations, made apparently with the greatest care and with competent instruments, the latitude is erroneous to the amount of at least 22 seconds. But these errors occur almost without exception in the localities where we should have expected them (on the principle of universal gravitation) from the disturbance of the adjacent mountains.

3. It is necessary also to suppose that in some cases the direction of gravity is altered to the East or West of that which it would have, were the Earth's Figure perfectly regular. These instances also occur in localities where (as above) we might *à priori* have expected them.

4. In consequence of these discordancies, the dimensions above given are liable to some uncertainty. We are, however, inclined to think that e cannot be so small as .00325, nor so great as .00345.

Observations of Pendulums

5. It appears (see Section 2, Art. 12 and 33) that if the whole attraction were directed to the Earth's centre, the ellipticity would be $\frac{1}{580}$; if the Earth were homogeneous, and the particles mutually attracted each other, the ellipticity would be $\frac{1}{230}$. The ellipticity given by the measures is between these. We are entitled then to suppose that the state of the Earth is intermediate to these two states: that is, to suppose that the interior of the Earth is more dense than the surface, but that the mutual attraction of the parts near the surface is sensible when compared with the attraction exerted by the parts near the centre.

Section 7.—Observations of Pendulums, for the purpose of measuring the Force of Gravity.

Necessity for using the pendulum.

The necessity of using the pendulum for measuring the force of gravity, will easily be seen if we consider the impossibility of ascertaining the magnitude of that force by any experiments on *single* descents of free masses. The quantity to be measured is the velocity which gravity creates in any freely descending body by its action continued during one second (or any other given duration) of time. This will be known, if we determine the space through which gravity draws the body in that time. But with all the contrivances that we can use for retarding the motion in a known proportion so as to make it measurable to our senses, it is impossible to make the measure of the space sufficiently accurate. Any one who has seen experiments with Atwood's or Smeaton's apparatus will allow that we cannot expect to measure the space described within a hundredth part of the whole. With the pendulum we can ascertain the same thing (in the opinion of some philosophers) within a four-hundred-thousandth part of the whole. This accuracy arises partly from the circumstance that the experiments with the pendulum may be continued for a very long time, with the certainty that there is no interruption between the end of one vibration and the beginning of the next, and partly from the very remarkable fact that the friction and other disturbing causes which ultimately put a stop to the experiment, (and which in the case of a descending body in Atwood's and other apparatus, would completely alter the results even if we had the power of observing them accurately,) do not injure its accuracy as long as it lasts.* We shall now proceed to describe the general principle, and the most approved methods of deducing the magnitude of gravity from observations of the pendulum.

General principle of the use of the pendulum.

In our Treatise on MECHANICS, § XV. it is shown that the time of vibration (expressed in seconds) of a simple pendulum whose length is l, in double the circular arc whose versed sine to that radius is b, will be nearly expressed by $\pi \sqrt{\frac{l}{2g}} \times \left(1 + \frac{b}{8l}\right)$, where g equals the space through which a body would fall freely in a second of time; the arc being supposed small. If the arc be extremely small the time will not differ much from $\pi\sqrt{\frac{l}{2g}}$. Suppose such a length to be given to the pendulum that the time of vibration may be one second. Then $\pi\sqrt{\frac{l}{2g}} = 1$, or $g = \frac{\pi^2 l}{2}$. The space g then will be found by multiplying the length of the pendulum by $\frac{\pi^2}{2}$. And if gravity is not the same on different parts of the Earth, the length proper for making the time of vibration equal to one second will not be the same in all. But when the lengths at the different places are determined, the proportion of the forces of gravity at those places (measuring the forces by the spaces through which bodies would fall freely in one second) will be the same as the proportion of these lengths.

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* For a demonstration of this we must beg to refer to the *Cambridge Transactions*, vol. iii. p. 111, &c.

In different places the seconds' pendulum has different lengths.

Found by observing the time of vibration of a given pendulum.

But it is seldom in our power to make a pendulum which shall vibrate exactly in one second. Instead of this we observe the time in which a pendulum of known length vibrates. Now, it is easily seen from the expression $t = \pi\sqrt{\frac{l}{2g}}$, or $l = \frac{2g \cdot t^2}{\pi^2}$, where t is the time of vibration, that the length of the pendulum at the same place is proportional to the square of its time of vibration. The length of the seconds' pendulum then bears to the known length the same ratio as unity to the square of the number of seconds in the observed time of vibration.

Compound pendulum may be used as well as simple pendulum.

All this, however, supposes the pendulum to be a material point suspended by a string without weight; a construction which evidently is imaginary. But it is shown in MECHANICS, § XX., that if a compound pendulum be given, we may calculate the length of a simple pendulum which would vibrate in the same time, whatever be the magnitude of gravity. A pendulum, therefore, such as may be made in practice, may be used in all respects for the same purposes as the imaginary simple pendulum.

The method which we have described, is in fact that which has been adopted in many of the most important experiments. The absolute length of the seconds' pendulum is thus determined afresh at every station of observation. But a method has also been used of finding the proportion between the force of gravity at each place, and the force of gravity at some place (London for instance, or Paris) where its absolute magnitude has been well ascertained. This is done by transporting the same pendulum, after having observed it at the place of reference, to all the different stations, and observing the time of vibration at all. Then, since $g = \frac{\pi^2 l}{2t^2}$, and l is the same at all the observations, (the pendulum being invariable,) g is inversely proportional to t^2. Consequently, the force of gravity at any new station, is to the force of gravity at the place of reference, inversely, as the square of the time of vibration at the new station to the square of the time of the same pendulum's vibration at the place of reference; or directly, as the square of the number of vibrations per day at the new station to the number of vibrations per day at the place of reference. The lengths of the seconds' pendulums are in the same ratio; consequently, that at the place of reference being known, that at the new station is found.

Method of transporting an invariable pendulum to different places.

Observations of Pendulums.

We shall now mention some additional considerations which must in all operations of great accuracy be taken into account.

Correction for the length of the arc of vibration.

First, we have supposed the arcs of vibration to be indefinitely small. When the arc is supposed small, but not so small that the term depending on it can be wholly neglected, the time of vibration is nearly $\pi\sqrt{\frac{l}{2g}} \cdot \left(1 + \frac{b}{8l}\right)$. If the arc of vibration be n degrees on each side of the vertical, then

$$b = l \,.\, \text{versin.}\, n^0 = 2l \,.\, \sin^2\frac{n^0}{2} = l \,.\, \frac{n^2 \sin^2 1^0}{2},$$

(nearly,) and the time of vibration is nearly

$$\pi\sqrt{\frac{l}{2g}} \cdot \left(1 + n^2 . \frac{\sin^2 1^0}{16}\right).$$

This is the time observed; and if the observations continue for so short a time that n does not sensibly alter, the observed time of vibration ought to be divided by $1 + n^2 . \frac{\sin^2 1^0}{16}$, or the number of vibrations per day ought to be multiplied by the same quantity, in order to represent the time or number of vibrations in indefinitely small arcs, (to which alone all our reasoning above will apply.) But if the observations continue for a long time it is necessary to know what is the law of decrease of the arc. The laws of friction and resistance of the air for small velocities being little known, it is best to recur to direct experiment. It was found by Borda (and we have found the same by our own observations) that the decreasing arcs form very nearly a geometrical series.* Putting m for the number of vibrations observed, (m being a large number,) p for the proportion of each arc to the preceding arc, (where $1 - p$ is extremely small,) n' for the number of degrees in the last arc $= n \,.\, p^{m-1}$, the degrees of the successive arcs are n, np, np^2, &c. . . . np^{m-1}, and the sum of all the times of vibration

$$\pi\sqrt{\frac{l}{2g}} \times \left\{m + \frac{n^2 \,.\, \sin^2 1^0}{16}\left(1 + p^2 + p^4 + \&c.. + p^{2m-2}\right)\right\}.$$

The sum of the geometrical series is

$$\frac{1 - p^{2m}}{1 - p^2} = \frac{1 - p^{2m-2}}{1 - p^2} \text{ very nearly;}$$

hence the sum of all the times

$$= \pi\sqrt{\frac{l}{2g}} \times \left\{m + \frac{n^2 - n'^2}{1 - p^2} \cdot \frac{\sin^2 1^0}{16}\right\}.$$

But $p^{m-1} = \frac{n'}{n}$, therefore $p^2 = \left(\frac{n'}{n}\right)^{\frac{2}{m-1}} = \left(\frac{n'}{n}\right)^{\frac{2}{m}}$ very nearly, therefore $\log p^2 = \frac{2}{m}(\log n' - \log n)$. But $\log p^2 = \text{modulus} \times \overline{p^2 - 1}$ nearly (as $p^2 - 1$ is very small;) hence $1 - p^2 = \frac{2}{\text{modulus} \times m} \cdot (\log n - \log n')$; consequently the sum of all the times

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* In our experiments the decrease of the arc at first was a very little more rapid than according to the geometric law, and at last a very little slower.

$$= \pi \sqrt{\frac{l}{2g}} \times \left\{ m + m . \frac{\text{modulus} \times (n^2 - n'^2)}{\log n - \log n'} . \frac{\sin^2 1^\circ}{32} \right\};$$

and the mean of the observed times

$$= \pi \sqrt{\frac{l}{2g}} \times \left\{ 1 + \frac{\text{modulus} \times (n^2 - n'^2)}{\log n - \log n'} . \frac{\sin^2 1^\circ}{32} \right\}.$$

The quantity within the brackets is that by which the observed mean time of vibration ought to be divided, or the number of vibrations per day ought to be multiplied, in order to reduce the vibrations to vibrations in indefinitely small arcs.

Correction for the temperature.

Secondly, we have supposed the length of the pendulum to continue invariable. But a metallic wire or metallic bar undergoes considerable changes in its length from changes in temperature, and it is necessary, therefore, to reduce the number of vibrations to the number which would have been made if the pendulum had been at some standard temperature. Suppose by heat the pendulum's length is increased above that at the standard temperature in the ratio of $1 : 1 + y$ (y being very small:) then the time of vibration is increased in the ratio of $1 : \sqrt{1+y}$, or $1 : 1 + \frac{y}{2}$ nearly, and the number of vibrations per day is diminished in that ratio. Consequently, the observed number of vibrations ought to be increased in that ratio in order to find the number which would have been made if the pendulum had been at the standard temperature. If the temperature during the observation be lower than the standard temperature, the observed number of vibrations ought to be diminished.

Correction for the weight of the air.

Thirdly, the vibrations have generally been observed in air. But the state of the air being variable, it is desirable to calculate the number of oscillations which would have been made in vacuum, all other circumstances being the same. For this we have only to observe that the effect of the air (like that of any other fluid upon a body immersed in it) is to diminish the weight of the pendulum by a quantity equal to the weight of the air displaced, or to diminish the apparent force of gravity in the same proportion.* If this diminution be in the ratio of $1 : 1 - z$, then the time of vibration (as will appear by changing g into $g(1-z)$ in the expressions above) will be increased in the ratio of $1 : \frac{1}{\sqrt{1-z}}$, and consequently, the number of vibrations per day diminished in the ratio of $1 : \sqrt{1-z}$, or $1 : 1 - \frac{z}{2}$ nearly. These are the vibrations observed; and therefore to get the number of vibrations in vacuum we must increase the observed number in the ratio of $1 : 1 + \frac{z}{2}$. As the weight of the air is nearly proportional to the height of the barometer, it is necessary for the calculation of this correction to observe the barometer.

We shall now describe some of the practical methods of observing the vibrations of the pendulum.

Observations of Pendulums.

Attached pendulum of a clock used as transportable pendulum.

In a few instances the pendulum of a clock has itself been used as an invariable pendulum to be transported from one station to another. This is the simplest of all methods, as the clock may be used as a transit clock, and its rate immediately determined by transits of the fixed stars: at all events comparison with another clock is all that is necessary. And though the accuracy is undoubtedly not so great as with the apparatus that we shall presently describe, yet very valuable results may be obtained in this manner. If the escapement is so constructed that the pendulum receives the impulse of the wheels when it is at the middle of its vibration, (which is nearly the case in the dead-beat construction,) the time of vibration is not altered by the maintaining power. The suspension of the pendulum is a matter of some difficulty. If it be suspended by a spring in the usual way, the time of vibration is not the same and does not vary in the same manner as the time of vibration in a circular arc, and the elasticity of the spring varies with variations of temperature. If it be suspended on knife-edges, (this is the term commonly used for a prismatic bar of very hard steel, on one edge of which, having an angle of from 20° to 90°, the pendulum turns during its vibration,) there is, perhaps, some fear of the edge slipping on its supports when it receives the pressure of the wheel-work. Satisfactory results have, however, been obtained with this construction.

Method of Bouguer, Lacondamine, Legentil, &c.

The method commonly used in the beginning of the last century was to suspend a small weight (commonly in the shape of two frustra of cones with their bases in contact) by a fibre of *pite*, (a preparation, we believe, of the leaf of a species of aloe,) and placing it before a clock to observe the number of vibrations which it made, not by counting them, but by observing when it had gained or lost two vibrations upon the clock pendulum. The clock was thus used to count the great number of vibrations; the observer counted only those which the detached pendulum gained or lost upon it. The number of vibrations which it would have gained or lost in a day being found by proportion, and the number of vibrations per day of the clock pendulum being ascertained by observations of the stars, the number of vibrations of the detached pendulum per day is found. From this, by the application of the corrections above-mentioned, the number of vibrations per day in indefinitely small arcs and in vacuum is found. The length of the pendulum was found by measuring the distance between the lower edge of the forceps that held the thread and the upper surface of the weight; or between the lower edge of the forceps and a plane on which the bottom of the weight would just rest. It is to be remarked, that the length of the seconds' pendulum thus found is always too great, as the curvature of the thread does not begin close to the suspending forceps.

Borda's method.

The method which was first used by the French, nearly at the time of measuring the arc between Dunkirk and Barcelona, was the following. A spherical ball of platinum, and a brass cap covering about one-fifth of its surface (represented in fig. 43) were ground together so as to fit accurately in every position of the ball. When this is the case, if the cap be suspended the ball will adhere to it provided a very small quantity of grease be interposed. The peculiar advantage sought in this construction was to destroy the effects of unequal Specific Gravity in different parts of the ball

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[*] We shall speak hereafter of an alteration which it has been found necessary to make in this proportion.

Observations of Pendulums.

by altering its position with regard to the direction of gravity. A fine metallic wire was attached to the cap and fastened at the top to a small apparatus connected with a knife-edge. (See fig. 44.) The upper part of this was a stalk cut with a screw-thread on which a small weight could be adjusted. The time of vibration of the pendulum being known very nearly before the observations were begun, the weight was adjusted so that the small apparatus of fig. 44, without the wire, would vibrate on the knife-edge in that time. Consequently, when the wire, cap, and ball were attached to it, the suspending apparatus might be wholly neglected in the calculation of the length of the equivalent simple pendulum. The knife-edge rested upon a steel plate. The number of vibrations per day was ascertained in the same way as in the first method: a shade with a vertical edge being placed so that when the pendulum was at rest, the vertical edge, as viewed in a small telescope, coincided with the pendulum wire. A cross was made on the bob of the clock pendulum, and the observation consisted in noting when the wire and cross disappeared at the same time behind the edge. All the corrections were applied which we have described. The length was measured by screwing up from below a horizontal plate of steel till it just touched the ball; then the pendulum being removed, a bar of known length with a T head (to the lower surface of which the end of the bar had been accurately fitted) was placed in such a manner that the lower surface of the T head rested on the upper steel plate, and a graduated slider on the bar was then made to touch the lower steel-plate. The ball was placed on the steel-plate and the same slider was used to determine its diameter; the wire and cap were weighed; then to determine the length of the isochronous simple pendulum, a formula was used founded on that given in Mechanics, § XX. At every station of observation it was necessary to repeat the whole of this process.

Kater's method.

The process used by Captain Kater for determining the length of the seconds' pendulum at London (for fixing the standard of English measures) was the following. The pendulum consisted of a bar A B of plate brass, (fig. 45,) about $1\frac{1}{2}$ inch broad, and $\frac{1}{8}$ inch thick. At A was a knife-edge of the hardest steel, its back bearing firmly against solid knees of brass; and at B a similar knife-edge. When the pendulum was in use, these edges rested on horizontal plates of agate. At C was a large flat bob, and at D and E two small slips or tails, the use of which we shall mention hereafter. At F was an adjustible weight, and at G a smaller weight, which by means of a screw could be adjusted with very great nicety. The principle of the operation was to observe the number of vibrations per day made by the pendulum when suspended on the knife-edge A, and again when suspended on B. If these were not equal, the sliding weights F and G were moved till they became equal. Then as A and B were at different distances from the centre of gravity, it was certain (see Mechanics, § XX.) that B was the centre of oscillation corresponding to the centre of suspension A, and therefore, that the distance between A and B was the length of the simple pendulum vibrating in the same time. The distance between the knife-edges was compared with the distance of two divisions on the standard scale by a microscopic apparatus furnished with internal micrometers.

The number of vibrations per day was determined by observing in the following manner the difference between the vibrations of the detached pendulum, and the vibrations of a clock pendulum. Suppose the pendulum suspended on the edge A. On the bob of the clock pendulum, which had been blackened, was placed a circle or lenticular figure of white paper, whose breadth was nearly equal to the breadth of the tail D. Between the two pendulums, or (which was the original construction) in the field bar of the telescope with which they were viewed, was placed an adjustible diaphragm, consisting of two perpendicular cheeks, which were placed at such a distance that they appeared to touch the edges of the tail D when the pendulum was at rest. Now suppose both pendulums to vibrate, the detached pendulum vibrating more slowly than the clock pendulum. The tail D is seen to cross the opening of the diaphragm, and is followed by the white patch on the clock pendulum. At every succeeding vibration the patch follows more closely, and at last is completely covered by the tail while it passes. This is called a *disappearance.* After a few vibrations it appears again, preceding the tail. This is called a *reappearance.* The time of disappearance was generally considered as the time of coincidence of vibrations, though in strictness the mean of the times of disappearance and reappearance ought to be taken, but the error produced by this mistake is seldom sensible. This is very far the most accurate way of comparing the vibrations of two pendulums. It is immaterial whether the detached pendulum vibrate in a greater or less time than the clock pendulum, but it is essential to the operation that it vibrate in a smaller arc; a condition which there is no doubt of securing in these experiments.

The English observations for ascertaining the length of the seconds' pendulum at different places have generally been made by transporting an invariable pendulum whose vibrations were observed before going and after returning at London or Greenwich. The construction was similar to that of fig. 40, supposing it deprived of the knife-edge B, the tail E, and the sliding weights F and G.

Bessel's method.

A method, proposed we believe in the last century, has lately been put in practice by M. Bessel for ascertaining the length of the seconds' pendulum at Königsberg. The pendulum consists of a heavy ball suspended by a fine wire, which is made to vibrate first with one length of wire and then with a different length. The advantage proposed in this construction was the following. It is easy to measure accurately the vertical distance between two points of suspension on which the pendulum successively vibrates, and therefore (as the ball in both experiments is depressed to the same place) the difference of length of the two pendulums is easily found. This distance too is not necessarily a fractional part of the standard, and in M. Bessel's experiments was, in fact, exactly one toise. And as the small corrections depending on the diameter of the ball, &c. are easily calculated from an approximate knowledge of the length of the pendulum, the difference between these corrections is easily found. Thus the difference of length of the simple pendulums isochronous to the pendulums observed is found with great accuracy. The times of vibration being also observed (which are proportional to the square roots of these lengths) we deduce from them the proportion of the lengths. From these two data the lengths are found. Now it was supposed that this measure could be made more accurately than those necessary to Borda's method, where the diameter of the ball must be ascertained, or than those of Kater's method, where the distance between two edges is to be measured. The sort of suspension used by M. Bessel was different from the others, consisting in unwrapping the wire from a horizontal cylinder round which the upper extremity was folded. And in the observation of coincidences, instead of placing the pendulum within a few inches of the clock pendulum, it was placed at a considerable distance, and the image of the clock pendulum formed by an interposed lens was thrown upon the detached pendulum: so that all disturbance of one pendulum by the other was effectually prevented.

Carlini's method.

In some observations by M. Carlini, the upper end of the wire was attached to a small apparatus represented in fig. 46. In the upper part of this is a wheel with a sharp edge, turning like the rowel of a spur; the edge serves as a knife-edge for suspension, and by turning the wheel the suspension is changed. The piece upon which it rests is represented in fig. 47; it is supposed to project from a wall. A microscope is placed in such a manner as to view the lower edge of the wheel directly, and likewise to view its image formed by reflection in the curved surface of fig. 47; the place where they appear to meet is very accurately the place of suspension. By means of a microscope placed at the bottom, the upper and lower surfaces of the ball were viewed even during the vibration. The vibrations were observed as by M. Bessel; a plane mirror, as well as a lens, being necessary for the formation of the image of the clock pendulum.

We shall now proceed to mention, individually, some of the principal pendulum observations.

Richer's observations.

In the Ancient *Mémoires de l'Académie*, vol. vii. is Richer's account of his observations at Cayenne; which we cite as a matter of curiosity* only, it being, we believe, the first of the kind, and that which seems to have attracted Newton's attention. The vibrations of his pendulum were very small and were sensible during 52 minutes of time; they were compared with those of an excellent clock; the observation was repeated several times every week for ten months; the result was that at Cayenne the seconds' pendulum was $1\frac{1}{2}$ lines shorter than at Paris. The latter he considered to be 3 feet $8\frac{3}{5}$ lines. This is the whole of his account.

Those of Deshayes.

In the *Mémoires* for 1701, is an abstract of the results obtained by Deshayes. At Cayenne, latitude 4° 56′, he found the length of the seconds' pendulum rather less than 3 feet $6\frac{1}{2}$ lines; at Grenada, latitude 12° 6′, at Martinique, latitude 14° 44′, the same; at St. Christopher, latitude 17° 19′, and several places in St. Domingo between latitude 18° 19′ and 19° 58′, 3 feet $6\frac{3}{4}$ lines; at another point in St. Domingo, latitude 19° 48′, 3 feet 7 lines. These results (about which no further account is given) are evidently worth little.

Observations by M. de la Croyère at Archangel.

In the *Petersburgh Transactions* for 1735, are recorded the observations made at Archangel by M. de la Croyère. The pendulum was a brass ball $14\frac{1}{2}$ lines in diameter supported by a thread of *pite* the top of which was held by a sort of forceps. An iron measure, copied from the standard in the Paris observatory, was used for making a pendulum of the length of the seconds' pendulum at Paris, (estimated at 3 feet $8\frac{1}{2}$ lines;) this length, however, was measured to the centre of the ball, and the error thence resulting was calculated. The vibrations were compared with those of a clock. The result was that the seconds' pendulum at Archangel was longer than that at Paris by $\frac{3}{20}$ of a line. Not a word is said of the temperature of the standard; and this leaves a small uncertainty as to the correct length.

By Campbell at Jamaica.

In the *Philosophical Transactions* for 1734, is an account by Bradley, of the transportation of a clock with an invariable pendulum (made by Graham) from London to Black-River in Jamaica, latitude 18°. The construction of the pendulum is not described. The observations in London, which lasted 10 days, were made by Graham, and those in Jamaica, for 26 days, by Mr. Colin Campbell. At London the clock gained $1^s.2$ daily on sidereal time; and at Black-River it lost daily $2^m\ 5^s.5$. The rates were found by transit observations. A spirit thermometer was observed; and the effect of the temperature was calculated on the rough observation that clocks of this construction did not generally alter their rate at London between summer and winter above 25^s or 30^s per day. The allowance made was $8^s.7$; and thus the difference of rate depending on the alteration of gravity was $1^m\ 58^s$. This may be considered a good determination, as the greatest uncertainty from the rough estimation of the effects of temperature could not amount to more than two or three seconds per day.

By Mairan at Paris.

In the *Mémoires* for 1735 is Mairan's account of his observations of the length of the seconds' pendulum at Paris. A horizontal plane, movable by screws, was placed exactly 1 toise from the lower surface of his forceps. The thread was of *pite;* it was fixed in the leaden ball (6 lines in diameter) by making a small hole and burnishing the edge upon the end of the thread. The distance between the lower surface of the ball and the horizontal plane was found by trying which of several pieces of glass would just fit under it; and measuring by a scale, with the assistance of a lens, the thickness of the glass. The diameter of the ball was measured in the same way by means of a parallel ruler. The ball was left to hang on the thread one day before the experiments were begun. At first Mairan tried to count the vibrations; but was obliged at last to resort to the method of coincidences with clock pendulum. For this purpose he was obliged to use a pendulum of little more than 3 feet in length; and the distance was now measured by placing the toise on one side, and screwing up the plane till it touched the sphere. Some observations were, however, made with pendulums of 6 and 9 feet; balls of brass, ivory, &c. were also used. The mean of twelve experiments gave for the length of the seconds' pendulum 3 feet $8\frac{17}{30}$ lines. There appears to be no correction for the weight of the air nor for temperature; the mean height of the thermometer was about 13° of Fahrenheit, which is so near the temperature commonly used in the application of the French toise, that the difference may be neglected. This appears to be an excellent determination, subject only to the error of Mairan's toise, and that common to the kind of suspension that he used. But, as an isolated experiment, it is nearly useless in the investigation of the Figure of the Earth.

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* See also the *Recueil d'Observations faites en plusieurs voyages, &c.* The length of the seconds' pendulum at Paris had been measured by Picard for the purpose of fixing a standard; and incidental observations had been made by Feuillée at Portobello and Martinique, by Halley at St. Helena, &c. They are worth very little.

By Godin at Paris.

By Godin at St. Domingo.

The same volume contains two determinations by Godin of the Paris pendulum, one with a machine by Graham, (which is not described,) the other with a simple pendulum whose bob was the frustra of two cones united by their bases. The latter determination agrees very nearly with Mairan's: the former gives a measure shorter by $\frac{1}{6}$th of a line. At Petit Goave in St. Domingo, latitude 18° 27′, with a brass ball an inch in diameter, suspended by a line of pite which was attached with sticking-plaster, he counted 7684 vibrations and compared them with a clock, and thus found for the length of the seconds' pendulum, 3 feet $7\frac{1}{3}$ lines. After this he used a double frustrum of a cone with a longer thread, and measured the distance to the upper surface by a sort of sliding rule. By counting 2012 vibrations he found for the length 3 feet $7\frac{8}{19}$ lines. Bouguer (see the same volume) found 3 feet $7\frac{1}{3}$ lines. Lacondamine used a 12-feet pendulum; but finding some difficulty in the measure, he adopted a pendulum of little more than 3 feet, which was attached to the slider of his beam-compasses; he found 3 feet $7\frac{14}{60}$ lines.

By Godin, Bouguer, and Lacondamine, near the Equator.

At Porto-bello, latitude 9° 33′, Godin and Bouguer found 3 feet $7\frac{7}{90}$ lines for the length of the seconds' pendulum; at Panama, latitude 8° 35′, they found 3 feet $7\frac{1}{3}$ lines; at Punta-Palmar, 2′ South latitude, Lacondamine found 3 feet 6.96 lines; at Riojama 9′ South, Bouguer found 3 feet 6.82 lines, and Lacondamine 3 feet 6.93 lines; at Quito, 25′ South, they found 3 feet 6.83 lines. The elevation of the last-mentioned place makes it almost unsafe to use such a measure in conjunction with those taken near the level of the sea. We copy these numbers from the table near the end of the IVth volume of the *Œuvres de* Maupertuis. The numbers given by Bouguer himself in his *Figure de la Terre,* for the length of the seconds' pendulum in vacuum, are: at the Equator, at an elevation of 2434 toises, 36 inches 6.69 lines; at an elevation of 1466 toises, 36 inches 6.88 lines; at the level of the sea, 36 inches 7.21 lines; at Porto-bello, 36 inches 7.30 lines; at Petit Goave, 36 inches 7.47 lines; at Paris, 36 inches 8.67 lines.

By Maupertuis in Sweden.

Maupertuis observed the rate of an invariable pendulum attached to a clock, at Paris and at Pello, latitude 66° 48′. This instrument was made by Graham: the pendulum was of brass, suspended by a knife-edge of steel which rested on steel planes. The temperature of the places of experiment at Paris and at Pello was the same, and the extent of the arcs of vibration the same. At Pello it gained $59^{s}.1$ per day on its Paris rate. At London it appeared from Graham's observations that it gained $7^{s}.7$ per day on its Paris rate. The observations at Pello lasted four days. As they feared that in a country full of iron and magnetic stones there might be some magnetic influence, they tried in another clock bobs of different metals, but the difference at Pello and Paris was the same in all. Great confidence may be placed in this kind of observation, where one of the principal difficulties (that of measuring the pendulum) is entirely avoided.

By Lacaille.

In 1752, (see *Mémoires* for 1751,) Lacaille observed the length of the seconds' pendulum at the Cape of Good Hope and the Isle of France, and on his return, at Paris. The length at the Cape, given in the *Mémoires,* is 3 feet 8.07 lines. In Delambre's *Histoire de l'Astronomie au XVIIIme. Siècle,* p. 478, M. Mathieu has given the results of a recalculation of all Lacaille's observations: they are as follows. Length of seconds' pendulum at the Isle of France, latitude 20° 10′ South, 3 feet 7.785 lines; at the Cape of Good Hope, 33° 55′ South, 3 feet 8.139 lines; at Paris, 48° 51′ North, 3 feet 8.79 lines. These results are comparable among themselves, but are not strictly comparable with others, because some corrections are introduced by M. Mathieu which were not introduced by observers of that age.

By Legentil.

In the *Voyage dans les Mers de l'Inde* of Legentil, published 1779, are contained observations of the pendulum at Pondicherry, latitude 11° 56′ North; at Manilla, latitude 14° 34′ North; and at Foul-point in Madagascar, latitude 17° 40′ South. They have been recalculated by M. Mathieu, (*Astronomie au XVIIIme. Siècle,* p. 698.) who observing that the standard was the same as Bouguer's, and therefore that Bouguer's and Lacaille's observation at Paris might be considered as comparable with these if the same reductions were made in all, has applied to all the same reductions, and thus found the following comparable lengths: Paris, 3 feet 8.582 lines; Pondicherry, 3 feet 7.345 lines; Manilla, 3 feet 7.517 lines; Foul-point, 3 feet 7.441 lines.

By Phipps, at Spitzbergen

In the account of Phipps's voyage towards the North Pole (made 1773) are some observations with an invariable clock pendulum made by Graham. The ball was a brass sphere 4 inches in diameter; the rod a steel wire $\frac{1}{10}$th of an inch thick; it turned upon knife-edges, which were not left quite sharp, the edge being rounded to a cylinder of $\frac{1}{400}$ inch radius. These knife-edges turned in notches made in two pieces of hardened steel; the notches were angular and rounded at the bottom. The rod was not fixed firmly to the knife-edges, but attached to a pin whose direction was parallel to the plane of vibration, in order that an equal bearing on both extremities might be ensured. The escapement was not absolutely a dead-beat, but slightly recoiling; as it is well known to artists that a pendulum on a knife-edge suspension may be thus made to vibrate in equal times under the action of very different weights. At London the clock went exactly mean solar time. On a small rocky island* near Spitzbergen, latitude 79° 50′ North, an equatorial was mounted as an altitude and azimuth instrument, and at about five in the afternoon the transit of the Sun's limb over the vertical wire was observed: this observation was repeated the next day. Besides this, the clock was compared with a chronometer, whose rate was well known. A correction was applied for the temperature. The conclusion was that the rate was accelerated $72^{s}.28$ per day. Again, in latitude 79° 44′ North, the clock was kept going for about three days, and compared with the chronometer, whose rate was determined by many altitudes of the Sun immediately before and after the observations. The acceleration, thus determined, was $73^{s}.06$ per day. On returning from the voyage, the clock was found to go true time at London. These results appear to be perfectly trustworthy.

By Borda, at Paris.

We have already described at great length the manner in which Borda's observations were made at Paris. We shall merely add that the pendulum was about 12 feet long, so that every vibration corresponded nearly to every second vibration of a clock. The whole apparatus was enclosed in a box, to prevent currents of air from affecting it. All the corrections which we have mentioned were applied, except that for elevation above the sea. The mean result from twenty experiments of four or five hours each, was 3 feet 8.5593 lines. See the *Base du Système Métrique*, tom. iii.

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* This is very near to the place where Captain Sabine afterwards observed the pendulum, but it does not appear to be the identical spot.

By Biot, on the French meridian.

The same methods were followed (see tom. iv. of the same Work) by Biot (except that he used a pendulum of about 3 feet in length) at Formentera, latitude 38° 40′; at Paris, latitude 48° 50′; at Bordeaux, latitude 44° 50′; at Figeac, latitude 44° 37′; at Clermont, latitude 45° 47′; at Dunkirk, latitude 51° 2′; at Leith Fort, latitude 55° 59′; and at Unst, in the Shetland Islands, latitude 60° 45′. The lengths were reduced to those which would have been observed at the level of the sea by increasing them in the duplicate ratio of the actual distance from the Earth's centre to the distance of the level of the sea from the Earth's centre; this amounts to the same as neglecting the attraction of the elevation on which they were placed. The results are respectively, (in mètres,) 0.7412520, 0.7419175, 0.74160863, 0.741612279, 0.7417052, 0.7420761, 0.994531014, and 0.742723136. All of these, it must be observed, are lengths of the decimal pendulum (making 100,000 oscillations in a mean solar day) except the last but one, which is the length of the sexagesimal seconds' pendulum. All these determinations must be considered as most valuable. Two invariable pendulums also were observed at Greenwich and at Paris; they appeared to show a retardation at Paris of $10^{s}.79$ daily; but it is expressly remarked, that the circumstances were very unfavourable to accuracy.

By Malaspina.

In the Additions to the *Connaissance des Temps* for 1816, is M. Mathieu's account of the observations made in an expedition under the command of Malaspina, undertaken by order of the Spanish Government, from 1789 to 1794. Their pendulum had a wooden rod; it was enclosed in a glass box; the vibrations were counted by two persons, who relieved each other at the end of every sixty; a third person stood at the clock. The following are their results for the length of the seconds' pendulum (taking for unit the length of the simple pendulum equivalent to their pendulum.)

Place.	Latitude.	Length.
Mulgrave........	59 34 20 N.	1.0082893
Nutka	49 35 15	68172
Montercy........	36 35 45	54463
Cadiz	36 31 46	57192
Macao	22 12	42378
Acapulco	16 50 49	43770
Manilla	14 35 49	45051
Umatag	13 17 52 N.	39539
Zamboanga	6 54 27 S.	40541
Lima.	12 4 38	41320
Isle Babao.......	18 35 45	45384
Port Jackson	33 51 20	56969
Monte Video	34 54 38	57973
Conception	36 42 32	57471
Port St. Helena ..	44 29 34	69083
Port Egmont.....	51 21 3 S.	71070

These results cannot be compared with any others, as scarcely any other observation (except Legentil's) has been made at any of these places for the determination of the absolute length of the pendulum.

By Kater in London.

In the *Phil. Trans.* for 1818, are contained Captain Kater's observations for determining the length of the seconds' pendulum at Portland Place, London. The principle we have already described: the result was corrected for extent of vibration by observing the arc at each coincidence and using the mean between two adjacent arcs for the vibrations between them; for temperature, by allowing 0.423 vibration for each degree of Fahrenheit; and for the weight of the air and elevation above the sea in the manner that we have described. They gave for the length in latitude 51° 31′ 8″, 39.13860 inches. But in the *Phil. Trans.* for 1819, Dr. Young remarked that the correction usually applied for the elevation above the sea was too great, as it neglected the attraction of the elevated mass. If we stood on a sphere of the same density as the Earth's mean density, the reduction would be but $\frac{3}{4}$ of that which is usually made; on a hemisphere, still less; on table land of mean density, the reduction would be $\frac{1}{4}$ of that usually made; on table land of density $= \frac{2}{3}$ mean density, it would be $\frac{1}{2}$; on table land whose density $= \frac{2.5}{5.5} \times$ mean density (which would probably represent pretty nearly the density of most rocks at the Earth's surface) the reduction would be $1 - \frac{3}{4}\,\frac{2.5}{5.5}$, or $\frac{66}{100} \times$ usual reduction. The last result thus reduced, and with other corrections, is 39.13929 inches. This is the number used in all the subsequent comparisons.

Dr. Young's correction for elevation.

Observations by Kater on English meridian.

The same Volume (for 1819) contains Captain Kater's determinations (by the transportation of an invariable pendulum) at Shanklin in the Isle of Wight, latitude 50° 37′; at Arbury Hill, latitude 52° 13′; at Clifton, latitude 53° 28′; at Leith Fort, latitude 55° 59′; at Portsoy, latitude 57° 41′; and at Unst, latitude 60° 45′. The results were 39.13614, 39.14250, 39.14600, 39.15554, (obtained under unfavourable circumstances,) 39.16159, and 39.17146 inches.

Observations in Ross's and Parry's voyages.

In the history of Captain Ross's voyage to Baffin's Bay (made in 1818) is an account of the going of a clock (with invariable pendulum vibrating on a knife-edge, which rested on hollow cylinders of agate) as determined by 7 days' observation at London, 3 days at Brassa in the Shetland Islands, latitude 60° 1′, and 4 days at Hare Island, latitude 70° 26′. And in Parry's first voyage are observations on the going of two similar clocks at London, before and after the voyage, and at Melville Island, latitude 74° 47′, during 38 days. The result[*] (see *Phil. Trans.* 1821) is, that at Brassa the length of the seconds' pendulum is 39.16929; at Hare Island 39.1984; and at Melville Island, 39.2070 inches.

By Hall near the Equator.

In the *Phil. Trans*, for 1823, are detailed accounts of observations made by Captain Basil Hall and Mr. Foster with an invariable detached pendulum at London; at Abingdon Island, one of the Galapagos, latitude 0° 32′, at San Blas de California, latitude 21° 30′; and at Rio de Janeiro, latitude 22° 55′ South. In this, as in

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[*] In the *Connaissance des Temps*, 1827, *Additions*, M. Arago has given his opinion that these observations would be rejected without scruple if they opposed any received theory, and that, therefore, they are useless when they favour it. With the greatest possible respect for the opinion of this illustrious philosopher, we must in this instance withhold our assent to it. The conclusion is not so accurate as if deduced from the use of invariable detached pendulums, but it is probably much superior to that which would be obtained in difficult circumstances by the French method. From our own experimental acquaintance with these clocks, we know them to be excellent.

Figure of the Earth.

other instances, we cannot enter into the details, and can only state that the operations appear quite satisfactory. The resulting lengths of the seconds' pendulum are 39.01717, 39.03829, (by a mean ;) and 39.04375.

By Brisbane in New South Wales.

In the same Volume is an account of observations made by Sir Thomas Brisbane, Mr. Rumker, and Mr. Dunlop, with an invariable detached pendulum at London and Paramatta in New South Wales, latitude 33° 49′ South. The length of the seconds' pendulum at the latter is 39.07696.

By Sabine in different latitudes.

Captain Sabine's *Account of Experiments to determine the Figure of the Earth*, &c. published in 1825, contains a very valuable* series of observations made with two detached invariable pendulums. The principal novelties in the manner of conducting the observations was, that for coincidences the mean of disappearances and reappearances was taken ; and that the correction for temperature was determined from experiments made at London in different temperatures ; the effect of an alteration equal to 1° of Fahrenheit was found to be 0.421 vibrations per day. Two clocks with invariable attached pendulums (the same which were used on Captain Parry's voyage) were also observed. The results, as finally corrected in the *Phil. Trans.* for 1828, will be given in our general table.

By Warren at Madras.

In 1809 experiments† were made at Madras observatory by Captain Warren, with a leaden ball $1\frac{1}{3}$ inch diameter, suspended by a fibre formed from the wild plantain tree. Its vibrations were counted, and the time of an astronomical clock noted by an assistant. For the measure, plates of glass were fixed to the wall, and paper attached to them, and upon these a scale was transferred from Colonel Lambton's scale. The beam compasses were applied to the lower surface of the support, and the lower surface of the ball, and were then taken to the scale. The diameter of the ball was inferred from its weight. From 10 days' observations the length of the seconds' pendulum was found to be 39.026273 inches (in air, and measured by Lambton's scale.) This determination is certainly inferior to one that we shall presently give.

By Goldingham near the Equator,

In 1821 an expedition under the direction of Mr. Goldingham, made observations at Pulo Gaunsah Lout (a small island in latitude 1′ 49″ North) with an invariable pendulum, which had been observed at London. The deduced length of the seconds' pendulum is 39.021260 inches. Every care appears to have been taken to make this accurate.

and at Madras.

The *Phil. Trans.* for 1822 contain Mr. Goldingham's determination, with the same pendulum, of the length of the seconds' pendulum at Madras, latitude 13° 4′ : it is found to be 39.026302, and appears to be very accurately determined. In this, and in the last, the reduction for temperature is applied in a very confusing manner ; but we have satisfied ourselves that it is correct.

By Foster at Port Bowen.

In the *Phil. Trans.* for 1826 is Lieutenant Foster's account of a very careful series of observations at Port Bowen, latitude 73° 14′, with an invariable pendulum which had been observed at Greenwich. The conclusion was that the length of the seconds' pendulum = 39.203472 inches. This, however, is founded on the supposition that 39.13911 is the length of the seconds' pendulum at Greenwich, reduced to the level of the sea. It is, however, nearly the length at Portland Place, London. We shall soon mention the difference between these lengths.

Observations of Pendulums.

By Freycinet in various latitudes

The French expedition sent out in 1817—1820, under Captain Freycinet, made observations at Paris ; at Rio de Janeiro, latitude 22° 55′ South ; at the Cape of Good Hope, latitude 33° 55′ South ; at the Isle of France, latitude 20° 10′ South ; at Rawak, near New Guinea, latitude 0° 2′ South ; at Guam, the capital of the Marian Islands, latitude 13° 28′ ; at Mowi, one of the Sandwich Islands, latitude 20° 52′ ; at Port Jackson, latitude 33° 52′ South ; and in the Isles Malouines, latitude 51° 35′ South. Three invariable brass pendulums were used, and one wooden one. When observed, they were enclosed in glass boxes, and a small clock placed near them had its pendulum adjusted, till it vibrated exactly in the same time. This clock (used only as a counter) was compared with a chronometer every half hour ; for determining the rate, the mean of five chronometers was used. The proportionate lengths of the seconds' pendulum are 1.00002271, 0.99783538, 0.99871582, 0.99794[illegible]5, 0.99709575, 0.99759331, 0.99792816, 0.99877424, 1.00022319. These determinations are probably pretty good. See the *Voyage autour du Monde, &c. par* Freycinet.

By Duperrey in the Pacific.

In a similar expedition made in 1822—1825 under Captain Duperrey, two of the same pendulums were employed. The method appears to have been the same as in the operations last described. The following are given as representing the lengths of the seconds' pendulum. At Paris, 1 ; at Toulon, latitude 43° 7′, 0.99950585 ; at the Malouines, 1.00025995 ; at Port Jackson, 0.99871430 ; at the Isle of France 0.99789022 ; at Ascension, latitude 7° 56′ South, 0.99729881. We extract these numbers from the *Additions* to the *Conn. des Temps*, 1830.

By Sabine at Paris.

The *Phil. Trans.* for 1828 contain Captain Sabine's observations on an invariable pendulum at London and at Paris. From an extensive series it appears that a pendulum which at London makes 85,940 vibrations per day, is retarded at Paris to the amount of 12.03 vibrations. As the length of the seconds' pendulum at London, uncorrected for elevation, is 39.13908 inches, the length at Paris is 39.12813. This is a very valuable determination, as it makes a great number of determinations comparable, which without it could not safely be brought together.

By Sabine at Greenwich.

In the *Phil. Trans.* for 1829 is a series of observations by the same gentleman for determining the difference of gravity at Portland Place, London, and the Royal Observatory, Greenwich. It appears that at Greenwich, no correction being applied for elevation, the pendulum is accelerated about 0.48 of a vibration per day, and therefore, the length of the seconds' pendulum at Greenwich is greater than at Portland Place by .000435. If the correction for elevation be applied, according to Dr. Young's rule, the difference is 0.60 of a vibration, or .000544 inch in the length. We do not ourselves believe that the pendulum, as commonly constructed, can be relied on to this degree of accuracy.

The same Volume contains experiments leading to a very remarkable result with regard to the correction for the weight of the air. They are, in fact, a repetition of

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* The publicity of some discussions respecting these observations compels us to give our own opinion. We shall decline entering into any inquiry respecting the original observations, and shall merely state that we have the most perfect confidence in the results.

† The accounts of this and the next set of experiments were presented by the East India Company to the Astronomical Society of London, by favour of whom we have been allowed to inspect them.

Experiments by Bessel and Sabine to determine the effect of the air on the vibrations of a pendulum.

experiments announced by Bessel in the *Astronomische Nachrichten*, No. 128, and published in the *Berlin Transactions* for 1826. It would seem that a quantity of air moves with the pendulum, and adds to its inertia, and therefore that besides a retardation arising from the diminished weight of the pendulum, there is another retardation arising from the increased mass of matter to be put in motion. A pendulum was made to vibrate in air, in vacuum, and in hydrogen gas, of elasticity equal to the elasticity of the atmosphere. It appears that the reduction to vacuum, as commonly made, ought to be increased in the ratio of 1 : 1.655. With a spherical bob, the reduction ought to be doubled. As, however, all the comparative observations have been made with little difference in the height of the barometer, the error in the reduction has very little effect on their results. It is remarkable that the correction for hydrogen gas appears to be double of what it would be if we supposed the amended corrections proportional to the Specific Gravities of the fluids. Bessel determined the atmospheric correction by observing the length of the seconds' pendulum in air, first with a bob of brass, and secondly, with a bob of ivory. The correction, being greater for the latter than for the former, in the proportion of the Specific Gravities, was determined with great accuracy from the difference of the observed lengths.

Observations by Biot in the Mediterranean and Italy.

The *Mémoires de l'Institut*, vol. viii. contain a paper by M. Biot, from which we extract the following determinations by himself of the length of the seconds' pendulum at Milan, latitude 45° 28′; at Padua, latitude 45° 24′; at Fiume, latitude 45° 19′; at Barcelona, latitude 41° 23′; at Formentera, latitude 38° 40′; and at Lipari, latitude 38° 29′. Expressed in millimètres they are 993.547642. 993.607294, 993.584075, 993.2321312, 993.0696597, 993.0791638. The former determination at Formentera, it appears, was not satisfactory. The details of these experiments are not given.

By Rumker in New South Wales.

The IIId Volume of the *Transactions of the Astronomical Society* contains a series of observations in Borda's method, made by Mr. Rumker at Paramatta. It appears that the length of the pendulum was measured by screwing up the steel plate, first till it touched the ball, then till it touched the measuring bar. The diameter of the ball was measured by winding a thread eleven times round it. The result in millimètres is 992.412801. Many details of the calculation are omitted.

By Fallows at the Cape of Good Hope.

In the *Phil. Trans.* for 1830 is a determination by Mr. Fallows, of the proportion of Gravity at London and at the Cape of Good Hope with an invariable pendulum. The number of vibrations per day at London was 86164.97, and at the Cape 86097.86. With the length 39.13929 inches for the seconds' pendulum at London, we find for the Cape 39.07836.

By transportation of an invariable pendulum, (from London, we believe,) Svanberg ascertained the length of the seconds' pendulum at Stockholm, latitude 59° 21′, to be 39.165414 inches. We are unable to refer to the original account of this experiment.

The length of the pendulum at Königsberg, latitude 54° 42′, reduced to the level of the sea, has been found by Bessel (by his method described above) to be 440.8179 lines. The two pendulums whose difference he measured, vibrated in 1″.7127 and 1″.0020. Observations of invariable pendulums transported from Paris gave a greater length. See the *Berlin Transactions* for 1826, or the *Quarterly Journal*, March, 1829.

The difference of Gravity at London and Altona has, we believe, been ascertained by Captain Sabine with an invariable pendulum. The results, so far as we know, have not yet been published.

Observations have been made in London by Mr. F. Baily with a pendulum with two knife-edges. The pendulum consists of a massy brass bar without any bob. This construction appears to be attended with the following source of inaccuracy; if the suspension planes are not very well levelled, the pendulum will not vibrate in a vertical plane, and the time of vibration will be too great. In Captain Kater's construction this was provided against by making the bar so thin that it would bend into the proper position. Mr. Baily's results differ sensibly from those of Captain Kater.* See the *Philosophical Magazine*, February, 1829.

Deduction from Pendulum Observations.

Section 8.—Deduction of the Figure of the Earth from Pendulum Observations.

Length of seconds' pendulum at Paris.

The first element which it is important to settle is, the length of the pendulum at Paris. In these reductions we shall suppose the French foot = 1.065766 × English foot; and the mètre = 39.37079 English inches.

	Eng. Inches.
Now Borda's experiments give 440.5593 lines =	39.12776
Biot's (uncorrected for elevation) give for the decimal pendulum $0^{m}.7419012$, or for the sexagesimal	39.12848
Sabine's	39.12813

The mean is 39.12812, and this is probably very accurate. If corrected for elevation by Dr. Young's rule it gives at the level of the sea 39.12877.

Manner of calculating all the other lengths.

From this length, and the length at London, (39.12908, or corrected by Dr. Young's rule for elevation 39.12929,) we have deduced the length at all the places where an invariable pendulum was observed. The absolute lengths measured by the French in the present century we have reduced to English measures. The absolute lengths measured in the last century we have compared with the Paris length given by the same observer, and have applied the difference to *our* value of the Paris length. Legentil's are compared as we have mentioned.

The observations may now be very well divided into first-rate and second-rate. We have reduced all of the first class to the level of the sea by Dr. Young's rule, using the multiplier .6. In the second class we have not attempted any alteration, except in Warren's, which we have corrected for the error of the standard, and reduced to vacuum.

Lengths assumed to correspond with the theoretical formula.

Now in Section 2, Art. 65, it appears that theory would lead us to expect that gravity at the latitude λ, or the length of the seconds' pendulum which is proportional to it, may be represented by $E\,(1 + F \sin^2 \lambda)$. Making this assumption for each of the lengths, and arranging them in the order of latitude, we have the following Table. Where two observers are mentioned, we have taken the mean of the lengths deduced in the way that we have described.

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* Mr. Baily remarks that Shuckburgh's scale seems to have sustained some injury. It appears, however, that the same divisions were used for the measure of the pendulum as for the comparison with the mètre, and nearly the same as for the comparison with other English standards; and thus the error in the result of measures which are only comparative is probably insensible.

Equations given by this assumption.

Deduction from Pendulum Observations.

First-rate Observations.

No.	Observers.	Place.	Latitude.	Length* of Seconds' Pendulum and Equation.
1	Sabine	Spitzbergen	79° 50′ N.	$39.21469 = E \times \{1 + F \times .9689\}$
2	Sabine	Melville Island	74 47	$39.2070 = E \times \{1 + F \times .9311\}$
3	Sabine	Greenland	74 32	$39.20335 = E \times \{1 + F \times .9289\}$
4	Foster	Port Bowen	73 14	$39.20419 = E \times \{1 + F \times .9168\}$
5	Sabine	Hammerfest	70 40	$39.19475 = E \times \{1 + F \times .8904\}$
6	Sabine	Drontheim	63 26	$39.17456 = E \times \{1 + F \times .8000\}$
7	Biot and Kater	Unst	60 45	$39.17162 = E \times \{1 + F \times .7612\}$
8	Svanberg	Stockholm	59 21	$39.16541 = E \times \{1 + F \times .7401\}$
9	Kater	Portsoy	57 41	$39.16159 = E \times \{1 + F \times .7142\}$
10	Biot and Kater	Leith Fort	55 59	$39.15546 = E \times \{1 + F \times .6870\}$
11	Bessel	Königsberg	54 42	$39.15072 = E \times \{1 + F \times .6661\}$
12	Kater	Clifton	53 28	$39.14600 = E \times \{1 + F \times .6456\}$
13	Kater	Arbury Hill	52 13	$39.14250 = E \times \{1 + F \times .6246\}$
14	Kater	London	51 31	$39.13929 = E \times \{1 + F \times .6127\}$
15	Sabine	Greenwich	51 29	$39.13983 = E \times \{1 + F \times .6122\}$
16	Biot	Dunkirk	51 2	$39.13773 = E \times \{1 + F \times .6045\}$
17	Kater	Isle of Wight	50 37	$39.13614 = E \times \{1 + F \times .5974\}$
18	Borda, Biot, and Sabine	Paris	48 50	$39.12851 = E \times \{1 + F \times .5667\}$
19	Biot	Clermont	45 47	$39.11615 = E \times \{1 + F \times .5137\}$
20	Biot	Milan	45 28	$39.11603 = E \times \{1 + F \times .5081\}$
21	Biot	Padua	45 24	$39.11896 = E \times \{1 + F \times .5070\}$
22	Biot	Fiume	45 19	$39.11788 = E \times \{1 + F \times .5055\}$
23	Biot	Bordeaux	44 50	$39.11296 = E \times \{1 + F \times .4971\}$
24	Biot	Figeac	44 37	$39.11215 = E \times \{1 + F \times .4933\}$
25	Duperrey	Toulon	43 7	$39.10952 = E \times \{1 + F \times .4672\}$
26	Biot	Barcelona	41 23	$39.10432 = E \times \{1 + F \times .4370\}$
27	Sabine	New York	40 43	$39.10120 = E \times \{1 + F \times .4255\}$
28	Biot, twice	Formentera	38 40	$39.09510 = E \times \{1 + F \times .3904\}$
29	Biot	Lipari	38 29	$39.09828 = E \times \{1 + F \times .3872\}$
30	Hall	California	21 30	$39.03829 = E \times \{1 + F \times .1343\}$
31	Freycinet	Sandwich Islands	20 52	$39.04690 = E \times \{1 + F \times .1269\}$
32	Sabine	Jamaica	17 56	$39.03503 = E \times \{1 + F \times .0948\}$
33	Freycinet	Marian Islands	13 28	$39.03379 = E \times \{1 + F \times .0542\}$
34	Goldingham	Madras	13 4	$39.02630 = E \times \{1 + F \times .0511\}$
35	Sabine	Trinidad	10 39	$39.01888 = E \times \{1 + F \times .0342\}$
36	Sabine	Sierra Leone	8 30	$39.01997 = E \times \{1 + F \times .0218\}$
37	Hall	Galapagos	0 32	$39.01717 = E \times \{1 + F \times .0001\}$
38	Sabine	St. Thomas	0 25	$39.02074 = E \times \{1 + F \times .0001\}$
39	Goldingham	Pulo Gaunsah Lout	0 2 N.	$39.02126 = E \times \{1 + F \times .0000\}$
40	Freycinet	Rawak	0 2 S.	$39.01433 = E \times \{1 + F \times .0000\}$
41	Sabine	Maranham	2 32	$39.01213 = E \times \{1 + F \times .0020\}$
42	Sabine and Duperrey	Ascension	7 55	$39.02363 = E \times \{1 + F \times .0190\}$
43	Sabine	Bahia	12 59	$39.02433 = E \times \{1 + F \times .0505\}$
44	Freycinet and Duperrey	Isle of France	20 10	$39.04684 = E \times \{1 + F \times .1189\}$
45	Hall and Freycinet	Rio de Janeiro	22 55	$39.04350 = E \times \{1 + F \times .1516\}$
46	Brisbane and Rumker	Paramatta	33 49	$39.07452 = E \times \{1 + F \times .3097\}$
47	Freycinet and Duperrey	Port Jackson	33 52	$39.07919 = E \times \{1 + F \times .3105\}$
48	Freycinet and Fallows	Cape of Good Hope	33 55	$39.07800 = E \times \{1 + F \times .3113\}$
49	Freycinet and Duperrey	Isles Malouines	51 35 S.	$39.13781 = E \times \{1 + F \times .6139\}$

Second-rate Observations.

No.	Observers.	Place.	Latitude.	Length* of Seconds' Pendulum and Equation.
50	Phipps	Spitzbergen	79° 47′ N.	$39.2046 = E \times \{1 + F \times .9686\}$
51	Sabine	Hare Island	70 26	$39.1984 = E \times \{1 + F \times .8878\}$
52	Maupertuis	Pello	66 48	$39.1837 = E \times \{1 + F \times .8448\}$
53	De la Croyère	Archangel	64 32	$39.1414 = E \times \{1 + F \times .8151\}$
54	Sabine	Brassa	60 1	$39.1693 = E \times \{1 + F \times .7502\}$
55	Godin and Bouguer	St. Domingo	18 27	$39.0239 = E \times \{1 + F \times .1002\}$
56	Campbell	Jamaica	18	$39.0322 = E \times \{1 + F \times .0955\}$
57	Legentil	Manilla	14 34	$39.0336 = E \times \{1 + F \times .0633\}$
58	Warren	Madras	13 4	$39.0312 = E \times \{1 + F \times .0511\}$
59	Legentil	Pondicherry	11 56	$39.0183 = E \times \{1 + F \times .0428\}$
60	Godin and Bouguer	Porto Bello	9 33 N.	$39.0011 = E \times \{1 + F \times .0275\}$
61	Legentil	Madagascar	17 40 S.	$39.0267 = E \times \{1 + F \times .0921\}$
62	Lacaille	Isle of France	20 10	$39.0389 = E \times \{1 + F \times .1189\}$
63	Lacaille	Cape of Good Hope	33 55 S.	$39.0703 = E \times \{1 + F \times .3113\}$

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* The whole of these (except Bessel's) are erroneous to the amount of the error in the correction for the density of the air. The magnitude of this error is nearly the same for all, and is about .0018.

Figure of the Earth.

Constants in the formula determined.

To determine E and F from the first-rate observations, the most favourable groups will be formed by taking for one the sum of the first 24 equations and the 49th, and for the other the sum of the 24 remaining equations. Thus we get

$$E \times \{25 + F \times 16.9070\} = 978.80528$$
$$E \times \{24 + F \times \ 3.8983\} = 937.18322.$$

The solution of these equations gives E = 39.01677, F* = .005133. These values give for the length of the Equatorial pendulum 39.01677, for the Polar pendulum 39.21704, and for the pendulum at latitude λ $39.01677 + .20027 \times \sin^2\lambda$.

Substituting these values of E and F in each of the equations, we get the following apparent errors in measurement of the pendulum, or in the number of vibrations per day.

First-rate Observations.

Comparison of calculated and measured lengths.

No	Place.	Latitude.	Observed Length.	Calculated Length.	Error in Length.	Error in Vibrations.
1	Spitzbergen ..	79° 50′ N.	39.21469	39.21081	+.00388	+ 4.3
2	Melville Island	74 47	.2070	.20324	+ 380	+ 4.2
3	Greenland ...	74 32	.20335	.20280	+ 55	+ 0.6
4	Port Bowen ..	73 14	.20419	.20038	+ 381	+ 4.2
5	Hammerfest..	70 40	.19475	.19509	— 34	— 0.4
6	Drontheim ...	63 26	.17456	.17699	— 243	— 2.7
7	Unst	60 45	.17162	.16922	+ 240	+ 2.7
8	Stockholm...	59 21	.16541	.16499	+ 42	+ 0.5
9	Portsoy......	57 41	.16159	.15981	+ 178	+ 2.0
10	Leith	55 59	.15546	.15436	+ 110	+ 1.2
11	Königsberg .	54 42	.15072	.15017	+ 55	+ 0.6
12	Clifton	53 28	.14600	.14607	— 7	— 0.1
13	Arbury Hill ..	52 13	.14250	.14186	+ 64	+ 0.7
14	London......	51 31	.13929	.13948	— 19	— 0.2
15	Greenwich ...	51 29	.13983	.13938	+ 45	+ 0.5
16	Dunkirk	51 2	.13773	.13784	— 11	— 0.1
17	Isle of Wight .	50 37	.13614	.13641	— 27	— 0.3
18	Paris	48 50	.12851	.13027	— 176	— 1.9
19	Clermont	45 47	.11615	.11965	— 350	— 3.9
20	Milan.......	45 28	.11603	.11853	— 250	— 2.8
21	Padua.......	45 24	.11896	.11831	+ 65	+ 0.7
22	Fiume	45 19	.11788	.11801	— 13	— 0.1
23	Bordeaux....	44 50	.11296	.11633	— 337	— 3.7
24	Figeac	44 37	.11215	.11557	— 342	— 3.8
25	Toulon......	43 7	.10952	.11034	— 82	— 0.1
26	Barcelona ...	41 23	.10432	.10429	+ 3	0.0
27	New York...	40 43	.10120	.10199	— 79	— 0.9
28	Formentera ..	38 40	.09510	.09496	+ 14	+ 0.2
29	Lipari.......	38 29	.09828	.09432	+ 396	+ 4.4
30	California....	21 30	.03829	.04367	— 538	— 6.0
31	Sandwich Isl.	20 52	.04690	.04219	+ 471	+ 5.2
32	Jamaica	17 56	.03503	.03576	— 73	— 0.8
33	Marian Islands	13 28	.03379	.02763	+ 616	+ 6.8
34	Madras......	13 4	.02630	.02700	— 70	— 0.8
35	Trinidad	10 39	.01888	.02362	— 474	— 5.2
36	Sierra Leone.	8 30	.01997	.02114	— 117	— 1.3
37	Galapagos....	0 32	.01717	.01679	+ 38	+ 0.4
38	St. Thomas...	0 25	.02074	.01679	+ 395	+ 4.4
39	PuloGaun.Lout	0 2 N.	.02126	.01677	+ 449	+ 5.0
40	Rawak......	0 2 S.	.01433	.01677	— 244	— 2.7
41	Maranham ...	2 32	.01213	.01717	— 504	— 5.6
42	Ascension ...	7 55	.02363	.02058	+ 305	+ 3.4
43	Bahia.......	12 59	.02433	.02688	— 255	— 2.8
44	Isle of France.	20 10	.04684	.04058	+ 626	+ 7.0
45	Rio de Janeiro	22 55	.04350	.04713	— 363	— 4.0
46	Paramatta ...	33 49	.07452	.07880	— 428	— 4.7
47	Port Jackson .	33 52	.07919	.07896	+ 23	+ 0.3
48	C. Good Hope	33 55	.07800	.07912	— 112	— 1.2
49	Isles Malouines	51 35 S.	.13781	.13972	— 191	— 2.1

Second-rate Observations.

Deduction from Pendulum Observations.

No	Place.	Latitude.	Observed Length.	Calculated Length.	Error in Length.	Error in Vibrations.
50	Spitzbergen ..	79° 47′ N.	31.2046	39.2108	—.0062	— 6.9
51	Hare Island ..	70 26	.1984	.1946	+ 38	+ 4.2
52	Pello	66 48	.1837	.1860	— 23	— 2.6
53	Archangel ...	64 32	.1414	.1800	— 386	—42.7
54	Brassa	60 1	.1693	.1670	+ 23	+ 2.6
55	St. Domingo .	18 27	.0239	.0372	— 133	— 14.7
56	Jamaica	18	.0322	.0359	— 37	— 4.1
57	Manilla......	14 34	.0336	.0295	+ 42	+ 4.7
58	Madras......	13 4	.0312	.0270	+ 42	+ 4.7
59	Pondicherry..	11 56	.0183	.0253	— 70	— 7.8
60	Porto Bello...	9 33 N.	.0011	.0223	— 212	—23.4
61	Madagascar ..	17 40 S.	.0267	.0352	— 85	— 9.4
62	Isle of France.	20 10	.0389	.0406	— 17	— 1.9
63	C. Good Hope	33 55 S.	.0703	.0791	— 88	— 9.8

In order to compare the lengths determined from Malaspina's observations with the others, we have calculated from the formula $39.01677 + .20027 \times \sin^2\lambda$ the length at each of his stations. We have divided the sum of all these by the sum of his numbers; and have used the quotient as a common multiplier by which we have multiplied each of his numbers. The length thus produced we have called the *observed length*, and we have compared it with our *calculated length* as in the other cases.

Spanish Observations.

No	Place.	Latitude.	Observed Length.	Calculated Length.	Error in Length.	Error in Vibrations.
64	Mulgrave	59° 34′ N.	39.18479	39.16565	+.01914	+21.1
65	Nutka	49 35	.12759	.13285	— 526	— 5.8
66	Montercy....	36 36	.07431	.08796	— 1365	—15.1
67	Cadiz	36 32	.08491	.08774	— 283	— 3.1
68	Macao	22 12	.02734	.04536	— 1802	—19.9
69	Acapulco	16 51	.03275	.03360	— 85	— 0.9
70	Manilla......	14 36	.03773	.02950	+ 823	+ 9.1
71	Umatag	13 18 N.	.01631	.02737	— 1106	—12.2
72	Zamboanga ..	6 54 S.	.02020	.01966	+ 54	+ 0.6
73	Lima........	12 5	.02323	.02555	— 232	— 2.6
74	Isle Babao ...	18 36	.03903	.03714	+ 189	+ 2.1
75	Port Jackson .	33 51	.08405	.07891	+ 514	+ 5.7
76	Monte Video .	34 55	.08795	.08238	+ 557	+ 6.2
77	Conception ..	36 43	.08600	.08835	— 235	— 2.6
78	Port St. Helena	44 30	.13112	.11516	+ 1596	+17.6
79	Port Egmont .	51 21 S.	.13885	.13892	— 7	— 0.1

Deductions from the comparison.

From an examination of these apparent errors, the following conclusions may be deduced.

1. The observations that we have called second-rate, and the Spanish observations, are perfectly useless for the investigation of the Earth's form.

2. Upon scrutinizing the errors of the first-rate observations, it would seem that, *cæteris paribus*, gravity is greater on islands than on continents. This is a conclusion, however, that cannot be adopted till the correction for elevation is ascertained with greater certainty. The stations of observation on continents, or large islands, are frequently more elevated than those on small islands; and, therefore, if our correction (the usual correction multiplied by 0.6) is too small, as it is additive, it will make the pendulum's length, reduced to the level of the sea, much too small at the continental stations and not much too small on islands. Yet we think our correction (adopted from Dr. Young) a good one, as it removes some apparent anomalies. At Clermont, for instance, an elevated station, with the usual

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* It is remarkable that when Newton found the decrease of gravity at the Equator to be greater than in a homogeneous spheroid, he supposed the ellipticity to be greater than for a homogeneous spheroid. We shall draw an inference exactly contrary to this.

correction, gravity appears too great; with ours there is no remarkable inequality. These considerations will show what caution must be used in adopting any theory* which rests upon very delicate differences of gravity.

Gravity at middle latitudes smaller than given by the formula.

3. But it is impossible to avoid remarking, that in the high North latitudes the greater number of errors have the sign +, and that about the latitude 45° they have the sign − ; those about the Equator being nearly balanced. No alteration of our numbers is sufficient to correct this. If we increased the multiplier of $\sin^2\lambda$, we might make the errors at high latitudes as nearly balanced as those at the Equator: but then those about latitude 45° would be still greater than at present. To destroy these without altering the others we may add a term $-A.\sin^2\lambda.\cos^2\lambda$. We have seen in Section 2. Art. 71. that such a term ought to exist, though we could only offer a conjecture as to its magnitude. From the table of errors it would seem that the length of the seconds' pendulum would be better expressed by $39.01677 + .20277 \times \sin^2\lambda - .00880 \times \sin^2\lambda.\cos^2\lambda$. The last term is ten times as great as our imperfect theory would lead us to expect. Still, as an empirical formula, this is the best that we can offer.

4. There does not seem to be any reason for thinking that gravity is different at different longitudes; as the irregularities on different meridians do not appear greater than those at places near one another. Nor does it appear that there is any difference between the Northern and Southern hemispheres.

Calculation of ratio of centrifugal force to gravity.

Now to determine the Earth's ellipticity we must refer to the formula of Section 2. Art. 65. It appears that gravity, or the length of the seconds' pendulum, being expressed by $E(1 + F\sin^2\lambda)$, F will $= \frac{5m}{2} - e$, and consequently $e = \frac{5m}{2} - F$; where m is the ratio of the centrifugal force at the Equator to the force of gravity there. To determine m, we have, Equatorial seconds' pendulum $= 39.01677$; hence the space through which the Equatorial gravity would make a body fall in 1^s is $\frac{\pi^2}{2} \times 39.01677$, and the velocity which it would generate in 1^s is $\pi^2 \times 39.01677$. Also (MECHANICS, § XIV.) the velocity which would be generated in 1^s by the uncounteracted centrifugal force produced by describing in T seconds a circle whose radius is R, is $\frac{4\pi^2.R}{T^2}$; or at the Earth's Equator $\pi^2 \times \frac{4 \times 251084556}{(86164.1)^2}$. Consequently $m = \frac{4 \times 251084556}{39.01677 \times (86164.1)^2} = .0034672$; and $= e\,.0086679 - F$.

Calculation of Earth's ellipticity.

The question now is, what value shall we choose for F? Shall we represent the length of the seconds' pendulum by $39.01677 + .20027\sin^2\lambda$, or by $39.01677 + .20277\sin^2\lambda - .00880\sin^2\lambda\cos^2\lambda$? The former gives $F = .005133$; the latter gives $F = .005197$. We are now in a case where our theory partly fails, and where it is impossible to say which of the empirical forms is more likely to be correct. If we adopt the former value of F we find $e = .003535$; if the latter, $e = .003471$. Either of these values of e is considerably greater than that which we found from the discussion of the geodetic measures (.003352).

Greater than that given by measures of arcs. No explanation of the difference.

We cannot offer any conjecture as to the cause of this difference. If the regions of the Earth in middle latitudes were composed of lighter materials than the other parts, it might possibly happen that the gravity at middle latitudes (from the resulting projection of the Earth at these parts) would be diminished, while at the Poles and Equator it would be increased. The same alteration of figure would cause the ellipticity, deduced principally from measures near the Equator and in middle latitudes, to appear less than it ought. The only way of verifying such an idea would be, to measure an extensive arc of meridian near the Pole, (for we consider the Swedish arcs to be almost useless,) a thing which is not likely to be ever effected. But is such an idea worth verifying? We cannot say that it is: and we have uttered it only in the effort to produce some conjectural explanation of an apparent anomaly.

We can find, however, another conclusion which it is more pleasing to dwell upon. If the form of the Earth's meridian were traced on paper the nicest eye would be unable to distinguish it from a circle. The ellipticity is so small that the closest inspection without measure could not judge which was intended for the greater and which for the smaller axis. The whole quantity in dispute is less than one-sixteenth of this ellipticity. Instead of being surprised that such a difference exists, we may well be astonished at the accuracy of modern measures of all kinds which make so small a quantity a subject of controversy.

Section 9.—Physical Theory of Precession and Nutation, and Deductions of the Earth's Ellipticity from their observed Magnitudes.

Difficulty in the fundamental explanation of precession.

To point out the peculiar difficulty which at first sight appears to exist in the explanation of these phenomena, we will consider the Earth in the situation which it has at the summer solstice. In fig. 48 let S be the Sun, B the pole of the Earth whose centre is A. If now the Earth were a sphere, the attraction of the Sun upon it would not tend to disturb its motion of rotation. But from the preceding discussions it appears, that the Earth is an oblate spheroid, and that, consequently, at K and L its form is protuberant above the spherical form. The attraction of the Sun upon K is more powerful than that on the principal mass of the Earth; the attraction on L is less powerful. Considering the effect of these forces in producing motion relative to A, the effect of the former is the same as if K were drawn towards S, that of the latter the same as if L were pushed away. The effect of both of these forces seems, to twist the spheroid so as to remove B further from S. But in point of fact the effect is, to lift up the pole B from the paper, while A remains in it; B preserving the same distance from S. This is the difficulty to be explained.

Precession explained by the composition of rotatory motions.

The explanation is to be sought in the composition of rotatory motions given in the Treatise on MECHANICS, Art. 143, &c. The Earth is revolving round the axis A B. The tendency of the forces that we have described is to twist the Earth round an axis which would be represented by sticking a pin perpendicular to the paper at A. The consequence is, that at the end of a short time the Earth is revolving about an axis lying in the plane which passes through the pin and A B, but shifted from A B towards the pin. (We say towards the pin and not the other way, because the direction of the shifting is determined by observing on which side of the old axis the motion impressed by the extraneous forces is opposed to the motion due to the original rotation; as it is plain that the position of the new axis must be in that part where the two motions destroy each other.)

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* Captain Sabine has observed, (we think correctly,) that in many instances the inequalities in the force of gravity appear to depend on the geological character of the stratum. On basaltic rocks, for instance, the number of vibrations is greater, and on alluvial soil less, than we should have inferred from the latitude of the station.

From this it appears, that the actual axis of rotation does in fact wander within certain small limits through the Earth. Since it continues to rise upwards from the paper at the extremity B while the Earth is revolving, it is easily seen that the pole describes on the Earth's surface a very small circle whose circumference = daily motion of the pole. The pole of the terrestrial ellipsoid then, whose form is determined by the rotation, will be in the centre of this small circle. And the motion will be the same as if the Earth revolved round one invariable axis, namely the axis of the ellipsoid, and this axis were lifted up from the paper.

The velocity of rotation is not altered.

Measure of the motion of the Earth's axis produced by an extraneous force.

Suppose the extraneous forces would impress in 1^s an angular velocity a round the axis represented by the pin; and suppose ω to be the angular velocity round the original axis. As the velocity a is not impressed at once, but by degrees, suppose the 1^s divided into n equal parts. At the end of the first part, the original velocity ω round A B is compounded with the velocity $\frac{a}{n}$ round the pin. The new axis (by the theorems referred to) is in a position dividing the angle between A B and the pin (a right angle) into two parts whose sines are in the proportion of ω and $\frac{a}{n}$; that is the angle which it makes with A B has for tangent $\frac{a}{n\,\omega}$. And the velocity of rotation round the new axis (see MECHANICS) is to that round the old one, as the sine of 90° to the sine of the complement of the last-mentioned angle; and consequently it $= \omega \sqrt{1 + \frac{a^2}{n^2\,\omega^2}} = \sqrt{\omega^2 + \frac{a^2}{n^2}}$. In the same manner, after the second portion of time, the angular velocity will be $\sqrt{\omega^2 + \frac{2\,a^2}{n^2}}$, &c.; and after 1^s or n portions of time, it will be $\sqrt{\omega^2 + \frac{n\,a^2}{n^2}} = \sqrt{\omega^2 + \frac{a^2}{n}}$. Now let the communication of the motion be gradual, that is, let n be indefinitely great, and this becomes ω. That is, the velocity of rotation is not altered. Taking up the investigation again, it will appear that in every n^{th} part of 1^s the axis is shifted through the angle whose tangent is $\frac{a}{n\,\omega}$: and, therefore, in 1^s the axis is shifted through n × angle whose tangent is $\frac{a}{n\,\omega}$, that is, making n indefinitely great, through the angle $\frac{a}{\omega}$ (as measured on the circumference of a circle whose radius = 1.)

Calculation of the momentum of the Sun's attraction.

We must begin then with calculating the value of a. Now to find the momentum of the impressed forces in producing rotation round A, we shall take the difference of the forces acting on each point P and on A (this amounts to the same as impressing on all the particles a force equal to that on A, which evidently would not alter the rotation.) The Sun's force on P $= \frac{S}{SP^2}$ (putting S for the acceleration produced by the Sun's action at distance 1): the resolved parts in the directions P N and N A are $\frac{S\,.\,PN}{SP^3}$ and $\frac{S\,.\,SN}{SP^3}$; or subtracting $\frac{S}{SA^2}$ the force on A, they are $\frac{S\,.\,PN}{SP^3}$ and $S\left(\frac{1}{SN^2} - \frac{1}{SA^2}\right)$, as S P is very nearly equal to S N. The values of these, supposing S A very great, are nearly $\frac{S}{SA^3} PN$ and $\frac{-2\,S}{SA^3} \times AN$. The momenta of these forces acting on a particle δm are $\frac{S}{SA^3} PN \times AN \times \delta m$ and $\frac{-2\,S}{SA^3} AN \times -PN \times \delta m$; their sum is $\frac{3\,S}{SA^3} AN\,.\,PN\,.\,\delta m$. Let A M parallel to the equator $= x$; M P parallel to the axis $= y$; B A S $= \theta$; then A N $= x \sin\theta - y \cos\theta$; N P $= x \cos\theta + y \sin\theta$; and the momenta of the forces is $\frac{3\,S}{SA^3} \times \{(x^2 - y^2) \sin\theta\,.\,\cos\theta + x\,y\,(\sin^2\theta - \cos^2\theta)\}\,\delta m$. Consequently the momenta of forces upon the whole spheroid is the sum of the quantities $\frac{3\,S}{SA^3} \sin\theta\,.\,\cos\theta \int x^2\,\delta m$, $-\frac{3\,S}{SA^3} \sin\theta\,.\,\cos\theta\,.\int y^2\,\delta m$, $\frac{3\,S}{SA^3} (\sin^2\theta - \cos^2\theta) \int x\,y\,\delta m$. But since for every value of x we can find pairs of particles, whose values of y are equal, and have different signs, $\int x\,y\,\delta m$ will be 0. Also, putting a and b for the major and minor semiaxes, and ρ for the density of the spheroid, it is easily seen that $\int x^2\,\delta m$ must be proportional to $a^2 \times$ mass of spheroid, or proportional to $a^2 \times a^2\,b\,.\,\rho$, or equal to $C\,.\,a^4\,b\,.\,\rho$; and that $\int y^2\,\delta m$ must be in the same proportion to $b^2 \times a^2\,b\,.\,\rho$, or equal to $C\,a^2\,b^3\,.\,\rho$: and therefore the whole momentum of forces $= \frac{3\,S}{SA^3} \sin\theta\,.\,\cos\theta\,.\,C\,.\,a^2\,b\,(a^2 - b^2)\,\rho$. If $a = b\,(1 + e)$ where e is small, this $= \frac{6\,S}{SA^3}\,.\,\sin\theta\,.\,\cos\theta\,.\,C\,b^5\,e\,.\,\rho$.

Physical Theory of Precession and Nutation, &c.

If, as in Section 2. Art. 44. we suppose the spheroid to consist of a series of spheroidal shells of different densities, the momentum of forces will be $\frac{6\,S}{S\,A^3} \,.\, \sin\theta \,.\, \cos\theta \,.\, C \int \rho \,.\, \frac{d\,(b^5\,e)}{d\,b} \,.\, d\,b$, or $\frac{6\,S}{S\,A^3} \sin\theta \,.\, \cos\theta \,.\, C \,.\, \psi\,(\mathrm{b})$.

Calculation of the momentum of inertia.

Now to find a (MECHANICS, § XIX.) we must divide this by the moment of inertia, or $\int (x^2 + y^2)\,\delta m$: which for a homogeneous spheroid $= C\,a^2\,b\,(a^2 + b^2)\,\rho = 2\,C \,.\, b^5 \,.\, \rho$ nearly; and for a heterogeneous spheroid $= 2\,C \int \rho \,.\, \frac{d\,(b^5)}{d\,b}\,d\,b$. Call the general integral $v\,(b)$, and the value which it has at the surface $v\,(\mathrm{b})$, and we have for the moment of inertia $2\,C\,v\,(\mathrm{b})$.

Calculation of the motion of the Earth's axis produced by the Sun.

Dividing by this the moment of the forces, we have $a = \frac{3\,S}{S\,A^3}\,\frac{\psi\,(\mathrm{b})}{v\,(\mathrm{b})} \sin\theta \,.\, \cos\theta$: and consequently the motion of the Earth's axis in $1^s = \frac{3\,S}{S\,A^3 \,.\, \omega} \,.\, \frac{\psi\,(\mathrm{b})}{v\,(\mathrm{b})} \sin\theta \,.\, \cos\theta$; which for convenience we may call $D \,.\, \sin\theta \,.\, \cos\theta$.

Direction of this motion.

Now this motion is perpendicular to the plane passing through the Sun and the Earth's axis. In fig. 49, let B represent the place of the Pole, and $\odot$ the place of the Sun as seen from the Earth's centre; let ♈ ♋ be the ecliptic, P the pole of the ecliptic, B b perpendicular to B $\odot$. Then the Earth's pole is moving with the velocity $D \,.\, \sin B\odot \,.\, \cos B\odot$ in the direction B b. This produces an alteration in the distance of B from P, and an alteration in the place of ♋. We shall investigate these separately.

Change in the inclination of the Earth's axis to the ecliptic produced by the Sun.

First, the resolved part of the velocity of B in the direction B P is $D \,.\, \sin B\odot \,.\, \cos B\odot \,.\, \sin \odot B\text{♋} = D \cos B\odot \,.\, \sin \odot\text{♋} = D \cos B\text{♋} \,.\, \cos \odot\text{♋} \,.\, \sin \odot\text{♋}$. Let $P\,B = I$: then $\cos B\text{♋} = \sin I$. Also let the Sun be supposed to go (apparently) through the ecliptic in a year or T seconds, with a uniform motion, (the effect of the inequality of motion is not sensible,) and let t be the time since the Sun was at ♈. Then $\text{♈}\odot = \frac{2\,\pi}{T}\,t$, and $\cos \odot\text{♋} \,.\, \sin \odot\text{♋} = \sin \frac{2\,\pi}{T}\,t \,.\, \cos \frac{2\,\pi}{T}\,t = \frac{1}{2} \sin \frac{4\,\pi}{T}\,t$; and the resolved velocity in the direction B P is $\frac{D}{2} \sin I \,.\, \sin \frac{4\,\pi}{T}\,t$. Integrating this with respect to t we find the motion in the direction B P $= -\frac{D \,.\, T}{8\,\pi} \,.\, \cos \frac{4\,\pi}{T}\,t = -\frac{D \,.\, T}{8\,\pi} \,.\, \cos 2\,\text{♈}\odot$. And the inclination is changed to $I + \frac{D \,.\, T}{8\,\pi} \cos 2\,\text{♈}\odot$. It is plain that the last term is periodical: and, therefore, though the inclination is changed in the course of a year, it is the same in all successive years. The variable term is called Solar Nutation.

Precession of equinoxes produced by the Sun.

Secondly, the resolved part of the velocity of B perpendicular to B ♋ is $D \,.\, \sin B\odot \,.\, \cos B\odot \,.\, \cos \odot B\text{♋}$. Now $\cos \odot B\text{♋} = \sin B\odot\text{♋} \,.\, \cos \odot\text{♋}$: the expression becomes then $D \,.\, \times (\sin B\odot \,.\, \sin B\odot\text{♋}) \times (\cos B\odot) \times \cos \odot\text{♋} = D \times (\sin B\text{♋}) \times (\cos B\text{♋} \,.\, \cos \odot\text{♋}) \times \cos \odot\text{♋} = D \,.\, \sin I \,.\, \cos I.\ \sin^2 \text{♈}\odot = \frac{D}{2} \sin I \,.\, \cos I \left(1 - \cos \frac{4\,\pi}{T}\,t\right)$. Integrating with respect to t we have for the motion of B perpendicular to P B, $\frac{D}{2} \,.\, \sin I \,.\, \cos I \,.\, t - \frac{D \,.\, T}{8\,\pi} \sin I \,.\, \cos I \sin \frac{4\,\pi}{T}\,t$. The motion of ♋ is found by dividing this by sin I, and it is, therefore, $\frac{D}{2} \,.\, \cos I \,.\, t - \frac{D \,.\, T}{8\,\pi} \cos I \,.\, \sin 2\,\text{♈}\odot$. The last term is periodic: it is considered a part of solar nutation. The first increases uniformly; it shows that ♋ is continually travelling towards ♈, or backwards: the regression in one year is $\frac{D}{2} \cos I \,.\, T$. This is called Solar Precession.

General explanatoin of lunar precession and lunar nutation.

The last investigations apply as well to the Moon as to the Sun. The only difference is that the Moon revolves in so short a time round the Earth that the periodic terms analogous to solar nutation are hardly sensible. So that putting M for the mass of the Moon, M A for her distance from the Earth's centre, $\frac{3\,M}{M\,A^3 \,.\, \omega} \times \frac{\psi\,(\mathrm{b})}{v\,(\mathrm{b})} = E$, I' the angular distance between the pole of the earth and the pole of the Moon's orbit, T' her periodic time; the motion of the Earth's pole during one revolution of the Moon is $\frac{E}{2} \sin I' \cos I' \,.\, T'$ in a direction perpendicular to the great circle joining the pole of the Earth, and the pole of the Moon's orbit. Now the pole of the Moon's orbit describes (nearly) a circle in $18\frac{1}{2}$ years round the pole of the ecliptic. And this change of place, altering the direction and magnitude of the motion of the Earth's pole, produces the inequality called Lunar Nutation.

In fig. 50, let p be the pole of the Moon's orbit. Then the motion of the Earth's pole in 1^s will be $\frac{E}{2} \sin B\,p \,.\, \cos B\,p$ in the direction B b. The resolved part of this velocity which increases the distance of B and P is $\frac{E}{2} \times (\cos B\,p) \times (\sin B\,p \,.\, \sin P\,B\,p) = \frac{E}{2} \times (\cos B\,P \,.\, \cos P\,p + \sin B\,P \,.\, \sin P\,p \,.\, \cos P) \times (\sin P\,p \,.\, \sin P)$. Let $P\,p = I''$, and let the periodic time of p round p in a retrograde direction be T'': and let t be the

Change in the inclination of the Earth's axis to the ecliptic produced by the Moon.

time since p passed the arc P B. Then the velocity above $= \frac{E}{2} \times \left\{ \cos I \,.\, \sin I'' \,.\, \cos I'' \,.\, \sin \frac{2\pi t}{T''} + \frac{1}{2} \sin I \,.\, \sin^2 I'' \sin \frac{4\pi t}{T''} \right\}$. Integrating this with respect to t, we have for the increase of P B, $\frac{E \,.\, T''}{4\pi} \times \left\{ -\cos I \,.\, \sin I'' \,.\, \cos I'' \,.\, \cos \frac{2\pi t}{T''} - \frac{1}{4} \sin I \,.\, \sin^2 I'' \,.\, \cos \frac{4\pi t}{T''} \right\}$. These terms are one part of lunar nutation. They are evidently periodical; and, therefore, the inclination of the Earth's axis to the axis of the ecliptic is not permanently altered.

Precession of the equinoxes produced by the Moon.

Again, the resolved velocity perpendicular to P B is $\frac{E}{2} \sin B p \,.\, \cos B p \,.\, \cos P B p \,.\, = \frac{E}{2} \sin B p \,.\, \cos B p \,.\, \frac{\cos P p - \cos B P \,.\, \cos B p}{\sin B P \,.\, \sin B p} = \frac{E}{2} \,.\, \cos B p \,.\, \frac{\cos P p - \cos B P \,.\, \cos B p}{\sin B P} = \frac{E}{2} (\cos I \,.\, \cos I'' + \sin I \,.\, \sin I'' \,.\, \cos P) \,.\, \frac{\sin^2 I \,.\, \cos I'' - \sin I \,.\, \cos I \,.\, \sin I'' \,.\, \cos P}{\sin I} = \frac{E}{2} \{ \sin I \,.\, \cos I \,.\, \cos^2 I'' - (\cos^2 I - \sin^2 I) \sin I'' \,.\, \cos I'' \,.\, \cos P - \sin I \cos I \,.\, \sin^2 I'' \,.\, \cos^2 P \}$. Putting $\frac{2\pi t}{T''}$ for P, and integrating with respect to t, we find for the motion perpendicular to P B, $\frac{E}{2} \left\{ \sin I \,.\, \cos I \,.\, (\cos^2 I'' - \frac{1}{2} \sin^2 I'') \,.\, t - \frac{T''}{2\pi} (\cos^2 I - \sin^2 I) \sin I'' \,.\, \cos I'' \,.\, \sin \frac{2\pi t}{T''} - \frac{T''}{8\pi} \,.\, \sin I \cos I \sin^2 I'' \,.\, \sin \frac{4\pi t}{T''} \right\}$. The two last terms are evidently periodical: they are the remaining part of lunar nutation. The retrograde motion of ♋ will be found by dividing the motion of the Pole by sin I; it is, therefore, $\frac{E}{2} \left\{ \cos I \,.\, (\cos^2 I' - \frac{1}{2} \sin^2 I') \, t - \frac{T''}{2\pi} \,.\, \frac{\cos 2 I}{\sin I} \,.\, \sin I'' \,.\, \cos I'' \,.\, \sin \frac{2\pi t}{T''} - \frac{T''}{8\pi} \,.\, \cos I \,.\, \sin^2 I'' \,.\, \sin \frac{4\pi t}{T''} \right\}$. The two last terms, which are periodical, constitute the equation of the equinoxes in longitude. The first term increases uniformly: it is called lunar precession.

Sum of solar and lunar precession.

The lunar precession in one year is found by putting T for t: it is therefore $\frac{E \,.\, T}{2} \,.\, \cos I \,.\, (\cos^2 I'' - \frac{1}{2} \sin^2 I'')$. The whole annual soli-lunar precession is, therefore, $\{ D + E (\cos^2 I'' - \frac{1}{2} \sin^2 I'') \} \frac{T}{2} \cos I$. Putting for D and E their values, this becomes $\left\{ \frac{S}{S A^3} + \frac{M}{M A^3} (\cos^2 I'' - \frac{1}{2} \sin^2 I'') \right\} \,.\, \frac{3 T \cos I}{2 \omega} \,.\, \frac{\psi(b)}{v(b)}$.

To reduce this to numerical calculation, we must remark that (Mechanics, Art. 65. where u corresponds to our S, and a to our S A,) $T^2 = \frac{4\pi^2 \,.\, S A^3}{S}$, therefore $\frac{S}{S A^3} = \frac{4\pi^2}{T^2}$. Also, by the same formula, as the Moon revolves round the Earth as a centre, $\frac{\text{Earth's mass}}{M A^3} = \frac{4\pi^2}{T''^2}$: and consequently, if the Moon $= \frac{1}{n}^{th}$ of the Earth, $\frac{M}{M A^3} = \frac{4\pi^2}{n T''^2}$.* This makes the expression $\{ 1 + \frac{T^2}{n T''^2} (\cos^2 I'' - \frac{1}{2} \sin^2 I'') \} \frac{6\pi^2 \cos I}{T \omega} \,.\, \frac{\psi(b)}{v(b)}$. But since ω is the angular velocity of the Earth, $T \omega$ is evidently the angle through which the Earth revolves in a year $= 366.26 \times 2\pi$. Thus the annual precession becomes $\{ 1 + \frac{T^2}{n T''^2} (\cos^2 I'' - \frac{1}{2} \sin^2 I'') \} \,.\, \frac{3\pi \,.\, \cos I}{366.26} \,.\, \frac{\psi(b)}{v(b)}$. This is measured by an arc of a circle whose radius is 1. If we express it in seconds of a degree, it must be multiplied by $\frac{360 \times 60 \times 60}{2\pi}$: this gives for the annual precession in seconds, $\{ 1 + \frac{T^2}{n T''^2} (\cos^2 I'' - \frac{1}{2} \sin^2 I'') \} \,.\, \frac{540 \times 60 \times 60}{366.26} \cos I \frac{\psi(b)}{v(b)}$. If we knew accurately the value of n, we might compare this expression with the observed value of annual precession (which is known very accurately) and might thence infer the value of $\frac{\psi(b)}{v(b)}$. This, it is plain, would give us considerable assistance in determining the form and constitution of the Earth.

Soli-lunar precession in a form for calculation.

But as n is not very certainly known, it will be better to use also another observable quantity depending on n The most easily observable is the first term of lunar nutation in the increase of P B. This coefficient we have found to be $\frac{E \,.\, T''}{4\pi} \,.\, \cos I \,.\, \sin I'' \,.\, \cos I''$, or $\frac{M}{M A^3} \,.\, \frac{3 T''}{4\pi \,.\, \omega} \cos I \,.\, \sin I'' \,.\, \cos I'' \frac{\psi(b)}{v(b)}$, or $\frac{3\pi T''}{n T''^2 \,.\, \omega}$

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* Instead of n we ought in strictness to have put $n + 1$. See Physical Astronomy, p. 653. The effect of this error is insensible.

$\cos I . \sin I'' . \cos I'' . \frac{\psi(b)}{v(b)} = \frac{3\pi}{n} . \frac{T''}{T} . \frac{T^3}{T'^3 . 366.26 . 2\pi} . \cos I . \sin I'' . \cos I'' . \frac{\psi(b)}{v(b)}$, which in seconds (as in the former case) $= \frac{1}{n} . \frac{T''}{T} . \frac{T^3}{T'^3} . \frac{270 \times 60 \times 60}{366.26 . \pi} . \cos I . \sin I'' . \cos I'' . \frac{\psi(b)}{v(b)}$.

Lunar nutation in a form for calculation.

Now suppose $I = 23° 28'$; $I'' = 5° 8' 38''$; $\frac{T''}{T} = 18.6$; $\frac{T}{T'} = \frac{366.26}{27.32}$; also call the whole annual precession a, and the term just mentioned b. Then on performing the calculations, we shall have these equations,

$$a = \left(4869 + \frac{864490}{n}\right) . \frac{\psi(b)}{v(b)}$$

$$b = \frac{231460}{n} . \frac{\psi(b)}{v(b)}.$$

Solving them,

$$n = 47.54 \times \frac{a}{b} - 177.56$$

$$\frac{\psi(b)}{v(b)} = .0002054 \times a - .0007671 \times b.$$

Function depending on the Earth's form deduced from observed values of precession and nutation.

The last value is the only one for which we have occasion at present.

Now a is very nearly 50.3; the value of b given by Bradley is 9.0; by Mayer 9.65; by Maskelyne 9.55; by Laplace 9.40; by Lindenau 8.99; by Brinkley 9.25. Then we get for $\frac{\psi(b)}{v(b)}$ the following values:

According to Bradley, $\frac{\psi(b)}{v(b)}$	$= .003428$
Mayer	$= .002930$
Maskelyne	$= .003006$
Laplace	$= .003121$
Lindenau	$= .003436$
Brinkley	$= .003236$.

The last value is perhaps the most probable; (it has been adopted as such by Mr. Baily in the construction of the Astronomical Society's Tables, and we think properly.) Now it must be observed that we cannot from this deduce the Earth's ellipticity without assuming some law for the density of the strata, and solving approximately a troublesome equation. For though $\psi(b) = \frac{5}{3}\left(e - \frac{m}{2}\right) b^2 . \phi(b)$, (Section 2. Art. 63.) and, therefore, $\frac{\psi(b)}{v(b)} = \frac{5}{3}\left(e - \frac{m}{2}\right) . \frac{b^3 \phi(b)}{v(b)}$, where $\phi(b)$ and $v(b)$ do not depend on e; yet the values both of e and of $\phi(b)$ and $v(b)$ will depend on the value of the constants in the law (whatever it may be;) and, therefore, when e is determined on the assumed law in terms of the constants by the formula of Section 2. and $\phi(b)$ and $v(b)$ are expressed in terms of them, there will, upon making $\frac{5}{3}\left(e - \frac{m}{2}\right)\frac{b^2 . \phi(b)}{v(b)}$ equal to one of the values above, be a complicated equation to solve. If, for instance, we assume the law mentioned in Section 2. Art. 68. and take Brinkley's value of nutation, we shall have this equation to solve

This function will not give the Earth's ellipticity without a knowledge of the law of density.

$$.003236 = \left\{\frac{5m}{2} . \frac{\frac{3f}{q^2 b^2} - 1}{\frac{q^2 b^2}{f} + f - 3} - \frac{m}{2}\right\} . \frac{f}{2 + f - \frac{6f}{q^2 b^2}},$$

Calculation with Brinkley's nutation.

where $f = 1 - \frac{qb}{\tan qb}$. Solving this by approximation, with the value $m = .0034672$, we find $q\,b = 143° 53'$; and substituting this in the expression

$$e = \frac{5m}{2} . \frac{\frac{3f}{q^2 b^2} - 1}{\frac{q^2 b^2}{f} + f - 3}, \text{ we find } e = .3323.$$

The smaller values of $\frac{\psi(b)}{v(b)}$ give smaller values for e.

We may remark that the mass of the Earth is $= \frac{4\pi}{3} \int \rho \frac{d . b^3}{db} db$ nearly, and its volume $= \frac{4\pi}{3} b^3$ nearly, therefore the mean density $= \frac{3 \int \rho . b^2 db}{b^3}$ nearly. With the law above mentioned, and the value

Figure of the Earth.

Earth's mean density compared with density at surface.

of $q\,\mathrm{b}$ just found, we find for the mean density $\frac{3\,\mathrm{A}}{q^2\,\mathrm{b}^3}\,.\,\sin q\,\mathrm{b}\,.f$, that at the surface being $\frac{\mathrm{A}\,.\,\sin q\,\mathrm{b}}{\mathrm{b}}$. The proportion of these is $\frac{3f}{q^2\,\mathrm{b}^2}$, or 2.20; a proportion that agrees pretty well with the results of experiments hereafter to be mentioned.

Section 10.—*Deduction of the Earth's Ellipticity from the observed Inequalities of the Moon's Motion.*

In Section 2. Art. 67. we have found for the inequality in the Moon's latitude depending on the Earth's non-sphericity, $-\left(\mathrm{e}-\frac{m}{2}\right)\times\frac{4\,.\,\overline{60}|^3}{\pi}\,\mathrm{M}\,.\left(\frac{\text{Earth's period}}{\text{Moon's period}}\right)\left(\frac{\text{Earth's radius}}{\text{Moon's distance}}\right)^2.\,\sin \mathrm{I}\,.\,\cos \mathrm{I}\,.\,\sin\theta$. This expression is, in fact, the same as Laplace's, (*Mécanique Céleste*, livre 7, No. 29.) observing that g is very nearly $=1+\frac{3}{4}\,m^2$ (in Laplace's notation.) If $\mathrm{I}=23°\,28'$, and $\frac{\text{Earth's radius}}{\text{Moon's distance}}=\sin 57'$, and $\frac{\text{Earth's period}}{\text{Moon's period}}=\frac{365.25}{27.25}$, and $\mathrm{M}=\frac{70}{71}$, this is equal to $-4891''\times\left(\mathrm{e}-\frac{m}{2}\right)\times\sin\theta$. Now this inequality has been found to exist, (it has the effect of increasing the apparent inclination of the Moon's orbit in one position of her nodes, and of diminishing it as much in the opposite position,) and its magnitude has been inferred from observation. The multiplier of $\sin\theta$ in Bürg's Tables is $-8''.0$; in Burckhardt's it is the same. This value gives $\mathrm{e}-\frac{m}{2}=\frac{8}{4891}=.001636$; and considering m (as before) $=.003467$, we have $\mathrm{e}=.001636+.001734=.003370$.

Ellipticity calculated from inequality in the Moon's latitude.

From inequality in the Moon's longitude.

We have mentioned in the same place, that there is an inequality in the Moon's longitude depending on the non-sphericity of the Earth. Laplace's expression (which our limits do not allow us to investigate) is equivalent to C . sin longitude of the ascending node of the Moon's orbit, where $\mathrm{C}=\frac{19}{2}\,.$ tan mean inclination of Moon's orbit × the coefficient of the last inequality. Making mean inclination $=5°\,8'\,38''$, C is found $=4183''\times\left(\mathrm{e}-\frac{m}{2}\right)$. In Bürg's Tables C is $6''.8$; in Burckhardt's it is $7''.0$ The first gives $\mathrm{e}-\frac{m}{2}=.001626$, $\mathrm{e}=.003360$; the second gives $\mathrm{c}-\frac{m}{2}$ $.001673$, $\mathrm{e}=.003407$. Any of these values agrees nearly enough with that deduced from the geodetic measures.

In estimating the reliance to be placed on these values, it must be observed that they are deduced from observations which are insufficient if they do not extend over a period of nearly 20 years. The same remark applies to the deductions of the last section. This always throws a very little doubt on the accuracy of such a determination, as in all probability the observations which are compared have been made by different persons and in different manners. The small lunar inequalities, besides, are involved among a mass of terms much greater than themselves; but an error in their determination has less influence on the value of e than an equal error in the determination of nutation. We mention these things merely to show that these deductions could not be put in competition with those derived from geodetic measures, or pendulum observations, if the discordancies among the latter were not so great as to make every confirmation of their results desirable. The coincidence of the results is however satisfactory, as it gives us a strong confidence that the result deduced from the measures is not far from the truth, and that our theory is in the main correct.

Inequalities of the Moon's Motion.

Doubts of the certainty of these conclusions.

Section 11.—*Observations which show that the Attraction of Masses comparatively small is sensible; and Determinations of the Earth's mean Density.*

Among these we may place the discrepancies which have been observed in comparing different parts of the same arc. Of these it must be confessed, that in some instances no distinct explanation can be given. At Arbury Hill, for instance, one of the stations nearly bisecting the English arc of meridian, the latitude was observed, and was found to differ about 5″ from any that could be admitted, on any supposition of the Earth's form; at Dodagoontah, on the great Indian arc, an unexplained disturbance to nearly the same amount was observed; at Takal Khera the same thing was observed; but Captain Everest appears to have accounted perfectly for this by the attraction of a range of mountains at the distance of 15 miles, (the range running Eastward from the termination of the Western Ghauts.) The magnitude of these disturbances, it must be observed, is much greater than any error that could possibly happen in the use of the astronomical instruments or in the geodetic measures, without the most unreasonable neglect. And in inferring the latitude of one place from the observed latitude of another not on the same meridian, through the medium of a geodetic measure, results have frequently been obtained which differ much from the observed latitudes. Thus (*Conn. des Temps*, 1827, *Additions*) the latitude of Turin deduced from that of Milan differs from the observed latitude by 8″.9; that of Venice by 9″.5; that of Rimini by 27″.4. We regret that in the statement which we have copied, there is no mention of the dimensions of the Earth which have been used in deducing the difference of latitudes from the geodetic measure.

Anomalies in comparing different parts of the same arc of meridian.

In comparing the latitudes of places at small distances.

Another class of observations leading to the same conclusion is the difference between the values of degrees given by different arcs passing over nearly the same ground. We have seen that in the Swedish arcs it is necessary to suppose one disturbance to the amount of 12″, or two disturbances whose sum = 12″. In the arc of parallel between Beachy Head and Dunnose, as compared with that between Dover and Falmouth, there is evidence (though less perfect) of disturbance nearly as great.

In different arcs on the same ground.

In the Piedmontese arc, if we wish to reconcile it in any degree with the arcs in greater and smaller latitudes, we must suppose that the effect of the disturbances is more than 40″. But no one who considers the situation of the alluvial basin of Piedmont, with the

In the Piedmontese arc.

*P. 503. 1st Co

highest part of the chain of the Alps to the North, and the Apennines (of no inconsiderable height) to the South, will doubt that this is perfectly explained by their attractions.

These observations, though they render the application of our complete theory doubtful, yet serve to give the strongest confirmation of its fundamental principle. We must not reject* them because they disagree with our theory; we must endeavour to ascertain whether they are consistent with the principle of universal gravitation, and if we find them to be consistent, we must examine what alteration must be made in the other suppositions of our theory to make it represent the facts of observation.

Suppositions necessary to account for these anomalies

Now all that is necessary is, to imagine that, after the Earth had assumed a form of equilibrium and become solidified, parts of the external crust were elevated by some internal force. And if successive coatings were deposited from a fluid covering the whole, we must suppose irregularities of deposition, in many cases connected with the former, to have taken place. These inequalities, on the principle of gravitation, might account for disturbances in the Earth's form, or rather in the form of the sea, great in themselves, but small in comparison with the magnitude of the Earth. The circumstance, however, of islands being found scattered over every part of the Ocean seems to justify us in the belief that the Earth was originally fluid, and that its form has not been much altered by posterior convulsions.

The disturbances alluded to have shown the expediency, and suggested the possibility, of determining by observation the attraction of mountains; and the desire of throwing some new light on the constitution of the Earth has prompted several experiments for ascertaining the mean density of the Earth. In the last century, too, it was doubtful whether the Earth's ellipticity was less than $\frac{1}{230}$, (the value assigned by Newton,) and the determination of the Earth's mean density might (assuming the principle of gravitation) assist in settling that point. For, as has been seen, if the principal attractive mass of the Earth were at its centre, the ellipticity would be not much greater than $\frac{1}{580}$; and therefore if the central part of the Earth were more dense than the parts near the surface, the ellipticity might be expected to be less than $\frac{1}{230}$. the value which it would have if the Earth were homogeneous. The first experiment of this sort was that of Bouguer, mentioned in our first Section; of which we take no further notice, as the person who made it did not think its results worthy of any confidence.

The knowledge of the Earth's mean density will assist in determining its form.

Determination of the Earth's mean Density.

Attraction of Schehallien found by astronomical observations.

The second was that of Dr. Maskelyne on the attraction of Schehallien. The history of this is given in our first Section;* the astronomical observations are in the volume of the *Phil. Trans.* for 1775. The sector was made by Sisson, with the plumb-line passing over a dot at the centre of the instrument; it was divided by taking an arc (7° 9′ 59″.917) whose chord = $\frac{1}{8}$th of the radius, and continually bisecting this arc: it was used with a micrometer-screw as we have described in Section 3. At each of the stations it was reversed but once. The error of collimation, as determined by a mean of the observations on the different stars, appears to have changed only a fraction of a second between making the observations on the North and on the South sides of the mountain. On the South side 76 observations were made, face East, and 93, face West; and on the North side 68 observations, face West, and 100, face East: the whole number of stars observed was 43. The situation of the observatory on each side was about half way up the hill. The difference of astronomical latitude was found to be 54″.6. The distance in feet between the parallels passing through the two observatories was found to be 4364.4 feet. This was determined by a survey of the mountain founded on two bases, one of 3012 feet and another of 5897 feet, measured with deal rods that were compared with the Royal Society's brass standard. From the extremities of these bases two cairns on the ridge of the mountain were observed, and the distance between them found. These cairns were not visible at the observatories, but signals were fixed at distant points where a cairn and an observatory appeared in the same vertical; then the angle between these signals, as seen from the observatory, was the same as that between the cairns. The same signals were observed at the cairns, instead of the observatories. The distance 4364.4 feet, at the rate of 101.64 feet to one second, (according to Bouguer's table,) gave for the difference of geodetic latitude 42″.94. The difference between this and the observed difference, or 11″.6, is to be attributed to the attraction of the mountain.

Survey of the mountain and calculation of its attraction.

For determining the figure and dimensions of the mountain, stations were chosen all round it; then poles were fixed in the hill-side in vertical planes, as determined by observations with a theodolite in one station, and the azimuth and altitude of each was observed at another station. A few, however, were placed in horizontal planes; and some in different manners. The calculation of the attraction was made by Dr. Charles Hutton. (*Phil. Trans.* 1778.) An accurate map being made, concentric circles were described with one observatory for the centre, and with radii in arithmetical progression, their common difference being 666⅔ feet. Each of the rings between these was divided into 48 parts, according to the following law: a line being drawn in the direction of the meridian, radii were drawn making angles with it whose sines were successively $\frac{1}{12}, \frac{2}{12}, \frac{3}{12}$, &c. Then it is easily seen that the attraction (in the direction of the meridian) of each of the prisms thus formed is a constant multiplied by the sine of the angular altitude of its top. This was determined by an easy practical method, for which, with many other details, we must refer to the original Memoir. Indeed the various contrivances* of calculation in this

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* In the *Phil. Trans.* 1812, is a paper by Don. J. Rodriguez, (we believe one of the gentlemen who assisted Delambre in his survey,) in which observed latitudes are compared with assumed dimensions of the Earth, and the difference is at once set down as an error of observation. The reader, who has taken the trouble to examine our Section on Meridian Measures, and its results, will judge how uncertain the dimensions of the Earth still are, and how certain it is that, using any dimensions whatever, we must still suppose some disturbing cause to affect the latitudes with errors far greater than the errors of observation. The paper in question has been severely criticised by Delambre in the *Conn. des Temps*, 1816, *Additions*. For our own part, we can scarcely imagine how any one who had been concerned with geodetic surveys, or who even knew their object, could compose such a memoir as that to which we allude.

* Most of these, it appears, were suggested by Mr. Cavendish (whose experiments we shall shortly describe.) And it appears that nearly all the preliminary calculations of the attraction of Skiddaw,

Paper will be found well worth the attention of any practical person. The attraction of 960 prisms was calculated at each observatory; the attraction at the North observatory was found to be to that at the South observatory nearly in the proportion of 7 : 9. And the sum of the attractions was found (supposing the density of the mountain equal to the mean density of the Earth) to be $\frac{1}{9933}$ of the Earth's attraction. But the sum of the disturbances in the direction of gravity as determined from the astronomical observations being $= 11''.6$, the sum of the attractions was, in fact, $=$ gravity $\times \tan 11''.6 = \frac{1}{17781} \times$ gravity $= \frac{1}{17804} \times$ Earth's attraction, (allowing for centrifugal force.) Consequently the density of the mountain was to the Earth's mean density as 9933 to 17804, or nearly as 5 : 9.

Mineralogical structure of the mountain.

But this supposed that the mountain was homogeneous. The geological characters of its rocks were examined by Professor Playfair, (*Phil. Trans.* 1811,) he found that the upper part consisted of quartz whose mean Specific Gravity $= 2.6398$; and the lower part of mica slate and hornblend slate, whose mean Specific Gravity $= 2.8326$, and of limestone whose mean Specific Gravity $= 2.7661$. Two separate calculations were made; one on the supposition that the rocks were separated by vertical surfaces, and another supposing them separated by surfaces nearly horizontal. The former gave for the Earth's mean density 4.559, the latter 4.867, that of water being 1. Dr. Hutton (*Phil. Trans.* 1821) says that the number should be rather greater.

Cavendish's apparatus for observing the attraction of leaden balls

The next experiment was that of Mr. Cavendish on the attraction of leaden balls. (*Phil. Trans.* 1798.) His apparatus is represented in fig. 51. *x*, *x*, are balls of lead about 2 inches in diameter, suspended to the ends of a light deal rod *h*, *h*. This is suspended by the wire *l g*, forming, in fact, a balance of torsion. The piece to which the top of the wire is attached, carries a wheel which is turned by the endless screw K F, so that the wire can be twisted till the resting-place of the balls is any required position. *n n* are small graduated scales carried by the rod. The whole of this apparatus is enclosed in a mahogany box, F E A B C D D C B A E F. At A and A are small glass windows, and near these are scales serving as a vernier for measuring the motion of the scales *n*. These are illuminated by lamps L and L, and viewed by telescopes T and T from the outside of the room. The leaden balls W, W (each weighing 2,439,000 grains) are suspended by copper rods attached to the piece *r r* which is suspended to a beam. By means of a rope passing round the pulley M M, the balls W, W can be moved without entering the room. The support of the balance of torsion and its cases is independent of the walls.

The first wire by which the deal rod was supported was of copper silvered, of which one foot weighed 2.4 grains. After a few experiments with this, it was found that the attraction of the large balls made the rod touch the sides of the case; and a stiffer wire was then used.

Disturbance produced by currents of air.

It was found at first, that a very small difference of temperature between the balls and the air produced currents of air within the mahogany case which materially disturbed the experiment. The cautions subsequently used seem to have prevented any bad effect from this cause.

Method of observation

The method of observing was the following. The large balls being in the *midway position* (their supporting rod at right angles to the deal rod) the position of the deal rod was read off from the scales *n*. The large balls were then brought so as nearly to touch the case, sometimes in the *positive position* (in which their attraction made the rod move so as to increase the number on the scales) and sometimes in the *negative position*. By their attraction on the small balls, the rod was immediately put in motion, and vibrated backwards and forwards. The greatest extent of vibration was observed; the mean of two consecutive extreme points on one side was taken, and the intermediate extreme point on the other side; and the point midway between these was considered to be the point at which the rod would rest under the action of the balls and the torsion of the wire. The time of passing the middle point was also ascertained, by observing the time of passing two points near the middle, and then (when the middle point was determined) calculating the time of passing it. This being done for vibrations separated by a considerable number, the time of vibration was accurately found. With the wire finally used, the change from the position of rest without the action of the lead balls to the middle position when they were applied, seldom exceeded 3 divisions, (each division $\frac{1}{20}$th of an inch,) and the time of vibration was about 7 minutes. With the first wire the change was about 14 divisions, and the time nearly 15 minutes.

The middle point is evidently the place where the attraction of the large balls is equal to the force of torsion of the wire. The time of vibration also depends on both of these forces. For suppose that at the middle point the distance of the small balls from the large ones was A, and the space through which they had been moved (to which the force of torsion is proportional) B; then putting W and T to represent these forces respectively at the distance 1, we shall have this equation $\frac{W}{A^2} = TB$. And at the distance x beyond this, and further from the balls, the whole force tending to bring the balls to the middle point is $T(B + x) - \frac{W}{(A - x)^2} = TB - \frac{W}{A^2} + \left(T - \frac{2W}{A^3}\right) x$ nearly, $= \left(T - \frac{2W}{A^3}\right) x$; and this is the force on which the time of vibration depends.

Method of calculation.

Thus there are, in fact, two equations to be solved, from which the attraction of the balls and the torsion of the wire could be determined. Besides this, the attraction of the mahogany case was calculated. The attraction of the leaden balls being thus determined, and compared with the attraction of the Earth, the proportion between the Earth's mean density and the density of lead was found; and thus the Earth's mean density is obtained. The result of 29 experiments (as corrected by Dr. Hutton, *Phil. Trans.* 1821) is 5.31, that of water being 1. The smallest number given by one experiment is 4.86, and the largest 5.79.

We are upon the whole inclined to prefer this result to that of the observations on Schehallien. It cannot

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&c. were made by Mr. Cavendish. This we have ascertained from an inspection of his papers, which we have had an opportunity of examining through the kind permission of his Grace the Duke of Devonshire.

Result probably preferable to the former.

be denied that the greatest delicacy was necessary to obtain any result, and that the determination is much less certain than the determination (such as it is) by the Schehallien experiment. But we consider that the quantity determined in the latter may be something very different from the attraction of the mountain. We have seen that at Arbury Hill, and some other places, there is evidence of disturbance to an amount nearly as great as the attraction of Schehallien, and without any known cause. May not such unknown causes have operated as well in the Schehallien experiment, and have increased or diminished the apparent effect of the mountain?

Considering the Earth's mean density as somewhat greater than 5, and the mean density of the rocks at the surface as 2.6, the proportion of the Earth's mean density to the superficial density is not very different from that of 2 : 1.

We have mentioned in our first Section the attempt made by the Baron de Zach to measure the attraction of a mountain near Marseilles. We have only to add that no calculation of the Earth's density was founded on these observations.

The attraction of a mountain might be found by observing the length of the seconds' pendulum on the top; if gravity should thus be found to be greater than gravity at the level of the sea in the same latitude diminished in the duplicate proportion of the distance from the Earth's centre, the excess would be attributable to the attraction of the mountain. Bouguer (as we have mentioned in Section 7.) observed the pendulum on one of the peaks of the Andes; but the circumstances were unfavourable, and we should have no confidence in the results. In the present century (see the *Additions* to the *Milan Ephemeris*) M. Carlini made similar observations in much more favourable circumstances at the hospice of Mont Cenis, at an elevation of 6375 feet above the level of the sea. The pendulum we have described at the beginning of Section 7: twenty experiments were made with it; they were reduced like the French measures, and corrected for elevation by the rule of inverse squares; the length at the level of the sea thus found was in mètres 0.993708. But the observations of Biot at Bordeaux, nearly at the level of the sea, corrected for the small difference of latitude, gave 0.993498. The difference is due to the attraction of the mountain mass. Representing this by a segment of a sphere, 1 geographic mile in height and 11 in diameter at the base, of Specific Gravity 2.66, the mean density of the Earth is calculated to be 4.39. Perhaps this experiment would have been more satisfactory if the pendulum had been made exactly like the French pendulums, or if an invariable pendulum had been used. As it stands, there is one considerable source of error,[*] namely, the erroneous reduction for the effect of the air (mentioned in Section 7.) The barometer at Mont Cenis being several inches lower than at Bordeaux, this error would be serious. Besides, we could hardly trust to a comparison between two places at so great a distance. On the whole, we do not think that any estimation of the Earth's density can be founded on this experiment.

Pendulum observations on Mont Cenis.

Conclusion.

The result erroneous and doubtful if corrected.

Section 12.—*Conclusion.*

1. The measures of the Earth, the observations of pendulums, and the lunar inequalities, agree in showing that the Earth's form does not differ much from that of an ellipsoid of revolution whose ellipticity is (we think certainly) greater than $\frac{1}{300}$, and whose major semiaxis is about 20,923,700 English feet.

2. The phenomena of precession and nutation give an ellipticity rather smaller; but as no result can be deduced from them except on an assumed law of density, this value cannot be put in opposition to the others.

3. As the results of the pendulum observations, the lunar inequalities, and the precessional phenomena, can only be used to determine the Earth's form by the intermediation of the principle of gravitation, the very near coincidence of the results is a strong argument in favour of the truth of that principle.

4. The same things make it highly probable that the Earth has once been in a fluid or semi-fluid state.

5. None of these results can be obtained without the admission of considerable anomalies, all of which, however, appear to be consistent with the principle of gravitation.

6. The mean density of the Earth considerably exceeds, and is probably double of the density of the superficial rocks.

7. The near agreement of the proportion between these as deduced from an assumed law with the proportion found by the experiments with leaden balls (where it is assumed in the calculation that the law of gravitation holds good at the distance of a few inches) makes it probable that the law is sensibly true to very small distances.

G. B. AIRY.

Observatory, Cambridge,
August 17, 1830.

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[*] If we correct this, by doubling the usual reduction, (the pendulum-balls being spherical,) the length at Bordeaux is 0.993553, and that at Mont Cenis, reduced for elevation, 0.993754; whence the mean density of the Earth = 4.59.

POSTSCRIPT.

The author has discovered a small error in the Table of the observed lengths of the seconds' pendulum. The reduction applied to the length of the Paris pendulum for elevation above the sea is that due to the *decimal* pendulum. In consequence, the length at Paris, and all the lengths depending on it, ought to be increased by .00013; a quantity which does not sensibly affect any of the results. This correction applies to Nos. 18, 25, 31, 33, 40, 44, 47, 49; and half of it to Nos. 42, 45, 48.

The multiplier 0.6 has been used (in reducing the foreign observations) in preference to 0.66, as it seems probable that the density at the Earth's surface is greater, and the mean density less, than Dr. Young supposed. The multiplier 0.66 is adopted for the English observations from the calculations of the respective observers; the effects of this inconsistency are not sensible.

In the determination of the length of the seconds' pendulum at Königsberg, a correction is included which is not applied to any of the other observations. The reason is that the effect of this correction is to increase the length of the seconds' pendulum; and it appears that the observations on which Bessel principally relied, give a smaller length than those made in the usual way. The difference depends probably on the difference of the apparatus employed.

Tides and Waves.

TIDES AND WAVES.

Introduction.

Introduction.

We propose, in this article, to enter at some length into the mathematical theories, and the experimental observations, applying to the two subjects of Tides and Waves of water. But we do not intend to treat them with the same extension. We shall give the various theories of Tides in detail sufficient to enable the reader to understand the present state of the science which regards them; and we shall advert to the principal observations which throw light either on the ordinary phænomena of tides, or on the extraordinary deviations that occur in peculiar circumstances. In thus treating the Tides, it will be necessary for us to enter largely into the theory of Waves. We shall take advantage of this circumstance for the introduction of several propositions, not applying to the theory of Tides, but elucidating some of the ordinary observations upon small Waves. But these investigations will be limited to that class which is most closely connected with tides, namely, that in which similar waves follow each other in a continuous series, or in which the same mathematical process may be used as when similar waves follow each other. In this class will be included nearly all the phænomena of waves produced by natural causes, and therefore possessing general interest. But it will not include the waves of discontinuous nature produced by the sudden action of arbitrary causes, which have been the subject of several remarkable mathematical memoirs, but which possess no interest for the general reader.

Plan of the Treatise.

The general plan of this Essay will be as follows:—

I. We shall describe cursorily the ordinary phænomena of Tides.

II. We shall explain the *Equilibrium-Theory of Tides*, including the first tidal theory given by Newton, and the more detailed theory of his successors, especially Daniel Bernoulli.

III. We shall give a sketch of Laplace's investigations, (founded essentially on the theory of the *motion* of water,) in the general form in which he first attempted the theory, as well as with the arbitrary limitations which he found it necessary to use for practical application.

IV. We shall give an extended Theory of Waves on water, applying principally to the motion of water in canals of small breadth, but with some indications of the process to be followed for the investigation of the motion of Waves in extended surfaces of water.

V. The results of a few Experiments on Waves will be given, in comparison with the preceding theory.

VI. We shall investigate the mathematical expressions for the Disturbing Forces of the Sun and Moon which produce the Tides, and shall use them in combination with the theory of Waves to predict some of the laws of Tides.

VII. We shall advert to the methods which have been used, or which may advantageously be used, for Observation of Tides, and for the Reduction of the Observations.

VIII. We shall give the results of extensive observations of the Tides, as well with regard to the change of the phænomena of tides at different times in the same place, as with respect to the relation which the time and height of tide at one place bear to the time and height at other places, and shall compare these with the results of the preceding theories, as far as possible.

And as Conclusion, we shall point out what we consider to be the present Desiderata in the Theory and Observations of Tides.

Section I.—Ordinary Phænomena of Tides.

Phænomena of River Tides.

(1.) If we suppose an observer stationed on the bank of a river,* at no great distance from the sea, (for instance, on the bank of the Thames any where below London Bridge, on the Humber below the mouth of the Trent, or on the Severn below the Passages,) he will remark the following changes in the state of the water.

Semidiurnal Tide.

Its time is related to the apparent position of the moon.

(2.) The first and most important change is, that the surface of the water rises and falls regularly twice in every day. A short series of observations will show however that this statement is not quite correct; the tides of each succeeding day are somewhat later than those of the preceding day: the average retardation from day to day being about 40 minutes. In a short time he will find that the times of occurrence of high water bear a very close relation to the time of the Moon's appearance in certain positions; and that the language of the persons who are most accustomed to observe the tides conveys at once this relation. Thus, at Ipswich, high water occurs when the moon is south nearly: at London Bridge high water occurs when the moon is nearly south-west: at Bristol, it takes place when the moon is E.S.E. These are rude statements, but they are sufficiently accurate for many purposes; and they show at once the close connection between the time of high water and the time of the moon's passage over the meridian. In fact, so completely is this recognized, that, in order to give the time of high water upon any day, it is usually thought sufficient to state the time of high water on the days of new moon and full moon, when the moon passes the meridian at twelve o'clock nearly. This time is called the *Establishment* of the port. Then to find (roughly) the time of high water on any other day, it is only necessary to add the Establishment to the time of the moon's passage.

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* We commence with this case, because, judging from the notions of sea-faring persons upon many points connected with the Tides, which are correct as regards rivers, but incorrect as regards the sea, (some of which will hereafter be indicated,) it is the case from which ideas of tidal movements have usually been taken with the greatest facility.

The rules, as we have mentioned them, indicate the time of only *one* high water in the day: but the reader must understand that there will always be *another* high water in the same day, preceding or following that which we have mentioned by 12 hours 20 minutes nearly. On those days, however, in which high water occurs within 20 minutes of noon, there is no other high water on the same civil day.

The interval between high water and moon's transit is variable.

(3.) On closer examination it will be found that the interval between the time of the moon's passage over the meridian and the time of high water varies sensibly with the moon's age. At new moon, full moon, first quarter, and third quarter, (or rather on the day following each of these phases,) the interval between the time of the moon's passage and the time of high water is nearly the same: but from new moon to first quarter, and from full moon to third quarter, the high water occurs earlier than would be inferred by using that same interval; and from first quarter to full moon, and from third quarter to new moon, it occurs later than the same interval would give it.

Spring and Neap Tides.

(4.) If the observer examines the height of the water, he will find that the height at high water and the depression at low water are not always the same. On the days following new moon and full moon, high water is higher and low water lower than at any other time: these are called *Spring Tides*. On the days following the first and third quarters, high water is lower and low water higher than at any other time: these are called *Neap Tides*. The whole variation of height at spring tides is nearly double that at neap tides. There are other variations of height depending on other circumstances; but they require, for the most part, very numerous observations to establish the fact of their existence, and to give a measure of their amount. In many places, however, the tide which occurs at one certain part of the day (the afternoon for instance) is, during one half of the year, sensibly higher than the other tide which occurs upon the same day, and, during the other half of the year, sensibly lower.

The duration of the fall is longer than the duration of the rise.

The water continues to run up the river after high water.

(5.) Upon examining the circumstances of a single tide, the following facts will attract notice. The interval from high water to low water is greater than that from low water to high water: the difference between these intervals is sensibly greater at spring tides than at neap tides. The current in the river runs upwards for some time after high water, and after changing its direction, continues to run downwards for some time after low water, when it again changes its direction and runs upwards. This phænomenon is often so much misrepresented in the language of nautical men, that the mistake deserves particular notice. From the habit of observing tides in places where the current ceases at high water and at low water, sailors conceive that high water may always be inferred from the cessation of the current; and therefore it is not unusual for persons on the banks of the Thames to say that "it is high water in the centre of the channel long after it is high water at the shore." The observer who is not convinced of the absurdity of supposing the water in the middle of the channel to stand at one time considerably higher and at another time considerably lower than at the shore, will satisfy himself most easily as to the general fact by stationing himself at one of the central piers of a bridge, (as London Bridge,) when he will see that the water continues to run upwards even after its surface has dropped nearly two feet.

High water occurs later for places higher up the river.

The progress of the tide is too rapid to be explained by the transmission of the same body of water.

The duration of fall increases and that of rise diminishes in ascending the river.

The rapid rise sometimes produces a bore, sometimes double or treble tides.

(6.) Now suppose that the observer examines the state of the tide in different parts of the same river. Commencing with the mouth of the river, (for instance Margate or Sheerness on the Thames, or Swansea or Cardiff on the Severn,) he will find that there is very little difference, or perhaps none which is appreciable, between the interval from high water to low water, and that from low water to high water. He will also find that the current runs up the channel for a long time (sometimes approaching to three hours) after high water, and runs down the channel for as long a time after low water. In going up the river, he will find that the time of high water occurs later and later, but yet that the velocity with which high water travels up the river is so great as entirely to banish the idea of explaining the Tide by supposing the same mass of water to have been moved all the way up the river. For instance, if at Margate the high water occurs on a certain day at twelve o'clock, it will occur at Sheerness at 24 minutes past one, at Gravesend at 15 minutes past two, and at London Bridge at a few minutes before three; having thus described in less than three hours a course of about 70 miles. He will also find that the interval from low water to high water diminishes as he goes up the river: thus, on the lower parts of the Severn, the rise and fall occupy little more than six hours each; but at Newnham on the Severn the whole rise of the water is effected in an hour and a half, the descent occupying nearly eleven hours. In cases like the last-mentioned, the first rise of the tide is sudden, and if the banks of the river are shoaly, the water spreads over the flat sands with a roaring surf, which travels rapidly up the river, presenting the phænomenon called a *bore* or *boar*, (sometimes *boar's-head*,) in French *barre* or *mascaret*. In other cases, however, when the difference of durations of rise and fall is considerable, there are in each high water two, or sometimes three distinct rises and falls of the water. The phænomena of bore and double tide are always much more conspicuous in spring tides than in neap tides.

In contracted estuaries the tide is very high.

In ascending rivers the tide diminishes

(7.) If the estuary or mouth of the river contracts very much, the elevation and depression of the water will become very great. Thus at the entrance of the Bristol Channel the whole rise at spring tides is about 18 feet, at Swansea about 30 feet, and at Chepstow about 50 feet. Similar high tides occur at St. Malo and other parts of the great bay formed on the northern coast of France by the projection of land towards Cherbourg, and tides still higher in the head of the Bay of Fundy (Baie Française) on the Eastern coast of North America. But when the tide has fairly entered a river, its range of elevation and depression generally diminishes. Thus at Newnham, on the Severn, the range is reduced to about 18 feet, and it is still less at Gloucester.

Bay tides are simple.

(8.) Quitting now the phænomena of river-tides; if observations are made in a bay communicating with the open sea, the results will be found to be much more simple. The water will rise during 6 hours 10 minutes

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and will fall during an equal time; the whole rise and fall will usually be less than in rivers; a very slight current will be directed towards the head of the bay during the rise, and from it during the fall of the water. The variations of spring and neap tides, and the relation of the time of high water to the time of the moon's passage over the meridian, follow the same general laws as in rivers.

Channel Tides nearly similar to those of rivers near the mouth.

(9.) In long and narrow seas (for instance the English Channel) the tide in mid-channel follows the same laws as at a station near the mouth of a river, rising and falling in equal times, and running in a direction which may be considered analogous to the direction up a river, for three hours before and three hours after high water; and in the opposite direction, for three hours before and three hours after low water. But near the sides of the channel, and especially near the mouths of bays or estuaries branching from the channel, the change of tide follows a very peculiar law.

Rotation of the direction of the Tide currents.

The water is never stationary, as in river-tides, when changing from flow to ebb, but the direction of the current changes in 12 hours 20 minutes through all the points of the compass. As a general rule, supposing the observer's face turned in the direction which is analogous to the direction up a river; near the shores on his left hand the course of the tide-current revolves in the same direction as the hands of a watch, and near the shores on the right hand it revolves in the opposite direction. Near the headlands which separate different bays, there is usually, at certain times of the tide, a very rapid current, called a *race*.

Tides small in open seas

(10.) The elevations and depressions of tides in the open seas are much smaller than in contracted seas or rivers, sometimes not exceeding one or two feet; the stream of the tide is generally insensible.

and in small seas.

(11.) In seas of small extent (as the Mediterranean) the Tide is nearly insensible.

Anomalous Tides in certain localities.

(12.) In some circumstances, phænomena which are scarcely perceptible in ordinary localities become paramount. Thus, in some positions near Behring's Straits, the difference of morning and evening tides, which is scarcely sensible in England, becomes so great that, in certain parts of the lunation, there appears to be only one tide in the day. Other phænomena peculiar to these localities, but less obvious to ordinary observation, will be noticed hereafter.

Spring and Neap Tides known to the ancients.

(13.) The phænomena which we have described must necessarily, for the most part, have been remarked by all nations dwelling on the borders of the ocean. Thus Cæsar, in his account of the invasion of Britain, (*De Bello Gallico*, lib. iv.) alludes to the nature of spring tides as perfectly well understood in connection with the moon's age. Some of the peculiarities of river tides, however, were not published in scientific works till the beginning of the last century; and some of the properties of the tides in the English and other channels were not known till the end of that century. Upon the whole, the statement above may be supposed to represent pretty well all that was known of the Tides about the year 1800: and it will serve to point out to the reader the leading facts, to whose explanation a Theory of the Tides ought to be directed. In the present century, the elaborate discussions of immense collections of accurate tide-observations by M. Laplace, Sir John W. Lubbock, and Professor Whewell, have brought to light and reduced to law many irregularities which were before that time unknown. We prefer, however, delaying the particular mention of these until we have discussed the various forms in which theory has been put for the purpose of explaining the grand facts of the Tides.

Section II.—Equilibrium-Theory of Tides.

Inadequacy of all theories of the Tides, and the cause of it.

(14.) Before entering upon either of the theories explaining the Tides, we must allude to their inadequacy, perhaps not to the *explanation* of the facts already observed, but certainly to the *prediction* of new ones. This inadequacy does not appear to arise from any defect in the principles upon which the theory is based, (although perhaps our ignorance of the laws of friction among the particles of water, and between water and the sides of the channels which contain it, may be considered a failure of this kind,) but from the extreme difficulty of investigating mathematically the motions of fluids under all the various circumstances in which the waters of the sea and of rivers are found. For the problem of the Tides, it is evident, is essentially one of the *motion* of fluids. Yet so difficult are the investigations of motion that, till the time of Laplace, no good attempt was made to determine, by theory, the laws of the Tides, except on the supposition that the water was *at rest*. Since that time theories of *motion* have been applied; and it is hoped that in the present Treatise it will be found that something has been added to the preceding investigations of motion, possessing in some degree a practical character. Yet the theory, even in this state, reaches very few cases. Indeed, throughout the whole of this subject, the selection of the proper theoretical ground of explanation is a matter of judgment. In some cases we may conceive that we are justified in using the Equilibrium-theory; in others the Wave-theory will apply, completely or partially; in a few cases, the results of observation in one locality will be considered as a fundamental set of experiments, upon which the explanation of the phænomena in other localities will be grounded without further reduction to theory; and as a last resource, in almost every case, we shall be driven to the same arbitrary suppositions which Laplace introduced. Nevertheless, we conceive that our mathematical theory, pursued into some degree of detail, will be far from useless. In the instances which it does not master completely, it will show that there are ample grounds for the arbitrary alterations of constants introduced by Laplace in his suppositions, to which we have more than once alluded. It will show that we are precluded from further advance, partly by our almost necessary ignorance of the forms of the bottom in deep seas, and partly by the imperfection of our mathematics. It will leave no doubt whatever that the first principles of our explanation are correct. Begging the reader to receive the first part of this paragraph as an apology on the part of mathematicians for applying to the motion of Tides a theory so evidently inadequate as the Equilibrium-theory, we shall now proceed to give that theory, nearly in the terms of its proposers.

Popular explanation of equilibrium-theory.

(15.) The popular explanation of the Equilibrium-theory is very simple. If we conceive the earth to be covered wholly or in a great degree with water, and consider that the attraction of the moon upon different particles (according to the law of gravitation) is inversely as the square of their distance, and is therefore greatest for those particles which are nearest to it; then it will be obvious that the moon attracts the water on that side which is next to her, more than she attracts the great mass of the earth, and therefore tends to raise the water from the earth on the side next to her; but she also attracts the great mass of the earth more than she attracts the water upon the side most distant from her, and therefore tends to draw the earth from the water on the side most distant from her; which will produce exactly the same effect as if a force tended to draw the water away from the earth on that side. Thus the moon's action tends to raise the water on two opposite sides of the earth; and similarly the sun's action tends to raise the water on two opposite sides. The close relation, however, which the times of high water bear to the times of the moon's passage, shows that the moon's influence in raising the tides must be much greater than the sun's. If the sun and moon are together, as seen from the earth, the elevations produced by these two bodies will coincide in place, and will therefore be added together. Thus Spring Tides will be produced. In other relative positions of the sun and moon, it may happen that the elevation produced by the sun will occur at a place where the moon causes depression: the action of the sun there tends to counteract that of the moon, and Neap Tides will be produced.

Newton's first theory of the motion of the sea.

His modified theory of motion.

(16.) The theory of Newton is rather a collection of hints for a theory than any thing else. In the *Principia*, lib. I. prop. 66, cor. 19, he has (by a remarkable deduction from the Lunar Theory) considered the motion of water in a canal passing round the earth in or near to the earth's equator, and has arrived at the singular conclusion that the water would be lowest in that part which is most nearly under the body (the sun or moon) whose attraction causes the motion of the water. This conclusion we shall find to be entirely supported by more complete investigations. In lib. III. prop. 24, he has modified this conclusion, and seems to suppose that in free seas the high water ought to follow the moon's transit over the meridian (conceiving, for the moment, the moon's attraction to be the sole exciting cause of the Tides) in three hours, or at least in less than six hours. To this he appears to have been led by erroneous reasoning of the same kind as that which, in lib. I. prop. 66, cor. 20, has introduced an incorrect inference as to the Solar Nutation of the Earth's axis. We shall find hereafter that the introduction of friction into our theories of the motion of water will lead to a conclusion somewhat similar. The only part in which he uses numerical calculation is in lib. III. prop. 36, and 37, the subjects of which are, "*Invenire vim Solis ad Mare movendum,*" "*Invenire vim Lunæ ad Mare movendum.*" The following is his method of computation (the demonstration of the different parts of which we defer till we treat of the more complete theory of Bernoulli). First he refers to the Lunar Theory for a calculation of the force which the sun exerts to draw the moon, when in quadratures, towards the earth, and he finds it to be $\frac{1}{638092}$ part of gravity at the earth's surface. Then he remarks that the similar force upon the water at the earth's surface, in the position distant 90° of terrestrial arc from the point to which the sun is vertical, is less than the force upon the moon, in the proportion in which the water's distance from the centre of the earth is less than the moon's distance from the centre of the earth, or in the proportion of 1 : 60·5 : and therefore the force which depresses the water, at the points 90° distant from those vertically under the sun, is $\frac{1}{38604600}$ of gravity. Then he observes that, in the points which are under the sun and opposite to the sun, the disturbing force of the sun tends to raise the water, and is twice as great as the depressing force already found. He then considers that the same general effect will be produced if we put away the depressing force entirely, and augment the elevating force by the same quantity, and thus we may consider that the sole cause of the disturbance of the water is an elevating force, at the point under the sun and the point opposite to the sun, equal to $\frac{1}{12868200}$ of gravity; the elevating force in other points being proportional to the versed sine of double the sun's altitude above the horizon of any point. In order to compute the effect of this force in raising the water, he compares this force with the centrifugal force ($\frac{1}{289}$ of gravity) at the earth's equator, produced by the diurnal rotation of the earth; it is therefore $\frac{289}{12868200}$ of the centrifugal force at the equator. Then, having found from his theory of the Figure of the Earth (supposed homogeneous) that the centrifugal force would raise the fluids at the equator 85820 Paris feet, and supposing the proportion of the elevations produced by the tidal force and the centrifugal force to be the same as the proportion of those forces, he obtains this result, that the action of the sun would raise the water, in the parts immediately under it and opposite to it, by 1 foot $11\frac{1}{8}$ inches Paris measure, or a little more than 2 feet English. Of the various steps of this process we shall here observe only that, though indirect, they are correct; and that the result (on the supposition of the earth's being homogeneous, and without rotation) represents correctly the elevation which the sun's action would produce.

Newton's calculation of the force of the Sun.

Newton's calculation of the elevation of the water upon the theory of equilibrium, produced by the Sun's action.

Newton's calculation of the proportion of the Sun's force to the Moon's.

(17.) In order to ascertain the effect which the moon's action would produce, it is necessary to know the mass of the moon. For this there were in Newton's time no direct means: and he was, therefore, obliged to refer to the phænomena of the Tides themselves, as observed in places where, from local causes, the rise of the tide is very considerable. He quotes the observations of Sturmy on the tides in the Severn, at the mouth of the Avon, which give 45 feet for equinoctial spring tides, 25 feet for equinoctial neap tides: and those of Colepresse, on the tides at Plymouth, which give 16 feet for the mean height (intermediate between spring and neap) and 9 feet difference between springs and neaps. Preferring the proportion deduced from the former, he considers the height of equinoctial spring tides to be to that of equinoctial neap tides as 9 : 5. These tides (as will be seen hereafter) are in one case the effect of the moon augmented by the effect of the sun; and in the other case the effect of the moon diminished by that of the sun. If no correction were needed, we should infer at once that the power of the moon is to that of the sun as 7 : 2. But Newton remarks, that the greatest tides at Bristol do not happen till 43 hours after syzygies, "*ob aquarum reciprocos motus,*" meaning, probably, that the oscillations, like the oscillations of a pendulum, have a kind of inertia, which (on purely mechanical principles) prevents them from attaining their greatest magnitude till the force which causes them has past its greatest magnitude. This we shall find, when we treat of Waves, to be incorrect, except we take account of friction. Assuming this, however, Newton proceeds

Corrections introduced by Newton apparently erroneous.

to correct for the position of the luminaries *at the instant* of Bristol high tide: remarking that, as the sun is 18½ degrees from the moon at spring tides, and $90° \pm 18\frac{1}{2}°$ at neap tides, it is not the whole force of the sun which in one case increases and in the other case diminishes the moon's effect, but the whole force of the sun $\times \cos 37°$: and also that, as the moon's declination, 43 hours after an equinoctial syzygy, is about 22°, it is not the whole force of the moon that is concerned, but the whole force of the moon $\times \cos^2 22°$. These corrections appear to us inconsistent with what has gone before: for if the tides are increasing from the accumulated action of the sun and moon during a long time, it seems clearly inaccurate to correct the results of observation for the places of those bodies at the very instant of observation. Then he observes that the moon is not, at syzygies, at her mean distance. All corrections applied, he finds that the force of the moon is to that of the sun as 4·4815 to 1: and, therefore, as the sun's force would raise the water 1 foot, 11⅛ inches, the moon's force would raise it 8 feet, 8 inches. This, he remarks, is amply sufficient to account for all the motions of the tides.

(18.) The proportion of the moon's tidal force to the sun's tidal force is used by Newton (as a different value found in nearly the same manner has been used by Laplace) as the basis on which he calculates the moon's mass for application to other parts of the theory of gravitation. We shall see grounds hereafter for questioning the propriety of this calculation.

Merit of Newton's theory.

(19.) Assuming that Newton intended here (as he has done in several parts of Optics) only to exhibit, as far as he was able, grounds for a numerical calculation relating to the subject of Tides, but not bearing directly upon any of its specific phænomena, we must allow that (in spite of the apparent inconsistency of his corrections) it is a wonderful first attempt. That it had no further meaning will be sufficiently evident, not only from the proposition already cited, lib. I., prop. 66, cor. 19, but also from an examination of his 24th proposition of the third book, and the first corollary of his 27th proposition. In these he has treated the general explanation of the Tides as a matter of Wave-theory entirely, (though not without errors,) particularly in regard to the interference of semidiurnal tides, and in explaining the small rise and fall at some islands in the open sea by the oscillation of the whole mass of water between the bounding continents. As a philosopher, we conceive Newton to have shown himself here superior to his successors.

General problem of the equilibrium-theory.

(20.) In explaining the more complete equilibrium-theory, we shall not confine ourselves to the methods of Daniel Bernoulli, or any other writer, but shall present the theory in the form which appears most convenient. The problem which we shall conceive to be presented to us for solution is this: suppose the earth to be a spherical solid nucleus, either homogeneous, or consisting of a series of spherical concentric strata, (each stratum having the same density and the same thickness in its whole extent,) which nucleus is covered with water: and suppose the disturbing forces of the sun and moon to act upon the water: to find the shape which the water will assume.

Tides the same on a spherical earth as on a spheroid.

(21.) We have designedly used the word *spherical* for the form of the earth, because the investigation of the alteration produced in the form which, if undisturbed, would be spheroidal, would prove rather troublesome, and would lead to no result which we shall not obtain without it. As the earth's ellipticity is small, (the difference between its major axis and its minor axis being only about $\frac{1}{300}$ of either,) and as the whole elevation of the water, on the equilibrium-theory, is but a few feet, the reader will have no difficulty in comprehending that the tidal elevation of the water on the spheroid, though without doubt theoretically different from that on a sphere, will practically differ by a quantity which is quite insensible. In the same manner the reader will understand that, supposing the water to be disturbed by the action of the sun, and supposing the action of the moon to be then introduced, the additional disturbance which it will cause will be (as far as the senses can discover) the same as it would have caused if it had acted on water not disturbed by the action of the sun. And thus the whole disturbance which the two luminaries will produce upon the water surrounding a spheroidal nucleus will be found with sufficient accuracy by investigating the disturbance which each of them, separately considered, would produce in the water surrounding a spherical nucleus, and by adding those two disturbances together.

The tide produced by each of the attracting luminaries the same as if the other did not exist.

(22.) Our first effort will now be directed to the estimation of the disturbing force of the sun upon the water. We shall use the following notation:—

K, the mean density of the earth's spherical nucleus: R, its radius.

k, the density of the water: r, the radius of the external spherical surface of the water when undisturbed by the sun and moon.

(The density is supposed to be estimated by the acceleration which a cubical unit of matter acting by its attraction during a unit of time will produce in a body whose distance is the unit of distance: the velocity and acceleration being referred to the same units.)

E, the whole mass of the earth and water.

g, the numerical expression, referred to the same units, for the acceleration which gravity at the earth's surface causes in bodies falling freely.

x, y, z, the rectangular coordinates of any point in the fluid, the centre of the spherical nucleus being the origin, and z being parallel to the line joining the centres of the sun and the earth.

D, the sun's distance: D_m, the sun's mean distance: P, the sun's parallax: P_m, the sun's mean parallax: T, the periodic time of the earth's revolution round the sun, or the length of a sidereal year: S, the sun's mass, estimated by the acceleration which it will produce (in the same manner as for the density, above).

D′, the moon's distance: D'_m, the moon's mean distance: P′, the moon's parallax: P'_m, the moon's mean parallax: T′, the periodic time of the moon's revolution round the earth: M, the moon's mass.

Actual forces of the Sun upon any particle of the water.

(23.) The distance of the sun from the point whose co-ordinates are x, y, z, is $\sqrt{\{x^2+y^2+(D-z)^2\}}$, and the attraction of the sun upon that point, according to the law of gravitation, is $\frac{S}{x^2+y^2+(D-z)^2}$. This force is in the direction of the line drawn from the point in question to the sun. Our expression for this force supposes it to be estimated as an accelerating force; the statical pressure which corresponds to it may be resolved into three pressures in the directions of x, y, z; and, by the principle that accelerations of a given particle are proportional to the pressures which cause them, the accelerating forces which act in these directions may be deduced from the given accelerating force by the same laws of resolution as those for statical pressures. Thus we find for the resolved parts of the sun's accelerating force on the particle in question,

In the direction of x $$\frac{-Sx}{\{x^2+y^2+(D-z)^2\}^{\frac{3}{2}}}$$

In the direction of y $$\frac{-Sy}{\{x^2+y^2+(D-z)^2\}^{\frac{3}{2}}}$$

In the direction of z $$\frac{S(D-z)}{\{x^2+y^2+(D-z)^2\}^{\frac{3}{2}}}.$$

Expansions of the expressions.

(24.) Now the proportion of the earth's radius to the distance of the sun is extremely small; and the value of $\frac{x}{D}$, $\frac{y}{D}$, or $\frac{z}{D}$, is necessarily smaller. It will be allowable, therefore, to expand these expressions approximately, retaining no higher powers of x, y, z, than the second. (Indeed these latter terms are wholly insensible for the sun; and we retain them only because, in the expressions which we shall infer by analogy for the forces of the moon, they may be considered sensible.) With this restriction, observing that

$$\frac{1}{\{x^2+y^2+(D-z^2\}^{\frac{3}{2}}}=\frac{1}{D^3}\left\{1-\frac{2z}{D}+\frac{x^2+y^2+z^2}{D^2}\right\}^{-\frac{3}{2}}=\frac{1}{D^3}\left\{1+\frac{3z}{D}+\frac{12z^2-3x^2-3y^2}{2D^2}\right\},$$

we have

$$\text{Sun's force in the direction of } x = \frac{-Sx}{D^3}-\frac{3Sxz}{D^4}$$

$$\text{Sun's force in the direction of } y = \frac{-Sy}{D^3}-\frac{3Syz}{D^4}$$

$$\text{Sun's force in the direction of } z = \frac{S}{D^2}+\frac{2Sz}{D^3}+\frac{S(6z^2-3x^2-3y^2)}{2D^4}.$$

Disturbing forces of the Sun upon every particle.

(25.) These expressions represent the *whole* force of the sun upon any particle. But it is evident that, to find the force which disturbs the form of the water in reference to the position of the earth, we must not use the *whole* force of the sun upon any particle, but the *excess* of the sun's force on the particle *above* the sun's force on the centre of gravity of the earth. In order to find the sun's force on the centre of gravity of the earth, we must multiply each particle of the earth by the force which acts upon it; we must add together all these products, and we must divide the sum by the sum of all the particles of the earth. Now, using the expressions above, (which apply to the earth as well as to the water,) we may easily see that, if we multiply each particle of the earth by the force $\frac{-Sx}{D^3}$, and add all the products together, the sum will be 0, because for every particle which has a certain positive value of x there will be another particle having an equal negative value of x, and their products will, when added, destroy each other. The same remark applies to the terms depending on y, z, xz, and yz. But it does not apply to the term $\frac{S}{D^2}$ or to that depending on x^2, y^2, and z^2.

(26.) Now for the term $\frac{S}{D^2}$ we have only to remark that, upon multiplying it by each of the particles, adding all the products, and dividing the sum by the sum of the particles, we again obtain $\frac{S}{D^2}$. For the other terms we may proceed thus:—The sum of all the products of each particle by its value of z^2, throughout the sphere, will be the same as the sum of the products of each particle by its value of x^2 or y^2, because, supposing the sphere at one time divided by planes perpendicular to z, and at another time by planes perpendicular to x or y, the sections for similar values of x, y, or z, will be similar. The sum, therefore, for $6z^2$ will be equal to that for $3x^2+3y^2$, and, therefore, that for $6z^2-3x^2-3y^2$ will be 0. The only remaining term, therefore, for the sun's force on the centre of gravity of the earth, is $\frac{S}{D^2}$, in the direction of z.

(27.) Subtracting this term, therefore, from the force in the same direction upon the particle under consideration, we have the following expressions for the sun's disturbing force,

In the direction of x $$\frac{-Sx}{D^3}-\frac{3Sxz}{D^4}$$

In the direction of y $$\frac{-Sy}{D^3}-\frac{3Syz}{D^4}$$

In the direction of z $$+\frac{2Sz}{D^3}+\frac{S(6z^2-3x^2-3y^2)}{2D^4}$$

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The density of the fluid supposed insignificant. Mathematical condition which determines the form of the fluid when in equilibrium.

(28.) We shall now proceed to investigate the form which the water covering the solid nucleus will receive from the action of these forces in addition to the attraction of the nucleus and the mutual attraction of the particles of water. And first we may remark that, if the attraction of the particles of the water is insensible, (or if the density of the water is insensible in comparison with that of the nucleus,) the problem is very simple. Referring to our Treatise on the FIGURE OF THE EARTH, section 2, article 7., we find that the condition for the possibility of equilibrium of the water is that $Xdx+Ydy+Zdz$ shall be a complete differential, or, in more correct language, that it shall be possible for us to find some function U, such that $\frac{dU}{dx}=X$, $\frac{dU}{dy}=Y$, $\frac{dU}{dz}=Z$; X, Y, Z, being the whole forces in the directions of x, y, z. In article 9. of the same Treatise it is shown that, when the forces are produced by attraction to any number of particles, this condition is always satisfied; and, therefore, it is satisfied here (which will also be easily seen on substituting the expressions which we shall immediately exhibit). In article 8. of the same Treatise, it is proved that the form of the external surface will be determined by making $Xdx+Ydy+Zdz=0$, or $U=C$. To apply this now, we must add, to the expressions above, the resolved parts of the attraction of the nucleus. That attraction is the same as if all the matter of the nucleus were collected at its centre; and it is, therefore, $\frac{4\pi}{3}.\frac{R^3K}{x^2+y^2+z^2}$. The resolved parts in the directions of x, y, z, are respectively

$$-\frac{4\pi}{3}.\frac{R^3Kx}{(x^2+y^2+z^2)^{\frac{3}{2}}},\qquad -\frac{4\pi}{3}.\frac{R^3Ky}{(x^2+y^2+z^2)^{\frac{3}{2}}},\qquad -\frac{4\pi}{3}.\frac{R^3Kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

Hence the whole forces acting on any particle of the water are

$$\frac{dU}{dx}=X=-\frac{4\pi}{3}.\frac{R^3Kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{Sx}{D^3}-\frac{3Sxz}{D^4}$$

$$\frac{dU}{dy}=Y=-\frac{4\pi}{3}.\frac{R^3Ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{Sy}{D^3}-\frac{3Syz}{D^4}$$

$$\frac{dU}{dz}=Z=-\frac{4\pi}{3}.\frac{R^3Kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{2Sz}{D^3}+\frac{S(6z^2-3x^2-3y^2)}{2D^4}.$$

From these we easily find

$$U=\frac{4\pi}{3}.\frac{R^3K}{(x^2+y^2+z^2)^{\frac{1}{2}}}+\frac{S(2z^2-x^2-y^2)}{2D^3}+\frac{S(2z^3-3x^2z-3y^2z)}{2D^4}$$

and the equation to the external surface of the water will therefore be

Equation to the surface of the water.

$$C=\frac{4\pi}{3}.\frac{R^3K}{(x^2+y^2+z^2)^{\frac{1}{2}}}+\frac{S(2z^2-x^2-y^2)}{2D^3}+\frac{S(2z^3-3x^2z-3y^2z)}{2D^4}.$$

We may remark, that the very same equation would have been obtained if we had considered only the disturbing force which acts in the direction of a tangent to the Earth's surface. For, the equation which we have used for the external surface amounts to this, that the whole force is perpendicular to the external surface. Therefore the inclination of the surface of the water to the surface of the sphere will depend entirely on the proportion of the tangential force to the force directed towards the centre of the sphere. The only tangential force is the tangential disturbing force, which must therefore be retained; but the force directed to the centre of the sphere consists of the attraction of the sphere and the minute disturbing force; and it is indifferent, for the inclination of which we have spoken, whether we retain that minute portion or not. If we retain it, we consider all the forces; if we omit it, we use no disturbing force but that which is tangential. We shall see hereafter that a similar rule is true when we consider the forces producing the *motion* of the sea.

Expansion supposing the form to differ little from a sphere

(29.) Since the difference of the form from a spherical form will be exceedingly small, we may for $(x^2+y^2+z^2)^{\frac{1}{2}}$, which is the distance of any point at the surface from the sphere's centre, put $r+q$ (then q is the elevation of the water above the height which it would have had if undisturbed by the attraction of the sun): and in substituting this expression in the first term on the right hand side of the equation we may neglect the square of q; and in substituting it in the factors of the other terms, which are exceedingly small, we may omit q entirely. In these small terms, therefore,

$$2z^2-x^2-y^2=2z^2-(r^2-z^2)=3z^2-r^2,$$

$$z(2z^2-3x^2-3y^2)=z\{2z^2-3(r^2-z^2)\}=z(5z^2-3r^2);$$

and in the larger term

$$\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}=\frac{1}{r+q}=\frac{1}{r}-\frac{q}{r^2}.$$

Substituting, the equation becomes

$$C=\frac{4\pi}{3}R^3K\left(\frac{1}{r}-\frac{q}{r^2}\right)+\frac{S(3z^2-r^2)}{2D^3}+\frac{Sz(5z^2-3r^2)}{2D^4},$$

or

$$q=r-\frac{3r^2C}{4\pi R^3K}+\frac{3r^2}{4\pi R^3K}\left\{\frac{S(3z^2-r^2)}{2D^3}+\frac{Sz(5z^2-3r^2)}{2D^4}\right\};$$

which will be more conveniently written

$$q=C'+\frac{3r^2}{4\pi R^3K}\left\{\frac{S(3z^2-r^2)}{2D^3}+\frac{Sz(5z^2-3r^2)}{2D^4}\right\}.$$

(30.) In order to determine the value of C', we must observe that the whole volume included within the external surface of the water is equal to the sphere whose radius is r, and therefore if we estimate the sum of all the quantities of water which are raised above the surface of the sphere, (depressions below the surface being considered negative,) that sum will $=0$. Now conceive that there is traced upon the surface of the sphere a series of circles at small distances, resembling the parallels of a terrestrial globe, the poles of all the circles being at the point which is nearest to the sun, or z having the same value through the whole circumference of each circle. Let θ be the angle made by the axis of z with the line drawn from the centre of the sphere to any point of one of these circles, $\theta+\delta\theta$ the similar angle for the next circle. The surface of the sphere included between these two circles will $=2\pi.r\sin\theta.r\delta\theta$ nearly, and therefore the volume of water elevated above that ring upon the sphere $=2\pi.q.r^2\sin\theta.\delta\theta$ nearly; which, since $z=r\cos\theta$, is $=-2\pi qr.\delta z$ nearly. We have, therefore, to find the sum of all the values of

$$-2\pi C'r\,\delta z-\frac{3r^3}{2R^3K}\left\{\frac{S(3z^2-r^2)}{2D^3}+\frac{S(5z^3-3rz)}{2D^4}\right\}\delta z,$$

or to find

$$-\int_z 2\pi C'r-\frac{3r^3}{2R^3K}\int_z\left\{\frac{S(3z^2-r^2)}{2D^3}+\frac{S(5z^3-3r^2z)}{2D^4}\right\}$$

through the whole extent of the sphere; that is, between the limits $z=-r$, $z=+r$. The value of this integral is $-4\pi C'r^2$. Making this $=0$, we have $C'=0$, and therefore

Expression for the elevation of the water.

$$q=\frac{3r^2}{4\pi R^3K}\left\{\frac{S(3z^2-r^2)}{2D^3}+\frac{Sz(5z^2-3r^2)}{2D^4}\right\},$$

or

$$=\frac{3r^4}{4\pi R^3K}\left\{\frac{S}{2D^3}(3\cos^2\theta-1)+\frac{r}{D}.\frac{S}{2D^3}.\cos\theta(5\cos^2\theta-3)\right\}.$$

(31.) In order to put this expression into a form adapted to numerical computation, we must deliver it from the quantities K and S. First, to remove K: since $\frac{4\pi R^3}{3}$ is the volume of the nucleus, $\frac{4\pi R^3K}{3}$ is its mass, expressed by the acceleration which it would cause at distance 1, as we have assumed in (22.); and, therefore, $\frac{4\pi R^3K}{3r^2}$ is the acceleration which it would cause at the surface of the water; but this acceleration, being that of ordinary falling bodies, is expressed by g: therefore $\frac{3r^4}{4\pi R^3K}=\frac{r^2}{g}$. Secondly, to remove S. In our treatise on Physical Astronomy, page 655, equation (24.), it is shown that $T=\frac{2\pi.a^{\frac{3}{2}}}{\sqrt{(M+m)}}$; which will be expressed in the notation of this Treatise if we remark that a is the mean distance of the revolving body $=D_m$; and that $M+m$ is the sum of the masses of the two bodies, which, as the earth is very small in comparison with the Sun, will not sensibly differ from S. T is the periodic time $=1$ year. Thus $T=\frac{2\pi.D_m^{\frac{3}{2}}}{\sqrt{S}}$, or $\frac{S}{D_m^3}=\frac{4\pi^2}{T^2}$; and therefore, $\frac{S}{D^3}=\left(\frac{D_m}{D}\right)^3.\frac{4\pi^2}{T^2}$. Also $\frac{r}{D}=\sin P$. Thus the expression for q becomes

Elevation expressed in a form fit for calculation.

$$\frac{2\pi^2r^2}{T^2g}.\left(\frac{D_m}{D}\right)^3.\{3\cos^2\theta-1+\sin P.\cos\theta.(5\cos^2\theta-3)\}.$$

This expression is in a form entirely fit for calculation; it is only necessary to remark that the same units must be used throughout: thus, if g expresses the acceleration in inches produced by gravity in one second of time, r must be expressed in inches, and T in seconds of time. To avoid the introduction of very large numbers, we may make use of the elements of the moon's motion. The same equation of Physical Astronomy (neglecting the perturbations of the moon) gives us $T'=\frac{2\pi.D'^{\frac{3}{2}}_m}{\sqrt{\left(\frac{4\pi R^3K}{3}+M\right)}}$. Let the moon's mass be $\frac{1}{n}$ part of the

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earth's, or $M=\frac{1}{n}.\frac{4\pi R^3 K}{3}$: then $T'=\frac{2\pi . {D'_m}^{\frac{3}{2}}}{\sqrt{\frac{4\pi R^3 K}{3}}\sqrt{\left(1+\frac{1}{n}\right)}}$; from which $\frac{3r^4}{4\pi R^3 K}=\frac{T'^2\left(1+\frac{1}{n}\right)r^4}{4\pi^2 {D'_m}^{\frac{3}{2}}}$,

and

$$q=\frac{T'^2}{2T^2}.\left(1+\frac{1}{n}\right).\frac{r^3}{{D'_m}^{\frac{3}{2}}}\,r.\left(\frac{D_m}{D}\right)^3.\{3\cos^2\theta-1+\sin P.\cos\theta\,(5\cos^2\theta-3)\}$$

$$=\frac{n+1}{2n}.\left(\frac{T'}{T}\right)^2.(\sin P'_m)^3.r.\left(\frac{D_m}{D}\right)^3.\{3\cos^2\theta-1+\sin P.\cos\theta\,(5\cos^2\theta-3)\}.$$

If we suppose the moon's mass $\frac{1}{80}$ of the earth, and give the values commonly adopted for the other quantities, this becomes

$$\frac{81}{160}.\left(\frac{27\cdot32}{365\cdot25}\right)^2.(\sin 57'.1'')^3\times 20900000\times\left(\frac{D_m}{D}\right)^3.\{3\cos^2\theta-1+\sin 8''\cdot7.\cos\theta\,(5\cos^2\theta-3)\},$$

where the earth's mean semidiameter is expressed in English feet. Performing the numerical computation upon either of these expressions, the formula becomes

Calculation of the elevation of the water.

$$q=0\cdot2710 \text{ foot} \times\left(\frac{D_m}{D}\right)^3.\{3\cos^2\theta-1+\sin 8''\cdot7.\cos\theta\,(5\cos^2\theta-3)\},$$

or

$$=0\cdot2710 \text{ foot} \times\left(\frac{D_m}{D}\right)^3\times(3\cos^2\theta-1),$$

the last part of the preceding formula being quite insensible.

(32.) Omitting for the present the consideration of the factor $\left(\frac{D_m}{D}\right)^3$, which never differs much from 1, we find that the greatest positive value of the formula (denoting elevation of the water) occurs when $\theta=0$, or $\theta=180°$, for which cases $3\cos^2\theta-1=2$, and the elevation of the water $=0\cdot542$ foot. Now θ is the angle contained between the line drawn from the earth's centre to the sun and the line drawn from the earth's centre to any point on the surface which is under consideration. Consequently, the value $\theta=0$ belongs to that point of the earth's surface, or of the surface of the water, which is immediately under the sun; and the value $\theta=180°$ belongs to that point of the surface of the water which is farthest from the sun. The sun's action, therefore, would raise the water 0·542 foot on that side which is next to the sun, and also on that side which is farthest from the sun.

Greatest elevation.

(33.) The greatest negative value of the formula (denoting depression of the water) occurs when $\theta=90°$, for which case $3\cos^2\theta-1=-1$, and the depression of the water is 0·2710 foot. Now θ is $=90°$ for all those parts of the surface of the water which are determined by making a plane to pass through the earth's centre perpendicular to the line joining the earth's centre with the sun. The sun's action, therefore, would depress the water 0·2710 foot in the zone, surrounding the earth, which is intermediate between the point under the sun and the point that is farthest from the sun.

Greatest depression.

(34.) It appears, therefore, that the elevation of the water produced by the sun on one part of the earth, where the elevation is greatest, is double of the depression produced on other parts where the depression is greatest. Suppose now that the water always assumes the form which we have found, and that the earth revolves within the coating of water. (This supposition, absurd as it is, is the only one upon which it is possible to apply the equilibrium-theory.) And suppose an observer to be stationed upon a small island projecting above the water, and to watch there the rise and fall of the surface of the water. To fix our ideas, suppose the earth's axis of revolution to be perpendicular to the line joining the sun and the earth, and suppose the observer to be at the earth's equator. Then, in the course of a revolution, he will be carried successively through the point which is nearest to the sun, through the zone intermediate between the point nearest to the sun and the point most distant from it, through the point most distant from the sun, again through the intermediate zone, and to the point which is nearest to the sun. He will, therefore, have been carried twice through the part where the elevation is greatest, and twice through the part where the depression is greatest. The greatest elevation, as we have found, is double of the greatest depression. From this circumstance many persons have imagined that, in all tides, under all local circumstances whatever, the line of mean water, or the line at which the surface of the water would stand if undisturbed by tidal action, is to be found by taking a line whose height above low water is one-third of the height of high water above low water; so that the elevation of high water above that line will be double the depression of low water below it. This (as we shall afterwards show) is inconsistent with the laws of Waves on deep water, upon which, without doubt, the phænomena of Tides depend entirely: but, moreover, it is inconsistent with the equilibrium-theory itself. For, to ascertain the *mean* height of the water, we must not suppose the sun and moon actually removed from

Elevation apparently double the depression.

Elevation above mean really equal to the depression below mean.

our system, but we must suppose an estimation to be made of the form which the water would assume under the action of their *mean* forces; and, considering this as the mean state of the water, we must compare the disturbed state with this, in order to ascertain the true value of the disturbance from mean state. Now the mean elevation of the water at our imaginary island, under the sun's mean action, will evidently be found by supposing the earth to turn uniformly round, by taking the actual elevation of the water in each momentary position, and by taking the mean of all those actual elevations. Putting A for the coefficient 0·2710 foot $\times\left(\frac{\mathrm{D}_m}{\mathrm{D}}\right)^3$, the elevation of the water, above the position which it would have had if no sun existed, is $\mathrm{A}\,(3\cos^2\theta-1)$: and if the sphere turns uniformly round, θ will pass uniformly through all the values from 0 to 2π: we have, therefore, to find the mean of all the values of $\mathrm{A}\,(3\cos^2\theta-1)$ when θ increases uniformly from 0 to 2π. Putting it under the form $\mathrm{A}\left\{\frac{3}{2}\,(1+\cos 2\theta)-1\right\}$, or $\mathrm{A}\left\{\frac{1}{2}+\frac{3}{2}\cos 2\theta\right\}$, and remarking that, while θ increases from 0 to 2π, $\cos 2\theta$ goes through equal positive and negative values, the mean is $\frac{\mathrm{A}}{2}$. And this is the elevation of the water under the action of the sun's mean force. Subtracting this from the actual elevation, or $\mathrm{A}\,(3\cos^2\theta-1)$, we have for the effect of the periodic tidal force $\mathrm{A}\left(3\cos^2\theta-\frac{3}{2}\right)$. The greatest tidal elevation is found by making $\theta=0$, or $180°$, or $\cos^2\theta=1$, and is therefore $\mathrm{A}\times\frac{3}{2}$: the greatest tidal depression is found by making $\theta=90°$, or $\cos\theta=0$, which gives tidal elevation $=-\mathrm{A}\times\frac{3}{2}$, or tidal depression $=\mathrm{A}\times\frac{3}{2}$. The greatest tidal elevation, and the greatest tidal depression, are therefore equal, even on the equilibrium-theory.

Elevation produced by the moon.

(35.) We shall now proceed to investigate the effect of the moon upon the water, still supposing the density of the water to be insignificant in comparison with that of the earth. The expression of (30.), *mutatis mutandis*, will apply to the elevation produced by the moon: supposing here that θ' is the angle between the line drawn from the earth's centre to the point under consideration, and the line drawn from the earth's centre to the moon. Putting q' then for the elevation produced by the moon,

$$q'=\frac{3r^4}{4\pi\,\mathrm{R}^3\,\mathrm{K}}\left\{\frac{\mathrm{M}}{2\,\mathrm{D}'^3}(3\cos^2\theta'-1)+\frac{r}{\mathrm{D}'}\cdot\frac{\mathrm{M}}{2\,\mathrm{D}'^3}\,.\cos\theta\ (5\cos^2\theta'-3)\right\}$$

To reduce this to calculation we must remark that $\frac{4\pi}{3}\,\mathrm{R}^3\,\mathrm{K}=$earth's mass$=n\times\mathrm{M}$; and therefore

$$\frac{3r^4}{4\pi\,\mathrm{R}^3\,\mathrm{K}}\times\frac{\mathrm{M}}{2\,\mathrm{D}'^3}=\frac{r^4}{2n\,\mathrm{D}'^3}=\frac{1}{2n}\cdot r\,.\left(\frac{r}{\mathrm{D}'_m}\right)^3\cdot\left(\frac{\mathrm{D}'_m}{\mathrm{D}'}\right)^3=\frac{1}{2n}.r.\left(\sin 57'.\ 1''\right)^3\cdot\left(\frac{\mathrm{D}'_m}{\mathrm{D}'}\right)^3.$$

The numerical value of this depends entirely upon the value of n, or the proportion of the earth's mass to the moon's. If (as we supposed before) $n=80$, and $r=20900000$ feet, the expression becomes

$$q'=\frac{20900000}{160}\left(\sin 57'.\ 1''\right)^3\cdot\left(\frac{\mathrm{D}'_m}{\mathrm{D}'}\right)^3\left\{3\cos^2\theta'-1+(\sin 57'.1'')\,\frac{\mathrm{D}'_m}{\mathrm{D}'}\cos\theta'.\ (5\cos^2\theta'-3)\right\}$$

$$=0\cdot5959\text{ foot}\times\left(\frac{\mathrm{D}'_m}{\mathrm{D}'}\right)^3\times\left(3\cos^2\theta'-1\right)+0\cdot0100\text{ foot}\times\left(\frac{\mathrm{D}'_m}{\mathrm{D}'}\right)^4\times\cos\theta'.(5\cos^2\theta'-3).$$

If we supposed the moon's mass to be $\frac{1}{40}$ of the earth, (which is very nearly the supposition of Newton,) the numerical coefficients would be respectively 1·1918 foot and 0·0200 foot. The phænomena of nutation (using the numbers in the FIGURE OF THE EARTH, page 235, and supposing $\frac{a}{b}=\frac{50\cdot3}{9\cdot25}$) give $n=82$.

Elevation a little greater on the side next to the moon than on the opposite side.

Whichever of these values of n, or of any intermediate to these two, we decide on adopting, it is clear that the last term of the formula is insignificant; its greatest numerical values, when $n=80$, being ± 0·02 foot. We shall therefore omit the consideration of it in future. We may remark that, theoretically considered, its meaning is that the water is raised a little higher on the side next to the moon than on the side most distant from the moon. For when $\theta'=0$, $\cos\theta'$ is positive, and the term $\cos\theta'\,(5\cos^2\theta'-3)$ is numerically added to $3\cos^2\theta'-1$, which is also positive: but when $\theta'=180°$, $\cos\theta'$ is negative, and the term $\cos\theta'\,(5\cos^2\theta'-3)$ is numerically subtracted from $3\cos^2\theta'-1$, which is positive. In the intermediate zone, $\cos\theta'=0$, and the height of the water is not altered by this term.

Investigation supposing the density of the water not insignificant

(36.) We shall defer the examination of the effects of the variations of $\frac{\mathrm{D}'_m}{\mathrm{D}'}$, of the composition of the effects of the sun and moon, and of their positions in declination, &c., and shall proceed to investigate the elevation of the water on the supposition that the density of the water is not insignificant in comparison with that of the earth. The modification which this produces in the theory is, that the attraction of the water upon its own particles must be found. The problem now becomes very similar to that of the Figure of the Earth, and we shall solve it in the same way, namely, synthetically, by supposing the form to be spheroidal, and showing that, with proper proportion of the axes, the equations of equilibrium will be satisfied. For the developments of the formulæ requisite, we shall refer, for the most part, to the article on the FIGURE OF THE EARTH.

Method of conducting this investigation.

(37.) We shall assume, then, that the form of the external surface of the water is that of a prolate spheroid, its axis being directed to the sun (omitting, for the present, the forces of the moon). And we shall consider that the forces which act on the water are, the attraction of the spherical nucleus, the attraction of the water, whose interior boundary is spherical and whose exterior boundary is spheroidal, and the disturbing forces of the sun found in (27.). As the attractions of the particles of water, as well as all other attractions, satisfy the condition $X=\frac{dU}{dx}$, $Y=\frac{dU}{dy}$, $Z=\frac{dU}{dz}$, it will only be necessary for us to determine the forces which act on the particles at the surface, to find from them the expression for U, to form the equation $U=C$ for the exterior surface, and to try whether this equation can be made to coincide with the assumed spheroidal equation.

Attraction of an entire spheroid of water.

(38.) The attraction of the water upon a point at its surface will be found by subtracting the attraction of a bulk equal to the spherical nucleus from the attraction of the spheroid of water supposed entire. Let the semiaxis of the spheroid in the direction of z, or towards the sun, be b; the semiaxes at right angles to this (or in the plane of x, y,) $b\,(1-\zeta)$. Comparing this with the assumption in the FIGURE OF THE EARTH, sec. ii. art. 30., it is evident that the formulæ of article 32. of that treatise will apply here, if we put $-\zeta$ for ε, and omit the terms depending on centrifugal force. Thus we have, for the attraction of the watery spheroid supposed entire,

In the direction of x, $\quad -\frac{4\pi}{3}k\left(1+\frac{2}{5}\zeta\right)x$

In the direction of y, $\quad -\frac{4\pi}{3}k\left(1+\frac{2}{5}\zeta\right)y$

In the direction of z, $\quad -\frac{4\pi}{3}k\left(1-\frac{4}{5}\zeta\right)z.$

The attraction of a sphere of the same density whose radius is R would be

In the direction of x, $\quad -\frac{4\pi}{3}k\,R^3\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

In the direction of y, $\quad -\frac{4\pi}{3}k\,R^3\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

In the direction of z, $\quad -\frac{4\pi}{3}k\,R^3\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$

Attraction of the water covering the spherical nucleus.

Subtracting these from the former we obtain for the real attractions of the water,

In the direction of x, $\quad -\frac{4\pi}{3}k\left(1+\frac{2}{5}\zeta\right)x+\frac{4\pi}{3}k\,R^3\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

In the direction of y, $\quad -\frac{4\pi}{3}k\left(1+\frac{2}{5}\zeta\right)y+\frac{4\pi}{3}k\,R^3\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

In the direction of z, $\quad -\frac{4\pi}{3}k\left(1-\frac{4}{5}\zeta\right)z+\frac{4\pi}{3}k\,R^3\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$

Attraction of the nucleus of earth.

(39.) The attractions of the spherical nucleus of earth will be

In the direction of x, $\quad -\frac{4\pi}{3}KR^3\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

In the direction of y, $\quad -\frac{4\pi}{3}KR^3\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

In the direction of z, $\quad -\frac{4\pi}{3}KR^3\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$

Whole force which acts on the water.

Combining these with the attractions of the water found above, and with the sun's disturbing forces found in (27.), omitting the last terms of the latter, we have

$$X=-\frac{4\pi}{3}k\left(1+\frac{2}{5}\zeta\right)x-\frac{4\pi}{3}(K-k)\,R^3\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{Sx}{D^3}$$

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$$Y=-\frac{4\pi}{3}k\left(1+\frac{2}{5}\zeta\right)y-\frac{4\pi}{3}(K-k)\,R^3\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{Sy}{D^3}$$

$$Z=-\frac{4\pi}{3}k\left(1-\frac{4}{5}\zeta\right)z-\frac{4\pi}{3}(K-k)\,R^3\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{2Sz}{D^3}.$$

(40.) The equation U=C becomes therefore

$$-\frac{4\pi}{3}k\left\{\frac{x^2+y^2+z^2}{2}+\zeta\,.\frac{x^2+y^2-2z^2}{5}\right\}+\frac{4\pi}{3}(K-k)\frac{R^3}{(x^2+y^2+z^2)^{\frac{1}{2}}}+\frac{S}{2D^3}(2z^2-x^2-y^2)=C$$

Equation to the surface deduced from the forces.

If we make $(x^2+y^2+z^2)^{\frac{1}{2}}=r+q$, and retain the first power of q, only in those terms which are not multiplied by ζ or $\frac{S}{2D^3}$, we obtain

$$-\frac{4\pi}{3}k\left\{\frac{r^2}{2}+rq+\zeta\frac{r^2-3z^2}{5}\right\}+\frac{4\pi}{3}\frac{(K-k)\,R^3}{r}-\frac{4\pi}{3}\frac{(K-k)\,R^3}{r^2}q+\frac{S}{2D^3}(3z^2-r^2)=C$$

or

$$q\times\frac{4\pi}{3}\left\{kr+\frac{(K-k)\,R^3}{r^2}\right\}=-\frac{4\pi}{3}k\frac{r^2}{2}+\frac{4\pi}{3}.\frac{(K-k)\,R^3}{r}-C+\left\{\frac{4\pi k}{15}\zeta+\frac{S}{2D^3}\right\}(3z^2-r^2).$$

The equation to the prolate spheroid is $\frac{x^2+y^2}{b^2(1-\zeta)^2}+\frac{z^2}{b^2}=1$, or $x^2+y^2+2\zeta(x^2+y^2)+z^2=b^2$. And the point now to be determined is, whether this equation can be made to coincide with that above. Putting $r+q$ for $(x^2+y^2+z^2)^{\frac{1}{2}}$, and omitting q in the term multiplied by ζ, this equation becomes

$$r^2+2r.q=b^2-2\zeta(x^2+y^2)=b^2+2\zeta(z^2-r^2)=b^2+\frac{2}{3}\zeta(3z^2-3r^2)=b^2-\frac{4}{3}\zeta r^2+\frac{2}{3}\zeta(3z^2-r^2),$$

or

$$2rq=b^2-\left(1+\frac{4}{3}\zeta\right)r^2+\frac{2}{3}\zeta(3z^2-r^2):$$

and therefore in the prolate spheroid

$$q\times\frac{4\pi}{3}\left\{kr+\frac{(K-k)\,R^3}{r^2}\right\}=\frac{2\pi}{3}\left\{k+\frac{(K-k)\,R^3}{r^3}\right\}\times\left\{b^2-(1+\frac{4}{3}\zeta)\,r^2\right\}+\frac{2\pi}{3}\left\{k+\frac{(K-k)\,R^3}{r^3}\right\}.\frac{2\zeta}{3}.(3z^2-r^2).$$

Spheroidal form found to be possible.

This expression is evidently of the same form as that found above from the condition of equilibrium: and it will exactly coincide with it if the two terms on the right hand side of one are respectively equal to the two terms on the right hand side of the other. The comparison of the first terms gives the value of C; the comparison of the second terms gives the equation

$$\frac{4\pi k}{15}\zeta+\frac{S}{2D^3}=\frac{4\pi}{9}\left\{k+\frac{(K-k)\,R^3}{r^3}\right\}\zeta.$$

From the latter,

$$\frac{S}{2D^3}=\zeta\times\left\{\frac{4\pi k}{9}-\frac{4\pi k}{15}+\frac{4\pi}{9}.\frac{(K-k)\,R^3}{r^3}\right\}=\frac{4\pi}{3}\zeta.\left\{\frac{2k}{15}+\frac{(K-k)\,R^3}{3r^3}\right\}$$

and

$$\zeta=\frac{S}{2D^3}.\frac{1}{\frac{4\pi}{3}.\left\{\frac{2k}{15}+\frac{(K-k)\,R^3}{3r^3}\right\}}.$$

Ellipticity determined

The spheroidal form is therefore a possible form of equilibrium, and the proportion of the axes must be 1 : 1 − ζ, where ζ has the value just found.

(41.) The value of q, found above from the properties of the spheroid, is $\frac{\zeta}{3r}(3z^2-r^2)+$ constant; which constant will be found, by the reasoning of (30.), to be $=0$; and therefore $q=\frac{\zeta}{3r}(3z^2-r^2)$. Substituting the value of ζ,

$$q=\frac{S}{2D^3}\cdot\frac{15r^2}{4\pi\{2kr^3+5(K-k)\,R^3\}}\cdot(3z^2-r^2)$$

$$=\frac{15r^4}{4\pi\{2kr^3+5(K-k)\,R^3\}}\cdot\frac{S}{2D^3}\cdot(3\cos^2\theta-1).$$

To reduce this to a form fit for calculation, we may, as before, put $\frac{S}{D^3}=\left(\frac{D_m}{D}\right)^3\cdot\frac{4\pi^2}{T^2}$: to remove K, we must remark that the attraction is nearly the same as if the whole mass were united at the centre of the sphere; but the whole mass (omitting small quantities) is $\frac{4\pi}{3}\left\{(r^3-R^3)\,k+R^3K\right\}$: its attraction on a point at the surface of the water is therefore $\frac{4\pi}{3}\cdot\left\{kr+(K-k)\frac{R^3}{r^2}\right\}$, which (as before) must be made$=g$: therefore $K=\frac{3g}{4\pi\left\{\frac{k}{K}r+\left(1-\frac{k}{K}\right)\frac{R^3}{r^2}\right\}}$, and the expression for q becomes

$$\frac{10\,\pi^2r^2}{T^2g}\cdot\left(\frac{D_m}{D}\right)^3\cdot\frac{\frac{k}{K}r^3+\left(1-\frac{k}{K}\right)R^3}{2\frac{k}{K}r^3+5\left(1-\frac{k}{K}\right)R^3}\,(3\cos^2\theta-1).$$

If the depth of fluid covering the solid nucleus be small, r may be considered $=R$, and then

Expression for elevation in the spheroid.

$$q=\frac{10\,\pi^2r^2}{T^2g}\left(\frac{D_m}{D}\right)^3\,\frac{1}{5-\frac{3k}{K}}\,(3\cos^2\theta-1).$$

(42.) When the density of the fluid is insignificant, $\frac{k}{K}=0$, and the expression for q becomes

$$\frac{2\pi^2r^2}{T^2g}\left(\frac{D_m}{D}\right)^3\cdot(3\cos^2\theta-1)$$

as we have found before. When the density of the fluid is equal to that of the solid nucleus, $\frac{k}{K}=1$, and

$$q=\frac{5\pi^2r^2}{T^2g}\left(\frac{D_m}{D}\right)^3\cdot(3\cos^2\theta-1)$$

or it is equal to the former result $\times\frac{5}{2}$. This remarkable difference is produced entirely by the attraction of the clevated portions of water and the diminution of attraction where the water is depressed. Converted into numbers, this value becomes 0·6775 foot $\times\left(\frac{D_m}{D}\right)^3\cdot(3\cos^2\theta-1)$: and the whole difference between the elevations of high and low water is 2·0325 feet. This result nearly coincides with Newton's, as it ought. The circumstances of the earth and water will be represented most nearly by taking $\frac{k}{K}=\frac{2}{11}$. The same results as to the proportion of the elevations found on different assumptions of the proportion of the densities of earth and water, which we have found for the effect of the sun, apply in all respects to the effect of the moon.

(43.) We shall now consider the effect of the combination of the tides produced by the Sun and the Moon; the place of observation being at any point on the earth's surface, which is supposed to be carried round by the earth's daily rotation, while the form of the water continues the same in respect of the positions of the Sun and Moon; and the positions of the Sun and Moon being any whatever. For simplicity of notation, we shall put S′ for the factor of $\left(\frac{D_m}{D}\right)^3\cdot(3\cos^2\theta-1)$ in the expression for the Solar Tide, and M′ for the factor of $\left(\frac{D'_m}{D'}\right)^3\cdot(3\cos^2\theta'-1)$ in the expression for the Lunar Tide; also we shall put $\frac{P}{P_m}$ for $\frac{D_m}{D}$, and $\frac{P'}{P'_m}$ for $\frac{D'_m}{D'}$, to which they are almost exactly equal.

(44.) In figure 1, let p be the pole of the earth, t the place at which the tide is observed, u the point to which the moon is vertical, v the point to which the sun is vertical. Then the elevation of the water at t is

Combination of Solar and Lunar Tides.

$$M'\left(\frac{P'}{P'_m}\right)^3\cdot(3\cos^2 tu-1)+S'\left(\frac{P}{P_m}\right)^3\cdot(3\cos^2 tv-1).$$

Draw a meridian pw fixed in space, and draw meridians through t, u, v; let s, m be the celestial right ascensions of the sun and moon, and l the terrestrial longitude of the place, as measured from that fixed meridian; σ, μ the celestial declinations of the sun and moon, and λ the terrestrial latitude of the place. Then $\cos tu=\sin\lambda.\sin\mu+\cos\lambda.\cos\mu.\cos\overline{l-m}$: $\cos tv=\sin\lambda.\sin\sigma+\cos\lambda.\cos\sigma.\cos\overline{l-s}$; and the elevation of the water is

$$M'\left(\frac{P'}{P'_m}\right)^3\cdot\left\{(1-3\sin^2\lambda).\left(\frac{3}{2}\cos^2\mu-1\right)+\frac{3}{2}\sin 2\lambda.\sin 2\mu.\cos\overline{l-m}+\frac{3}{2}\cos^2\lambda.\cos^2\mu.\cos 2.\overline{l-m}\right\}$$
$$+S'\left(\frac{P}{P_m}\right)^3\cdot\left\{(1-3\sin^2\lambda).\left(\frac{3}{2}\cos^2\sigma-1\right)+\frac{3}{2}\sin 2\lambda.\sin 2\sigma.\cos\overline{l-s}+\frac{3}{2}\cos^2\lambda.\cos^2\sigma.\cos 2.\overline{l-s}\right\}$$

of which the different parts may be more advantageously combined in the following form:

$$(1-3\sin^2\lambda).\left\{M'\left(\frac{P'}{P'_m}\right)^3\cdot\left(\frac{3}{2}\cos^2\mu-1\right)+S'\left(\frac{P}{P_m}\right)^3\cdot\left(\frac{3}{2}\cos^2\sigma-1\right)\right\}$$
$$+\frac{3}{2}\sin 2\lambda.\left\{M'\left(\frac{P'}{P'_m}\right)^3\cdot\sin 2\mu.\cos\overline{l-m}+S'\left(\frac{P}{P_m}\right)^3\cdot\sin 2\sigma.\cos\overline{l-s}\right\}$$
$$+\frac{3}{2}\cos^2\lambda.\left\{M'\left(\frac{P'}{P'_m}\right)^3\cdot\cos^2\mu.\cos 2.\overline{l-m}+S'\left(\frac{P}{P_m}\right)^3\cdot\cos^2\sigma.\cos 2.\overline{l-s}\right\}.$$

We shall consider the different terms of this expression in order.

Tides of long period.

(45.) The first line does not depend upon l, m, or s, and is therefore independent of the right ascensions of the sun and moon or the hour of the day. It does not therefore represent a *Tide* in the ordinary sense of the word. Nevertheless it depends upon μ and σ, and therefore will vary with the variation of the declination of the sun and moon. But it is indifferent whether the declination is north or south. As the moon's declination goes through all its changes in respect of magnitude (without regard of sign) in half a lunation, the term depending on M′ will produce a slow tide, going through all its changes in 14 days nearly. There will also be a slow tide produced by the term depending on S′, going through its changes in half a year. These Tides do not exist for any place at which the sine of latitude $=\sqrt{\frac{1}{3}}$: near the equator, the water is high (as depending on this cause) when the moon and sun are in the equator; near the earth's pole, it is low at the same time; the greatest change of surface at the equator, supposing the sine of the greatest declination to be $\frac{2}{5}$, will be $\frac{6}{25}\left\{M'\left(\frac{P'}{P'_m}\right)+S'\left(\frac{P}{P_m}\right)^3\right\}$ nearly; the greatest change at the pole will be $\frac{12}{25}\left\{M'\left(\frac{P'}{P'_m}\right)^3+S'\left(\frac{P}{P_m}\right)^3\right\}$.

Diurnal Tides.

(46.) The second line consists of two parts, each of which is a multiple of the cosine of $l-m$ or $l-s$. This then represents two tides; of which one goes through all its changes, while the distance of the meridian passing through the place from the meridian passing through the moon (or the moon's hour-angle) changes by 360°; and the other goes through all its changes while the sun's hour-angle changes by 360°. Each of these then produces a *diurnal* tide; and their combined effect will be represented by a single *diurnal* tide of varying extent, and which follows the moon's transit at a variable interval. For, the line may be put in this form:

$$+\frac{3}{2}\sin 2\lambda\left\{M'\left(\frac{P'}{P'_m}\right)^3\sin 2\mu.\cos\overline{l-m}\quad+S'\left(\frac{P}{P_m}\right)^3\sin 2\sigma.\cos(\overline{l-m}+\overline{m-s})\right\}$$

Observing that $\cos(\overline{l-m}+\overline{m-s})=\cos\overline{m-s}.\cos\overline{l-m}-\sin\overline{m-s}.\sin\overline{l-m}$, this may be thus expressed:

$$+\frac{3}{2}\sin 2\lambda\left\{M'\left(\frac{P'}{P'_m}\right)^3\sin 2\mu+S'\left(\frac{P}{P_m}\right)^3\cdot\sin 2\sigma.\cos\overline{m-s}\right\}\cos\overline{l-m}$$
$$-\frac{3}{2}\sin 2\lambda.S'\left(\frac{P}{P_m}\right)^3\cdot\sin 2\sigma.\sin\overline{m-s}.\sin\overline{l-m}.$$

Remarking that $A.\cos\overline{l-m}-B\sin\overline{l-m}=\sqrt{A^2+B^2}\left\{\frac{A}{\sqrt{A^2+B^2}}\cos\overline{l-m}-\frac{B}{\sqrt{A^2+B^2}}\sin\overline{l-m}\right\}$

$$=\sqrt{A^2+B^2}\left\{\cos E.\cos\overline{l-m}-\sin E.\sin\overline{l-m}\right\},\text{ where }\tan E=\frac{B}{A};$$
$$=\sqrt{A^2+B^2}.\cos\overline{l-m+E};$$

this may be put in the following form:

$$+\frac{3}{2}\sin 2\lambda\sqrt{\left\{M'^2\left(\frac{P'}{P'_m}\right)^6\cdot\sin^2 2\mu+2M'S'\left(\frac{P'P}{P'_mP_m}\right)^3\sin 2\mu.\sin 2\sigma.\cos\overline{m-s}+S'^2\left(\frac{P}{P_m}\right)^6\sin^2 2\sigma\right\}}\times\cos\overline{l-m+E}$$

where

$$\tan \mathrm{E} = \frac{\mathrm{S}'\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3 . \sin 2\sigma . \sin \overline{m-s}}{\mathrm{M}'\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^3 \sin 2\mu + \mathrm{S}'\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3 \sin 2\sigma . \cos \overline{m-s}}.$$

On this diurnal tide we may make the following remarks.

(47.) 1st. It is 0 at the equator (where $\lambda = 0$), and at the pole (where $\lambda = 90°$), and greatest for places at latitude 45°. For these the greatest value (which occurs when $m - s = 0$, or at new moon) is

$$+\frac{3}{2}\left\{\mathrm{M}'\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^3 \sin 2\mu' + \mathrm{S}\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3 \sin 2\sigma\right\} \cos \overline{l-m}:$$

and as the greatest value of $\sin 2\mu$ and $\sin 2\sigma$ will be about $\frac{3}{4}$, the greatest diurnal tide will be about $\frac{9}{8}\left\{\mathrm{M}'\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^3 + \mathrm{S}'\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3\right\} \cos \overline{l-m}$: and the greatest oscillation produced by the diurnal tide will be $\frac{9}{4}\left\{\mathrm{M}'\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^3 + \mathrm{S}'\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3\right\}$.

Changes in the diurnal Tide.

(48.) 2d. The position of the meridian passing through the place of observation, with reference to the position of the meridian passing through the place of the moon, at the time of high water as produced by the diurnal tide, is determined by the equation $l - m + \mathrm{E} = 0$, or $m - l = +\mathrm{E}$. This is the moon's hour-angle to the east of the meridian at the time of high diurnal tide. If now we examine the expression for tan E, we find that E undergoes the following changes. If $\sin \overline{m-s}$ is positive, and μ and σ are both positive, E is a positive angle and not very great, since $\frac{\mathrm{S}'}{\mathrm{M}'}$ is a fraction, as will be inferred from the remark on the relative influence of the sun and moon in (15.) But if, while σ retains the same sign, μ changes its sign (in consequence of the moon crossing the equator), the denominator will successively become small, 0, and negative. E therefore increases suddenly to 90°, and to an angle not much less than 180°. This change of angle shows that the phænomena of this Tide are very rapidly reversed when the moon crosses the equator: that, instead of its high water occurring when the moon is a little way east of the upper meridian, it occurs when the moon is a little way east of the lower meridian. At the time of the moon's crossing the equator the tide will be small: for it will depend on S′ only. This supposes that $\sin \overline{m-s}$ has not changed sign; if it diminishes to 0 and then becomes negative, tan E gradually becomes a smaller negative quantity and then becomes positive, or E exceeds 180°; and the high water takes place when the moon is west of the lower meridian. The reader can easily examine the changes which occur in other positions of the two luminaries; but the following rules will be found to comprehend nearly the whole. If we consider the sidereal *day* to be defined as the time between the first point of Aries passing the meridian and the first point of Libra passing the meridian; and the sidereal *night* to be the time between the passages of the first point of Libra and of the first point of Aries; and suppose the latitude of the place of observation to be north; then the high water produced by diurnal tide always occurs *in* the sidereal day, if we neglect the sun's action, or *in or near* the sidereal day if the sun's action be small; and its time is that of the moon's passage over the upper or lower meridian (according as her longitude is less or greater than 180°) *exactly* if the sun's action be neglected, or *nearly* if the sun's action be small. The sidereal time of the high tide is later and later each successive day; the magnitude of the tide is greatest when the high tide occurs nearly at the middle of the sidereal day; and when the sudden shift takes place from the end of sidereal day to the beginning of sidereal day, the diurnal tide is extremely small.

Rule for the time of diurnal high water.

Semidiurnal Tide.

(49.) We shall now consider the third line of the expression in (44.). It consists of two parts, each of which goes through its changes twice, while $l - m$ for one, or $l - s$ for the other, increases by 360°. Each of these, therefore, produces a semidiurnal tide (the term semidiurnal being, for one, referred to the length of the lunar day, and, for the other, being referred to the length of the solar day). These two tides may be compounded into one semidiurnal tide, by the same process as that in (46.); and the expression for the elevation of the water from this cause becomes

$$+\frac{3}{2}\cos^2 \lambda \sqrt{\left\{\mathrm{M}'^2\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^6 \cos^4 \mu + 2\mathrm{M}'\mathrm{S}'\left(\frac{\mathrm{P}'\mathrm{P}}{\mathrm{P}'_m\mathrm{P}_m}\right)^3 \cos^2 \mu . \cos^2 \sigma . \cos 2.\overline{m-s} + \mathrm{S}'^2\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^6 \cos^4 \sigma\right\}} \times \cos \overline{2(l-m)+\mathrm{F}}$$

where

$$\tan \mathrm{F} = \frac{\mathrm{S}'\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3 \cos^2 \sigma . \sin 2.\overline{m-s}}{\mathrm{M}'\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^3 \cos^2 \mu + \mathrm{S}'\left(\frac{\mathrm{P}}{\mathrm{P}_m}\right)^3 \cos^2 \sigma . \cos 2.\overline{m-s}}.$$

We shall proceed to discuss several cases of this formula.

(50.) 1st. The coefficient is greatest for places at the equator, and is 0 at the poles; for all parts of the earth, therefore, it may be considered to have a sensible value. In the succeeding statements of value we shall express the values corresponding to the equator: those for other points will be found by multiplying by $\cos^2\lambda$.

Spring Tides.

(51.) 2nd. Suppose that $m-s$ is 0° or 180° (that is, that it is new moon or full moon): the expression becomes

$$\frac{3}{2}\left\{M'\left(\frac{P'}{P_m}\right)^3\cos^2\mu+S'\left(\frac{P}{P_m}\right)^3\cos^2\sigma\right\}\times\cos 2.\overline{l-m}.$$

The extent of oscillation of the surface of the water will here be

$$3\left\{M'\left(\frac{P'}{P'_m}\right)^3\cos^2\mu+S'\left(\frac{P}{P_m}\right)^3\cos^2\sigma\right\}.$$

The value of $\cos\sigma$ will not much differ from that of $\cos\mu$; therefore the expression will be nearly

$$3\cos^2\mu\left\{M'\left(\frac{P'}{P'_m}\right)^3+S'\left(\frac{P}{P_m}\right)^3\right\}.$$

This will be greatest, *cæteris paribus*, when $\cos\mu=1$, that is, for new or full moon, when the sun is at either equinox. For a solstitial new or full moon, $\cos^2\mu=\frac{21}{25}$ nearly, and the expression is

$$3.\frac{21}{25}\left\{M'\left(\frac{P'}{P'_m}\right)^3+S'\left(\frac{P}{P_m}\right)^3\right\}.$$

Neap Tides.

(52.) 3rd. Suppose that $\overline{m-s}$ is 90° or 270° (that is, that the moon is in quadratures): the expression becomes

$$\frac{3}{2}\left\{M'\left(\frac{P'}{P'_m}\right)^3\cos^2\mu-S'\left(\frac{P}{P_m}\right)^3\cos^2\sigma\right\}\times\cos 2.\overline{l-m};$$

and the whole extent of oscillation is

$$3\left\{M'\left(\frac{P'}{P'_m}\right)^3\cos^2\mu-S'\left(\frac{P}{P_m}\right)^3\cos^2\sigma\right\}.$$

If the sun is at either equinox, the moon will be at solstice, and the whole extent of oscillation is nearly $3\left\{M'\left(\frac{P'}{P'_m}\right)^3.\frac{21}{25}-S'\left(\frac{P}{P_m}\right)^3\right\}$. If the sun is at either solstice, the moon will be at equinox, and the whole extent of oscillation is $3\left\{M'\left(\frac{P'}{P'_m}\right)^3-S'\left(\frac{P}{P_m}\right)^3.\frac{21}{25}\right\}$. Combining these remarks with those of the last paragraph, we find that the syzygial tides are greatest, and the quadratural tides are least, at the equinoxes, and that these are respectively the greatest and least of all the tides.

(53.) 4th. If we put $M'+N$ for $M'\left(\frac{P'}{P'_m}\right)^3\cos^2\mu$, and $S'+T$ for $S'\left(\frac{P}{P_m}\right)^3\cos^2\sigma$, and consider that, in consequence of the small variation of the factors $\left(\frac{P'}{P'_m}\right)^3\cos^2\mu$ &c., N and T will be small, and their powers above the first may therefore be neglected without great error, the whole extent of oscillation is expressed by

$$3\sqrt{\left\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2+(2M'+2S'\cos 2.\overline{m-s})N+(2S'+2M'\cos 2.\overline{m-s})T\right\}}$$

$$=3\sqrt{\left\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2\right\}}+3\frac{M'+S'\cos 2.\overline{m-s}}{\sqrt{\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2\}}}N+3\frac{S'+M'\cos 2.\overline{m-s}}{\sqrt{\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2}}T.$$

If $P'=P'_m+p'=P'_m\left(1+\frac{p'}{P'_m}\right)$, then $\left(\frac{P'}{P'_m}\right)^3\cos^2\mu=1+3\frac{p'}{P'_m}-\sin^2\mu$ nearly; and if $P=P_m+p$, then $\left(\frac{P}{P_m}\right)^3\cos^2\sigma=1+\frac{3p}{P_m}-\sin^2\sigma$ nearly. Consequently $N=\frac{3M'p'}{P'_m}-M'\sin^2\mu$, and $T=\frac{3S'p}{P_m}-S'\sin^2\sigma$. Substituting these in the last formula, we find the following approximate expression for the whole extent of oscillation:

Approximate expression for the oscillation of the

$$3\sqrt{\left\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2\right\}}$$

$$+9\frac{M'^2+M'S'\cos 2.\overline{m-s}}{P'_m\sqrt{\{M'^2+2M'S'\cos 2.m-s+S'^2\}}}p'$$

water produced by semidiurnal Tide.

$$-3\frac{M'^2+M'S'\cos 2.\overline{m-s}}{\sqrt{\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2\}}}\sin^2\mu$$

$$+9\frac{S'^2+M'S'\cos 2.\overline{m-s}}{P_m\sqrt{\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2\}}}p$$

$$-3\frac{S'^2+M'S'\cos 2.\overline{m-s}}{\sqrt{\{M'^2+2M'S'\cos 2.\overline{m-s}+S'^2\}}}\sin^2\sigma.$$

This form is convenient for calculation. For the first or principal term depends upon no variable except $\overline{m-s}$, the angle by which the moon's right ascension exceeds the sun's right ascension, and may therefore be expanded in a single table to any extent: the other terms depend each upon two variable quantities (of which one is $\overline{m-s}$, and the other is the difference of parallax from the mean, or the declination), and will therefore be contained each in a table of double entry, which, however, needs not to be extensive. It is nearly in this form that the tables have been constructed which have lately been used for the prediction of tides.

Retard of high water upon the moon's transit.

(54.) 5th. The time of high water is determined by making $2(l-m)+F=0$, or $l-m=-\frac{F}{2}$. Now $l-m$ is evidently the angle by which the place of observation has travelled to the east of the meridian passing through the moon, or it is the time (expressed as an hour-angle) by which the moon has apparently passed the meridian of the place. Remarking now that the difference between F and tan F is so small (for these investigations) that we may use one for the other, and also that, as tan F or F is expressed in parts of radius, it must, in order to give an expression in minutes of time, be multiplied by $\frac{24\times 60}{2\pi}$, we find for the retardation of high water after the moon's passage over the meridian, expressed in minutes of time,

$$-\frac{24\times 60}{4\pi}\times\frac{(S'+T).\sin 2.\overline{m-s}}{M'+N+(S'+T)\cos 2.\overline{m-s}}.$$

These minutes, it is to be remarked, are strictly minutes of *lunar* time (that is, of time in which the measure of 24 hours is the interval between two transits of the moon): but they will not sensibly differ from solar minutes. Expanding to the first power of N and T, the retardation is

$$-\frac{360^m}{\pi}\times\left\{\frac{S'\sin 2.\overline{m-s}}{M'+S'\cos 2.\overline{m-s}}-\frac{S'\sin 2.\overline{m-s}}{(M'+S'\cos 2.\overline{m-s})^2}N+\frac{M'\sin 2.\overline{m-s}}{(M'+S'\cos 2.\overline{m-s})^2}T\right\};$$

Approximate expression for the retardation.

which, as before, will be conveniently expressed for calculation in the following form:

$$-\frac{360^m}{\pi}.\frac{S'\sin 2.\overline{m-s}}{M'+S'\cos 2.\overline{m-s}}$$

$$+\frac{1080^m}{\pi}.\frac{M'S'.\sin 2.\overline{m-s}}{P'_m(M'+S'\cos 2.\overline{m-s})^2}p'$$

$$-\frac{360^m}{\pi}.\frac{M'S'.\sin 2.\overline{m-s}}{(M'+S'.\cos 2.\overline{m-s})^2}\sin^2\mu$$

$$-\frac{1080^m}{\pi}.\frac{M'S'.\sin 2.\overline{m-s}}{P_m(M'+S'\cos 2.\overline{m-s})^2}p$$

$$+\frac{360^m}{\pi}.\frac{M'S'.\sin 2.\overline{m-s}}{(M'+S'.\cos 2.\overline{m-s})^2}\sin^2\sigma.$$

Principal changes in the retardation.

(55.) The changes in the value of this expression which are most interesting to the general reader are principally those of the first term. It appears that when $m-s$ is greater than 0° and less than 90°, or greater than 180° and less than 270°, this term is negative; that is, high water should precede the moon's transit while the moon is passing from syzygy to quadrature. While the moon is passing from quadrature to syzygy, high water should follow the moon's transit. The greatest value of this acceleration or retard for the mean values M' and S' will be found by differentiating the first term with respect to $m-s$, and making the differential coefficient $=0$: this gives $\cos 2.\overline{m-s}=-\frac{S'}{M'}$, from which tan F at its maximum value $=\pm\frac{S'}{\sqrt{(M'^2-S'^2)}}$, and

Expression for greatest retard.

$\sin F = \pm \frac{S'}{M'}$. This expression is useful, because, in a long series of observations, it is possible to determine the value of F with considerable accuracy, and that determination will immediately give us the value of the fraction $\frac{S'}{M'}$, or the proportion of the Sun's mean force to the Moon's mean force.

Expressions referred to mean declination.

(56.) In the expressions for N and T, we have proceeded on the supposition, that it is most convenient to refer the actual values of $M'\left(\frac{P'}{P'_m}\right)^3 \cos^2\mu$ and $S'\left(\frac{P}{P_m}\right)^3 \cos^2\sigma$ to the values which they would have, if the respective parallaxes had their mean values, and if the declinations were 0. With regard to the parallaxes, there is no need for alteration; but with regard to the declinations, it has sometimes been found more convenient to refer to a declination Δ, which gives for $\cos^2\Delta$ a value that is nearly a mean among all the values of $\cos^2\mu$ or $\cos^2\sigma$ (conditions that are sensibly equivalent) which occur in observation. This gives for Δ a value of 16° 35′ nearly. Then the term $M'\left(\frac{P'}{P'_m}\right)^3 \cos^2\mu$ must be expressed in the form $M'\left(\frac{P'}{P'_m}\right)^3 . (\cos^2\Delta + \overline{\cos^2\mu - \cos^2\Delta})$ or $M'\cos^2\Delta\left\{1 + \frac{3p'}{P'_m} + \frac{\cos^2\mu - \cos^2\Delta}{\cos^2\Delta}\right\}$. If we put M_Δ for $M'\cos^2\Delta$, and call the last expression $M_\Delta + N$, we obtain $N = \frac{3M_\Delta p'}{P'_m} - M_\Delta\frac{\sin^2\mu - \sin^2\Delta}{\cos^2\Delta}$. A similar change must be made in the value of T. And thus it appears, that the expressions of (53.) and (54.) may still be retained, provided that we put M_Δ and S_Δ instead of M' and S', and that instead of $\sin^2\mu$ and $\sin^2\sigma$ we put $\frac{\sin^2\mu - \sin^2\Delta}{\cos^2\Delta}$ and $\frac{\sin^2\sigma - \sin^2\Delta}{\cos^2\Delta}$. Then the greatest value of F which we have spoken of in (55.) will be the mean of all the greatest values of acceleration or retard produced in a very long series of observations; since then, not only will the positive and negative values of p' and p, (the excesses and defects of actual parallax as referred to mean parallax,) destroy each other, but also the positive and negative values of $\sin^2\mu - \sin^2\Delta$ and $\sin^2\sigma - \sin^2\Delta$, (the excesses and defects of the actual values of $\sin^2\mu$ and $\sin^2\sigma$, as referred to their mean values,) will destroy each other; and thus a value of F will be obtained, from which $\frac{S_\Delta}{M_\Delta}$ or its equal $\frac{S'}{M'}$ can immediately be found.

(57.) With regard to the three principal parts into which we have resolved the lunisolar tide, we may make the following remarks:—

The first part treated in (45.) &c., is so slow in its period that it will generally be lost among the others whose recurrence is so much more frequent. If, however, we wished to examine its law, we should remark, that its variations depend chiefly on the variations of $\sin^2\mu$ and $\sin^2\sigma$. And we see that $\sin^2\mu$ will be (for a few lunations) nearly proportional to the square of the sine of the moon's longitude from a given point, and therefore nearly proportional to the square of the sine of an angle increasing proportionally to the time (which we will call $At+B$, putting t for the time), or that $\sin^2\mu \propto \sin^2(At+B) \propto \frac{1}{2} - \frac{1}{2}\cos(2At+2B)$. From this it appears that the fluctuation of the surface of the water, dependent on this cause, would be expressed (omitting the constant terms) by a multiple of $\cos(2At+2B)$. In like manner the fluctuation depending on $\cos^2\sigma$ would be expressed by $\cos(2Ct+2D)$.

(58.) The fluctuations depending on the second and third parts, treated of from (46.) to (53.) are, as we have seen, expressed by cosines of multiples of $l-m$ and $l-s$, which for a few tides can be expressed nearly enough by cosines of multiples of $l-m$. This angle is nearly proportional to the time, and thus these fluctuations can be expressed in the same form as those above.

(59.) If we considered the variations which the factors of these different terms undergo, arising from the change of parallax, declination, &c., it would be found that they would all be expressed by series of cosines of multiples of the time; which combined as factors with the cosines of the former would produce cosines of new multiples of the time.

All the fluctuations may be expressed by cosines.

(60.) Thus it appears that the fluctuation of the water will be expressed by a multitude of terms, each of which will be of the form $C\cos(2At+2B)$. If any one of these terms existed alone, the following curious law would be true. Suppose the lines of the extreme elevation and extreme depression of the surface of the water to be marked upon a wharf-wall; and suppose a circle to be described upon the wall, touching those two lines; then if the circumference of that circle be divided into equal parts, the fall of the water will expose the successive equal parts in successive equal times. For, since the whole fluctuation is 2C, C is the radius of our circle; and the elevation of any point of the circumference above the mean (or above the centre of the circle) is C × cosine angle from the top; but the tidal term gives for that elevation C × cosine $(2At+2B)$, and therefore $2At+2B$ must be the same thing as that angle from the top; and, therefore, that angle, and the circumference which is proportional to it, must have increased proportionally to the time. But this law does not hold for an assemblage of a multitude of such terms.

(61.) We have now given a tolerably complete investigation of the equilibrium-theory. But before quitting this section we will point out roughly how far it agrees with observation.

General comparison of the equilibrium-theory with observation.

(62.) The most conspicuous tide, on the coasts of Europe at least, is the semidiurnal. The acceleration or retard of this tide on the moon's transit, does not at one port in a hundred agree in any measure with the result of this theory. The extreme differences of acceleration or retard (F of article 55., &c.) agree better, but are not exactly the same at all ports. They do not occur on the days on which this theory predicts them, but always later. The absolute elevation of the tide is great at one port and small at another, without any relation to the quantity calculated from the theory. The proportions of the elevations however at the same port, in different stages of the lunation, agree pretty well with the theory (though not equally at all ports); yet the critical phænomena (spring and neap tides) occur later than the theory gives them, and that by a quantity which is not the same as the delay of extreme values of F, mentioned above. The peculiar phænomena of river tides are not touched by this theory.

(63.) The diurnal tide ought to be discovered, in observation, in one of the following ways. If the diurnal tide were much greater than the semidiurnal, there would appear, to common observation, to be only one tide in the day with some irregularities. If it were much smaller, its effect would be shown in either or both the following ways. If its high water occurred nearly at one high water of semidiurnal tide, its low water would occur nearly at another high water of semidiurnal tide; and one of the semidiurnal tides would be increased and the other would be diminished. If its high water occurred between two semidiurnal high waters, then at the first semidiurnal high water the surface would still be rising in consequence of diurnal tide, and the compound high water would be later; and at the second semidiurnal high water the surface would be falling in consequence of diurnal tide, and the compound high water would therefore be past; consequently the interval between these two high waters would be less than it ought to be on the usual laws of semidiurnal tide. The diurnal tide ought, in these latitudes, to be equal or nearly equal to the semidiurnal tide. Yet in the Thames it is absolutely insensible; and in other ports, as well of England as of other parts of Europe and America, though discoverable, it is not notorious, and has only been found from the observations made by men of science. It has been found to be very conspicuous at some places near the equator and some places near the pole, where it ought not to be discoverable or scarcely discoverable.

The Tides of longer period have scarcely been observed.

Inadequacy of the equilibrium-theory.

(64.) Combining these remarks with those which we made at the introduction of this theory (14.), it must be allowed that it is one of the most contemptible theories that was ever applied to explain a collection of important physical facts. It is entirely false in its principles, and entirely inapplicable in its results. Yet, strange as it may appear, this theory has been of very great use. It has served to show that there are forces in nature following laws which bear a not very distant relation to some of the most conspicuous phænomena of the Tides; and, what is far more important, it has given an algebraic form to its own results, divided into separate parts analogous to the parts into which the tidal phænomena may be divided, admitting easily of calculation and of alteration, and thus at once suggesting the mode of separating the tidal movements, and affording numerical results of theory with which they are to be compared. The greatest mathematicians and the most laborious observers of the present age have agreed equally in rejecting the *foundation* of this theory and comparing all their observations with its *results*. And, till theories are perfect (a thing scarcely to be hoped for in any subject, and less in the Tides than in any other), this is one of the most important uses of theory.

Its historical utility.

Section III.—Laplace's Theory of Tides.

Laplace's theory is a theory of motion.

(65.) In the theory which we are now about to describe, a prodigious step was made towards a rational explanation, on mathematical principles, of the tidal phænomena. The idea of a state of equilibrium was entirely laid aside, and the motion of the water was legitimately investigated, on the supposition that it is in motion, and subject to all the laws of fluids in motion. It was found necessary, however, in order to make the application of mathematics practicable, to start with two suppositions, which are inapplicable to the state of the earth. These are: that the earth is covered with water; and that the depth of this water is the same through the whole extent of any parallel of latitude. Under these suppositions it is evident that the theory is far from being one of practical application; though it clearly approaches much nearer to truth than the theory of equilibrium which we have already described.

Suppositions by which it is limited.

(66.) It would be useless to offer this theory in the same shape in which Laplace has given it; for the part of the *Mécanique Céleste*, which contains the Theory of Tides, is perhaps on the whole more obscure than any other part of the same extent in that work. We shall give the theory in a form equivalent to Laplace's, and, indeed, so nearly related to it, that a person familiar with the latter will perceive the parallelism of the successive steps. The results at which we shall arrive are the same as those of Laplace.

(67.) We shall commence with a few considerations of a general nature, based upon the suppositions that we have already enunciated, and the additional supposition that the depth of the sea is small compared with the radius of the earth; and taking for granted a knowledge of the principal results of the equilibrium theory.

(68.) The motion of the water which forms the variable elevations of the Tides at different parts of the earth must be conceived to be principally a horizontal oscillation, the water on both sides of the highest point at any time having run towards that point in order to raise the surface there, and, consequently, (as the highest point occupies different positions at different times,) the water at any particular place running sometimes in one direction and sometimes in another. Combining this with the general result of the equilibrium-theory as to semidiurnal Tides, (namely, that the water is equally raised at two opposite points,) it will easily be seen that, if a canal were traced through the water, forming a great circle of the earth, it would (in certain positions at least of the sun and moon) be divided into four parts, in two of which the water is running in one direction, and in the other two it is running in the opposite direction. Suppose that in one of these parts

A small vertical motion of the water implies a large horizontal motion.

Extraneous vertical forces, and forces depending on vertical motion, may be omitted.

Vertical force of gravity on the small elevations of water must not be omitted.

the length is 1000 times as great as the depth, and suppose that the water is depressed one foot through its whole extent. It is evident that the volume of the water (omitting the factor depending on the breadth of the canal), for which a new place is to be found, is = the length of the canal × 1 foot, which = 1000 × depth of the canal × 1 foot, or = depth of the canal × 1000 feet. Consequently the water at one end of the canal, if that at the other end remained unmoved in horizontal place, must have moved 1000 feet, or 1000 times as far as the whole vertical motion of any part. The whole of the extraneous *vertical* forces then which act upon the particles of the water may be omitted in our investigations. For these forces are of two kinds. One is that which depends upon the acceleration or retardation of the particles of water in their upward or downward direction: thus, if the water has been raised 1 foot in 6 hours, the force of which we speak is the pressure which must have acted, in order, by its action continued for 6 hours, to produce a motion of 1 foot. It is clear that this is insignificant in comparison with that force which in the same time has produced a motion of 1000 feet. The force of the other kind is the disturbing force of the sun or moon: now the pressure which this causes among the particles of the water depends not only upon the magnitude of the force, but also upon the depth of the column or the length of the canal through which it acts: and, therefore, though its resolved vertical part may be as great as its resolved horizontal part, yet as the vertical part acts only upon a column of water 5 or 6 miles deep (at the utmost), and the horizontal part acts along the whole length of a canal 5000 or 6000 miles long, the pressure among the particles of water which is caused by the former will be insignificant in comparison with that caused by the latter. As regards both these kinds of force then, the vertical force may be put out of consideration. But the same remark does not apply to the vertical force of gravity, nor to the difference in the pressure which it produces, depending on the small tidal difference of elevation of the water at different places. For the force of gravity, as we have seen in (16.), is nearly forty millions of times as great as the sun's disturbing force, and therefore the force of gravity acting on one foot of additional elevation of water would cause as great an additional pressure among the particles of water as the sun's disturbing force acting along a canal whose length is 8000 miles.

(69.) From this we gather that the only forces which we shall have to consider are: the vertical force of gravity, the resolved disturbing forces of the sun and moon in the horizontal direction at each place, and the forces arising from the horizontal accelerations or retardations of the water in each place. We shall suppose the density of the water insignificant, and gravity therefore will depend simply upon the attraction of the solid earth, and the centrifugal force arising from the earth's rotation.

Particles originally in a vertical line may be supposed to be always in a vertical line.

(70.) Now as the depth of the water is very small compared with the radius of the earth, the horizontal disturbing force will be very nearly the same at the bottom of the water as at the top. Consequently we may conceive the whole motion of the water to be of such a kind that particles which were originally in a vertical line remain in a vertical line, although that vertical line may have had a motion on the earth's surface. The elevation of the water must be supposed to be produced merely by the approach of different vertical lines, (arising from their difference of horizontal velocities,) and the consequent forcing up of the water between them: the depression, by their separation, and the consequent drop of the water to fill up the space between them. The reader will remark, that this is not the most general supposition that we can make in regard to the motion of water, but it is one which is possible, and which is sufficient for our theory. Algebraically speaking, it is not our object to obtain a *general solution* of the equations applying to fluids, but a *particular integral* adapted to the case under consideration.

(71.) We shall now proceed to put the theory into a mathematical form. In figure 2, let P be the pole of the earth, PA a meridian fixed in space, PS a meridian fixed upon the earth's surface, and therefore travelling away from PA with a uniform angular velocity which we will call n; so that, at the end of the time t, PS will make with PA the angle nt: PT the meridian passing through the original place T of any point of the water whose motion is required. We shall suppose that in the quiescent state of the water the angle SPT would have been ϖ, but that in consequence of the tidal disturbance the particle of water is moved to T′, and the angle ϖ is altered by the variable angle v, so that SPT′ $=\varpi+v$, and APT′ $=nt+\varpi+v$. We shall conceive that the original polar distance of the water at T was θ; but that in consequence of the tidal disturbance it is now $\theta+u$. Also we shall put γ for the depth of the water at T, supposed quiescent, and w for its tidal elevation above the quiescent state when at T′. Here γ, in conformity with the supposition made in (65.), is to be considered a function of θ only: u, v, and w are all to be considered as functions of θ, π, and t.

Investigation of the equation of continuity.

(72.) The equation which we shall first form is that which expresses that, however any part of the fluid is transported by the tidal and rotatory motions, it occupies still the same volume. Take another point U′ corresponding to another particle of water U, which was originally on the same meridian with T, and whose polar distance was originally $\theta+\delta\theta$, and therefore is now $\theta+u+\left(1+\frac{du}{d\theta}\right)\delta\theta$ nearly. The angle APU′ has the value which $nt+\varpi+v$ receives when $\theta+\delta\theta$ is used instead of θ in forming the value of v: it is therefore $nt+\varpi+v+\left(\frac{dv}{d\theta}\right)\delta\theta$ nearly. Also take two points, V′ and W′, corresponding to two points V and W, whose polar distances were originally θ, and $\theta+\delta\theta$, but which were upon a meridian making with PS the angle $\varpi+\delta\varpi$. The polar distances of these two points have now respectively the values $\theta+u+\frac{du}{d\varpi}\delta\varpi$ and $\theta+u+\left(1+\frac{du}{d\theta}\right)\delta\theta+\frac{du}{d\varpi}\delta\varpi$, and their angular distances from PA have respectively the values $nt+\varpi+v+\left(1+\frac{dv}{d\varpi}\right)\delta\varpi$ and $nt+\varpi+v+\left(1+\frac{dv}{d\varpi}\right)\delta\varpi+\frac{dv}{d\theta}\delta\theta$. The distance, therefore, from T′ to U′ (if r be the radius of the spherical or spheroidal surface of the water) is sensibly equal to $r\left(1+\frac{du}{d\theta}\right)\delta\theta$, and that from V′ to W′ is the same. Also the difference of angular distances from PA, between T′ and V′, is sensibly equal to $r\left(1+\frac{dv}{d\varpi}\right)\delta\varpi$, and that between U′ and W′ is sensibly the same. Now the area of the surface upon which the water originally stood is seen without difficulty to be $r\delta\theta\times r\sin\left(\theta+\frac{\delta\theta}{2}\right).\delta\varpi$: and, therefore, its original volume

$$=r\delta\theta\times r\sin\left(\theta+\frac{\delta\theta}{2}\right).\delta\varpi\times\gamma.$$

The area upon which it now stands is not estimated so readily, because its sides are inclined; namely, one pair making with the meridian the small angle $\frac{1}{r\delta\theta}\times r.\sin\theta.\frac{dv}{d\theta}\delta\theta=\sin\theta.\frac{dv}{d\theta}$, and the other pair making with the parallel the small angle $\frac{1}{r\sin\theta\delta\varpi}\times r\frac{du}{d\varpi}\delta\varpi=\frac{1}{\sin\theta}.\frac{du}{d\varpi}$. These small inclinations do not, however, produce any sensible effect on the area. The reader will perceive this most readily by estimating the area of a parallelogram *abed*, figure 3, whose sides are inclined to the sides of the rectangle *acgf*. For the area of *abed*

$$=ab\times ad\times\sin bad=ab\times ad\times\sin(90^\circ-bac-daf)=ab\times ad\times\cos(bac+daf)$$
$$=ab\times ad\times(\cos bac.\cos daf-\sin bac.\sin daf)$$
$$=(ab.\cos bac)\times(ad.\cos daf)\times(1-\tan bac.\tan daf$$
$$=ac.af.(1-\tan bac.\tan daf),$$

which it is evident differs from $ac.af$ only by the product of $ac.af$ into a small quantity of the second order. In like manner we may estimate the area T′U′W′V′ without regard to the inclination of its sides; and it becomes, therefore,

$$r\left(1+\frac{du}{d\theta}\right)\delta\theta\times r\sin\left\{\theta+u+\left(1+\frac{du}{d\theta}\right)\frac{\delta\theta}{2}\right\}\times\left(1+\frac{dv}{d\varpi}\right)\delta\varpi.$$

The depth of the water, which at the original place was γ, is now altered, from two causes: first, because $\theta+u$, upon which it depends, differs from θ; and, secondly, because there is the tidal elevation of the surface; it is, therefore, $\gamma+\frac{d\gamma}{d\theta}u+w$. The volume of the water standing upon T′V′W′U′ is, therefore,

$$r\left(1+\frac{du}{d\theta}\right)\delta\theta\times r\sin\left\{\theta+u+\left(1+\frac{du}{d\theta}\right)\frac{\delta\theta}{2}\right\}\times\left(1+\frac{dv}{d\varpi}\right)\delta\varpi\times\left(\gamma+\frac{d\gamma}{d\theta}u+w\right)$$
$$=r\delta\theta\times r\sin\left(\theta+\frac{\delta\theta}{2}\right).\delta\varpi\times\gamma\times\left(1+\frac{du}{d\theta}\right)\times\left\{1+\text{cotan}\,\overline{\theta+\frac{\delta\theta}{2}}.\overline{u+\frac{du}{d\theta}.\frac{\delta\theta}{2}}\right\}\times\left(1+\frac{dv}{d\varpi}\right)\times\left(1+\frac{u}{\gamma}.\frac{d\gamma}{d\theta}+\frac{w}{\gamma}\right).$$

Making this equal to the original volume, or

$$r\delta\theta\times r\sin\left(\theta+\frac{\delta\theta}{2}\right)\delta\varpi\times\gamma,$$

we have

$$1=\left(1+\frac{du}{d\theta}\right)\times\left\{1+\text{cotan}\,\overline{\theta+\frac{\delta\theta}{2}}.\overline{u+\frac{du}{d\theta}.\frac{\delta\theta}{2}}\right\}\times\left(1+\frac{dv}{d\varpi}\right)\times\left(1+\frac{u}{\gamma}.\frac{d\gamma}{\delta\theta}+\frac{w}{\gamma}\right);$$

or, multiplying out these quantities, rejecting products of the small terms, and rejecting the insignificant terms depending on $\delta\theta$,

$$0=\frac{du}{d\theta}+u\,\text{cotan}\,\theta+\frac{dv}{d\varpi}+\frac{u}{\gamma}.\frac{d\gamma}{d\theta}+\frac{w}{\gamma}.$$

If we multiply this by γ, it becomes

$$0=\frac{d}{d\theta}(u\gamma)+u\gamma.\text{cotan}\,\theta+\gamma\frac{dv}{d\varpi}+w;$$

and if we multiply again by $\sin\theta$, it becomes

$$0=\frac{d}{d\theta}(u\gamma\sin\theta)+\gamma\sin\theta.\frac{dv}{d\varpi}+w.\sin\theta.$$

This is the equation expressing that the same water always occupies the same volume: it is frequently called the *equation of continuity*.

Explanation of the pressure of fluids.

(73.) We now proceed to investigate the equations applying strictly to the *motion* of the water. We will first allude in a few words to the general equations of motion of incompressible fluids, referring to our article Hydrodynamics, page 276, or to other treatises, for a more detailed exposition. Let the places of the particles of a fluid at any instant of time be defined by three rectangular coordinates: suppose x, y, z, to be the coordinates of one point, and p the pressure among the particles at that point. The pressure may perhaps be most easily conceived in the following manner:—Suppose a plane to be inserted in the fluid, and suppose the fluid on one side of the plane to be removed; it will be necessary to apply a pressure to this plane, in order to maintain the remaining fluid in the same state (whether of repose, or of motion or change of motion); and the pressure for every square unit of surface on this plane is our quantity p. This pressure ought in strictness to be estimated as a statical pressure by the number of pounds and ounces under the action of gravity, at a given place on the earth's surface: but it will be preferable to take, instead, a quantity which bears a constant ratio to it, namely, the acceleration which this pressure will, by acting for one unit of time, cause in a cubic unit of the fluid.

Investigation of the rectangular equations of the motion of fluids.

(74.) Conceive, then, a small parallelopiped to be inclosed by planes corresponding to the coordinates x, $x+h$; $y, y+k$; $z, z+l$; h, k, and l being extremely small. The pressure per unit of surface on that end whose ordinate is x, is p; but the area of that end is kl; therefore the actual pressure is pkl. The actual pressure on the other end is $kl\times$the value of p corresponding to $x+h$, or it is $kl\times\left(p+\frac{dp}{dx}h\right)$. The former of these tends to push the parallelopiped forward in the direction of x: the latter tends to push it backward. The actual pressure, then, tending to push it backward, is $\frac{dp}{dx}hkl$; and as the volume of the parallelopiped is hkl, the accelerating force in the direction opposite to that in which x is measured is $\frac{dp}{dx}$. If there is acting an extraneous accelerating force, tending to urge the particles in the direction of x, and represented by X, then the real accelerating force in the direction of x is $\mathrm{X}-\frac{dp}{dx}$. Consequently, (Mechanics, page 91,) $\frac{d^2x}{dt^2}=\mathrm{X}-\frac{dp}{dx}$. Similarly, $\frac{d^2y}{dt^2}=\mathrm{Y}-\frac{dp}{dy}$; $\frac{d^2z}{dt^2}=\mathrm{Z}-\frac{dp}{dz}$.

Equality of pressure in moving fluids requires investigation.

(75.) It is to be remarked, that here we conceive the quantity p to be the same in the three equations which we have just found. That this is true when a fluid is in equilibrium (or that fluids press equally in all directions) there is no doubt: indeed it can be shown to be a necessary consequence of the possibility of division of the fluid by planes in all directions, and of the perpendicularity of the pressure of the fluid on any such plane. It is not so self-evidently certain for fluids in variable motion. Without expressing any doubt of its truth, we wish at the same time to call the reader's attention to the difference of evidence for the principle in the different cases.

Investigation of the polar equations of the motion of fluids.

(76.) To apply this to the movement of the sea: let x be parallel to the axis of revolution of the earth, y directed towards the first point of Aries, and z at right angles to these. And put θ' for $\theta+u$, ϖ' for $\overline{\varpi+v}$: r for the distance of any particle from the earth's centre. Then $x=r.\cos\theta'$, $y=r.\sin\theta'.\cos\overline{nt+\varpi'}$, $z=r.\sin\theta'.\sin\overline{nt+\varpi'}$. Our equations of motion, which depend on x, y, z, must be transformed into others depending on r, θ', ϖ'. We must, then, instead of $\frac{dp}{dx}$, put $\frac{dp}{dr}.\frac{dr}{dx}+\frac{dp}{d\theta'}.\frac{d\theta'}{dx}+\frac{dp}{d\varpi'}.\frac{d\varpi'}{dx}$: for $\frac{dp}{dy}$ we must put $\frac{dp}{dr}.\frac{dr}{dy}+\frac{dp}{d\theta'}.\frac{d\theta'}{dy}+\frac{dp}{d\varpi'}.\frac{d\varpi'}{dy}$; and for $\frac{dp}{dz}$ we must put $\frac{dp}{dr}.\frac{dr}{dz}+\frac{dp}{d\theta'}.\frac{d\theta'}{dz}+\frac{dp}{d\varpi'}.\frac{d\varpi'}{dz}$. And, in forming the quantities $\frac{dr}{dx}$, $\frac{d\theta'}{dx}$, &c., we must express r, θ', and ϖ', in terms of no variable quantities whatever, except x, y, and z. For then it is clear that we have $\frac{dp}{dx}$, &c. exactly in the same manner as if p was explicitly expressed in terms of x, y, and z. Now

$$r^2=x^2+y^2+z^2;$$

therefore

$$\frac{dr}{dx}=\frac{x}{r};\quad \frac{dr}{dy}=\frac{y}{r};\quad \frac{dr}{dz}=\frac{z}{r}.$$

Also

$$\tan\theta'=\frac{\sqrt{y^2+z^2}}{x}$$

differentiating, and observing that $\cos^2\theta'=\frac{x^2}{r^2}$,

$$\frac{d\theta'}{dx}=\frac{-\sqrt{y^2+z^2}}{r^2}\,;\quad \frac{d\theta'}{dy}=\frac{xy}{r^2\sqrt{y^2+z^2}}\,;\quad \frac{d\theta'}{dz}=\frac{xz}{r^2\sqrt{y^2+z^2}}.$$

And

$$\tan\overline{nt+\varpi'}=\frac{z}{y}\,;\quad \cos^2\overline{nt+\varpi'}=\frac{y^2}{y^2+z^2}:$$

therefore

$$\frac{d\varpi'}{dx}=0\,;\quad \frac{d\varpi'}{dy}=\frac{-z}{y^2+z^2}\,;\quad \frac{d\varpi'}{dz}=\frac{y}{y^2+z^2}.$$

Substituting these, the equations of motion become

$$\begin{aligned}
&\frac{dp}{dr}\cdot\frac{x}{r}-\frac{dp}{d\theta'}\cdot\frac{\sqrt{y^2+z^2}}{r^2} &&=\mathrm{X}-\frac{d^2x}{dt^2}\\
&\frac{dp}{dr}\cdot\frac{y}{r}+\frac{dp}{d\theta'}\cdot\frac{xy}{r^2\sqrt{y^2+z^2}}-\frac{dp}{d\varpi'}\cdot\frac{z}{y^2+z^2} &&=\mathrm{Y}-\frac{d^2y}{dt^2}\\
&\frac{dp}{dr}\cdot\frac{z}{r}+\frac{dp}{d\theta'}\cdot\frac{xz}{r^2\sqrt{y^2+z^2}}+\frac{dp}{d\varpi'}\cdot\frac{y}{y^2+z^2} &&=\mathrm{Z}-\frac{d^2z}{dt^2}.
\end{aligned}$$

(77.) From these we obtain

$$\begin{aligned}
\frac{dp}{dr}&=\mathrm{X}\frac{x}{r}+\mathrm{Y}\frac{y}{r}+\mathrm{Z}\frac{z}{r}-\frac{x}{r}\cdot\frac{d^2x}{dt^2}-\frac{y}{r}\frac{d^2y}{dt^2}-\frac{z}{r}\cdot\frac{d^2z}{dt^2}\\
\frac{dp}{d\theta'}&=-\mathrm{X}\sqrt{y^2+z^2}+\frac{\mathrm{Y}xy+\mathrm{Z}xz}{\sqrt{y^2+z^2}}+\frac{1}{\sqrt{y^2+z^2}}\left\{(y^2+z^2)\frac{d^2x}{dt^2}-xy\frac{d^2y}{dt^2}-xz\frac{d^2z}{dt^2}\right\}\\
\frac{dp}{d\varpi'}&=\mathrm{Z}y-\mathrm{Y}z+z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}.
\end{aligned}$$

Now

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}=\frac{d}{dt}\left(x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)-\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

$$=\frac{1}{2}\cdot\frac{d^2(r^2)}{dt^2}-\left(\frac{dr}{dt}\right)^2-r^2\left(\frac{d\theta'}{dt}\right)^2-r^2\sin^2\theta'\left(n+\frac{d\varpi}{dt}\right)^2:$$

therefore

$$-\frac{x}{r}\cdot\frac{d^2x}{dt^2}-\frac{y}{r}\cdot\frac{d^2y}{dt^2}-\frac{z}{r}\cdot\frac{d^2z}{dt^2}=-\frac{1}{2r}\cdot\frac{d^2(r^2)}{dt^2}+\frac{1}{r}\left(\frac{dr}{dt}\right)^2+r\left(\frac{d\theta'}{dt}\right)^2+r\sin^2\theta'\left(n+\frac{d\varpi}{dt}\right)^2.$$

And

$$(y^2+z^2)\frac{d^2x}{dt^2}-xy\frac{d^2y}{dt^2}-xz\frac{d^2z}{dt^2}=y\frac{d}{dt}\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)+z\frac{d}{dt}\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)$$

$$=r\sin\theta'\cos\overline{nt+\varpi'}\frac{d}{dt}\left\{-r^2\cos\overline{nt+\varpi}.\frac{d\theta'}{dt}+r^2\sin\theta'\cos\theta'.\sin\overline{nt+\varpi}.\left(n+\frac{d\varpi'}{dt}\right)\right\}$$

$$+r\sin\theta'\sin\overline{nt+\varpi'}.\frac{d}{dt}\left\{-r^2\sin\overline{nt+\varpi'}\frac{d\theta'}{dt}-r^2\sin\theta'\cos\theta'.\cos\overline{nt+\varpi'}\left(n+\frac{d\varpi'}{dt}\right)\right\};$$

therefore

$$\frac{1}{\sqrt{y^2+z^2}}\left\{(y^2+z^2)\frac{d^2x}{dt^2}-xy\frac{d^2y}{dt^2}-xz\frac{d^2z}{dt^2}\right\}$$

$$=-\cos\overline{nt+\varpi'}.\frac{d}{dt}\left(r^2\cos\overline{nt+\varpi'}.\frac{d\theta'}{dt}\right)-\sin\overline{nt+\varpi'}\frac{d}{dt}\left(r^2\sin\overline{nt+\varpi'}.\frac{d\theta'}{dt}\right)$$

$$+\cos\overline{nt+\varpi'}.\frac{d}{dt}\left(r^2\sin\theta'\cos\theta'\left(n+\frac{d\varpi'}{dt}\right)\times\sin\overline{nt+\varpi'}\right)$$

$$-\sin\overline{nt+\varpi'}.\frac{d}{dt}\left(r^2\sin\theta'.\cos\theta'\left(n+\frac{d\varpi'}{dt}\right)\times\cos\overline{nt+\varpi'}\right).$$

If for the moment we put U for $r^2\cos\overline{nt+\varpi'}.\ \frac{d\theta'}{dt'}$, V for $r^2\sin\overline{nt+\varpi'}.\ \frac{d\theta'}{dt}$, W for $r^2\sin\theta'.\cos\theta'\left(n+\frac{d\varpi'}{dt}\right)$, this expression becomes

$$-\frac{1}{r^2\frac{d\theta'}{dt}}\times\left\{U\frac{dU}{dt}+V\frac{dV}{dt}\right\}$$

$$+\frac{1}{r^2\sin\theta'\cos\theta'.\left(n+\frac{d\varpi'}{dt}\right)}\left\{W.\cos\overline{nt+\varpi'}.\frac{d}{dt}\left(W.\sin\overline{nt+\varpi'}\right)-W.\sin\overline{nt+\varpi}.\frac{d}{dt}\left(W.\cos\overline{nt+\varpi'}\right)\right\}$$

$$=-\frac{1}{2r^2\frac{d\theta'}{dt}}\times\frac{d}{dt}\left(U^2+V^2\right)+\frac{1}{r^2\sin\theta'.\cos\theta'.\left(n+\frac{d\varpi'}{dt}\right)}\times W^2.\frac{d}{dt}\left(nt+\varpi'\right)$$

$$=-\frac{1}{2r^2\frac{d\theta'}{dt}}\cdot\frac{d}{dt}\left(r^4\left(\frac{d\theta'}{dt}\right)^2\right)+\frac{1}{r^2\sin\theta'.\cos\theta'.\left(n+\frac{d\varpi'}{dt}\right)}\times r^4.\sin^2\theta'.\cos^2\theta'.\left(n+\frac{d\varpi'}{dt}\right)^2\times\left(n+\frac{d\varpi'}{dt}\right)$$

$$=-\frac{d}{dt}\left(r^2\frac{d\theta'}{dt}\right)+r^2.\sin\theta'.\cos\theta'.\left(n+\frac{d\varpi'}{dt}\right)^2.$$

And

$$z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}=\frac{d}{dt}\left(z\frac{dy}{dt}-y\frac{dz}{dt}\right)$$

$$=\frac{d}{dt}\left\{r\sin\theta'.\sin\overline{nt+\varpi'}\frac{d}{dt}\left(r\sin\theta'.\cos\overline{nt+\varpi'}\right)-r\sin\theta'.\cos\overline{nt+\varpi'}\frac{d}{dt}\left(r\sin\theta'\sin\overline{nt+\varpi}\right)\right\}$$

$$=-\frac{d}{dt}\left\{r^2\sin^2\theta'\left(n+\frac{d\varpi'}{dt}\right)\right\}$$

$$=-2r\frac{dr}{dt}\sin^2\theta'\left(n+\frac{d\varpi'}{dt}\right)-2r^2\sin\theta'.\cos\theta'\frac{d\theta'}{dt}\left(n+\frac{d\varpi'}{dt}\right)-r^2\sin^2\theta'.\frac{d^2\varpi'}{dt^2}.$$

Polar equations of the motion of fluids.

Substituting these, our equations become

$$\frac{dp}{dr}=X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}-\frac{1}{2r}\cdot\frac{d^2(r^2)}{dt^2}+\frac{1}{r}\left(\frac{dr}{dt}\right)^2+r\left(\frac{d\theta'}{dt}\right)^2+r\sin^2\theta'\left(n+\frac{d\varpi'}{dt}\right)^2$$

$$\frac{dp}{d\theta'}=-X\sqrt{y^2+z^2}+\frac{Yxy+Zxz}{\sqrt{y^2+z^2}}-\frac{d}{dt}\left(r^2\frac{d\theta'}{dt}\right)+r^2\sin\theta'\cos\theta'\left(n+\frac{d\varpi'}{dt}\right)^2$$

$$\frac{dp}{d\varpi'}=Zy-Yz-2r\frac{dr}{dt}\sin^2\theta'\left(n+\frac{d\varpi'}{dt}\right)-2r^2\sin\theta'.\cos\theta'.\frac{d\theta'}{dt}\left(n+\frac{d\varpi'}{dt}\right)-r^2\sin^2\theta'\frac{d^2\varpi'}{dt^2}.$$

Limited equations applicable to the motion of the sea.

(78.) The equations as we have just given them are complete. But, in conformity with the remarks in (69.), we may considerably reduce the number of terms. For, if we omit the motion in the direction perpendicular to the surface, we may entirely omit $\frac{dr}{dt}$ and $\frac{d^2r}{dt^2}$. Also, as the tidal oscillation is very small in proportion to the earth's dimensions, we may in every case omit the squares of $\frac{d\theta'}{dt}$ and $\frac{d\varpi'}{dt}$ and their product. Thus the equations become

$$\frac{dp}{dr}=X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}+r\sin^2\theta'\left(n^2+2n\frac{d\varpi'}{dt}\right)$$

$$\frac{dp}{d\theta'}=-X\sqrt{y^2+z^2}+\frac{Yxy+Zxz}{\sqrt{y^2+z^2}}-r^2\frac{d^2\theta'}{dt^2}+r^2\sin\theta'.\cos\theta'.\left(n^2+2n\frac{d\varpi'}{dt}\right)$$

$$\frac{dp}{d\varpi'}=Zy-Yz-2r^2\sin\theta'\cos\theta'\frac{d\theta'}{dt}\cdot n-r^2\sin^2\theta'\frac{d^2\varpi'}{dt^2}.$$

But θ' enters only as the representative of $\theta+u$: so that, supposing u small, $\frac{dp}{d\theta'}$ will not sensibly differ from $\frac{dp}{d\theta}$; $\sin\theta'$ and $\cos\theta'$ in the factors will not sensibly differ from $\sin\theta$ and $\cos\theta$; and $\frac{d\theta'}{dt}$ is $=\frac{du}{dt}$, because θ does not depend on the time. Also $\frac{dp}{d\varpi'}$ will not sensibly differ from $\frac{dp}{d\varpi}$, and $\frac{d\varpi'}{dt}=\frac{dv}{dt}$. Thus the equations of motion become

$$\frac{dp}{dr}= X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}+r\sin^2\theta\left(n^2+2n\frac{dv}{dt}\right)$$

$$\frac{dp}{d\theta}=-X\sqrt{y^2+z^2}+\frac{Yxy+Zxz}{\sqrt{y^2+z^2}}-r^2\frac{d^2u}{dt^2}+r^2\sin\theta.\cos\theta.\left(n^2+2n\frac{dv}{dt}\right)$$

$$\frac{dp}{d\varpi}= Zy-Yz-2r^2\sin\theta.\cos\theta.\frac{du}{dt}n-r^2\sin^2\theta\frac{d^2v}{dt^2}.$$

The forces divided into two parts.

The pressures divided into three parts.

Centrifugal force is not to be introduced.

(79.) We shall now proceed to express the forces X, Y, Z. For convenience we shall divide each into two parts, $X=X'+X''$, $Y=Y'+Y''$, $Z=Z'+Z''$: where X', Y', Z', relate only to the attractions of the earth, and X'', Y'', Z'' only to those of the disturbing bodies. We shall also suppose p divided into three parts, p', p'', p''', of which the first is that part which depends on the earth's attraction and the factor n^2, the second depends only on the disturbing forces, and the third only on the differentials of u and v. And we shall put r' for the distance from the earth's centre to the surface of the undisturbed watery spheroid; so that $r'+w$ is the value of r, at the surface of the water as disturbed by the Tide. First we will remark, as a point particularly worthy of the reader's attention, that here we must not introduce what is usually called *centrifugal force*. For our equations proceed at once from the equations by rectangular co-ordinates, and in these the whole relations between the forces and the motions are absolutely included. The assertion of the existence of *centrifugal force* is in fact merely an assertion that, in order to maintain bodies or fluids in an assumed state of circular motion, a *centripetal* force must be applied. We shall soon see that the effects of centrifugal force, as regards the figure of the earth, are fully taken into account without introducing that expression at all. The point deserves a little more notice than has commonly been given to it in theories of the Figure of the Earth and similar subjects.

Forces depending on the earth's attraction and on rotation.

(80.) The expressions for X', Y', Z', or the attractions of the terrestrial spheroid on a particle of the sea, will be derived from those for Z', X', Y', in Article 66. of the FIGURE OF THE EARTH, observing that here (in conformity with the notation used by Laplace in this investigation) we have taken x for the axis of rotation, instead of z, as in the FIGURE OF THE EARTH. Those expressions suppose the form of the terrestrial spheroid to be the form of equilibrium of its strata supposed fluid; which supposition is not infringed by our assumption that the depth of the sea is unequal, because the depth of the sea and its inequalities are very small in comparison with the earth's ellipticity. Taking the expressions referred to, and observing that m, by Article 62. of that treatise, $=\frac{3\pi}{T^2}\cdot\frac{b^3}{\phi(b)}$, and that ϕ (b) in Article 66.$=\frac{3M}{4\pi}$, so that $m=\frac{4\pi^2b^3}{T^2M}$; and remarking that, as T is the time of rotation, and n the angular velocity of rotation, $nT=2\pi$, so that $\frac{4\pi^2}{T^2}=n^2$, and m therefore $=\frac{n^2b^3}{M}$; we obtain

$$X'=-\frac{Mx}{r^3}-x\frac{b^2(3y^2+3z^2-2x^2)}{r^7}\left(Me-\frac{n^2b^3}{2}\right)$$

$$Y'=-\frac{My}{r^3}-y\frac{b^2(y^2+z^2-4x^2)}{r^7}\left(Me-\frac{n^2b^3}{2}\right)$$

$$Z'=-\frac{Mz}{r^3}-z\frac{b^2(y^2+z^2-4x^2)}{r^7}\left(Me-\frac{n^2b^3}{2}\right).$$

Substituting these in the expressions for $\frac{dp}{dr}$, &c. in (78.), and combining the terms which depend on n^2, we have

$$\frac{dp'}{dr}=-\frac{M}{r^2}+\frac{b^2(2-3\sin^2\theta)}{r^4}\left(Me-\frac{n^2b^3}{2}\right)+n^2r\sin^2\theta$$

$$\frac{dp'}{d\theta}=+\frac{2b^2\sin\theta\cos\theta}{r^3}\left(Me-\frac{n^2b^3}{2}\right)+n^2r^2\sin\theta.\cos\theta$$

$$\frac{dp'}{d\varpi}=0.$$

Integrating these,

$$p'=\frac{M}{r}+\frac{b^2\left(\sin^2\theta-\frac{2}{3}\right)}{r^3}\left(Me-\frac{n^2b^3}{2}\right)+\frac{n^2r^2}{2}\sin^2\theta+C.$$

If we conceive $r=r'+r''$, r' being the value of r at the surface of the undisturbed watery spheroid, and r'' a variable quantity (which at the surface$=w$), it will be unnecessary to insert r'' in any term but the first or largest term. Thus

$$p'=\frac{\mathrm{M}}{r'}-\frac{\mathrm{M}r''}{r'^2}+\frac{\mathrm{Mb^2e}}{r'^3}\left(\sin^2\theta-\frac{2}{3}\right)+\frac{n^2}{2}\left(r'^2\sin^2\theta-\frac{\mathrm{b}^5}{r'^3}\sin^2\theta+\frac{2\mathrm{b}^5}{3r'^3}\right)+\mathrm{C}.$$

Pressure depending on the earth's attraction and on rotation

Putting for r' in the first term its approximate value b $(1+\mathrm{e}\sin^2\theta)$, and in the other terms b only, it becomes

$$p'=\frac{\mathrm{M}}{\mathrm{b}}-\frac{\mathrm{M}}{\mathrm{b}^2}r''-\frac{2}{3}\frac{\mathrm{Me}}{\mathrm{b}}+\frac{n^2\mathrm{b}^2}{3}+\mathrm{C}.$$

The whole of this, except the second term, is constant: and we may therefore write it, $p'=\mathrm{I}-\frac{\mathrm{M}}{\mathrm{b}^2}r''$. And as $\frac{\mathrm{M}}{\mathrm{b}^2}$ =attraction of the earth nearly, $=g$ nearly, we may write it

$$p'=\mathrm{I}-gr''.$$

(81.) Before leaving this part of the investigation we will remark that, if at any one place where $r''=0$ the pressure is 0 (that is, if the surface be free there), it will $=0$ at every place where $r''=0$, or where $r=r'$, or where the fluid has the external boundary which we have taken from the investigations of the Figure of the Earth. And thus our process in this article, conducted without mention of centrifugal force, leads to the same result as that which was based upon it.

(82.) The equations determining p'' will be

$$\frac{dp''}{dr}=\mathrm{X}''\frac{x}{r}+\mathrm{Y}''\frac{y}{r}+\mathrm{Z}''\frac{z}{r}$$

$$\frac{dp''}{d\theta}=-\mathrm{X}''\sqrt{y^2+z^2}+\frac{\mathrm{Y}''xy+\mathrm{Z}''xz}{\sqrt{y^2+z^2}}$$

$$\frac{dp''}{d\varpi}=\mathrm{Z}''y-\mathrm{Y}''z.$$

Relative co-ordinates of the sun or moon.

Now let S be the mass of the sun; D its distance from the earth's centre; D′ its distance from the particle at x, y, z; s its right ascension; σ its declination. Then x' for the sun$=\mathrm{D}\sin\sigma$; $y'=\mathrm{D}\cos\sigma.\cos s$; $z'=\mathrm{D}\cos\sigma.\sin s$;

$$x'-x=\mathrm{D}\sin\sigma-r\cos\theta,$$

$$y'-y=\mathrm{D}\cos\sigma.\cos s-r\sin\theta.\cos\overline{nt+\varpi},$$

$$z'-z=\mathrm{D}\cos\sigma.\sin s-r\sin\theta.\sin\overline{nt+\varpi};$$

$$\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}=\mathrm{D}'=\sqrt{\{\mathrm{D}^2-2\mathrm{D}r(\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})+r^2\}};$$

and, preserving only the first power of r,

$$\frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}=\frac{1}{\mathrm{D}'^3}=\frac{1}{\mathrm{D}^3}\times\left\{1+\frac{3r}{\mathrm{D}}(\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})\right\}.$$

Now the whole force which the sun exerts upon the particle at x, y, z, in the direction of x, is $\frac{x'-x}{\mathrm{D}'}\times\frac{\mathrm{S}}{\mathrm{D}'^2}=\frac{\mathrm{S}(x'-x)}{\mathrm{D}'^3}$. But the whole of this is not disturbing force upon the water in regard to its movement upon the earth. In order to find this, we must, as in (25.), subtract the force which the sun exerts upon the earth's centre of gravity, which is the same as if the whole mass of the earth were collected at its centre, and is therefore $=\frac{\mathrm{S}x'}{\mathrm{D}^3}$ in the direction of x. Subtracting this, we have

$$\mathrm{X}''=\frac{\mathrm{S}(x'-x)}{\mathrm{D}'^3}-\frac{\mathrm{S}x'}{\mathrm{D}^3}=\mathrm{S}x'\left(\frac{1}{\mathrm{D}'^3}-\frac{1}{\mathrm{D}^3}\right)-\frac{\mathrm{S}x}{\mathrm{D}'^3}$$

Disturbing forces of the sun or moon.

Similarly

$$\mathrm{Y}'=\mathrm{S}y'\left(\frac{1}{\mathrm{D}'^3}-\frac{1}{\mathrm{D}^3}\right)-\frac{\mathrm{S}y}{\mathrm{D}'^3}$$

$$\mathrm{Z}''=\mathrm{S}z'\left(\frac{1}{\mathrm{D}'^3}-\frac{1}{\mathrm{D}^3}\right)-\frac{\mathrm{S}z}{\mathrm{D}'^2}.$$

(83.) The expressions at the beginning of last article therefore become

$$X''\frac{x}{r}+Y''\frac{y}{r}+Z''\frac{z}{r}=\frac{S}{r}(x'x+y'y+z'z)\left(\frac{1}{D^3}-\frac{1}{D^3}\right)-\frac{Sr}{D'^3}$$

$$-X''\sqrt{y^2+z^2}+\frac{Y''xy+Z''xz}{\sqrt{y^2+z^2}}=\frac{S}{\sqrt{y^2+z^2}}\left(x.\overline{y'y+z'z}-x'.\overline{y^2+z^2}\right)\left(\frac{1}{D'^3}-\frac{1}{D^3}\right)$$

$$Z''y-Y''z \qquad =S.\overline{z'y-y'z}.\left(\frac{1}{D'^3}-\frac{1}{D^3}\right).$$

Substituting in these the values of x, y, z, in terms of r, θ, and ϖ, and those of x', y', z', in terms of D, σ, s; we find (retaining only the principal power of r),

$$\frac{dp''}{dr}=\frac{Sr}{D^3}\{3(\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})^2-1\}$$

$$\frac{dp''}{d\theta}=\frac{3Sr^2}{D^3}\{\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s}\}.\{-\sin\sigma.\sin\theta+\cos\sigma.\cos\theta.\cos\overline{nt+\varpi-s}\}$$

$$\frac{dp''}{d\varpi}=-\frac{3Sr^2}{D^3}\{\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s}\}\cos\sigma.\sin\theta.\sin\overline{nt+\varpi-s}.$$

Pressure produced by the sun or moon.

Integrating, $$p''=\frac{Sr^2}{2D^3}\{3(\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})^2-1\}.$$

It is unnecessary to add an arbitrary constant, since one has been attached to the expression for p', to which this is to be added.

(84.) In conformity with the remarks in (68.), no regard is to be given, in this expression, to the variation in the values of r arising from the difference in the depth of particles of water. Instead of r, therefore, we may put the elliptic value r', or even the polar radius b. (This amounts to the assumption, that the Tides upon a spheroid, so nearly spherical as the earth, will not sensibly differ from those upon a sphere.) Putting b for r, and adding p' and p'', we find,

Equations for the pressure depending on the motion of the sea.

$$p=I-gr''+\frac{3Sb^2}{2D^3}\{(\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})^2-\tfrac{1}{3}\}+p'''$$

where $$\frac{dp'''}{dr}=2nr.\sin^2\theta.\frac{dv}{dt}$$

$$\frac{dp'''}{d\theta}=-r^2\frac{d^2u}{dt^2}+2nr^2\sin\theta.\cos\theta.\frac{dv}{dt}$$

$$\frac{dp'''}{d\varpi}=-2nr^2\sin\theta.\cos\theta\frac{du}{dt}-r^2\sin^2\theta.\frac{d^2v}{dt^2}.$$

At the surface of the water, $p=0$. But at the surface $r''=w$; and, by the equation of continuity (72.) $w=-\frac{d}{d\theta}(u\gamma)-u\gamma.\text{cotan}\,\theta-\gamma\frac{dv}{d\varpi}$. Hence the tidal equation for the surface of the water becomes

$$0=I+\frac{3Sb^2}{2D^3}\{(\sin\sigma.\cos\theta+\cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})^2-\tfrac{1}{3}\}-gw+p''';$$

where w and p''' are subject to the equations expressed above.

(85.) If we expand the middle term of the tidal equation, that equation becomes

$$0=I+\frac{Sb^2}{2D^3}(\tfrac{3}{2}\cos^2\sigma-1)(1-3\cos^2\theta)+\frac{3Sb^2}{4D^3}\sin 2\sigma.\sin 2\theta.\cos\overline{nt+\varpi-s}$$
$$+\frac{3Sb^2}{4D^3}\cos^2\sigma.\sin^2\theta.\cos\overline{2nt+2\varpi-2s}-gw+p'''.$$

For most purposes, we may consider the variation of the terms depending on S to be produced only by the variation of $\overline{nt+\varpi-s}$, and we may consider $\frac{d}{dt}(nt+\varpi-s)$ or $n-\frac{ds}{dt}$ as a constant $=n'$, or we may represent $nt+\varpi-s$ by $n't+\varpi$ (by changing, if necessary, the origin of the time). Some of the terms therefore of the equation, depending on the disturbing force, are constant: some multiply $\cos\overline{n't+\varpi}$; and some multiply $\cos\overline{2n't+2\varpi}$. In strictness we ought to consider that σ and D are both variable; but they may be expressed in series of sines and cosines of multiples of the time; and when combined with $\cos\overline{n't+\varpi}$ and $\cos\overline{2n't+2\varpi}$ they will produce cosines of other arcs in which the coefficient of t is different, but that of ϖ is the same. The most general form therefore for one of the terms will be

General term of the pressure produced by the disturbing forces.

$$\Theta . \cos \overline{it+k\varpi},$$

where, for the first of the important terms, $\Theta=\mathrm{A}.\sin 2\theta$, $k=1$, and $i=n'$, or differs little from n; for the second $\Theta=\mathrm{B}.\sin^2\theta$, $k=2$, and $i=2n'$; and where, in the terms in general, k has no values but 0, 1, 2, but i may have any value between wide limits, and Θ may have any multiples of $\sin 2\theta$ and $\sin^2\theta$ added together.

(86.) The equation between w, u, and v; those between p''', u, and v; and that between w, p''', and the terms arising from the disturbing force; being all linear, we may take the terms arising from the disturbing force separately, and, finding the solution for each term, we may add all together. It will be sufficient, therefore, to proceed with the solution of the equation

$$0=\Theta \cos \overline{it+k\varpi}-gw+p'''.$$

Limited equations for the pressure, as depending on all causes.

(87.) Now we may remark, that the equation for $\frac{dp'''}{dr}$ may be neglected, not because its terms are small, but because the variation of r to which they apply is insignificant. The reasoning of (68.) applies entirely to this; but the reader may see the same thing in the solution of the equations for p''. For, after having obtained the value of p'' by means of three equations $\left(\text{for } \frac{dp''}{dr}, \frac{dp''}{d\theta}, \text{and } \frac{dp''}{d\varpi}\right)$, we have put b for r, thereby implying that the variations of r are insensible; and we have thus obtained exactly the same expression as if we had put b for r in the equations for $\frac{dp''}{d\theta}$ and $\frac{dp'}{d\varpi}$ and had neglected the equation for $\frac{dp''}{dr}$; although the terms of $\frac{dp''}{dr}$ are comparable to those of $\frac{dp''}{d\theta}$ and $\frac{dp''}{d\varpi}$. In like manner we shall here put b for r and neglect $\frac{dp'''}{dr}$. Thus we have

$$\frac{dp'''}{d\theta}=-\mathrm{b}^2\frac{d^2u}{dt^2}+2n\mathrm{b}^2.\sin\theta.\cos\theta.\frac{dv}{dt}$$

$$\frac{dp'''}{d\varpi}=-2n\mathrm{b}^2.\sin\theta.\cos\theta\,\frac{du}{dt}-\mathrm{b}^2\sin^2\theta\,\frac{d^2v}{dt^2}$$

$$w=-\frac{d}{d\theta}(u\gamma)-u\gamma.\text{cotan}\,\theta-\gamma\frac{dv}{d\varpi}$$

$$0=\Theta\cos\overline{it+k\varpi}-gw+p'''$$

Popular explanation of the terms introduced by considering the earth's rotation.

(88.) The terms multiplying n, it is to be remarked, are produced entirely by introducing the consideration of the earth's rotation. Of the origin of these terms the following popular explanation (which, however, is sufficiently accurate to be used as a basis of calculation) may be given. Suppose a particle of water to be running towards the equator (that is, suppose that u is increasing). By this motion, it is proceeding from a small parallel towards a large one, or from a place in which the movement from west to east (produced by the earth's rotation) is small, to one in which the movement from west to east is large. It is, therefore, advancing to a part of the earth whose movement towards the east is more rapid than its own, and therefore it tends to lag behind the movement of that part of the earth. Consequently there will be, in the expression for $\frac{d^2v}{dt^2}$, (which is the acceleration towards the east,) a term depending on u, which denotes that there is a retarding force when u is increasing: that is, $\frac{d^2v}{dt^2}$ will be expressed by terms of which one is a negative multiple of $\frac{du}{dt}$. Again, suppose a particle of water to be running towards the east (that is, suppose that v is increasing). The angular velocity of this particle is greater than the earth's angular velocity: therefore the centrifugal force is greater than the centrifugal force would have been if it had had no such motion towards the east; there is therefore, from this cause, an addition of centrifugal force; and this addition (like the original centrifugal force) is in the direction of the radius of the parallel, and is perpendicular to the earth's axis; resolving this into a vertical and a horizontal force, the horizontal force is directed towards the equator, or tends to increase u: that is, $\frac{d^2u}{dt^2}$ will be expressed by terms of which one is a positive multiple of $\frac{dv}{dt}$.

(89.) A general solution of the equations of (87.) is scarcely to be hoped for; it is a matter of difficulty to find, in a very limited case, a particular integral which will satisfy them. We shall begin, in the manner of Laplace, by trying whether they cannot be satisfied by expressions of the following form:

Assumption of the form of solution as regards the longitude.

$$w=a.\cos\overline{it+k\varpi}$$
$$u=b.\cos\overline{it+k\varpi}$$
$$v=c.\sin\overline{it+k\varpi}$$
$$p'''=a'''.\cos\overline{it+k\varpi}$$

where a, b, c, a''', are functions of θ only.

(90.) Substituting these expressions, the equations become

$$\cos\overline{it+k\varpi}\cdot\frac{da'''}{d\theta}=\quad+\mathrm{b}^2 b.i^2\cos\overline{it+k\varpi}\quad+2n\,\mathrm{b}^2.\sin\theta.\cos\theta.c.i.\cos\overline{it+k\varpi}$$
$$-k.a'''.\sin\overline{it+k\varpi}=\quad+2n\,\mathrm{b}^2\sin\theta.\cos\theta.b.i.\sin\overline{it+k\varpi}\quad+\mathrm{b}^2.\sin^2\theta.c.i^2.\sin\overline{it+k\varpi}$$
$$a.\cos\overline{it+k\varpi}=\quad-\frac{d\,(b\gamma)}{d\,\theta}\cos\overline{it+k\varpi}\quad-b\gamma.\text{cotan}\,\theta.\cos\overline{it+k\varpi}\quad-\gamma ck.\cos\overline{it+k\varpi}$$
$$0\qquad=\Theta.\cos\overline{it+k\varpi}\quad-ga.\cos\overline{it+k\varpi}\quad+a'''.\cos\overline{it+k\varpi}$$

or

$$\frac{da'''}{d\theta}=\mathrm{b}^2 i^2.b+2n\,\mathrm{b}^2 i.\sin\theta.\cos\theta.c$$
$$-ka'''=2n\,\mathrm{b}^2 i.\sin\theta.\cos\theta.b+\mathrm{b}^2 i^2.\sin^2\theta.c$$
$$a=-\frac{d\,(b\gamma)}{d\,\theta}-b\,\gamma\;\text{cotan}\,\theta-\gamma\,c\,k$$
$$0=\Theta-g\,a+a'''.$$

The assumption is sufficient.

It is clear that our assumptions have satisfied the equations as far as regards ϖ; for every term containing ϖ has vanished upon making the substitution. Moreover we have four differential equations by which to determine four unknown quantities, and these (whatever practical difficulties we may find) are, theoretically, sufficient for their determination. It is therefore certain that a solution may be found, in the form which is expressed by the four assumptions for w, u, v, and p'''.

Inferences from the form assumed.

(91.) Before proceeding further with the solution, we may draw a few inferences from the form of the expressions that we have already found. First, since the velocity of the water in the direction of the meridian is $\mathrm{b}\frac{du}{dt}$ or $-\mathrm{b}bi\sin\overline{it+k\varpi}$, and the velocity perpendicular to the meridian is $\mathrm{b}\sin\theta.\frac{dv}{dt}$, or $\mathrm{b}c.\sin\theta.i.\cos\overline{it+k\varpi}$, the water at any given place, except in certain limited cases, is never at rest, but the direction in which it runs changes perpetually. The expression for its whole velocity at any instant is $\mathrm{b}\sqrt{\left(\left(\frac{du}{dt}\right)^2+\sin^2\theta\left(\frac{dv}{dt}\right)^2\right)}$, or $\mathrm{b}\sqrt{(b^2 i^2.\sin^2\overline{it+k\varpi}+c^2\sin^2\theta.i^2.\cos^2\overline{it+k\varpi})}$ and the tangent of the angle which its course makes with the meridian at that instant is $\frac{\sin\theta\frac{dv}{dt}}{\frac{du}{dt}}$, or $\frac{-c\sin\theta}{b}\,\text{cotan}\,\overline{it+k\varpi}$. This is the same kind of rotation to which we have alluded in (9.): but we shall see hereafter that the latter arises from a different cause.

The direction of the tide-current revolves.

High or low water in solar or lunar tide will occur at the transit of the sun or moon.

(92.) Secondly, since the term $\Theta.\cos\overline{it+k\varpi}$ produces a term w of the form $a.\cos\overline{it+k\varpi}$ (where a may be positive or negative), and since, for the principal tidal terms in p'', $\cos\overline{it+k\varpi}$ will have the form $\cos\overline{nt+\varpi-s}$, or $\cos\overline{2nt+2\varpi-2s}$, which terms have their maximum or minimum values when $s=nt+\varpi$, or when the sun's distance from the fixed meridian = the distance of the place from the fixed meridian, or when the sun is on the meridian of the place: it follows that either the high or the low water of the solar tide must occur, in conformity with this theory, exactly when the sun is on the meridian. A similar law applies to the time of the lunar tide.

Time of high water altered by the variation of the forces.

(93.) Thirdly, if a factor of Θ (which depends on the sun's linear distance and declination) is expressed by $1+f\cos i't$, where i' is small (or where the term $f\cos i't$ varies slowly), then $\Theta\cos\overline{it+k\varpi}$ will become

$$\Theta\cos\overline{it+k\varpi}+\frac{\Theta f}{2}\cos\overline{(i+i')\,t+k\varpi}+\frac{\Theta f}{2}\cos\overline{(i-i')\,t+k\varpi}:$$

and since a in the expression for w depends on i, we must use $a+\frac{da}{di}i'$ in the second term, and $a-\frac{da}{di}i'$ in the third term; and the expression for w will be

$$a \cos \overline{it+k\varpi}+\frac{f}{2}\left(a+\frac{da}{di}i''\right) \cos \overline{(i+i')\,t+k\varpi}+\frac{f}{2}\left(a-\frac{da}{di}i'\right) \cos \overline{(i-i')\,t+k\varpi}$$

or

$$a\,(1+f \cos i'\,t) \cos \overline{it+k\varpi}-i'\frac{da}{dt}.\,f \sin i'\,t.\, \sin \overline{it+k\varpi}.$$

The time of high water on any given day will not now be the same as before; for, before, it occurred when $it+k\varpi=0$ or 180°, but now it occurs when

$$\tan \overline{it+k\varpi}=\frac{i'.\frac{da}{dt}.\,f.\,\sin i'\,t}{a\,(1+f \cos i'\,t)} \text{ nearly.}$$

Day of highest tide corresponds to the day of greatest force.

But the highest tide of all (the height being expressed by the square root of the sum of the squares of the coefficients of $\cos \overline{it+k\varpi}$ and $\sin \overline{it+k\varpi}$, as in (46.),) occurs when $\sqrt{\left\{a^2\,(1+f \cos i't)^2+i'^2.\left(\frac{da}{dt}\right)^2.f^2.\sin^2 i't\right\}}$ is maximum; and this is when $\cos i't=1$, or when the force which causes the tide is greatest. As far, therefore, as this theory applies, the highest solar tide ought to occur on the very day on which the solar force is greatest; and similarly the highest lunar tide ought to occur on the very day on which the lunar force is greatest.

Assumption of expression for the depth of the sea.

Laplace's method of solving the equations depending on the latitude.

(94.) Returning now to our equations of (90.). The only way in which Laplace has attempted to solve them is the following. He has assumed, as the only law of depth on which a solution appears to be practicable, $\gamma=l-lq\cos^2\theta$. He has then shown that, taking separately the terms independent of $\overline{nt+\varpi-s}$, those depending on $\cos \overline{nt+\varpi-s}$, and those depending on $\cos \overline{2nt+2\varpi-2s}$, it is possible to determine for each the value of q for which the equations can be solved, (the solution, in Laplace's manner, not being practicable for all values of q). He has shown that, to a certain degree of approximation, the same value of q will apply to the terms of different kinds. And in the simplest case (when $q=0$, or the depth is the same in all parts) he has shown that the terms in the height of the water depending on $\cos \overline{nt+\varpi-s}$ will be insensible, or the diurnal tide will be insensible. We now proceed (although by a method different in its principal parts from Laplace's) to obtain equivalent results.

(95.) From the first and second of the equations in (90.) we find

$$b=\frac{-i.\,\sin\theta.\frac{da'''}{d\theta}-2kn \cos\theta.\,a'''}{\mathrm{b}^2\,i.\sin\theta.(4n^2\cos^2\theta-i^2)}, \qquad c=\frac{2n.\,\sin\theta.\,\cos\theta.\,\frac{da''}{d\theta}+i\,k.a'''}{\mathrm{b}^2\,i.\,\sin^2\theta.\,(4n^2.\cos^2\theta-i^2)}.$$

Substituting these values in the third, which may be put in the form

$$a.\,\sin\theta=-\frac{d}{d\theta}\,(\gamma.b.\,\sin\theta)-\gamma.k.c.\,\sin 0$$

we obtain

$$a.\,\sin\theta=\frac{d}{d\theta}\left\{\gamma.\frac{i.\,\sin\theta.\frac{da'''}{d\theta}+2kn.\,\cos\theta.\,a''}{\mathrm{b}^2\,i.\,(4n^2.\,\cos^2\theta-i^2)}\right\}-\gamma.\frac{2kn.\,\sin\theta.\,\cos\theta.\frac{da''}{d\theta}+ik^3.\,a''}{\mathrm{b}^2\,i.\,\sin\theta.\,(4n^2.\,\cos^2\theta-i^2)}.$$

The equations reduced to one differential equation.

But from the fourth equation,

$$\Theta.\,\sin\theta=g.a.\,\sin\theta-a'''.\,\sin\theta.$$

Substituting in this last the value of $a.\sin\theta$ just found, we finally obtain

$$\Theta\,\mathrm{n}\,\theta=\frac{g}{\mathrm{b}^2\,i}.\frac{d}{d\theta}\left\{\gamma.\frac{i.\,\sin\theta.\frac{da''}{d\theta}+2kn.\,\cos\theta.\,a'''}{4n^2.\,\cos^2\theta-i^2}\right\}-\frac{g}{\mathrm{b}^2 i}.\gamma.\frac{2kn.\,\sin\theta.\,\cos\theta.\frac{da''}{d\theta}+ik^2.\,a'''}{\sin\theta.\,(4n^2\cos^2\theta-i^2)}-a'''\sin\theta$$

a linear differential equation of the second order, in which the only unknown quantity is a''.

(96.) If we perform the differentiation and multiply by $\sin\theta.\,(4n^2.\,\cos^2\theta-i^2)^2$, the equation becomes

$$\begin{aligned}\Theta.\,\sin^2\theta.\,(4n^2.\,\cos^2\theta-i^2)^2= {} & \frac{g}{\mathrm{b}^2 i}.\frac{d\gamma}{d\theta}.\,(4n^2\cos^2\theta-i^2)\,.\,(i.\,\sin^2\theta.\,\frac{da'''}{d\theta}+2kn.\,\sin\theta.\,\cos\theta.\,a''') \\ & +\frac{g}{\mathrm{b}^2 i}.\gamma.\left\{\begin{aligned}& i.\,\sin^2\theta.\,(4n^2.\,\cos^2\theta-i^2).\frac{d^2a'''}{d\theta^2}+i.\,\sin\theta.\,\cos.\,\theta\,(8n^2\sin^2\theta+4n^2\cos^2\theta-i^2)\,\frac{da'''}{d\theta} \\ & +\,(i^3\,k^2+2i^2\,kn.\,\sin^2\theta-4i\,k^2n^2.\,\cos^2\theta+8kn^3.\,\sin^2\theta.\,\cos^2\theta)\,.\,a''\end{aligned}\right\} \\ & -\sin^2\theta.\,(4n^2.\,\cos^2\theta-i^2)^2.\,a''\end{aligned}$$

If, with Laplace, we suppose $\gamma = l.\ (1-q.\ \cos^2\theta)$, $\frac{d\gamma}{d\theta} = 2lq.\ \sin\theta.\ \cos\theta$; and the equation takes the following form; in which we may remark that the factors of Θ, a''', and $\frac{d^2a'''}{d\theta^2}$, contain only even powers of $\cos\theta$; and that the factor of $\frac{da'''}{d\theta}$ contains only odd powers of $\cos\theta$, multiplied by $\sin\theta$.

Expansion of differential equation with assumed expression for the depth.

$$\Theta \times \{i^4 + (-8i^2n^2 - i^4)\cos^2\theta + (16n^4 + 8i^2n^2)\cos^4\theta - 16n^4\cos^6\theta\} =$$

$$\frac{d^2a'''}{d\theta^2} \times \frac{g}{\mathrm{b}^2} \times \left\{-i^2l + (4ln^2 + i^2l + i^2lq)\cos^2\theta + (-4ln^2 - 4ln^2q - i^2lq)\cos^4\theta + 4ln^2q\cos^6\theta\right\}$$

$$+\frac{da'''}{d\theta} \times \frac{g}{\mathrm{b}^2} \times \cos\theta.\ \sin\theta \times \left\{(8ln^2 - 2i^2lq - i^2l) + (3i^2lq - 4ln^2)\cos^2\theta - 4ln^2q\cos^4\theta\right\}$$

$$+a''' \times \frac{g}{\mathrm{b}^2} \times \left\{(i^2k^2l + 2ikln) + \left(\frac{8kln^3}{i} - 2ikln - 4k^2ln^2 - 6iklnq - i^2k^2lq\right)\cos^2\theta\right.$$

$$\left. + \left(-\frac{8kln^3}{i} + \frac{8kln^3q}{i} + 4k^2ln^2q\ .\ + 6iklnq\right)\cos^4\theta - \frac{8kln^3q}{i}\cos^6\theta\right\}$$

$$+a''' \times \{-i^4 + (8i^2n^2 + i^4)\cos^2\theta + (-16n^4 - 8i^2n^2)\cos^4\theta + 16\,n^4\cos^6\theta\}.$$

It is evident that this equation can always be solved by assuming for a''' a form similar to that for Θ multiplied by a series of powers of $\cos^2\theta$ with indeterminate coefficients. The same would have held if we had assumed for γ an expression consisting of a greater number of terms with even powers of $\cos\theta$.

We shall now proceed with some steps of the solution of this equation.

(97.) Omitting the first or constant term in the equation of (85.), which, in fact, is only an arbitrary constant, to be so determined with reference to the term gw that the whole mass of the fluid will not be altered, the next term is $\frac{\mathrm{S}\,\mathrm{b}^2}{2\mathrm{D}^3}\left(\frac{3}{2}\cos^2\sigma - 1\right)(1 - 3\cos^2\theta)$. This term does not contain ϖ at all, and therefore we must make $k=0$. But as D is slowly variable, and as σ also is variable, the factor $\frac{\mathrm{S}\,\mathrm{b}^2}{2\mathrm{D}^3}\left(\frac{3}{2}\cos^2\sigma - 1\right)$ would, if expanded, consist of a series of terms like $\mathrm{A}\cos it$, where i is small. Comparing $\mathrm{A}\cos it\ (1-3\cos^2\theta)$ with $\Theta\cos\overline{it + k\varpi}$ we have $\Theta = \mathrm{A}\ (1-3\cos^2\theta)$, $k=0$, i a small number. Substituting for Θ and k, and retaining only the lowest power of i in each term, the equation becomes

Equation corresponding to slow changes of the disturbing force.

$$\mathrm{A} \times (16n^4\cos^4\theta - 64n^4\cos^6\theta + 48n^4\cos^8\theta) =$$

$$\frac{d^2a'''}{d\theta^2} \times \frac{g}{\mathrm{b}^2} \times \left\{4ln^2\cos^2\theta + (-4ln^2 - 4ln^2q)\cos^4\theta + 4ln^2q\cos^6\theta\right\}$$

$$+\frac{da'''}{d\theta} \times \frac{g}{\mathrm{b}^2} \times \cos\theta.\sin\theta \times (8ln^2 - 4ln^2\cos^2\theta - 4ln^2q\cos^4\theta)$$

$$+a''' \times (-16n^4\cos^4\theta + 16n^4\cos^6\theta).$$

Suppose

$$a''' = \mathrm{B}_2\cos^2\theta + \mathrm{B}_4\cos^4\theta + \mathrm{B}_6\cos^6\theta + \&\mathrm{c}.$$

Forming the expressions for $\frac{da''}{d\theta}$ and $\frac{d^2a'''}{d\theta^2}$, and expanding all the even powers of $\sin\theta$ in terms of $\cos\theta$, and after substituting in the equation, making the coefficient of each power of $\cos\theta = 0$, we find

Solution in powers of the cosine of the latitude.

$$a'' = \mathrm{A} \times \left\{\frac{\mathrm{b}^2n^2}{gl}\cos^4\theta + \frac{2q}{3}.\frac{\mathrm{b}^2n^2}{gl}\cos^6\theta + \frac{\mathrm{b}^2n^2}{gl}\left(\frac{\mathrm{b}^2n^2}{10gl} + \frac{q^2}{2}\right)\cos^8\theta\right.$$

$$\left. + \left(\frac{\mathrm{b}^2n^2}{gl} \times \left(\frac{2}{25} + \frac{62}{15.35}q\right) + \frac{14}{35}q^3\right)\frac{\mathrm{b}^2n^2}{gl}\cos^{10}\theta + \&\mathrm{c}.\right\}.$$

And

$$a = \frac{\Theta}{g} + \frac{a''}{g}$$

$$= \frac{\mathrm{A}}{g}(1 - 3\cos^2\theta) + \frac{\mathrm{A}}{g}\ \text{multiplied by the series just found.}$$

This is the coefficient of $\cos it$ in the expression for w, the tidal elevation of the water (89.).

(98.) If we remark that the centrifugal force at the equator is expressed by $\mathrm{b}n^2$, it is evident that $\frac{\mathrm{b}n^2}{g}$ is the proportion of that centrifugal force to gravity, or is the quantity which in the Treatise on the

FIGURE OF THE EARTH we have called m. The fraction $\frac{\mathrm{b}^2 n^2}{gl}$ is therefore $=\frac{m}{\frac{l}{\mathrm{b}}}$, and is not small, therefore, except m or $\frac{1}{289}$ be smaller than the proportion of the depth of the sea to the earth's radius, which is expressed by $\frac{l}{\mathrm{b}}$. The form of solution adopted above will not, therefore, strictly apply if the depth is very small, as the coefficients of the series will not converge. In that case, it will be advantageous to put the solution in this form:

$$a'''=\mathrm{C}_0+\mathrm{C}_1 . l+\mathrm{C}_2 l^2+\mathrm{C}_3 l^3+\&\mathrm{c}.,$$

where C_0, C_1, C_2, C_3, &c. are functions of θ: and to substitute in the equation, and make the coefficient of each power of $l=0$. Thus we shall have

Solution in powers of the depth of the sea.

$$+\mathrm{A}\,(1-3\cos^2\theta)=-\mathrm{C}_0$$

$$0=\frac{d^2\mathrm{C}_0}{d\theta^2}\times\frac{g}{\mathrm{b}^2}\times\{4n^2\cos^2\theta+(-4n^2-4n^2q)\cos^4\theta+4n^2q\cos^6\theta\}$$

$$+\frac{d\mathrm{C}_0}{d\theta}\times\frac{g}{\mathrm{b}^2}\times\cos\theta . \sin\theta\times(8n^2-4n^2\cos^2\theta-4n^2q\cos^4\theta)$$

$$+\mathrm{C}_1\times(-16n^4\cos^4\theta+16n^4\cos^6\theta);$$

from which

$$\mathrm{C}_1\, l=\frac{-\mathrm{A}gl}{\mathrm{b}^2 n^2}.\frac{3}{2}.\frac{1+\cos^2\theta+q\cos^2\theta-3q\cos^4\theta}{\cos^2\theta},$$

and so on, the determination of each successive term being effected in the same manner. Hence a''' is expressed by the series

$$-\mathrm{A}\,(1-3\cos^2\theta)-\mathrm{A}.\frac{gl}{\mathrm{b}^2 n^2}.\frac{3}{2}.\frac{1+\cos^2\theta+q\cos^2\theta-3q\cos^4\theta}{\cos^2\theta}+\&\mathrm{c}.$$

and

$$a=\frac{\Theta}{g}+\frac{a'''}{g}=\frac{\mathrm{A}\,(1-3\cos^2\theta)}{g}+\frac{a'''}{g}=$$

$$-\frac{\mathrm{A}}{g}.\frac{l}{\mathrm{b}m}.\frac{3}{2}.\frac{1+\cos^2\theta+q\cos^2\theta-3q\cos^4\theta}{\cos^2\theta}+\&\mathrm{c}.$$

The series converges by powers of $\frac{l}{\mathrm{b}m}$, and diverges by powers of $\cos^2\theta$. Algebraically speaking, therefore, it fails near the equator: but it is probable that it would apply so near to it that, in regard to physical interpretation, the failure would be unimportant.

(99.) The fluctuation of which we have treated includes all those which Laplace denominates "*les oscillations de la première espèce.*"

(100.) The second term depending on the disturbing force, in the equation of (85.), is $+\frac{3\mathrm{Sb}^2}{4\mathrm{D}^3}\sin 2\sigma . \sin 2\theta . \cos\overline{nt+\varpi-s}$, or $\frac{3\mathrm{Sb}^2}{2\mathrm{D}^3}\sin 2\sigma . \sin\theta . \cos\theta . \cos\overline{nt+\varpi-s}$, which we will call $\mathrm{E} . \sin\theta . \cos\theta . \cos\overline{nt+\varpi-s}$. Comparing this with the general term assumed in (85.), we have

$$\Theta=\mathrm{E} . \sin\theta . \cos\theta$$

$$k=1$$

$$i=n-\frac{ds}{dt}.$$

In the investigation on which we are about to enter, we shall consider E as constant, (its variation depending only on the changes of distance and declination of the luminary, which are slow,) and $\frac{ds}{dt}$ as insignificant. The last assumption gives us

$$i=n.$$

Substituting these values of k and i, in the equation of (96.), and dividing by n^4, it becomes $\left(\text{since} . \frac{gl}{\mathrm{b}^2 n^2}=\frac{l}{\mathrm{b}m}\right)$

$$\Theta\times(1-\cos^2\theta)\times(1-4\cos^2\theta)^2=$$

Equation corresponding to diurnal Tide.

$$\frac{d^2 a'''}{d\theta^2}\times\frac{l}{\mathrm{b}m}\times\{-1+(5+q)\cos^2\theta+(-4-5q)\cos^4\theta+4q\cos^6\theta\}$$

$$+\frac{da'''}{d\theta}\times\frac{l}{\mathrm{b}m}\times\cos\theta.\sin\theta\times\{(7-2q)+(-4+3q)\cos^2\theta-4q\cos^4\theta\}$$

$$+a'''\times\frac{l}{\mathrm{b}m}\times\{3+(2-7q)\cos^2\theta+(-8+18q)\cos^4\theta-8q\cos^6\theta\}$$

$$-a'''\times(1-\cos^2\theta)\times(1-4\cos^2\theta)^2.$$

We shall solve this by the same process as that used in (98.)

(101.) Suppose then

$$a'''=\mathrm{F}_0+\mathrm{F}_1\frac{l}{\mathrm{b}m}+\mathrm{F}_2\frac{l^2}{\mathrm{b}^2m^2}+\&\mathrm{c}.;$$

F_0, F_1, F_2, &c. being functions of θ. Substituting this in the equation, and making the coefficient of each power of $l=0$, the first comparison (or the comparison of terms independent of l) evidently gives

$$\Theta\times(1-\cos^2\theta)\times(1-4\cos^2\theta)^2=-\mathrm{F}_0\times(1-\cos^2\theta)\times(1-4\cos^2\theta)^2,$$

or

$$\mathrm{F}_0=-\Theta=-\mathrm{E}.\sin\theta.\cos\theta.$$

From this we have

$$\frac{d\mathrm{F}_0}{d\theta}=\mathrm{E}.(1-2\cos^2\theta),\quad\frac{d^2\mathrm{F}_0}{d\theta^2}=+4\mathrm{E}.\sin\theta.\cos\theta.$$

The second comparison (or that of terms containing the first power of l) gives

$$0=\frac{d^2\mathrm{F}_0}{d\theta^2}\times\frac{l}{\mathrm{b}m}\times\{-1+(5+q)\cos^2\theta+(-4-5q)\cos^4\theta+4q\cos^6\theta\}$$

$$+\frac{d\mathrm{F}_0}{d\theta}\times\frac{l}{\mathrm{b}m}\times\cos\theta.\sin\theta\times\{(7-2q)+(-4+3q)\cos^2\theta-4q\cos^4\theta\}$$

$$+\ \mathrm{F}_0\ \times\frac{l}{\mathrm{b}m}\times\{3+(2-7q)\cos^2\theta+(-8+18q)\cos^4 0-8q\cos^6\theta\}$$

$$-\ \mathrm{F}_1\ \times\frac{l}{\mathrm{b}m}\times(1-\cos^2\theta)\times(1-4\cos^2\theta)^2.$$

Substituting, in the three first lines, the expressions found for F_0, $\frac{d\mathrm{F}_0}{d\theta}$, and $\frac{d^2\mathrm{F}_0}{d\theta^2}$, the equation becomes

$$0=\mathrm{E}\times\frac{l}{\mathrm{b}m}\times\sin\theta.\cos\theta\times\{-2q+18q\cos^2\theta-48q\cos^4\theta+32q\cos^6\theta\}-\mathrm{F}_1\times\frac{l}{\mathrm{b}m}\times(1-\cos^2\theta)\times(1-4\cos^2\theta)^2,$$

or

$$0=-\mathrm{E}.\sin\theta.\cos\theta\times 2q.(1-\cos^2\theta).(1-4\cos^2\theta)^2-\mathrm{F}_1\times(1-\cos^2\theta).(1-4\cos^2\theta)^2;$$

from which

$$\mathrm{F}_1=-2q.\mathrm{E}.\sin\theta.\cos\theta=+2q\,\mathrm{F}_0.$$

The form of the equation between F_1 and F_2, that between F_2 and F_3, that between F_3 and F_4, &c. will be exactly the same as the form of that between F_0 and F_1: and since the form of F_1 (considered as a function of θ) is exactly the same as that of F_0, the nature of the substitution in the equation will be the same, and therefore the relation of all the successive coefficients F_1, F_2, &c. will be the same. Thus we shall have

$$\mathrm{F}_2=2q\mathrm{F}_1=-4q^2.\mathrm{E}.\sin\theta.\cos\theta$$

$$\mathrm{F}_3=2q\mathrm{F}_2=-8q^3.\mathrm{E}.\sin\theta.\cos\theta,$$

and so on. The expression for a''', therefore, is

$$-\mathrm{E}\sin\theta.\ \cos\theta-\mathrm{E}\ \sin\theta.\cos\theta\left(\frac{2ql}{\mathrm{b}m}+\frac{4q^2l^2}{\mathrm{b}^2m^2}+\frac{8q^3l^3}{\mathrm{b}^3m^3}+\&\mathrm{c}.\right)$$

$$=-\mathrm{E}\sin\theta\cos\theta-\frac{2q\frac{l}{\mathrm{b}m}\mathrm{E}\sin\theta.\cos\theta}{1-\frac{2ql}{\mathrm{b}m}};$$

and, therefore,

$$a=\frac{\Theta}{g}+\frac{a'''}{g}=-\frac{2q\frac{l}{\mathrm{b}m}}{1-\frac{2ql}{\mathrm{b}m}}.\frac{\mathrm{E}}{g}.\sin\theta\cos\theta,$$

Finite solution corresponding to diurnal tide.

an expression in a finite form. The expression for the elevation of the water is, therefore,

$$-\frac{2q\dfrac{l}{\mathrm{b}m}}{1-\dfrac{2ql}{\mathrm{b}m}}\cdot\frac{\mathrm{E}}{g}.\sin\theta.\cos\theta.\cos\overline{nt+\varpi-s}.$$

When the depth is uniform, there is no diurnal tide in height.

(102.) In article (96.) we assumed the depth of the water to be represented by $l(1-q\cos^2\theta)$. If, then, the depth of the water be uniform over the whole globe, $q=0$. In that case the expression just found for the elevation of the water vanishes for every value of θ and ϖ, or for every point on the earth's surface. Thus we obtain the most remarkable and most unexpected result, that if the depth of the sea is uniform, the diurnal tide (or that depending on $nt+\varpi-s$) is insensible. For, the expression for its elevation absolutely vanishes when the depth of the water is everywhere the same.

But there is diurnal tide in motion.

(103.) This evanescence of the tide applies, however, only to the *elevation* of the water. The *horizontal motion* is not destroyed. For, the displacement of the water in the direction of the meridian is u, and therefore its velocity in the direction of the meridian is $\frac{du}{dt}=-ib\sin\overline{it+k\varpi}$, (89.), $=-nb.\sin\overline{nt+\varpi-s}$: its displacement in longitude is v, and therefore its velocity in the direction of the parallel is $\sin\theta.\frac{dv}{dt}=ic.\sin\theta.\cos\overline{it+k\varpi}$ $=n.c.\sin\theta.\cos\overline{nt+\varpi-s}$. Now, taking the expressions for b and c in (95.), and assuming $a'''=-\mathrm{E}.\sin\theta.\cos\theta$, (to which it is reduced when $q=0$,) and $\frac{da'''}{d\theta}=\mathrm{E}(1-2\cos^2\theta)$, we obtain $b=\frac{\mathrm{E}}{\mathrm{b}^2n^2}$, $c=-\frac{\mathrm{E}\cos\theta}{\mathrm{b}^2n^2\sin\theta}$: from which the velocity in the direction of the meridian $=\frac{-\mathrm{E}}{\mathrm{b}n}\sin\overline{nt+\varpi-s}$, and that in the direction of the parallel $=\frac{-\mathrm{E}}{\mathrm{b}n}\cos\theta.\cos\overline{nt+\varpi-s}$: and the whole velocity of the water, which is the square root of the sum of the squares of these quantities, $=\frac{\mathrm{E}}{\mathrm{b}^2n}\sqrt{1-\sin^2\theta.\cos^2\overline{nt+\varpi-s}}$, and the tangent of the angle which its course makes with the parallel $=\frac{1}{\cos\theta}\tan\overline{nt+\varpi-s}$. Thus we obtain the result, that at the equator the water moves only north and south, resting for an instant at the change of motion: on every other part of the earth the water is always moving with some velocity, but the current is perpetually changing its direction: at the pole, the velocity is constant, and the direction is always transverse to the meridian which passes through the luminary. The same remarks hold when the depth of the water is not uniform: as the expressions for a''', b, and c differ from those which apply when the depth is uniform, only by having the factor $\frac{\mathrm{E}}{1-\dfrac{2ql}{\mathrm{b}m}}$ instead of E.

Rules for the time of high and low water in the diurnal tide.

(104.) If the place of observation is north of the equator, $\sin\theta.\cos\theta$ is positive: and if the declination of the luminary is north, $\frac{3\mathrm{Sb}^2}{2\mathrm{D}^3}\sin 2\sigma$ or E is positive: and if the luminary is on the meridian of the place, $\cos\overline{nt+\varpi-s}$ is positive, and has its greatest value. Hence, when the depth is variable, the sign of the elevation at the transit of the luminary, if on the same side of the equator as the place of observation, will be the same as that of $-q$. If the water be shallower at the poles than at the equator, q in the expression $l(1-q\cos^2\theta)$ must be positive, and therefore low water occurs at the transit of the luminary. If the water be deeper at the poles, q is negative, and the high water occurs at transit.

(105.) The numerical values of all these quantities will be computed by the process of (31.) and (35.) The mean value of $\frac{\mathrm{Sb}^2}{2\mathrm{D}^3g}$ for the Sun is 0·2710 foot: for the Moon, 0·5959 foot. If $l=\frac{7}{5}$ of a mile, $\frac{l}{\mathrm{b}m}=\frac{1}{10}$.

(106.) The fluctuations of which we have now treated are called by Laplace "*les oscillations de la seconde espèce.*"

(107.) The third term depending on the disturbing force, in the equation of (85.), is $\frac{3\mathrm{Sb}^2}{4\mathrm{D}^3}\cos^2\sigma.\sin^2\theta.\cos\overline{2nt+2\varpi-2s}$, which we shall call $\mathrm{G}.\sin^2\theta.\cos\overline{2nt+2\varpi-2s}$. Comparing this with the general term assumed in (85.), we have

$$\Theta = \mathrm{G}\sin^2\theta$$
$$k = 2$$
$$i = 2n-2\frac{ds}{dt}=2n \text{ nearly.}$$

Equation corresponding to semi-diurnal tide,

Substituting these values in the equation of (96.), and dividing by n^4, it becomes

$$\begin{aligned} 16.\mathrm{G}.\sin^8\theta = {} & \frac{d^2a'''}{d\theta^2}\times\frac{l}{\mathrm{b}m}\times\{-4+(8+4q)\cos^2\theta+(-4-8q)\cos^4\theta+4q\cos^6\theta\} \\ & +\frac{da'''}{d\theta}\times\frac{l}{\mathrm{b}m}\times\cos\theta.\sin\theta\times\{(4-8q)+(-4+12q)\cos^2\theta-4q\cos^4\theta\} \\ & +a'''\times\frac{l}{\mathrm{b}m}\times\{24+(-16-40q)\cos^2\theta+(-8+48q)\cos^4\theta-8q\cos^6\theta\} \\ & -16.a'''.\sin^6\theta. \end{aligned}$$

Dividing by 4 $\sin^4\theta$, this becomes

$$\begin{aligned} 4.\mathrm{G}.\sin^4\theta = {} & \frac{d^2a'''}{d\theta^2}\times\frac{l}{\mathrm{b}m}\times\{-1+q\cos^2\theta\} \\ & +\frac{da'''}{d\theta}\times\frac{\cos\theta}{\sin\theta}\times\frac{l}{\mathrm{b}m}\times\{(1-2q)+q\cos^2\theta\} \\ & +\frac{a'''}{\sin^2\theta}\times\frac{l}{\mathrm{b}m}\times\{6+(2-10q)\cos^2\theta+2q\cos^4\theta\} \\ & -4a'''.\sin^2\theta. \end{aligned}$$

Let

$$a'''=\mathrm{H}_0+\mathrm{H}_1\frac{l}{\mathrm{b}m}+\ \&\mathrm{c}.$$

Substituting this in the equation, and comparing the coefficients of the same power of $\frac{l}{\mathrm{b}m}$, the comparison of terms independent of l gives

Solution in powers of the depth of the sea.

$$4\mathrm{G}.\sin^4\theta=-4\mathrm{H}_0\sin^2\theta,$$

or

$$\mathrm{H}_0=-\mathrm{G}\sin^2\theta.$$

This gives

$$\frac{d\mathrm{H}_0}{d\theta}=-2\mathrm{G}\sin\theta\cos\theta\,;\quad \frac{d^2\mathrm{H}_0}{d\theta^2}=\mathrm{G}(2-4\cos^2\theta).$$

The comparison of terms multiplying the first power of l gives

$$\begin{aligned} 0 = {} & \frac{d^2\mathrm{H}_0}{d\theta^2}\times\frac{l}{\mathrm{b}m}\times\{-1+q\cos^2\theta \\ & +\frac{d\mathrm{H}_0}{d\theta}\times\frac{\cos\theta}{\sin\theta}\times\frac{l}{\mathrm{b}m}\times\{(1-2q)+q\cos^2\theta\} \\ & +\frac{\mathrm{H}_0}{\sin^2\theta}\times\frac{l}{\mathrm{b}m}\times\{6+(2-10q)\cos^2\theta+2q\cos^4\theta\} \\ & 4\mathrm{H}_1\times\frac{l}{\mathrm{b}m}\times\sin^2\theta. \end{aligned}$$

Substituting for H_0, $\frac{d\mathrm{H}_0}{d\theta}$, and $\frac{d^2\mathrm{H}_0}{d\theta^2}$, the values found above, we obtain

$$0=\mathrm{G}\times\{-8+16q.\cos^2\theta-8q.\cos^4\theta\}-4\mathrm{H}_1.\sin^2\theta,$$

or

$$\mathrm{H}_1=\mathrm{G}\times\frac{-8+16q.\cos^2\theta-8q.\cos^4\theta}{4\sin^2\theta}=2\mathrm{G}\times\frac{-1+2q.\cos^2\theta-q.\cos^4\theta}{\sin^2\theta}.$$

By a similar process H_2 will be formed from H_1. Then

$$a'''=-\mathrm{G}\sin^2\theta-2\mathrm{G}\times\frac{1-2q.\cos^2\theta+q.\cos^4\theta}{\sin^2\theta}.\frac{l}{\mathrm{b}m}+\ \&\mathrm{c}.$$

and

$$a=\frac{\Theta}{g}+\frac{a'''}{g}=-\frac{2\mathrm{G}}{g}\times\frac{1-2q.\cos^2\theta+q.\cos^4\theta}{\sin^2\theta}.\frac{l}{\mathrm{b}m}+\ \&\mathrm{c}.$$

If the depth is uniform, or $q=0$, a will $=-\frac{2\mathrm{G}}{g}.\frac{1}{\sin^2\theta}.\frac{l}{\mathrm{b}m}+\ \&\mathrm{c}.$

Law of depth which gives a finite solution.

(108.) In general, the value of a will be expressed by an infinite series, in which the terms converge by powers of $\frac{l}{\mathrm{b}m}$ and diverge by powers of $\sin^2\theta$. But in one case the solution can be put in a finite form. Suppose $q=1$, or the depth of the water, which $=l(1-q\cos^2\theta)$, to be expressed by $l.\sin^2\theta$. Then

$$H_1=-2G.\frac{1-2\cos^2\theta+\cos^4\theta}{\sin^2\theta}=-2G\frac{(1-\cos^2\theta)^2}{\sin^2\theta}=-2G.\sin^2\theta.$$

Now since the term $H_0=-G\sin^2\theta$ gave us $H_1=-2G\sin^2\theta$, the term $H_1=-2G\sin^2\theta$ would in like manner give us $H_2=-4G\sin^2\theta$; and so on. Thus we have

$$a'''=-G\sin^2\theta-2G\sin^2\theta.\frac{l}{bm}-4G\sin^2\theta\left(\frac{l}{bm}\right)^2-\&c.$$

$$=-G\sin^2\theta-G\sin^2\theta\frac{2\frac{l}{bm}}{1-\frac{2l}{bm}}$$

and

$$a=\frac{\Theta}{g}+\frac{a'''}{g}=-\frac{2G}{g}.\frac{l}{bm}.\frac{1}{1-\frac{2l}{bm}}\sin^2\theta.$$

The expression for the elevation of the water in this case is therefore

$$-\frac{2G}{g}.\frac{l}{bm}.\frac{1}{1-\frac{2l}{bm}}.\sin^2\theta.\cos\overline{2nt+2\varpi-2s}$$

and b and c, from the expressions in (95.), are respectively $\frac{1}{b^2n^2}.\frac{G}{1-\frac{2l}{bm}}.\text{cotan}\,\theta$ and $\frac{1}{b^2n^2}.\frac{G}{1-\frac{2l}{bm}}.\frac{1+\cos^2\theta}{2\sin^2\theta}$;

and the velocity of the water in the direction of the meridian $=-\frac{2}{bn}.\frac{G}{1-\frac{2l}{bm}}.\text{cotan}\,\theta.\sin\overline{2nt+2\varpi-2s}$, and that

in the direction of the parallel $=\frac{2}{bn}.\frac{G}{1-\frac{2l}{bm}}.\frac{1+\cos^2\theta}{2\sin\theta}.\cos\overline{2nt+2\varpi-2s}$.

In this case, low water occurs at the transit of the luminary.

(109.) When the luminary is on the meridian, either above or below the horizon, $\cos\overline{2nt+2\varpi-2s}=1$, and the elevation of the water is expressed by $-\frac{2G}{g}.\frac{l}{bm}.\frac{1}{1-\frac{2l}{bm}}.\sin^2\theta$; or the elevation has its maximum negative value; that is, it is low water. We have already (16.) alluded to Newton's anticipation of this result; and we shall find it confirmed by investigations in our next section.

Laplace's solution in powers of the sine of latitude.

(110.) Laplace has solved the equation, on the supposition that the depth is uniform, in a manner equivalent to the following. Make $q=0$ in the equation of (107.), and put $1-\sin^2\theta$ for $\cos^2\theta$, and it becomes

$$\begin{aligned}4G.\sin^4\theta=&-\frac{d^2a'''}{d\theta^2}\times\frac{l}{bm}\\&+\frac{da'''}{d\theta}\times\frac{\cos\theta}{\sin\theta}\times\frac{l}{bm}\\&+\frac{a'''}{\sin^2\theta}\times\frac{l}{bm}\times(8-2\sin^2\theta)\\&-4a'''.\sin^2\theta.\end{aligned}$$

Since $ga=\Theta+a'''=G\sin^2\theta+a'''$,

assume $ga=K_2\sin^2\theta+K_4\sin^4\theta+\&c.+K_{2k}\sin^{2k}\theta+K_{2k+2}\sin^{2k+2}\theta+\&c.$

or $a'''=(K_2-G)\sin^2\theta+K_4\sin^4\theta+\&c.+K_{2k}\sin^{2k}\theta+\&c.$

Substitute this in each term of the equation above, reducing the even powers of $\cos\theta$ into expressions depending on $\sin\theta$; then comparing the coefficients of successive powers of $\sin\theta$, we have

$$8(K_2-G)=0$$
$$12(K_4-K_4)=0$$

$$(-16K_6+10K_4)\frac{l}{bm}-4K_2=0$$

$$(-40K_8+28K_6)\frac{l}{bm}-4K_4=0$$

and generally, after the two first,

$$\left\{-(4k^2+12k)K_{2k+4}+(4k^2+6k)K_{2k+2}\right\}\frac{l}{bm}-4K_{2k}=0.$$

The first equation determines K_2; the second leaves K_4 absolutely indeterminate; the other equations determine in succession each of the succeeding coefficients from those preceding.

Explanation of the indeterminateness of one coefficient.

(111.) The indeterminateness of K_4 is a circumstance that admits of very easy interpretation. It is one of the arbitrary constants in a complete solution of the equation. It shows that we may give to K_4 any value that we please, even if $G=0$; and then, provided that we accompany our arbitrary K_4 with the corresponding values of K_6, K_8, &c., we shall have a series which expresses a value of ga that will satisfy the equation *when there is no external disturbing force whatever*, and which therefore may be added, multiplied by any number, to the expression determined as corresponding to a given force. In the next Section we shall find several instances exactly similar to this. Yet this obvious view of the interpretation of this circumstance appears to have escaped Laplace, and he has actually persuaded himself to adopt the following process. Putting the general equation among the coefficients into the form

$$\frac{K_{2k+2}}{K_{2k}}=\frac{2\dfrac{bm}{l}}{(2k^2+3k)-(2k^2+6k)\dfrac{K_{2k+4}}{K_{2k+2}}}$$

Error in Laplace's process.

he has unwarrantably conceived that this must apply when $k=1$ for the determination of K_4; and thus, applying the same equation to each quotient of terms which occurs in the denominator of the fraction, he finds

$$\frac{K_4}{K_2}=\cfrac{2\dfrac{bm}{l}}{2.1^2+3.1-(2.1^2+6.1)\times\cfrac{\dfrac{2bm}{l}}{2.2^2+3.2-(2.2^2+6.2)\times\cfrac{2\dfrac{bm}{l}}{2.3^2+3.3-\&c.}}}$$

in an infinite continued fraction. And upon this he founds some numerical calculations, adapted to different suppositions of the depth of the sea. We state, as a thing upon which no person, after examination, can have any doubt, that this operation is entirely unfounded.

Correction of Laplace's process.

(112.) In conformity with the remarks that we have just made, Laplace ought to have determined the series for ga which will satisfy the equations

$$0=-\frac{d^2a'''}{d\theta^2}\times\frac{l}{bm}+\frac{da'''}{d\theta}\times\frac{\cos\theta}{\sin\theta}\times\frac{l}{bm}+\frac{a'''}{\sin^2\theta}\times\frac{l}{bm}\times(8-2\sin^2\theta)-4a'''\sin^2\theta$$

$$ga=0+a'''$$

having a constant K_4 indeterminate; and then he ought to have added this series, either to a solution of the equation in (110). found with $K_4=0$, or to a solution in which any definite value has been used for K_4. The series which satisfies the equations above is easily found to be

$$K_4\times\left\{\sin^4\theta+\frac{5}{8}\sin^6\theta+\left(\frac{7}{16}-\frac{bm}{10.l}\right)\sin^8\theta+\left(\frac{189}{576}-\frac{79}{720}.\frac{bm}{l}\right)\sin^{10}\theta+\&c.\right.$$

Now when $\frac{bm}{l}=10$, (or the depth $=\frac{1}{2890}$ of the earth's radius) Laplace has found

$$a=\frac{G}{g}\{\sin^2\theta+20\cdot1862.\sin^4\theta+10\cdot1164.\sin^6\theta-13\cdot1047\sin^8\theta-15\cdot4488\sin^{10}\theta-\&c.\}.$$

He ought to have found

$$a=\frac{G}{g}\{\sin^2\theta+20\cdot1862.\sin^4\theta+10\cdot1164.\sin^6\theta-13\cdot1047\sin^8\theta-15\cdot4488\sin^{10}\theta-\&c.\}$$
$$+\frac{K_4}{g}\left\{\sin^4\theta+\frac{5}{8}\sin^6\theta+\left(\frac{7}{16}-\frac{10}{10}\right)\sin^8\theta+\left(\frac{189}{576}-\frac{790}{720}\right)\sin^{10}\theta+\&c.\right\}$$

where K_4 is absolutely arbitrary.

When $\frac{bm}{l}=\frac{5}{2}$, (or the depth $=\frac{1}{722}$ of the earth's radius,) Laplace has found

$$a=\frac{G}{g}\{\sin^2\theta+6\cdot1960.\sin^4\theta+3\cdot2474.\sin^6\theta+0\cdot7238.\sin^8\theta+0\cdot0919.\sin^{10}\theta+\&c.\}.$$

He ought to have found

$$a=\frac{G}{g}\{\sin^2\theta+6\cdot1960.\sin^4\theta+3\cdot2474.\sin^6\theta+0\cdot7238\sin^8\theta+0\cdot0919\sin^{10}\theta+\&c.\}$$
$$+\frac{K_4}{g}\left\{\sin^4\theta+\frac{5}{8}\sin^6\theta+\left(\frac{7}{16}-\frac{5}{2.10}\right)\sin^8\theta+\left(\frac{189}{576}-\frac{79}{720}.\frac{5}{2}\right)\sin^{10}\theta+\&c.\right\}.$$

When $\frac{bm}{l}=\frac{5}{4}$, (or the depth $=\frac{1}{361}$ of the earth's radius,) Laplace has found

$$a=\frac{G}{g}\{\sin^2\theta+0\cdot7504.\sin^4\theta+0\cdot1566.\sin^6\theta+0\cdot0157.\sin^8\theta+0\cdot0009.\sin^{10}\theta+\&c.\}.$$

He ought to have found

$$a=\frac{G}{g}\{\sin^2\theta+0\cdot7504\sin^4\theta+0\cdot1566.\sin^6\theta+0\cdot0157.\sin^8\theta+0\cdot0009.\sin^{10}\theta+\&c.\}$$
$$+\frac{K_4}{g}\left\{\sin^4\theta+\frac{5}{8}\sin^6\theta+\left(\frac{7}{16}-\frac{5}{10.4}\right)\sin^8\theta+\left(\frac{189}{576}-\frac{79}{720}.\frac{5}{4}\right)\sin^{10}\theta+\&c.\right\}.$$

It is needless to observe that Laplace's numerical calculations of the heights of the tides in certain latitudes, and his inferences, as to the latitude where there is no tide, &c., fall to the ground.

This solution is applicable to a sea bounded on its north or south side.

(113.) If, using the more complete values of a that we have just found, we proceed to form the values of a''', b, and u, we find that u will contain a series of terms multiplied by the indeterminate K_4. We may determine K_4 so that, for a given value of θ, u shall $=0$; that is to say, so that, in a given latitude, the water shall have no north and south motion. We might therefore suppose an east and west barrier (following a parallel of latitude) to be erected in the sea, and the investigation would still apply. Thus then we have a complete solution for a sea which is bounded by a shore whose course is east and west.

(114.) The fluctuations of which we have last treated are those which Laplace has called "*les oscillations de la troisième espèce.*" They constitute the ordinary semidiurnal tide.

Separate effects of the Sun and Moon are to be combined by algebraical addition.

(115.) We have throughout this Section spoken of the luminary as if only the Sun or only the Moon were efficient in producing the tides. But the reader will easily understand that the same investigations, *mutatis mutandis*, apply to the moon as to the sun; and that, when the effect of each is found separately, their compound effect will be the algebraic sum of their separate effects. For our equations throughout comprehend only the first power of the terms depending on disturbing force, and the first power of the unknown p''' and the terms originating from it; and thus, if a term f_m, depending on the Moon's action, produces the term p_m, and a term f_s, depending on the Sun's action, produces the term p_s, then, by simple addition of the equations, the equation will be produced which shows that the combined terms f_m+f_s will produce p_m+p_s. This would not be true if the squares of terms of p''' entered, because the sum of the squares of p_m and p_s is not the same as the square of their sum: and so for higher powers.

(116.) With this we shall terminate our account of Laplace's Theory of the Tides; but, before closing this Section, we must call our reader's attention, as well to the important points of the investigation in general, as to the modification which we have thought desirable to introduce into Laplace's methods. We will first advert to the latter. Laplace has commenced with the equations of motion of fluids in their most general, and (we may be permitted to say) in their most repulsive form. Proceeding from these in a way which appears at first not easy to understand, but in which, nevertheless, the same operations may be traced as in the investigations of this Essay, he arrives at the equations of (87.), the assumptions of (89.), and the equations of (90.). From this point he proceeds by a method totally different from that which we have used. Partly for the generality of an investigation applying to all degrees of density of the sea, and partly (it would seem) for the sake of introducing his own favourite equation for the attraction of bodies nearly spherical, Laplace has embarrassed his process with investigations, nowhere fully explained, and sometimes only hinted at, applying to the different terms into which (for substitution in his own equation above alluded to) the disturbing forces and other quantities are to be resolved. In asserting that this is the most obscure of the investigations of the Mécanique Céleste, we trust that we may consider ourselves supported by the circumstance that no following mathematician has entered into Laplace's method for

Criticism on Laplace's process in the whole investigation.

the tides, or has even verified his most remarkable results; although Laplace has frequently alluded to one of these (the evanescence of diurnal tide in height when the depth is uniform) in a way that shows that he considered it one of the happiest of his discoveries. We add that this investigation is *unnecessarily* obscure. Although the introduction of the density of the sea has apparently enabled Laplace to give greater generality to his symbols, it has given none to his final results. Of the latter class there are two, namely, the evanescence of the diurnal tide when the depth is uniform, and the numerical calculation of the semi-diurnal tide. The former is obtained by the methods of this Treatise; and, moreover, is obtained with the utmost generality as to density of fluid: for, if the depth is nowhere altered, the attraction is not altered, and therefore it is indifferent whether we treat the attraction of the fluid *per se*, or consider it included in the attraction of the rigid body. In the latter, Laplace has himself neglected the density of the water. We trust that these remarks will be thought sufficiently to explain our abandonment of Laplace's method, and our substitution for it of a method which is (we should hope) intelligible to almost every student of the differential calculus.

Merit of Laplace's theory as an application of mathematics to physics.

(117.) If now, putting from our thoughts the details of the investigation, we consider its general plan and objects, we must allow it to be one of the most splendid works of the greatest mathematician of the past age. To appreciate this, the reader must consider, first, the boldness of the writer who, having a clear understanding of the gross imperfection of the methods of his predecessors, had also the courage deliberately to take up the problem on grounds fundamentally correct (however it might be limited by suppositions afterwards introduced); secondly, the general difficulty of treating the motions of fluids; thirdly, the peculiar difficulty of treating the motions when the fluids cover an area which is not plane but convex; and, fourthly, the sagacity of perceiving that it was necessary to consider the Earth as a revolving body, and the skill of correctly introducing this consideration. The last point alone, in our opinion, gives a greater claim for reputation than the boasted explanation of the long inequality of Jupiter and Saturn.

Negative results of Laplace's theory.

(118.) If we look to the results of the theory, it will be found that they are rather of a negative than of a positive kind. They show that, without a far more complete knowledge of the form of the bottom of the sea than we can hope to possess, it will be impossible, even with more powerful mathematics, to calculate tides *à priori*. They show that the calculations founded on the equilibrium-theory cannot be good for anything. In proving that (with sea at least of a certain shallowness) the part of the equator next to the sun or moon would be a place of *low* water, they destroy all hope of using an equilibrium-theory, even as an approximation. In establishing the remarkable result as to the non-existence of diurnal tide in height when the depth is uniform, they show that no inference can be drawn from the mere magnitude of a force as to the magnitude of its effects.

Extension required to make the theory applicable to observations.

(119.) The results of this theory, however, would give us a knowledge of the physics of tides of no contemptible kind, if, upon any supposition whatever as to depth, we were able to introduce the horizontal limitation of the sea. This implies that we should be able to solve generally the equations of (87.) without the term $\Theta . \cos \overline{it+k\varpi}$. Such a solution, in a very limited case, is given by the indeterminateness of K_4 in (112.). We see not the smallest prospect of succeeding in this with the degree of generality required (that is, of expressing the solution by $A \cos it + B \sin it$, where A and B are the most general functions of θ and ϖ): we have, however, no hesitation in pointing out this as the subject which, in the present state of theory, is most especially worthy of the attention of the theoretical investigator of tides. It is principally in this respect that the theory (in other respects imperfect) of the next Section is superior to this. [The principle of introducing such limitations in general will be found in (291.) &c.] As it is, Laplace's theory fails totally in application, from the impossibility of introducing in it the consideration of the boundaries of the sea.

(120.) It is almost unnecessary to remark that this theory gives no assistance in explaining the peculiarities of river or channel tides: and it gives no idea whatever of the difference in the proportion of the effects produced by bodies (as the Sun and Moon) whose motions in right ascension are not precisely equal; a matter which we shall find to be very important.

Laplace's final assumption.

(121.) After considering the negative nature of the results of his theory, and the degree in which any accurate conclusions must depend upon the precise knowledge and correct mathematical treatment of a number of circumstances which are wholly unknown, Laplace at last takes refuge in the assumption that all that we are certain of is, that the disturbances of the sea will be periodical as the forces that cause those disturbances, but that their times of maximum or minimum are not necessarily the same as the times of maximum or minimum of the forces, and that their coefficients are not necessarily in the same proportion as the forces which cause them, unless the periods of the forces are exactly the same. That is to say, if the forces acting in any given manner are represented by

$$A_{\prime} \cos (i_{\prime}t + B_{\prime}) + A_{\prime\prime} \cos (i_{\prime\prime}t + B_{\prime\prime}) + A_{\prime\prime\prime} \cos (i_{\prime\prime\prime}t + B_{\prime\prime\prime}) + \&c. ;$$

then the elevation of the tide will be represented by

$$C_{\prime} \cos (i_{\prime}t + B_{\prime} + E_{\prime}) + C_{\prime\prime} \cos (i_{\prime\prime}t + B_{\prime\prime} + E_{\prime\prime}) + C_{\prime\prime\prime} \cos (i_{\prime\prime\prime}t + B_{\prime\prime\prime} + E_{\prime\prime\prime}) + \&c.,$$

where the quantities $C_{\prime}$, $C_{\prime\prime}$, $C_{\prime\prime\prime}$, $E_{\prime}$, $E_{\prime\prime}$, $E_{\prime\prime\prime}$, can only be determined from experience. And that, in the case of a canal communicating with two tidal seas, any variation in the value of $i_{\prime}$ ($A_{\prime}$ and $B_{\prime}$ remaining unvaried) will be accompanied by a variation of $C_{\prime}$ and $E_{\prime}$. On this point the reader is referred to (312.).

(122.) We will terminate this Section by a brief demonstration of two of Laplace's supplementary propositions which are closely related to the subject before us.

The equilibrium of the sea is

(123.) The first is, that the equilibrium of the sea is stable (that is, if its relative position with the land is disturbed, it will have a tendency to return to its former position), if the density of the land is greater stable if its density is less than that of the earth.

than that of the sea; out is unstable if the density of the land is less than that of the sea. In fig. 4, suppose the land to be more dense than the water, and suppose that at any instant each is in the form of a sphere, the centre of the terrestrial sphere being at A, and the centre of the aqueous sphere being at B. The attraction of the whole may be found by conceiving the whole large sphere to be filled with matter of the same density as water, and conceiving the small sphere to have, in addition, the density equal to the excess of the density of earth over that of water. The attractions of these spheres will be directed to their respective centres. Thus, a particle p on the surface will be drawn by one of these attractions (that of the aqueous sphere) in the direction pB, and by the other (the excess of the sphere of earth over a similar sphere of water) in the direction pA. Representing these forces by pa and pb, their compound effect will be that of a force in the direction pc. It is evident that this force is not perpendicular to the surface at p (the normal being pB), but is inclined in such a direction that it tends to carry the particle p towards e. A similar result will be found for every particle at the surface, namely, that the force on the particle tends to carry it in that direction where the water is at present deficient; and tends, therefore, to restore the equality of distribution of the water.

The equilibrium of the sea is unstable if its density is greater than that of the earth.

(124.) But if the land is less dense than the water, then, in fig. 5, we may conceive the whole of the larger sphere to be occupied with matter of the same attractive power as water, but then we must suppose the small sphere to be occupied with a repulsive substance. The particle p will then be drawn by the complete aqueous sphere in the direction pB or pb, but will be repelled by the imaginary repulsive sphere in the direction pa. The whole force, therefore, on p will be represented by pc acting in the direction inclined to the normal pB on the side opposite to that in the last article, and therefore it will tend to carry the particles of water towards that side f where there is already a redundance, and, therefore, to cause the distribution of water to become more unequal than it is at present.

Sect. III. Laplace's Theory of Tides.

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Investigation of the dependence of precession upon the tides.

(125.) The second supplementary proposition is, that the amount of precession, and its subordinate portions included under the general term *nutation*, are not affected by the tidal motions of the sea. To demonstrate this, we must refer to our Treatise on the FIGURE OF THE EARTH, section 9. It will there be seen that, in consequence of the action of the Sun and Moon upon the earth, supposed to be a solid, there is impressed upon the earth a tendency to revolve round an axis, which is in the plane of the equator; and that the result of the composition of this impressed motion of rotation with the motion of rotation about its polar axis is, that the real axis of rotation (a revolution about which will represent the real motion of every particle of the earth under the effect of the two rotations) will change its position in space, in that direction which exactly corresponds to precession (including nutation). The quantity, then, upon which the amount of precession will immediately depend, is the angular velocity which the action of the sun and moon tends to give round the equatorial axis of which we have spoken; and this angular velocity is represented by the fraction, whose numerator is the moment of all the impressed forces tending to produce rotation round that equatorial axis, and whose denominator is the moment of inertia of the earth about the same axis. If the water were united in one solid mass with the earth, we should, for the moment of impressed forces, merely consider the effect of the sun's and the moon's attraction upon the earth and upon the water. But, as the water is not rigidly connected with the earth, but has a fluctuating motion upon it, and acts upon the earth by a pressure which is modified as well by its elevation as by the circumstances of its motion, it is necessary to take into account the effect of the pressure of the water upon the earth, and the corresponding reaction of the earth upon the water. We must also consider that the same particles of water, in their tidal motion, always oscillate about nearly the same part of the earth.

Principle of the conservation of areas.

(126.) The most general way of considering this is the easiest. The particles of water act upon one another and upon the particles of earth, and the particles of earth act upon one another and upon the particles of water, either by the pressure of contact, or by attraction; and each of these forces produces an equal reaction upon the acting particle. Now there is a well known principle, called that of the *conservation of areas*, which is thus enunciated:—"The sum of the products of the mass of each particle into the area which it describes round a given axis is not altered by the mutual action of the particles." The demonstration of this (which is not given in our Treatise MECHANICS) may be shortly stated as follows. Let x, y be the co-ordinates of a particle m which is acted on by the accelerating forces X, Y; x', y' those of another particle m', which is acted on by the accelerating forces X′, Y′ (z being the axis of rotation); D the distance between these particles; and let F be the force, estimated as a pressure, with which these two particles attract each other. Then.

$$\frac{d^2x}{dt^2}=\mathrm{X}+\frac{\mathrm{F}}{m}\cdot\frac{x'-x}{\mathrm{D}}$$

$$\frac{d^2y}{dt^2}=\mathrm{Y}+\frac{\mathrm{F}}{m}\cdot\frac{y'-y}{\mathrm{D}}$$

$$\frac{d^2x}{dt^2}=\mathrm{X}'+\frac{\mathrm{F}}{m'}\cdot\frac{x-x'}{\mathrm{D}}$$

$$\frac{d^2y'}{dt^2}=\mathrm{Y}'+\frac{\mathrm{F}}{m'}\cdot\frac{y-y'}{\mathrm{D}}.$$

Therefore

$$m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=m\,(x\mathrm{Y}-y\mathrm{X})+\mathrm{F}\,\frac{xy'-yx}{\mathrm{D}}.$$

But if r is put for the distance of m from the axis, and θ for the angle made by r with the plane of xz, $x=r\cos\theta$, $y=r\sin\theta$, and $x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$. Put A for the projection, on the plane xy, of double the area described by m round the axis in the unit of time; then $\mathrm{A}=r^2\frac{d\theta}{dt}$: and this equation becomes

$$m\frac{d\mathrm{A}}{dt}=m(x\mathrm{Y}-y\mathrm{X})+\mathrm{F}.\frac{xy'-yx'}{\mathrm{D}}.$$

Similarly for the other particle,

$$m'\frac{d\mathrm{A}'}{dt}=m'(x'\mathrm{Y}'-y'\mathrm{X}')+\mathrm{F}.\frac{x'y-y'x}{\mathrm{D}}.$$

Adding these,

$$m\frac{d\mathrm{A}}{dt}+m'\frac{d\mathrm{A}'}{dt}=m(x\mathrm{Y}-y\mathrm{X})+m'(x'\mathrm{Y}'-y'\mathrm{X}');$$

or is the same as it would have been if there had been no mutual action of the particles. The same would be found to be true if there had been any number of particles. Then, integrating, we find that $m\mathrm{A}+m'\mathrm{A}'+$&c. is the same as if there had been no mutual action. It is, however, to be remarked, that if the ordinates x, y, x', y', &c. are sensibly altered by that mutual action, the products $x\mathrm{Y}$, $y\mathrm{X}$, &c. are altered; and, therefore, in this indirect way the expressions for $m\frac{d\mathrm{A}}{dt}+m\frac{d\mathrm{A}'}{dt}+$&c. may be altered.

(127.) It follows from this, that the sum of the products of the mass of each particle of the earth and sea by the area which it describes round the equatorial axis is unaltered by the fluctuation of the sea (except so far as the surface is raised or depressed by that fluctuation, and the moment of the impressed forces is thereby increased or diminished; but this increase or diminution must be utterly insensible, and we shall give no further attention to it). Now, if the earth and sea were so entirely disconnected that one of them could revolve for any length of time with any velocity, increasing or diminishing in any manner, while the other could revolve with any other velocity changing in any other manner, we could pronounce nothing as to the effect of the fluctuations on precession. But the assumption on which we are to proceed is, that the tidal motion is not great, and is of an oscillatory kind. If, then, there were but a single sea upon the earth, and if in consequence of fluctuation this sea received a considerable velocity in the same direction in which the impressed forces tend to carry the earth, the rotation of the solid earth in that direction round the equatorial axis would thereby be *diminished* (in virtue of the *conservation of areas*). But as soon as that water came to a state of rest, the rotation of the earth round the equatorial axis would acquire the same value as if the water were rigidly connected with the earth; and when the water had a movement in the opposite direction, the earth's velocity of rotation round the equatorial axis would be *increased*. And this would be true whether or not the sea had had in the mean time, from the earth's diurnal revolution, a great motion parallel to z (for z does not enter into the formula), and whether or not the sea had been carried to the opposite side in respect of x or y (for the formula is perfectly general as regards changes of magnitude and sign of x and y). The same applies to any number of seas, of any forms. Thus, though the angular motion round the equatorial axis, and the consequent momentary precession, may be irregular, yet its irregularity will not extend beyond a single complete tidal oscillation; and the whole precession during one tide will be the same as if the water had been fixed.

Precession is not affected by the tidal motion of the sea, provided that the motion is oscillatory.

(128.) The reader will perceive that, although through the greater part of this Section we have not taken into account the density of the water, we have taken it fully into account in these two supplementary propositions. And, in the last of them, we have departed from all hypotheses as to the symmetrical disposition of the water, and have supposed it to be distributed and bounded in any way whatever.

SECTION IV.—THEORY OF WAVES IN CANALS.

(129.) We have already stated (64.) that the Equilibrium-Theory of Tides, though curious in its relation to the history of the science, and valuable for the coincidence of the algebraic form of its results (under certain modifications) with those of more accurate theories, and with the laws deduced from observations, does not deserve the smallest attention as representing the state of the ocean at any time. We have also stated (65.) that Laplace's theory of the movement of the sea, supposing the globe completely covered by water, whose depth is uniform, or follows a very simple geographical law, though based upon sounder principles, has far too little regard to the actual state of the earth to serve for the explanation of the principal phænomena of tides. We now come to a third theory: that of the motion of the tidal waters, supposing them to run in the manner of ordinary waves in canals. It is evident that this theory will not apply to every part of the sea, and therefore it must, to a certain extent, be considered imperfect. Still it will apply strictly to many cases (to rivers without exception; and to arms of the sea where their breadth is smaller than their length, and where the irregularities of the coasts are not very remarkable), and it will apply without sensible modification to other cases of open seas, where the whole may be conceived divided into parallel canals in which the circumstances are nearly similar. For these reasons we are inclined to think that this mode of considering the subject, in the present imperfection of mathematics, deserves special notice among the various Theories of the Tides.

(130.) It is necessary for our present purpose to enter into a pretty general investigation of the Theory of Waves of water; and we shall therefore commence without any obvious reference to the subject of Tides.

(131.) We shall, for convenience, divide this Section into the following parts:

- Subsection 1.—General explanation of waves; and general theory of waves, supposing the motion of the particles small.
- Subsection 2.—Theory of waves in canals of uniform depth and uniform breadth, whether the waves be short or long, the motion of the particles being supposed small.
- Subsection 3.—Theory of long waves in which the elevation of the water bears a sensible proportion to the depth of the canal.
- Subsection 4.—Theory of waves when the water is acted on by horizontal and vertical forces, the motions of the particles being small; including also the theory of a single wave, and the theory of waves in canals of variable depth and variable breadth; with the introduction of the ideas of *free-wave* and *forced-wave*.
- Subsection 5.—Method of introducing the limits of the canal in general; and application of the doctrine of *free-wave* and *forced-wave*.
- Subsection 6.—Theory of waves, as affected by friction.
- Subsection 7.—Theory of waves in water of three dimensions, or where the horizontal extent of the surface in two dimensions is taken into account.

Subsection 1.—General Explanation of Waves, and general Theory of Waves, supposing the motion of the particles small.

(132.) Without citing the explanations in other essays, it may be desirable here to call the reader's attention to the meaning of the term *wave*, and to the form of the mathematical expression which must be used to represent the motions of the particles of water in wave-motions. The same general ideas attach to the term *wave* or *undulation* in the sciences of Acoustics and Optics: and a clear conception of those ideas may be considered as one of the most important steps in the understanding of many important physical sciences.

A wave in continuous motion does not imply that the water is continuously moving in the same direction.

(133.) In watching the waves of the sea (we allude not to the breaking of the surf, which will be distinctly considered hereafter), the reader may perhaps have imagined that, in each wave, a quantity of water, equal in bulk to that wave, was advancing towards the shore. A very little attention, however, would show that this notion is incorrect. A cork, or a particle of foam, floating on the water, is not carried towards the shore; if watched narrowly, it will be found that it moves towards the shore while the crest of each wave is under it, and from the shore while it is in the hollow of each wave, but these motions are scarcely greater than its vertical motions, and the advance and the regress sensibly balance each other.

It is only the motion of a shape.

(134.) Thus it will appear, as a visible fact, that the continued motion of a wave in one direction is not a continued motion of the water in that direction, but may be described as a continued motion of a *shape*, or of an *arrangement of the particles of the water*. It is necessary now to show that a very small reciprocating motion of each particle of water is sufficient to account for unlimited motion of the *wave* or *shape* continued constantly in one direction.

Sect. IV Theory of Waves.
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Subsect 1. General Explanation of Waves.

(135.) In figure 6, suppose that ABCDEFG represents the outline of a succession of waves at one instant of time, *abcdefg* the outline at a second instant; it is required to show how the waves can have advanced from the position ABCDEFG to *abcdefg* by a small oscillating motion of each particle of water.

Explanation of motion of the wave by oscillatory motion of the particles.

(136.) Draw vertical lines from the surface to the bottom of the water; conceive that all the particles in each line are subject to motion in the direction represented by the small arrows in the figure; that is, that all the particles below the crest of the wave are moving forwards; that all the particles below the hollow of the wave are moving backwards; and that all below the midway-points (A, C, E, G,) are for the moment stationary. And suppose that the velocity of the horizontal motion of the particles in vertical lines intermediate to those drawn in the figure is intermediate to the velocities of the particles in the lines drawn in the figure. This supposition will account for the motion of the wave or *shape*.

(137.) For, take points B_0, B_1 near to B: C_0, C_1 near to C, &c.: draw lines from them to the bottom, and consider the horizontal motion of the particles in those lines. B_0 and B_1 are both between the point of principal backward motion and the point of rest; therefore the particles below B_0 and those below B_1 will be moving backwards, and with nearly the same speed: and therefore the intermediate surface at B will not be sensibly elevated or depressed, inasmuch as the vertical boundaries $B_0\,B_0'$ and $B_1\,B_1'$ of the included column of water will, after a short time, be at the same distance from each other as at present. But the particles in the line $C_0\,C_0'$ are between a point of rest and a point of backward motion, and therefore are moving backwards; those in the line $C_1\,C_1'$ are between a point of rest and a point of forward motion, and therefore are moving forwards; consequently the vertical boundaries, $C_0\,C_0'$, $C_1\,C_1'$, of the included column are separating more widely apart, and therefore the surface at C will drop, and will, after a short time, be found depressed to *c*. In like manner it will be found that in both the lines $D_0\,D_0'$ and $D_1\,D_1'$ the particles are moving forward with nearly the same velocity, and therefore in the intermediate part at D the elevation of the surface is not sensibly altered. But in $E_0\,E_0'$ the particles are moving forward, and in $E_1\,E_1'$ they are moving backward; the horizontal space between these boundaries is therefore diminished, and therefore the surface of the water between them is raised; and it will therefore, after a short time, be found at *e* instead of E. Pursuing this reasoning it will be evident that the continued horizontal motion of the wave or *shape* forwards is entirely accounted for by the rising of some portions of the surface and the falling of others, and that these risings and fallings may be considered as the effect of small horizontal motions of the particles of the water, some forwards and others backwards.

(138.) And as, in the progress of the waves, the same particles are alternately on the crest and in the hollow of the wave, every particle will be alternately moving forwards and backwards, and alternately upwards and downwards; the wave meanwhile will be advancing continually in the same direction.

(139.) The reader must consider the above as merely a general *geometrical* explanation of the motion of a wave by means of oscillating motions of every particle of the water. Whether such motions are consistent with the various forces acting upon water (which forces themselves depend in part upon the motions of the particles) will shortly be a most important subject of inquiry. But from the tenor of this explanation, we may at once collect what must be the form of the mathematical expressions which will be necessary to represent the motions of the particles. We shall call the horizontal ordinate of any particle in its position of rest, measured in the direction of the canal's length from some fixed point, x: and the vertical ordinate of any particle in its position of rest, measured upwards from some horizontal plane (which, when the depth is uniform, may be the bottom of the water) y.

(140.) Now, first, it appears from our explanation that each particle is disturbed in the horizontal as well as in the vertical direction. Consequently, to represent the position of any particle at any time, we must use for co-ordinates

$$x'=x+\mathrm{X}$$

$$y'=y+\mathrm{Y},$$

where both X and Y depend on, or are functions of, x, y, and t (t being used to express the time, as measured from some arbitrary epoch.)

Algebraic characteristic of a wave.

(141.) Secondly. The characteristic of a wave is this: that though, at any one instant, different particles are displaced in different ways; yet, the state of any particle in advance (that is, a particle for which x is large) will be, at some future time, the same as the state of a particle in arrear (that is, a particle for which x is small) is now, provided we wait during a time proportioned to the interval of space between these two particles. In other words: supposing the velocity of the wave to be v, so that in the time t' it will move through $v\,t'$: then the characteristic of a wave is, that the particle whose ordinate is $x+vt'$ will have the same disturbance at the time $t+t'$, which the particle whose ordinate is x has at the time t. Mathematically expressed, putting $\phi(x,\,t)$ for the disturbance at the time t of a particle whose ordinate was x,

$$\phi(x,t)=\phi(\overline{x+vt'},\ \overline{t+t'})\ :$$

expanding this latter to the first power of t',

$$\frac{d.\phi(x,t)}{dx}vt'+\frac{d.\phi(x,t)}{dt}t'=0\ :$$

a well known equation, whose solution is

$$\phi(x,t)=\chi(vt-x),$$

χ being any arbitrary function. As v is not known, it will be equally convenient to put this under the form

$$\chi(nt-mx)$$

where n and m are constants: the function χ being still arbitrary (that is, capable of adaptation to any physical assumption) though not precisely the same as in the former expression. If we suppose the horizontal movements of the particles in different points of the same vertical to be different, we may express that supposition by multiplying this expression by a function of y, which will give

$$\mathrm{X}\text{ or }\phi(x,\ t)=\mathrm{F}(y)\times\chi(nt-mx).$$

And if we suppose that the horizontal movements of the particles in different points of the same vertical are not simultaneous; for instance, that their times of rest or their times of greatest motion do not all occur at the same instant; we may use the expression

$$\mathrm{X}=\mathrm{F}(y)\times\chi(nt-mx-\mathrm{Q}),$$

where Q depends on y only.

Characteristic when the motion of particles is oscillatory.

(142.) Thirdly. The motion of each particle of water was supposed to be *reciprocating* or *oscillatory*. There is no kind of expression so convenient for representing oscillatory motion as one depending on sines or cosines. And by combination of several terms of that kind, any kind of oscillation may be represented. Thus, the formula

$$\cos(nt-mx)$$

would represent a regular oscillation going through all its changes while nt increased by 2π, or while t increased by $\frac{2\pi}{n}$: but the following formulæ

$$\cos\,(nt-mx)+a\cos\,(2nt-m'x)$$

$$\cos\,(nt-mx)+a.\cos\,(2nt-m'x)+b.\cos\,(3nt-m''x)$$

would represent regular oscillations of a different kind, but still going through all their changes while t increased by $\frac{2\pi}{n}$. It will even be seen hereafter that, by proper management of these terms, we may investigate the motions of a *discontinuous* wave, that is where a single wave, and no more, passes along a canal. For these reasons, we shall always suppose the function χ to have the form of a series of sines or cosines of multiples of $nt-mx$. But as each of these terms will be treated separately in the same manner, we shall omit all but the first (for convenience only), and shall therefore assume

$$X=F(y).\cos(nt-mx-Q).$$

The form of Y, as will appear in the investigation, will necessarily follow from that of X.

Characteristic when the form of the wave changes.

(143.) Fourthly. In some very important cases, we shall find it necessary to depart from our original simple suppositions. Circumstances may be conceived, in which, without at all losing the character of a single determinate series of waves, the elevation of each wave alters as it proceeds; and the linear interval between the crest of one wave and the crest of the next wave alters, (the interval of time remaining the same as before,) or the velocity of the wave alters. Thus, in figure 7, the wave, which while at ABC was long and flat, may at IKL become short and steep. This case will be fully represented by introducing a variable coefficient depending on x, and by conceiving that the factor m, or $\frac{n}{v}$, by which the periodic function depends on x, is itself a function of x, and that, instead of a product, an integral is to be used. Thus we shall have

$$X=F(y).G(x).\cos(nt-\textstyle\int_x m).$$

We shall now proceed with that part of the investigation which depends on the properties of fluids.

(144.) In figure 8, let oO represent the bottom of a canal of variable depth: *abc* the surface of the water in a state of rest, ABC the surface at a certain instant of time when in wave-motion: and suppose that the very narrow column of water, which when the whole was at rest had the form *oc*, has at this instant the form OC. Suppose the column *oc* divided into a very great number of small parts by horizontal planes, and let *ps* be one of these parts, and PS the corresponding part of the column OC in its displaced state at the instant under consideration. Let the coordinates of the point o at the bottom be x and η: the vertical coordinate of the surface *abc* be k: the coordinates of p be x and y: and suppose $pq=h$, $pr=l$. And put X for the horizontal displacement, and Y for the vertical displacement, of the particle whose coordinates are x and y, at the instant under consideration. Let Ξ be the value of X at the bottom, and K the value of Y at the top. Then, considering the disturbance so small that all quantities beyond the first order may be omitted,

The horizontal ordinate of $P=x+X$,

The horizontal ordinate of $Q=x+h+X+\frac{dX}{dx}h$,

therefore the horizontal distance between P and Q $=h\times\left(1+\frac{dX}{dx}\right)$ nearly.

And the vertical ordinate of $P=y+Y$,

The vertical ordinate of $R=y+l+Y+\frac{dY}{dy}l$,

therefore the vertical distance between P and R $=l\times\left(1+\frac{dY}{dy}\right)$ nearly.

Investigation of equation of continuity.

(145.) Now, conceiving that the water, which occupied the volume *ps* without any vacant space, does now occupy the volume PS without any vacant space, or remains *continuous*, it is evident that we must have

$$\text{area } ps=\text{area PS}.$$

But, as in (72.), the area PS, considered as a rhomboid, $=$PQ × PR × sin RPQ,

$=$PQ × PR × cosine (inclination of PQ to horizontal line + inclination of PR to vertical)

$=$(PQ × cos incl. PQ to horiz. line) × (PR × cos incl. PR to vertic. line) × (1 − tan incl. PQ . tan incl. PR)

$=$(hor. dist. between P and Q) × (vert. dist. between P and R) × (1 − tan incl. PQ . tan incl. PR).

Now the inclination of PQ to the horizontal line is small; and that of PR to the vertical is small: therefore the product of their tangents is exceedingly small, and may be neglected. Hence we obtain

$$\text{area PS}=(\text{hor. dist. between P and Q})\times(\text{vert. dist. between P and R})$$

$$=h\times\left(1+\frac{dX}{dx}\right)\times l\times\left(1+\frac{dY}{dy}\right)$$

$$=hl\times\left(1+\frac{dX}{dx}+\frac{dY}{dy}\right)\text{ nearly.}$$

Making this equal to area $ps=hl$, we obtain

$$0=\frac{dX}{dx}+\frac{dY}{dy},$$

or

$$Y=-\int_y \frac{dX}{dx}+\text{ a function of } x.$$

As there is a fixed boundary to the water at the bottom, we must commence our integration there. Now it is evident that at the bottom the value of Y is $\Xi\times\frac{d\eta}{dx}$, inasmuch as the lowest particles of the water are pushed through the horizontal space Ξ in contact with the sloping bottom whose vertical ordinate is η, and the tangent of whose inclination to the horizon is, therefore, $\frac{d\eta}{dx}$. Thus we have

Equation of continuity.

$$Y=\Xi.\frac{d\eta}{dx}-\int_y \frac{dX}{dx} \text{ (from } y=\eta \text{ to } y).$$

This is the equation of continuity.

(146.) Now let us consider the relation, between the forces which act on the different points of the water, and the motion of the water. Continue the horizontal plane *pq* to TV, and from T and V draw the vertical lines TW, VX, to the disturbed surface. The points W and X do not coincide with B and C, but in regard to the accuracy of any expression depending merely on the wave-disturbance they may be used indifferently; because WB and XC depend upon the extent of disturbance, and the slope of WB and XC also depend on the extent of disturbance, so that the difference of elevations of W and B, or of X and C, will depend on the square of the disturbance. But the difference of elevations of W and X (which is the difference that we shall shortly use) depends on the extent of disturbance; and, therefore, when the disturbance is made very small, it is a much greater quantity than the difference of elevations of W and B, which depends on the square of the disturbance. And this is entirely independent of the length of the wave. Thus, in a wave whose length from crest to crest is many feet, the whole value of WB may be an inch: by diminishing the violence of the motion, or making the wave flatter, while its length remains the same, WB may be diminished to a tenth of an inch, and then the difference of elevation of W and B will be a hundredth part of what it was before, while that of W and X will be a tenth part of what it was before. Thus, when the motion of the particles is small, instead of using the value of K corresponding to the point W, we may use that which corresponds to the point B: and instead of using the difference between the values of K for W and X, we may use the difference between the values of K corresponding to B and C.

Investigation of the equation of equal pressure.

(147.) Now let p be the pressure at any point in the line PW, estimated by the velocity which that pressure acting on the surface 1 would produce in the volume of water 1, by acting during the time 1. Then p will be a function of x, y, and t. Let g' be the force of gravity, estimated by the velocity which it will produce by its action during the time 1: for reasons which will hereafter appear we shall consider g' as varying from one point of the earth's surface to another, or as being a function of x. Then the pressure at the point whose elevation is y' is p; that at the point whose elevation is $y'+\delta y'$ is $p+\frac{dp}{dy'}\delta y'$: the excess of the upper pressure above the lower is $\frac{dp}{dy'}\delta y'$: and this excess, acting on the column whose length is $\delta y'$, will tend to urge it downwards with an acceleration represented by $\frac{dp}{dy'}$. Adding to this the effect of gravity, we shall have the whole acceleration downwards $=\frac{dp}{dy'}+g'$. Therefore

$$\frac{d^2y'}{dt^2}=-\frac{dp}{dy'}-g'.$$

But $y'=y+Y$, therefore $\frac{d^2y'}{dt^2}=\frac{d^2Y}{dt^2}$ (as y does not depend on t). Consequently

$$\frac{dp}{dy'}=-g'-\frac{d^2Y}{dt^2}, \text{ and } p=\int_{y'}\left(-g'-\frac{d^2Y}{dt^2}\right).$$

Performing the integration from y to $k+K$, ($k+K$ being the value of y' at the surface of the water,) so that the pressure at the surface is zero,

$$p=g'(k+K-y)+\int_{y'}\frac{d^2Y}{dt^2} \text{ (from } y'=y \text{ to } y'=k+K).$$

But, as $\frac{d^2Y}{dt^2}$ is itself a small quantity depending on the motion of the particles, we shall incur no sensible error by integrating with respect to y instead of y', and by taking for superior limit k instead of $k+\mathrm{K}$. Thus we obtain for the pressure at T

$$p=+g'(k+\mathrm{K}-y)+\int_y \frac{d^2\mathrm{Y}}{dt^2}\,(y \text{ to } k).$$

(148.) The particle at V corresponds to a point whose ordinate in the state of rest did not sensibly differ from $x+h$, and the pressure at V is, therefore, $p+\frac{dp}{dx}h$ nearly.

(149.) These pressures have been found from consideration of the motions of the particles of water in a vertical direction, and are therefore vertical pressures. Now the peculiar property of fluids is, that the pressure is equal in all directions, or that the expressions which we have found for the vertical pressures at T and V represent also the horizontal pressures at those points. (On this subject the reader is referred to article 75.) Thus we have

Horizontal pressure at T tending to push TV forwards $=p$,

Horizontal pressure at V tending to push TV backwards $=p+\frac{dp}{dx}h$.

Difference, tending to push TV forwards $=-\frac{dp}{dx}h$.

The length of the column TV on which it acts is nearly h: therefore the pressure tends to urge it forwards with an acceleration represented by $-\frac{dp}{dx}$. If, besides, any force depending on extraneous causes is acting, which would produce an acceleration represented by F, the whole acceleration will be $\mathrm{F}-\frac{dp}{dx}$. Thus we have

$$\frac{d^2(x+\mathrm{X})}{dt^2}=\mathrm{F}-\frac{dp}{dx}.$$

But x does not depend on t, therefore $\frac{d^2(x+\mathrm{X})}{dt^2}=\frac{d^2\mathrm{X}}{dt^2}$. And, putting for p its value, we finally obtain this equation,

Equation of equal pressure.

$$\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g'(k+\mathrm{K}-y)-\int_y \frac{d^2\mathrm{Y}}{dt^2}\,(y \text{ to } k)\right\}.$$

This may be called the equation of equal pressure.

This equation, in conjunction with the equation

$$\mathrm{Y}=\Xi.\frac{d\eta}{dx}-\int_y \frac{d\mathrm{X}}{dx}\,(\eta \text{ to } y),$$

contains the whole theory of the motion of fluids in canals, of uniform breadth, but of uniform or variable depth. the motion being supposed to be entirely longitudinal and vertical.

Equation of equal pressure when gravity is uniform.

(150.) If gravity be considered uniform, and $=g$, the term $\frac{d}{dx}\{-g'(k+\mathrm{K}-y)\}$ becomes $\frac{d}{dx}(-gk+gy-g\mathrm{K})$ $=\frac{d}{dx}(-g\mathrm{K})$: and the equation of equal pressure becomes

$$\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}-\int_y \frac{d^2\mathrm{Y}}{dt^2}\,(y \text{ to } k)\right\}.$$

Equation of continuity when the depth is uniform.

If the depth of the canal be uniform, $\frac{d\eta}{dx}=0$, and the equation of continuity, taking $\eta=0$, or assuming y to be measured from the bottom, becomes

$$\mathrm{Y}=-\int_y \frac{d\mathrm{X}}{dx}\,(0 \text{ to } y).$$

(151.) In the treatment of these equations, different methods must be used, according to the demands of the problem. If the nature of the motion in the direction of x be assumed; that is, if the expression for X be known, and if it be required to find what force is necessary to maintain the fluid in that state of motion; then we must, from the given expression for X, find the expression for Y by the equation of continuity; and, substituting both in the equation of equal pressure, we shall obtain F. But if F be given, and X and Y be required, we can only eliminate Y by means of the equation of continuity, and then solve the equation for X by methods depending on the form of the resulting equation.

(152.) If there be no motion of the particles, $0=\mathrm{F}-\frac{d}{dx}(g'.\overline{k+\mathrm{K}-y})$; or, if gravity be constant, $0=\mathrm{F}-g\frac{d\mathrm{K}}{dx}$.

Equations limited to oscillatory motion.

(153.) Hitherto we have not, in the investigation, limited ourselves in any way to the case of oscillating motion of the particles: the equations are, in fact, perfectly general as to the kind of longitudinal and vertical motion, and require only that the motion be small in extent. We shall now assume that the motion is oscillating; and, for the reason mentioned in (142.), we shall confine the expression for X or Y to a single term represented by a cosine or sine. Now suppose X to be represented by $\mathrm{A}\cos(nt-\mathrm{B})$, where A and B are any functions whatever of x and y. (This is the most general form that can be assumed: it does not even imply that the form or magnitude of the waves is uniform, or that the motions of particles, originally in the same vertical column, are constantly in the same direction.) Then $\frac{d^2\mathrm{X}}{dt^2}=-n^2\mathrm{A}\cos(nt-\mathrm{B})=-n^2\mathrm{X}$. And as, in consequence of the equation of continuity, Y will necessarily depend on the sines or the cosines of the same angle $nt-\mathrm{B}$, we shall also have $\frac{d^2\mathrm{Y}}{dt^2}=-n^2\mathrm{Y}$. Thus the equation of equal pressure becomes

$$-n^2\mathrm{X}=\mathrm{F}+\frac{d}{dx}\{-g'(k+\mathrm{K}-y)+n^2\textstyle\int_y\mathrm{Y}\ (y\text{ to }k)\};$$

or, if gravity is considered as uniform,

$$-n^2\mathrm{X}=\mathrm{F}+\frac{d}{dx}\{-g\mathrm{K}+n^2\textstyle\int_y\mathrm{Y}\ (y\text{ to }k)\}.$$

We shall now proceed with some applications of these equations.

(154.) Problem.—To examine whether it is possible that a system of waves, depending upon oscillatory motion of the particles of water, can move along a canal of uniform breadth, but of variable depth: gravity being supposed uniform, and no other force being supposed to act.

Investigation of the possibility of unbroken waves when the depth is variable.

(155.) The equations to be satisfied are

$$\mathrm{Y}=\Xi.\frac{d\eta}{dx}-\int_y\frac{d\mathrm{X}}{dx}(\eta\text{ to }y)$$

$$-n^2\mathrm{X}=\frac{d}{dx}\{-g\mathrm{K}+n^2\textstyle\int_y\mathrm{Y}\ (y\text{ to }k)\}.$$

Differentiating the first with respect to y and to x,

$$\frac{d\mathrm{Y}}{dy}=-\frac{d\mathrm{X}}{dx}$$

$$\frac{d^2\mathrm{Y}}{dy\,dx}=-\frac{d^2\mathrm{X}}{dx^2}.$$

The second equation, observing that the limits of integration of $\int_y\mathrm{Y}$ are independent of x, may be put under this form,

$$-n^2\mathrm{X}=-g.\frac{d\mathrm{K}}{dx}-n^2\int_y\frac{d\mathrm{Y}}{dx}(k\text{ to }y).$$

Differentiating with respect to y,

$$-n^2\frac{d\mathrm{X}}{dy}=-n^2\frac{d\mathrm{Y}}{dx},\quad\text{or}\ \frac{d\mathrm{X}}{dy}=\frac{d\mathrm{Y}}{dx}.$$

Differentiating again with respect to y,

$$\frac{d^2\mathrm{X}}{dy^2}=\frac{d^2\mathrm{Y}}{dx\,dy}.$$

Eliminating $\frac{d^2\mathrm{Y}}{dx\,dy}$ by means of the former equation,

$$\frac{d^2\mathrm{X}}{dy^2}+\frac{d^2\mathrm{X}}{dx^2}=0,$$

the general solution of which equation is

$$\mathrm{X}=\phi(y+x\sqrt{-1})+\psi(y-x\sqrt{-1}),$$

where ϕ and ψ express functions chosen to satisfy any conditions required by the circumstances of the problem.

If, instead of ϕ and ψ, we use two other functions χ and ω, we may put the solution under this form,

$$X=\{\chi(y+x\sqrt{-1})+\chi(y-x\sqrt{-1})\}+\frac{1}{\sqrt{-1}}\{\omega(y+x\sqrt{-1})-\omega(y-x\sqrt{-1})\}$$

the whole of which is real, whatever be the form of the functions χ and ω, provided that they contain, in their form, nothing imaginary.

(156.) But the expression just found is the solution of the *derived* equation. We have now to ascertain whether it will satisfy the *original* equations. For convenience we will confine our substitution to a single term (as the result produced by another term will be easily inferred from its result), and will make

$$X=v'(y+x\sqrt{-1}),$$

conceiving v' to be the derived function of some function v. Then

$$\frac{dX}{dx}=\sqrt{-1}\,.v''(y+x\sqrt{-1})$$

$$\int_y\frac{dX}{dx}=\sqrt{-1}\,.v'(y+x\sqrt{-1})$$

therefore $-\int_y\frac{dX}{dx}$ from η to y is $\sqrt{-1}\;v'(\eta+x\sqrt{-1})-\sqrt{-1}\,.v'(y+x\sqrt{-1})$;

Also Ξ, or the value of X when $y=\eta$, $=v'(\eta+x\sqrt{-1})$,

therefore $$\Xi\frac{d\eta}{dx}=v'(\eta+x\sqrt{-1})\frac{d\eta}{dx}.$$

Hence $$Y=\Xi\frac{d\eta}{dx}-\int_y\frac{dX}{dx}(\eta\text{ to }y)=v'(\eta+x\sqrt{-1})\left(\frac{d\eta}{dx}+\sqrt{-1}\right)-\sqrt{-1}.\;v'(y+x\sqrt{-1})$$
$$=\frac{d}{dx}.\;v(\eta+x\sqrt{-1})-\sqrt{-1}\,.\;v'(y+x\sqrt{-1}).$$

And K, or the value of Y when $y=k$, $=\frac{d}{dx}.\;v(\eta+x\sqrt{-1})-\sqrt{-1}\,.\;v'(k+x\sqrt{-1})$;

therefore $$g\frac{dK}{dx}=g\frac{d^2}{dx^2}v(\eta+x\sqrt{-1})+g.v''(k+x\sqrt{-1}).$$

And $$\frac{dY}{dx}=\frac{d^2}{dx^2}v(\eta+x\sqrt{-1})+v''(y+x\sqrt{-1});$$

therefore $\int_y\frac{dY}{dx}(y\text{ to }k)$ is $(k-y)\,\frac{d^2}{dx^2}v(\eta+x\sqrt{-1})+v'(k+x\sqrt{-1})-v'(y+x\sqrt{-1})$.

Substituting these in the equation $-n^2X=-g\frac{dK}{dx}+n^2\int_y\frac{dY}{dx}(y\text{ to }k)$,

$$-n^2.\,v'(y+x\sqrt{-1})=-g\frac{d^2}{dx^2}v(\eta+x\sqrt{-1})-gv''(k+x\sqrt{-1})+n^2(k-y)\frac{d^2}{dx^2}v(\eta+x\sqrt{-1})$$
$$+n^2v'(k+x\sqrt{-1})-n^2v'(y+x\sqrt{-1}).$$

Removing the terms which mutually destroy each other, and conceiving the same substitution to be made for a term $v'(y-x\sqrt{-1})$, we shall have this equation,

$$0=(n^2k-n^2y-g)\frac{d^2}{dx^2}\{v(\eta+x\sqrt{-1})+v(\eta-x\sqrt{-1})\}-g\{v''(k+x\sqrt{-1})+v''(k-x\sqrt{-1})\}$$
$$+n^2\{v'(k+x\sqrt{-1})+v'(k-x\sqrt{-1})\}$$

(157.) It is evident, from the form of the multipliers, and the perfect independence of x and y, that this equation implies the existence of the separate equations.

$$0=\frac{d^2}{dx^2}\left\{v(\eta+x\sqrt{-1})+v(\eta-x\sqrt{-1})\right\}$$
$$0=-g\{v''(k+x\sqrt{-1})+v''(k-x\sqrt{-1})\}+n^2\{v'(k+x\sqrt{-1})+v'(k-x\sqrt{-1})\}$$

When the depth is variable, that is, when η is a function of x, it does not appear possible to satisfy the first of these equations by any form of v. It would appear, therefore, that when the depth is variable, it is impossible that there can be a series of waves which consist of oscillatory motion of the particles, and which satisfy the two equations of continuity and of equal pressure.

Waves consisting of oscillatory motion not possible when the depth is not uniform.

(158.) The following physical interpretation of this mathematical result appears to be correct, and is worthy of attention. It appears that, if the water is moving in the manner of waves, one at least of the two conditions (continuity and equal pressure) must fail. While the continuity holds, the equal pressure will exist, from the nature of the fluid. Therefore the continuity must cease, or the water must become *broken*. This appears to be the explanation of the broken water which is usually seen upon the edge of a shoal or a ledge of rocks, although the whole is covered, perhaps deeply, by the water. We shall advert again to this subject.

(159.) When the depth is uniform, or $\eta=0$, there is no difficulty in satisfying the equation. For instance, if $v(\theta)=\varepsilon^{m\theta}-\varepsilon^{-m\theta}$, the expression $v(y+x\sqrt{-1})+v(y-x\sqrt{-1})$ becomes $2(\varepsilon^{my}-\varepsilon^{-my}).\cos mx$, which when $y=0$ is always$=0$. Any other form of $v(\theta)$ expressed by odd powers only of θ would do equally well.

Subsection 2.—Investigation of the Motion of Waves in a Canal of uniform Depth, the Motions being small.

(160.) We shall assume (as a hypothesis to be proved or disproved by substitution in the equations) that the motion of the waves is uniform, and that all the motions of the particles, horizontal and vertical, are oscillatory; but we shall not, in the first instance, assume that the motions of all the particles in the same vertical line are of the same kind at the same instant. This will be expressed by assuming

$$X=P.\cos(nt-mx-Q)$$

where m is constant and P and Q may be functions of y.

Expanding the cosine, and making $P.\cos Q=R$, $P.\sin Q=S$, this becomes

$$X=R.\cos(nt-mx)+S.\sin(nt-mx).$$

(161.) Now, as we have found (155.), $\frac{d^2X}{dy^2}+\frac{d^2X}{dx^2}=0$: which in the present instance becomes

$$\left(\frac{d^2R}{dy^2}-m^2R\right)\cos(nt-mx)+\left(\frac{d^2S}{dy^2}-m^2S\right)\sin(nt-mx)=0.$$

Therefore $$\frac{d^2R}{dy^2}-m^2R=0,\qquad \frac{d^2S}{dy^2}-m^2S=0;$$

whence $$R=C.\varepsilon^{my}+D.\varepsilon^{-my},\qquad S=C'.\varepsilon^{my}+D'.\varepsilon^{-my},$$

and $$X=(C.\varepsilon^{my}+D.\varepsilon^{-my})\cos(nt-mx)+(C'.\varepsilon^{my}+D'.\varepsilon^{-my})\sin(nt-mx).$$

And as $Y=-\int_y\frac{dX}{dx}$ (0 to y), we easily find

$$Y=-(C\varepsilon^{my}-D\varepsilon^{-my}-C+D)\sin(nt-mx)+(C'\ \varepsilon^{my}-D'.\varepsilon^{-my}-C'+D)\cos(nt-mx)$$

$$K=-(C\varepsilon^{mk}-D\varepsilon^{-mk}-C+D)\sin(nt-mx)+(C'.\varepsilon^{mk}-D'.\varepsilon^{-mk}-C'+D')\cos(nt-mx)$$

$$\frac{dY}{dx}=+m(C\varepsilon^{my}-D\varepsilon^{-my}-C+D)\cos(nt-mx)+m(C'.\varepsilon^{my}-D'.\varepsilon^{-my}-C'+D')\sin(nt-mx)$$

$$\int_y\frac{dY}{dx}(y\text{ to }k)=$$

$$-\{C\varepsilon^{my}+D\varepsilon^{-my}-C\varepsilon^{mk}-D\varepsilon^{-mk}+m.\overline{y-k}\,.\,\overline{-C+D}\}\cos(nt-mx)$$
$$-\{C'.\varepsilon^{my}+D'.\varepsilon^{-my}-C'.\varepsilon^{mk}-D'.\varepsilon^{-mk}+m.\overline{y-k}.\overline{-C'+D'}\}\sin(nt-mx)$$

Substituting in the equation $-n^2X=-g\frac{dK}{dx}+n^2\int_y\frac{dY}{dx}(y\text{ to }k)$

$$-n^2(C\varepsilon^{my}+D\varepsilon^{-my})\cos(nt-mx)-n^2(C'.\varepsilon^{my}+D'.\varepsilon^{-my})\sin(nt-mx)=$$
$$-mg(C\varepsilon^{mk}-D\varepsilon^{-mk}-C+D)\cos(nt-mx)-mg(C'.\varepsilon^{mk}-D'\varepsilon^{-mk}-C'+D')\sin(nt-mx)$$
$$-n^2\{C\varepsilon^{my}+D\varepsilon^{-my}-C\varepsilon^{mk}-D\varepsilon^{-mk}+m.\overline{y-k}.\overline{-C+D}\}\cos(nt-mx)$$
$$-n^2\{C'.\varepsilon^{my}+D'.\varepsilon^{-my}-C'.\varepsilon^{mk}-D'.\varepsilon^{-mk}+m.\overline{y-k}.\overline{-C'+D'}\}\sin(nt-mx)$$

Removing the terms which destroy each other, equating separately the coefficients of cos $(nt-mx)$ and of sin $(nt-mx)$, and observing that y is entirely independent of k, we obtain in the first place

$$-C+D=0 \qquad -C'+D'=0: \qquad \text{or } D=C, \qquad D'=C'.$$

Substituting these values of D and D′ in the other equations, they become

$$-mg\,C\,(\varepsilon^{mk}-\varepsilon^{-mk})+n^2\,C.\,(\varepsilon^{mk}+\varepsilon^{-mk})=0$$

$$-mg\,C'\,(\varepsilon^{mk}-\varepsilon^{-mk})+n^2\,C'\,(\varepsilon^{mk}+\varepsilon^{-mk})=0.$$

These equations leave C and C′ indeterminate; and they agree in giving the following relation between m and n,

$$n^2=mg.\,\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}.$$

Then, substituting in R and S the values of D and D′,

$$X=(\varepsilon^{my}+\varepsilon^{-my}).\{C\cos(nt-mx)+C'\sin(nt-mx)\};$$

Expressions for displacement of particles in wave-motion when the depth is uniform.

or if $A=\sqrt{C^2+C'^2}$, $\tan B=\frac{C'}{C}$; then $C=A\cos B$, $C'=A\sin B$, and the expression becomes

$$X=A.\,(\varepsilon^{my}+\varepsilon^{-my}).\,\cos(nt-mx-B)$$

where A and B are independent of x and y, and m depends upon n by the equation above. From this we easily find

$$Y=-A.\,(\varepsilon^{my}-\varepsilon^{-my}).\,\sin(nt-mx-B).$$

(162.) From these expressions it appears, as B is constant, that the mathematical results compel us to admit (what we did not at first assume) that the motion of all the particles originally in the same vertical line is, at any instant, of the same kind, although the extent of the motion of the different particles is very different. In future we shall generally omit B from the expression, as the same effect will be produced in the value of the expression by altering the origin of the time as by retaining B.

Explanation of terms.

(163.) It will be convenient to explain here a few terms which will hereafter be occasionally used.

Phase.

(164.) The angle $nt-mx-B$, or $nt-mx$, upon which the expressions for X and Y depend, is called the *phase* of the wave. It is to be considered as an angle, or rather as part of a circle (in the sense in which trigonometrical expressions depend upon an angle or part of a circle) which admits of indefinite increase.

Length of wave.

(165.) The expressions for X and Y are not altered in value if we increase or diminish mx by 2π, 4π, 6π, &c.: that is, if we increase or diminish x by $\frac{2\pi}{m}$, $\frac{4\pi}{m}$, $\frac{6\pi}{m}$, &c., while t is unaltered: but the same cannot be asserted of any other value of x. Hence $\frac{2\pi}{m}$ is the value of each of the successive distances at which we find the water in the same state of disturbance, or $\frac{2\pi}{m}$ is the distance between one wave and the next. We shall call this the *length of a wave*, or the *amplitude of a wave*, and shall denote it by the symbol λ.

Period of wave.

(166.) The expressions for X and Y are not altered in value if we increase or diminish nt by 2π, 4π, 6π, &c., that is, if we increase or diminish t by $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, &c., while x is unaltered: but the same cannot be asserted of any other value of t. Hence $\frac{2\pi}{n}$ represents each of the intervals of time at which the water at any given place is successively in the same state of disturbance: we shall call this the *period* of a wave and shall denote it by the symbol τ.

(167.) We shall now proceed to develope some of the practical interpretations of the equations just found.

(168.) Since X and Y both depend on $nt-mx$, if we put $t+t'$ for t, and $x+x'$ for x, we shall find new values X′ and Y′ depending on $nt-mx+nt'-mx'$; and these will be the same as X and Y if $nt'-mx'$ be zero: that is, at the end of the interval t', we shall find the water in a similar state of displacement, provided we examine it at the distance x' in advance: that is, in the time t' the wave will have travelled through the space x', or the velocity of the wave will be expressed by $\frac{x'}{t'}$; which, by virtue of the equation $nt'-mx'=0$, is the same as $\frac{n}{m}$. Hence

$$\text{Velocity of the wave}=\frac{n}{m}.$$

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Subsect. 2. Motion of Waves in a Canal of uniform Depth.

But we have found $n^2=gm.\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}$: hence the velocity $=\sqrt{\frac{n^2}{m^2}}=\sqrt{\left\{\frac{g}{m}\cdot\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}\right\}}$. This expression is not constant, but depends on m and k. From this it appears—

Inferences with regard to the velocity of waves.

1st. In water of given depth, the velocity of waves of all kinds is not the same, but depends upon m, or upon $\frac{2\pi}{m}$, or upon λ the length of the wave. (In this respect the waves of water are analogous to those of light, but are not analogous to those of sound.)

2nd. It follows immediately that, in water of given depth, the velocity of waves depends upon the interval of time at which they follow each other.

3rd. If the distance between the crests of the waves $\left(\lambda \text{ or } \frac{2\pi}{m}\right)$ is given, the velocity varies with the depth of the water.

4th. If the interval of time between successive waves $\left(\tau \text{ or } \frac{2\pi}{n}\right)$ is given, the velocity varies with the depth of the water.

(169.) The circumstances of waves of different lengths will easily be reduced to numerical calculation, if for n we put $\frac{2\pi}{\tau}$, and for m we put $\frac{2\pi}{\lambda}$; the equation between n and m is thus brought to the following form:

$$\tau^2=\frac{2\pi\lambda}{g}\cdot\frac{\varepsilon^{\frac{4\pi k}{\lambda}}+1}{\varepsilon^{\frac{4\pi k}{\lambda}}-1};$$

where it is to be remarked that $\varepsilon^{\frac{4\pi k}{\lambda}}$ is the number whose common logarithm is $\frac{4\pi k}{\lambda}\times 0\cdot434294$. If the time is expressed in seconds, and the space in feet, g (or the velocity which gravity communicates to a free body in one second), expressed in the same manner, is 32·16. From these data the following table is computed:

Table of the periods of waves.

Table I.

Depth of water, in feet.	Length of the wave, in feet.								
	1	10	100	1000	10,000	100,000	1000,000	10,000,000	100,000,000
	Corresponding period of wave, in seconds.								
1	0·442	1·873	17·645	176·33	1763·3	17633	176330	1763300	17633000
10	0·442	1·398	5·923	55·800	557·62	5576·2	55762	557620	5576200
100	0·442	1·398	4·420	18·730	176·45	1763·3	17633	176330	1763300
1000	0·442	1·398	4·420	13·978	59·230	558·00	5576·2	55762	557620
10,000	0·442	1·398	4·420	13·978	44·201	187·30	1764·5	17633	176330
100,000	0·442	1·398	4·420	13·978	44·201	139·78	592·30	5580	55762
1000,000	0·442	1·398	4·420	13·978	44·201	139·78	442·01	1873	17645

(170.) From these numbers the velocities are easily computed by dividing the length of the wave by the period. The following table contains the results:—

Table of the velocities of waves.

Table II.

Depth of water, in feet.	Length of the wave, in feet.									
	1	10	100	1000	10,000	100,000	1000,000	10,000,000	100,000,000	Infinite.
	Corresponding velocity of wave per second, in feet.									
1	2·2624	5·3390	5·6672	5·6710	5·6710	5·6710	5·6710	5·6710	5·6710	5·6710
10	2·2624	7·1543	16·883	17·921	17·933	17·933	17·933	17·933	17·933	17·933
100	2·2624	7·1543	22·624	53·390	56·672	56·710	56·710	56·710	56·710	56·710
1000	2·2624	7·1543	22·624	71·543	168·83	179·21	179·33	179·33	179·33	179·33
10,000	2·2624	7·1543	22·624	71·543	226·24	533·90	566·72	567·10	567·10	567·10
100,000	2·2624	7·1543	22·624	71·543	226·24	715·43	1688·3	1792·1	1793·3	1793·3
1000,000	2·2624	7·1543	22·624	71·543	226·24	715·43	2262·4	5339·0	5667·2	5671·0

Tides and Waves.

Tides and Waves.

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The velocity of short waves does not depend on the depth.

The velocity of long waves depends only on the depth.

(171.) From these numbers it appears that,

1st. When the length of the wave is not greater than the depth of the water, the velocity of the wave depends (sensibly) only on its length, and is proportional to the square root of its length.

2nd. When the length of the wave is not less than a thousand times the depth of the water, the velocity of the wave depends (sensibly) only on the depth, and is proportional to the square root of the depth. It is, in fact, the same as the velocity which a free body would acquire by falling from rest, under the action of gravity, through a height equal to half the depth of the water.

3rd. For intermediate proportions of the length of the wave and the depth of the water, the velocity of the wave can be found only by means of the general equation.

(172.) The values in the extreme cases will easily be found by expansion of the general formula. Thus, when λ is small, or $\frac{k}{\lambda}$ large, $\varepsilon^{\frac{4\pi k}{\lambda}}$ is very large: $\frac{\varepsilon^{\frac{4\pi k}{\lambda}}+1}{\varepsilon^{\frac{4\pi k}{\lambda}}-1}$ does not sensibly differ from unity, and $\tau^2=\frac{2\pi\lambda}{g}$: whence the velocity $=\frac{\lambda}{\tau}=\sqrt{\frac{g\lambda}{2\pi}}$. When λ is large, or $\frac{k}{\lambda}$ small, $\varepsilon^{\frac{4\pi k}{\lambda}}=1+\frac{4\pi k}{\lambda}$ nearly, and $\tau^2=\frac{\lambda^2}{gk}$; whence the velocity $=\frac{\lambda}{\tau}=\sqrt{gk}$.

(173.) There is one numerical value of τ which, in reference to future applications, deserves special consideration—it is that corresponding to the wave originally produced by the action of the Sun and Moon. This wave may with propriety be called the tide-wave. But for clearness we may remark (although anticipating another part of this Section), that there are two distinct classes of waves bearing the name *tide-wave.* One of them is produced by the immediate action of the forces of the Sun and Moon, and its highest or its lowest point is always at a determinate distance from that place in the canal at which the disturbing forces vanish; the velocity, therefore, of this wave does not depend at all upon the depth of the water. This we shall call the *forced tide-wave.* Coexistent with this there may be, and generally is, a wave in which the period τ is the same, but in which λ is different, and the velocity consequently different; produced originally by the action of the Sun and Moon, but not affected by their action in regard to its velocity of propagation: this kind of tide-wave is the only one which is sensible in narrow seas and rivers. This we shall call the *free tide-wave.* Our present computations belong to the *free* tide-wave only.

(174.) The interval of time (τ) for the semidiurnal tide-wave is not constant; but it may be considered as 12 hours 24 minutes, or 44,640 seconds. Now we have found, Table II., that with a depth of water of 100,000 feet, or 20 miles nearly, a wave of any length exceeding 100,000,000 feet will travel with a velocity sensibly independent of its length; and that even if its length be only 10,000,000 feet, the velocity will not differ from that of a longer wave by $\frac{1}{1509}$ part. The former wave, by Table I., would have the period 55,762 seconds; the latter would have the period 5580 seconds. It is clear that the period of the tide-wave is sufficient to justify us in assuming the velocity of the *free* tide-wave as equal to that which belongs properly to a wave of indefinite length in water of the same depth. On this principle the following table is computed:—

Table for the semidiurnal free tide-wave.

TABLE III.

Depth of water, in feet.	Velocity of free tide-wave per second, in feet.	Length of free tide-wave, in miles.	Space described by free tide-wave in one hour, in miles.	Depth of water, in feet.	Velocity of free tide-wave per second, in feet.	Length of free tide-wave, in miles.	Space described by free tide-wave in one hour, in miles.
1	5·671	47·946	3·8666	5000	401·00	3390·2	273·41
4	11·342	95·891	7·7332	6000	439·27	3713·8	299·50
10	17·933	151·62	12·227	7000	474·47	4011·4	323·50
20	25·361	214·42	17·292	8000	507·23	4288·3	345·84
40	35·867	303·24	24·455	9000	538·00	4548·5	366·82
60	43·927	371·38	29·950	10000	567·10	4794·6	386·66
80	50·723	428·83	34·584	15000	694·54	5872·1	473·55
100	56·710	479·46	38·666	20000	802·00	6780·5	546·82
200	80·200	678·05	54·682	25000	896·66	7580·9	611·36
400	113·42	958·91	77·332	30000	982·25	8304·4	669·71
600	138·91	1174·4	94·711	35000	1060·9	8969·7	723·37
800	160·40	1356·1	109·36	40000	1134·2	9589·1	773·32
1000	179·33	1516·2	122·27	45000	1203·0	10171	820·22
2000	253·61	2144·2	172·92	50000	1268·1	10721	864·59
3000	310·62	2626·1	211·78	55000	1330·0	11244	906·80
4000	358·67	3032·4	244·55	60000	1389·1	11744	947·11

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(175.) The diurnal tide-wave, and others connected with the tides, of which we shall have occasion to speak, so far as they are *free* tide-waves, may all be considered as moving with the same velocity. The table will, therefore, apply to them, excepting only the column of the *length of the tide-wave*, the numbers in which must be doubled for the diurnal tide-wave.

(176.) We shall now proceed to examine the circumstances of the motion of each particle of water in wave-motion. And, first, we shall consider the values of the extreme extent of motion of each particle.

Formulæ for extreme motion of individual particles.

(177.) We have found (161.) that the horizontal displacement X of any particle, whose height above the bottom is y, is represented by $A(\varepsilon^{my}+\varepsilon^{-my})\cos(nt-mx-B)$. The extreme values of this quantity are $-A(\varepsilon^{my}+\varepsilon^{-my})$ when $nt-mx-B$ is an odd multiple of 180°, and $+A(\varepsilon^{my}+\varepsilon^{-my})$ when $nt-mx-B$ is an even multiple of 180°. And (161.) the vertical displacement Y of the same particle is represented by $-A(\varepsilon^{my}-\varepsilon^{-my})\sin(nt-mx-B)$; the extreme values of which are $-A(\varepsilon^{my}-\varepsilon^{-my})$ when $nt-mx-B=90°+$ an even multiple of 180°, and $+A(\varepsilon^{my}-\varepsilon^{-my})$ when $nt-mx-B=90°+$ an odd multiple of 180°. Hence $A(\varepsilon^{my}+\varepsilon^{-my})$ and $A(\varepsilon^{my}-\varepsilon^{-my})$ may be considered as representing the greatest extent of horizontal and vertical motions; or, omitting A (as our object is only to show the proportion of the various motions), and putting $\frac{2\pi}{\lambda}$ for m, the horizontal and vertical motions will be represented respectively by $\varepsilon^{\frac{2\pi y}{\lambda}}+\varepsilon^{-\frac{2\pi y}{\lambda}}$ and $\varepsilon^{\frac{2\pi y}{\lambda}}-\varepsilon^{-\frac{2\pi y}{\lambda}}$. At the bottom, where $y=0$, the extreme horizontal motion will be represented by $\varepsilon^{0}+\varepsilon^{0}=2$. By means of these formulæ the following table is computed :—

TABLE IV.

Greatest extent of the horizontal and vertical displacements of the particles at different depths, for different proportions of the length of the wave to the whole depth: the greatest horizontal displacement of the particles at the bottom being represented by 2.

Table of proportional extent of horizontal and vertical motions.

Depth below the surface.	Values of $\frac{k}{\lambda}$.					
	10000		1000		100	
	Corresponding displacements of the particle.					
	Horizontal.	Vertical.	Horizontal.	Vertical.	Horizontal.	Vertical.
At surface	1(27287 fig.)	1(27287 fig.)	5(2728 fig.)	5(2728 fig.)	7(272 fig.)	7(272 fig.)
$\frac{1}{10}$ whole depth	2(24558 fig.)	2(24558 fig.)	6(2455 fig.)	6(2455 fig.)	3(245 fig.)	3(245 fig.)
$\frac{2}{10}$,, ,,	4(21829 fig.)	4(21829 fig.)	9(2182 fig.)	9(2182 fig.)	1(218 fig.)	1(218 fig.)
$\frac{3}{10}$,, ,,	8(19100 fig.)	8(19100 fig.)	1(1910 fig.)	1(1910 fig.)	1(191 fig.)	1(191 fig.)
$\frac{4}{10}$,, ,,	1(16372 fig.)	1(16372 fig.)	1(1637 fig.)	1(1637 fig.)	5(163 fig.)	5(163 fig.)
$\frac{5}{10}$,, ,,	3(13643 fig.)	3(13643 fig.)	2(1364 fig.)	2(1364 fig.)	2(136 fig.)	2(136 fig.)
$\frac{6}{10}$,, ,,	6(10914 fig.)	6(10914 fig.)	3(1091 fig.)	3(1091 fig.)	1(109 fig.)	1(109 fig.)
$\frac{7}{10}$,, ,,	1(8186 fig.)	1(8186 fig.)	4(818 fig.)	4(818 fig.)	7(81 fig.)	7(81 fig.)
$\frac{8}{10}$,, ,,	2(5457 fig.)	2(5457 fig.)	5(545 fig.)	5(545 fig.)	3(54 fig.)	3(54 fig.)
$\frac{9}{10}$,, ,,	5(2728 fig.)	5(2728 fig.)	7(272 fig.)	7(272 fig.)	1(27 fig.)	1(27 fig.)
At bottom	2	0	2	0	2	0

Depth below the surface.	Values of $\frac{k}{\lambda}$.					
	10		1		$\frac{1}{10}$	
	Corresponding displacements of the particle.					
	Horizontal.	Vertical.	Horizontal.	Vertical.	Horizontal.	Vertical.
At surface	1(27 fig.)	1(27 fig.)	535·429	535·425	2·40792	1·34094
$\frac{1}{10}$ whole depth	3(24 fig.)	3(24 fig.)	285·651	285·643	2·32838	1·19220
$\frac{2}{10}$,, ,,	6(21 fig.)	6(21 fig.)	152·398	152·384	2·25802	1·04816
$\frac{3}{10}$,, ,,	1(19 fig.)	1(19 fig.)	81·3122	81·2876	2·19658	0·90826
$\frac{4}{10}$,, ,,	2(16 fig.)	2(16 fig.)	43·3961	43·3499	2·14381	0·77195
$\frac{5}{10}$,, ,,	4(13 fig.)	4(13 fig.)	23·1825	23·0961	2·09951	0·63869
$\frac{6}{10}$,, ,,	8(10 fig.)	8(10 fig.)	12.4257	12·2637	2·06349	0·50795
$\frac{7}{10}$,, ,,	1(8 fig.)	1(8 fig.)	6·73766	6·43398	2·03564	0·37922
$\frac{8}{10}$,, ,,	286690	286690	3·79812	3·22888	2·01581	0·25199
$\frac{9}{10}$,, ,,	535·4	535·4	2·40793	1·34093	2.00395	0·12575
At bottom	2·0	0·0	2·00000	0·00000	2·00000	0·00000

TABLE IV.—*continued.*

Depth below the surface.	Values of $\frac{k}{\lambda}$.					
	$\frac{1}{100}$		$\frac{1}{1000}$		$\frac{1}{10000}$	
	Corresponding displacements of the particle.					
	Horizontal.	Vertical.	Horizontal.	Vertical.	Horizontal.	Vertical.
At surface	2·003950	0·125744	2·000040	0·012566	2·000001	0·001257
$\frac{1}{10}$ whole depth	2·003199	0·113155	2·000032	0·011310	2·000001	0·001131
$\frac{2}{10}$,, ,,	2·002527	0·100571	2·000025	0·010053	2·000001	0·001005
$\frac{3}{10}$,, ,,	2·001935	0·087991	2·000019	0·008797	2·000000	0·000880
$\frac{4}{10}$,, ,,	2·001421	0·075415	2·000014	0·007540	2·000000	0·000754
$\frac{5}{10}$,, ,,	2·000987	0·062841	2·000010	0·006284	2·000000	0·000628
$\frac{6}{10}$,, ,,	2·000632	0·050270	2·000006	0·005026	2·000000	0·000502
$\frac{7}{10}$,, ,,	2·000355	0·037701	2·000004	0·003770	2.000000	0·000378
$\frac{8}{10}$,, ,,	2·000158	0·025132	2·000002	0·002514	2·000000	0·000252
$\frac{9}{10}$,, ,,	2·000040	0·012566	2·000001	0·001257	2·000000	0·000126
At bottom	2·000000	0·000000	2·000000	0·000000	2·000000	0·000000

In this table, by 1(27287 figures) is meant the figure 1 followed by 27287 significant figures before the decimal point; and similarly 2(24558 figures) is intended to express the figure 2 followed by 24558 figures before the decimal point; and so for the others.

(178.) From inspection of this table the following points are evident:

When the wave is short, the motion is not sensible except near the surface.

1st. When the depth is great in comparison with the length of the wave (as in the case of ordinary waves in the open sea), the motion of the water at any great depth below the surface is wholly insignificant in comparison with that at the surface. The following rule may be convenient. As the depth below the surface proceeds in arithmetical progression, the motion diminishes in geometrical progression; and at a depth equal to the length of the wave the motion is diminished to $\frac{1}{535\cdot4}$ of that at the surface. This rule will be demonstrated by remarking that when $\frac{k}{\lambda}$ is large and therefore $\frac{y}{\lambda}$ is large (the depths below the surface not being extremely great) $\varepsilon^{mk}+\varepsilon^{-mk}$, or $\varepsilon^{\frac{2\pi}{\lambda}}+\varepsilon^{-\frac{2\pi k}{\lambda}}$, and $\varepsilon^{my}+\varepsilon^{-my}$, or $\varepsilon^{\frac{2\pi y}{\lambda}}+\varepsilon^{-\frac{2\pi y}{\lambda}}$, will, as to sense, be reduced to their first terms $\varepsilon^{\frac{2\pi k}{\lambda}}$ and $\varepsilon^{\frac{2\pi y}{\lambda}}$: and therefore the motion at the depth $k-y$ will $=$ motion at surface $\times\frac{\varepsilon^{\frac{2\pi y}{\lambda}}}{\varepsilon^{\frac{2\pi k}{\lambda}}}=$ motion at surface $\times\,\varepsilon^{-\frac{2\pi}{\lambda}(k-y)}$, or the motion diminishes in geometrical progression, as the depth $k-y$ increases in arithmetical progression. And if $k-y=\lambda$, the motion at the depth $\lambda=$motion at surface $\times\,\varepsilon^{-2\pi}=$ motion at surface $\times\frac{1}{535\cdot4}$.

The horizontal and vertical motions are equal.

(179.) 2d. On the same supposition, the greatest horizontal motion of any particle is equal to its greatest vertical motion: except for those particles very near to the bottom, where the whole motion is insensible.

When the wave is long, the horizontal motion is nearly the same at all depths.

(180.) 3d. When the length of the wave is great in comparison with the depth of the water, (as in the case of tide-waves,) the horizontal motion is sensibly the same from the surface to the bottom, and the vertical motion for different particles varies in the same proportion as their height above the bottom.

The vertical motion is much less than the horizontal motion.

(181.) 4th. On the same supposition the vertical motion of the superficial particles is very much less than their horizontal motion. The proportion of the vertical motion to the horizontal motion is $\frac{\varepsilon^{\frac{2\pi k}{\lambda}}-\varepsilon^{-\frac{2\pi k}{\lambda}}}{\varepsilon^{\frac{2\pi k}{\lambda}}+\varepsilon^{-\frac{2\pi k}{\lambda}}}$: and when $\frac{2\pi k}{\lambda}$ is small, $\varepsilon^{\frac{2\pi k}{\lambda}}-\varepsilon^{-\frac{2\pi k}{\lambda}}$ is nearly equal to $\frac{4\pi k}{\lambda}$, and $\varepsilon^{\frac{2\pi k}{\lambda}}+\varepsilon^{-\frac{2\pi k}{\lambda}}$ is nearly equal to 2: and thus the proportion of the vertical motion of the superficial particles to the horizontal motion is $\frac{4\pi k}{2\lambda}$ $=6\cdot2832\times\frac{k}{\lambda}$.

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When the waves are short, each particle at the surface moves uniformly in a circle.

(182.) We shall now examine the curve described by any individual particle. And first, the motion of a particle at the surface in ordinary small waves. We have seen that, in this case, the horizontal displacement $X = A . \varepsilon^{\frac{2\pi k}{\lambda}} . \cos(nt - mx - B)$ $\left\{\text{neglecting the insignificant term } \varepsilon^{-\frac{2\pi k}{\lambda}}\right\}$, which we will call $C . \cos(nt - mx - B)$. In like manner, the vertical displacement $Y = -A . \varepsilon^{\frac{2\pi k}{\lambda}} . \sin(nt - mx - B) = -C . \sin(nt - mx - B.)$ Therefore $\sqrt{X^2 + Y^2}$, or the particle's whole distance from the point whose co-ordinates are x and y (which do not vary with the time) is $= C$. That is, any particle is, at any instant whatever, somewhere in the circumference of a circle, whose centre is the fixed point defined by the co-ordinates x and y, and whose radius is C or $A . \varepsilon^{\frac{2\pi k}{\lambda}}$. Let θ be the angle made with the vertical by the radius joining the particle and the point whose ordinates are x, y; considered positive when the particle is *above* that point, and *advanced before* it. Then $\tan\theta = \frac{X}{Y} = -\frac{C . \cos(nt - mx - B)}{C . \sin(nt - mx - B)} = -\cot(nt - mx - B) = \tan(nt - mx - B + 90°)$, or $\theta = nt - mx - B + 90°$. As the time t increases uniformly, the angle θ increases uniformly. Consequently the particle revolves uniformly in the circle of which we have spoken; moving forwards when at the highest part of the circle, and backwards when at the lowest part. The same applies to particles below the surface, except those nearly in contact with the bottom.

In all other cases, each particle moves in an ellipse.

(183.) In all other cases, and especially when the length of the wave is great in proportion to the depth of the water, or $\frac{k}{\lambda}$ is small, the coefficients entering into the expressions (which are the same as the greatest motions, whose proportions are given by the numbers in Table IV.) are very different. Representing X by $C . \cos\overline{nt - mx - B}$ and Y by $-c . \sin\overline{nt - mx - B}$, we shall have $\frac{X^2}{C^2} + \frac{Y^2}{c^2} = 1$: the equation to an ellipse whose semiaxes are C and c. The particle therefore will always be found in the circumference of an ellipse whose centre is the point of which the co-ordinates are x, y. In the case of the tide-wave, this ellipse is extremely flat.

The water at the top of the wave moves most rapidly forward.

(184.) But in the tide-wave, and every other wave, which travels along a channel in the manner which we have supposed in this subsection, this law is universal: that the water is travelling forward with its greatest speed at the time of high water, or at the top of the wave. This follows clearly enough from the theorems that we have laid down (that in small waves every particle moves uniformly in a circle, and therefore each particle has its most rapid motion forwards when it is highest: and that in long waves the motions in the two ordinates are expressed by the same law with different coefficients): but it is so important that this should be clearly understood, that we think it necessary expressly to call the reader's attention to it. It will be easily shown by remarking that the horizontal velocity or $\frac{dX}{dt} = -n C . \sin\overline{nt - mx - B}$, and therefore the horizontal velocity forwards is in a constant proportion to Y the elevation of the water above its mean place. Thus when the water is highest, the velocity forward is greatest: when the water is at its mean height (whether before high water or after high water), Y is 0 and the water's velocity is 0, or it is still water: when the water is at its greatest depression, or Y has its greatest negative value, the water is running backwards with its greatest velocity. We shall consider this as the standard case in the theory of waves; and shall occasionally refer to it in noting the deviations from these laws which occur in particular circumstances.

That in the hollow of the wave moves most rapidly backwards.

(185.) Reverting now to the general equations of (150.) for shallow waves, supposing $F = 0$, namely, $Y = -\int_y \frac{dX}{dx}$ (0 to y): $\frac{d^2X}{dt^2} = \frac{d}{dx}\left\{-gK - \int_y \frac{d^2Y}{dt^2} (y \text{ to } k)\right\}$ we may make the following remarks.

First: having found that this equation is satisfied by a function of $+nt$, as $A(\varepsilon^{my} + \varepsilon^{-my}) . \cos(nt - mx)$, we can at once assert that it will be satisfied by the same function of $-nt$, or $A(\varepsilon^{my} + \varepsilon^{-my}) \cos(-nt - mx)$, or

$$A(\varepsilon^{my} + \varepsilon^{-my}) \cos(nt + mx).$$

Opposite waves.

For the only way in which the multiplier n affects the substitution of these quantities in the equation is by its influence in the second differential coefficients $\frac{d^2X}{dt^2}$ and $\frac{d^2Y}{dt^2}$: and these, by differentiation, obtain the same factor whether $+n$ or $-n$ be the multiplier of t. Thus, if $X = A(\varepsilon^{my} + \varepsilon^{-my}) \cos(nt - mx)$, $\frac{d^2X}{dt^2} = -n^2X$: if $X = A(\varepsilon^{my} + \varepsilon^{-my}) \cos(-nt - mx)$, $\frac{d^2X}{dt^2}$ still $= -n^2X$. Now an expression depending on $\cos(-nt - mx)$ or $\cos(nt + mx)$ will imply a wave, in the same manner as one depending on $\cos(nt - mx)$, but with this difference: that whereas in the latter case the wave moves in the direction of x, on the contrary, in the former case, the wave moves in the opposite direction, but with the same speed. This will be seen from the same reasoning as in (168.).

Combination of several waves.

(186.) Secondly: Having found any number of functions X′, X″, X‴, &c., with their corresponding functions Y′, Y″, Y‴, &c., K′, K″, K‴, &c., each of which satisfies the equations

$$Y' = -\int_y \frac{dX'}{dx}, \quad \frac{d^2X'}{dt^2} = \frac{d}{dx}\left\{ -gK' - \int_y \frac{d^2Y'}{dt^2}\,(y \text{ to } k) \right\}, \text{ \&c.}$$

we see at once that the sum X′+X″+X‴+ &c., will satisfy the same equation. For upon adding together all the separate equations formed from X′, X″, X‴, &c. we produce the very same equation which would be formed by substituting the whole quantity X′+X″+X‴+, &c. Now each of the functions X′, X″, &c., with its related functions Y′, K′, Y″, K″, &c, represents a wave or system of waves; and our theorem therefore amounts to this: " If there are any number of systems of shallow waves which can exist separately on water, these waves can all exist together upon the water: the elevations being added at the places where they coincide, or the depression of one being subtracted from the elevation of another where the place of depression of one wave corresponds to the place of elevation of another, &c."

Combination of equal opposite waves.

(187.) One instance of this coexistence of waves may be mentioned as a matter of interest. Suppose two waves, equal in period, and equal in magnitude, to move in opposite directions upon a canal. In one of these, the expression for X′ will be

$$A.(\varepsilon^{my}+\varepsilon^{-my})\cos(nt-mx+B);$$

in the other, the expression for X″ will be

$$A.(\varepsilon^{my}+\varepsilon^{-my})\cos(nt+mx+B').$$

The sum of these gives for the compounded wave X=X′+X″=

$$2A.(\varepsilon^{my}+\varepsilon^{-my}).\cos\left(nt+\frac{B+B'}{2}\right).\cos\left(mx+\frac{B'-B}{2}\right).$$

And the corresponding value of $Y = -\int_y \frac{dX}{dx} =$

$$2A.(\varepsilon^{my}-\varepsilon^{-my}).\cos\left(ni+\frac{B+B'}{2}\right).\sin\left(mx+\frac{B'-B}{2}\right),$$

from which K or the elevation of the surface =

$$2A.(\varepsilon^{mk}-\varepsilon^{-mk}).\cos\left(nt+\frac{B+B'}{2}\right).\sin\left(mx+\frac{B'-B}{\ }\right).$$

From inspection of these expressions we may see that

Produces a stationary wave.

(188.) First, as neither X nor Y depends upon such a quantity as $pt-qx$, there is no appearance whatever of a progressive wave.

Each particle moves in a straight line.

(189.) Secondly; the displacement of any particle in the direction of x depends upon $\cos\left(nt+\frac{B+B'}{2}\right)$, (the factor $\cos\left(mx+\frac{B'-B}{2}\right)$ being constant for that particle,) and the displacement of the same particle in the direction of y depends also upon $\cos\left(nt+\frac{B+B'}{2}\right)$. Therefore, for any particle, the horizontal and vertical displacements are always in the same proportion, or each particle moves in a straight line.

(190.) Thirdly; at the places where $\cos\left(mx+\frac{B'-B}{2}\right)=0$, or where $mx+\frac{B'-B}{2}=\frac{\pi}{2}$, or $\frac{3\pi}{2}$, or $\frac{5\pi}{2}$, &c., there is no horizontal motion whatever, but there is great vertical motion.

(191.) Fourthly; at the places where $\sin\left(mx+\frac{B'-B}{2}\right)=0$, or where $mx+\frac{B'-B}{2}=0$, or π, or 2π, &c., there is no vertical motion whatever, but there is great horizontal motion.

Subsection 3.—Theory of Long Waves, in which the Elevation of the Water bears a sensible proportion to the depth of the Canal.

(192.) Hitherto we have considered the displacements of the particles to be so small that the squares and higher powers of those displacements might be neglected. In the following problem we shall take into account the higher orders of the displacements.

Problem.—To investigate the motion of a very long wave, as the tide-wave, in a canal whose depth is so small that the range of elevation and depression of the surface bears a considerable proportion to the whole depth.

Limitations permitted when the wave is very long.

(193.) We have seen, (180.) and (181.), that, for the tide-wave, the horizontal motion of the particles at different depths in the same vertical line is sensibly the same, and that the vertical motion is extremely small in comparison with the horizontal motion. We shall, therefore, assume that the horizontal motion is uniform for all particles in the same vertical; or that particles which are once in a vertical line remain in a vertical line. And we shall assume that, in estimating the pressure at any point, the height of water above that point is all that needs to be considered: the part represented in the former equations by $\frac{d^2 \mathrm{Y}}{dt^2}$ being insignificant on account of the smallness of the vertical motion. These assumptions are similar to those in (68.).

Equation of continuity, supposing the displacement to be large.

(194.) Let x be the horizontal ordinate corresponding to one vertical line of particles in their state of rest: $x+h$ that corresponding to another vertical line of particles; k the depth of the water in a state of rest. In the state of wave-motion, let the first-mentioned vertical line be displaced at the time t through the horizontal space X, so that its horizontal ordinate is $x+\mathrm{X}$ (x being independent of t); and let V or $k+\mathrm{K}$ be the corresponding depth of the water. Then, the horizontal ordinate which was x has now become $x+\mathrm{X}$; that which was $x+h$ has now become $x+h+\mathrm{X}+\frac{d\mathrm{X}}{dx}h$ nearly: therefore the intermediate horizontal space which was h has now become $h\left(1+\frac{d\mathrm{X}}{dx}\right)$. And the vertical elevation which was k has now become $k+\mathrm{K}$ or V. Consequently, the volume which was expressed by $h \times k$ is now expressed by $h\left(1+\frac{d\mathrm{X}}{dx}\right)\times \mathrm{V}$. These expressions must be equal: therefore we have $h \times k = h\left(1+\frac{d\mathrm{X}}{dx}\right)\times \mathrm{V}$, or

$$\mathrm{V}=\frac{k}{1+\frac{d\mathrm{X}}{dx}}.$$

This is the equation of continuity for this case.

Equation of equal pressure.

(195.) Now if we consider the pressure upon any particle whose coordinates are $x+\mathrm{X}$ and y, as produced by the elevation of the water above it, we shall find it to be represented by the weight of a column of water whose length is $\mathrm{V}-y$. In like manner, the pressure upon the particle whose ordinates are $x+h'+\mathrm{X}+\frac{d\mathrm{X}}{dx}h'$ and y, is represented by the weight of a column of water whose length is $\mathrm{V}+\frac{d\mathrm{V}}{dx}h'-y$. And as the pressures are equal in all directions, the horizontal pressures on the included horizontal column whose length is $h'\left(1+\frac{d\mathrm{X}}{dx}\right)$, are the weights of columns of water, one represented by $\mathrm{V}-y$, tending to push it forwards, and one represented by $\mathrm{V}+\frac{d\mathrm{V}}{dx}h'-y$, tending to push it backwards. Or, the pressure tending to push forwards the column $h'\left(1+\frac{d\mathrm{X}}{dx}\right)$ is the weight of a column $\frac{-d\mathrm{V}}{dx}h'$; and, therefore, the acceleration produced in the direction of x is $-g\frac{d\mathrm{V}}{dx}\cdot\frac{1}{1+\frac{d\mathrm{X}}{dx}}$; or $\frac{d^2(x+\mathrm{X})}{dt^2}=-g\frac{d\mathrm{V}}{dx}\cdot\frac{1}{1+\frac{d\mathrm{X}}{dx}}$. Observing that x is independent of t, and putting for V its value found above, we obtain the equation

$$\frac{d^2\mathrm{X}}{dt^2}=gk\,\frac{\frac{d^2\mathrm{X}}{dx^2}}{\left(1+\frac{d\mathrm{X}}{dx}\right)^3},$$

in which equation $\left(\text{as the variations of V are not to be neglected, and as those variations depend on } \frac{d\mathrm{X}}{dx}\right)$ we must not neglect $\frac{d\mathrm{X}}{dx}$, although we may consider it small.

(196.) This equation may be solved to any degree of approximation by successive substitution. Expanding the fraction, and (for convenience) putting v^2 for gk, the equation may be thus expressed:

$$\frac{d^2X}{dt^2}-v^2.\frac{d^2X}{dx^2}=v^2.\frac{d^2X}{dx^2}\times\left\{-3\frac{dX}{dx}+6\left(\frac{dX}{dx}\right)^2-\&c.\right\}.$$

First we may obtain an approximate value for X by neglecting all the terms on the second side of the equation; then, substituting that value in the first term on the second side, we shall have a more approximate value; and so on.

Solution to first approximation.

(197.) First approximation; $\frac{d^2X}{dt^2}-v^2\frac{d^2X}{dx^2}=0$. To solve this equation let $vt-x=u$, $vt+x=w$; and consider X as a function of u and w. Then $\frac{dX}{dt}=\frac{dX}{du}.\frac{du}{dt}+\frac{dX}{dw}.\frac{dw}{dt}=v\left(\frac{dX}{du}+\frac{dX}{dw}\right)$. Call this X′. Then $\frac{d^2X}{dt^2}=\frac{dX'}{dt}=v\left(\frac{dX'}{du}+\frac{dX'}{dw}\right)=v^2\left(\frac{d^2X}{du^2}+2\frac{d^2X}{du\,dw}+\frac{d^2X}{dw^2}\right)$. Also $\frac{dX}{dx}=\frac{dX}{du}.\frac{du}{dx}+\frac{dX}{dw}.\frac{dw}{dx}=-\frac{dX}{du}+\frac{dX}{dw}$. Call this X″. Then $\frac{d^2X}{dx^2}=\frac{dX''}{dx}=-\frac{dX''}{du}+\frac{dX''}{dw}=\frac{d^2X}{du^2}-2\frac{d^2X}{du\,dw}+\frac{d^2X}{dw^2}$. Substituting these in the equation, we have $4v^2.\frac{d^2X}{du\,dw}=0$, or $\frac{d^2X}{du\,dw}=0$. Integrating successively with regard to u and w, and remarking that the arbitrary constant at each integration may be an arbitrary function of the other variable, $\frac{dX}{dw}=\psi'(w)$ $X=\phi(u)+\psi(w)$. Restoring the expressions for u and w, $X=\phi(vt-x)+\psi(vt+x)$, where ϕ and ψ may be any functions whatever. The second term represents a wave rolling backwards, but as we do not at present consider such a wave, we shall make that term $=0$. The first expresses that we may assume any law whatever, provided that $vt-x$ is the subject of it; but in the Theory of Tides we must assume that, in the mouth of the channel connected with the open sea, the disturbance depends upon $\cos(nt+A)$, n and A being independent of the time: and, therefore, we must assume the form of X here to be

$$a\cos(mvt-mx),$$

where $mv=n$. For convenience, we omit to add a constant to the arc $mvt-mx$, as the same effect may be produced by altering the epoch from which t is measured. The assumption of this form for X will not prevent us from adding to it a small term of the form $a^2.\cos(2mvt-2mx)$, or $a^2.\sin(2mvt-2mx)$, if we should find it necessary, as those terms also satisfy the equation $\frac{d^2X}{dt^2}-v^2\frac{d^2X}{dx^2}=0$.

Solution to second approximation.

(198.) Second approximation. With the value $X=a.\cos(mvt-mx)$, we have $\frac{dX}{dx}=ma.\sin(mvt-mx)$, $\frac{d^2X}{dx^2}=-m^2a.\cos(mvt-mx)$; and substituting these values in the first term on the second side of the equation,

$$\frac{d^2X}{dt^2}-v^2\frac{d^2X}{dx^2}=\frac{3}{2}a^2v^2m^3.\sin(2mvt-2mx),$$

or

$$4v^2\frac{d^2X}{du\,dw}=\frac{3}{2}a^2v^2m^3.\sin 2mu,$$

or

$$\frac{d^2X}{du\,dw}=\frac{3}{8}a^2m^3.\sin 2mu.$$

Integrating with respect to u,

$$\frac{dX}{dw}=\psi'(w)-\frac{3}{16}a^2m^2.\cos 2mu.$$

Integrating again with respect to w,

$$X=\phi(u)+\psi(w)-\frac{3}{16}a^2m^2.w.\cos 2mu;$$

or

$$X=\phi(vt-x)+\psi(vt+x)-\frac{3}{16}a^2m^2.(vt+x).\cos(2mvt-2mx).$$

The arbitrary functions must be so chosen that the result of this solution may be consistent, to the first order of the small quantity a, with the first approximate solution; and may also enable us to satisfy the tidal condition at the mouth of the river. For this purpose the term $vt.\cos(2mvt-2mx)$ must be destroyed; because its existence would imply that the rise and fall of the water at the river's mouth are constantly increasing from tide to tide. We shall, therefore, take for $\phi(vt-x)+\psi(vt+x)$ the following quantity, in which every term is a function of $vt-x$:

$$a\cos(mvt-mx)+\frac{3}{16}a^2m^2(vt-x).\cos(2mvt-2mx)+ca^2.\cos(2mvt-2mx)$$
$$+c'a^2.\sin(2mvt-2mx),$$

which gives for X

$$a.\cos(mvt-mx)-\frac{3}{8}a^2m^2.x.\cos(2mvt-2mx)$$
$$+\,c\,a^2.\cos(2mvt-2mx)$$
$$+\,c'a^2.\sin(2mvt-2mx)\,;$$

where c and c' may be hereafter chosen so as to satisfy the condition of communicating with the tide of the open sea.

(199.) Taking the expression found for X, we obtain

$$\frac{dX}{dx}=am.\sin(mvt-mx)-\frac{3}{4}a^2m^3.x.\sin(2mvt-2mx)$$
$$+\;2a^2m\,c.\sin(2mvt-2mx)$$
$$-a^2\left(\frac{3}{8}m^2+2mc'\right)\cos(2mvt-2mx)\,;$$

and, substituting this in the expression $V=\frac{k}{1+\frac{dX}{dx}}$, we find

$$V=k\left\{\begin{array}{l}1-am\sin(mvt-mx)+a^2\frac{m^2}{2}\\ \qquad+\frac{3}{4}a^2m^3.x.\sin(2mvt-2mx)\\ \qquad-\;2a^2mc.\sin(2mvt-2mx)\\ \qquad+a^2\left(-\frac{m^2}{2}+\frac{3m^2}{8}+2mc'\right)\cos(2mvt-2mx)\end{array}\right\};$$

This, however, is the expression for the height of the water at the present place of those particles whose horizontal ordinate was originally $=x$. If we wish to find the height of the water at the place whose ordinate, measured on the bank of the canal, is x', we must consider that the original ordinate of the particles now there was x, where $x+X=x'$, or $x=x'-X=x'-X'$ nearly $=x'-a.\cos(mvt-mx')$ nearly; and this quantity must be substituted for x in the expression for V. In the term multiplied by a, instead of $\sin(mvt-mx)$, we shall have

$$\sin\{mvt-mx'+ma.\cos(mvt-mx')\}=\sin(mvt-mx')+\frac{ma}{2}+\frac{ma}{2}\cos(2mvt-2mx')\text{ nearly.}$$

The terms multiplied by a^2 require no alteration, as the alteration would introduce terms depending on a^3, to which order we have not proceeded in our approximation. Thus we find for the elevation of the water at the point whose ordinate is x',

$$V=k\left\{\begin{array}{l}1-am\sin(mvt-mx')+\frac{3}{4}a^2m^3x'.\sin(2mvt-2mx')\\ \qquad-\;2a^2mc.\sin(2mvt-2mx')\\ \qquad+a^2\left(-\frac{m^2}{2}-\frac{m^2}{2}+\frac{3m^2}{8}+2mc'\right)\cos(2mvt-2mx')\end{array}\right\}.$$

(200.) Now suppose x' to be measured from the point where the canal communicates with the open sea. The expression for the elevation of the water at that point will depend upon $\sin nt$ or $\sin mvt$, (by the laws of the tides which we shall hereafter investigate,) but not upon $\sin 2mvt$ or $\cos 2mvt$: and, therefore, when $x'=0$, the terms depending on $\sin(2mvt-2mx')$ and $\cos(2mvt-2mx')$ must vanish. We must, therefore, assume

$$c=0,\quad -\frac{m^2}{2}-\frac{m^2}{2}+\frac{3m^2}{8}+2mc'=0.$$

Thus we find at length,

$$V=k\left\{1-am.\sin(mvt-mx')+\frac{3}{4}a^2m^3.\,x'.\,\sin(2mvt-2mx')\right\};$$

or, putting b for am,

$$V=k\left\{1-b\,.\,\sin(mvt-mx')+\frac{3}{4}b^2.\,mx'.\,\sin(2mvt-2mx')\right\}:$$

Expression for tidal elevation in the river.

and the elevation of the water above its mean height is expressed by

$$K = -bk.\sin(mvt - mx') + \frac{3}{4} b^2.k.mx'.\sin(2mvt - 2mx').$$

Peculiarity of the form of the expression.

The first term of this expression is similar to those which we have found for the elevation when the displacement of the particles is small in comparison with the depth of the water. The second is entirely different, inasmuch as it contains the multiplier x' on the outside of the periodical function: it may be conceived to represent a wave whose magnitude continually increases as it travels along the canal.

(201.) To represent to the eye the form of the wave produced by the combination of the two terms, we have constructed the curve in figure 9. The horizontal line represents the level line of the mean height of water: the elevation or depression of the curve represents (on an enormously exaggerated scale) the elevation or depression above the mean height, given by the expression above. The value of x' is supposed to increase from the left to the right: on which supposition the quantity $mvt - mx'$, representing the phase of the wave, diminishes from the left to the right (mvt being constant).

(202.) To exhibit to the eye the law of the ascent and descent of the surface of the water at different points of the canal, the figures 10, 11, 12, and 13 are constructed. The first of these is intended for the point where the canal communicates with the sea: the others for points successively more and more distant from the sea. The horizontal line is used as a measure of time, or rather of the phase $mvt - mx'$: in which, for each station, x' is constant: the elevation or depression of the corresponding point of the curve represents the corresponding elevation or depression of the water above its mean height, as given by the expression above.

An inspection of these diagrams will suggest the following remarks:—

Peculiarity in the form of the wave.

(203.) When the wave leaves the open sea, its front slope and its rear slope are equal in length, and similar in form. But as it advances in the canal, its front slope becomes short and steep, and its rear slope becomes long and gentle. In advancing still further, this remarkable change takes place in the rear slope: it is not so steep in the middle as in the upper and the lower parts; at length it becomes horizontal at the middle; and, finally, slopes the opposite way, forming in fact two waves (figure 9).

(204.) At the station near the sea, (see figure 10,) the time occupied by the rise of the water is equal to the time occupied by the descent: at a station more removed from the sea (figure 11) the rise occupies a shorter time than the descent: the rise is steady and rapid throughout, but the descent begins rapid, then becomes more gentle, then becomes rapid again: at stations still farther from the sea, (figures 12 and 13,) the descent, after having begun rapid, is absolutely checked, or is even changed for a rise, to which another rapid descent succeeds: in this case there will be at that station two unequal tides corresponding to one tide at the mouth of the canal.

(205.) The greatest elevation and depression of the water are not much altered from one station to another.

Investigation of the duration of rise and fall of the water.

(206.) We may investigate the duration of the rise and fall in the following manner. The elevation of the water above its mean height, or $-bk\left\{\sin(mvt - mx') - \cos(mvt - mx').\frac{3}{2}bmx'.\sin(mvt - mx')\right\}$, may be put under the form $-bk\sin\left\{mvt - mx' - \frac{3}{2}bmx'.\sin(mvt - mx')\right\}$, considering the second term as small in comparison with the first. The phase of low water will be represented by giving to the sine its greatest positive value, or by making $mvt - mx' - \frac{3}{2}bmx'.\sin(mvt - mx') = \frac{\pi}{2}$. The first approximation (neglecting the second term) gives $mvt - mx' = \frac{\pi}{2}$, and $\sin(mvt - mx') = 1$. Substituting this in the second term, $mvt - mx' - \frac{3}{2}bmx' = \frac{\pi}{2}$: therefore

$$\text{at low water,} \qquad t = \frac{1}{mv}\left\{\frac{\pi}{2} + mx' + \frac{3}{2}bmx'\right\}.$$

The next low water will be found by making $mvt - mx' - \frac{3}{2}bmx'.\sin(mvt - mx') = \frac{5\pi}{2}$; the next by making it $= \frac{9\pi}{2}$, &c.; and the corresponding values of t are $\frac{1}{mv}\left\{\frac{5\pi}{2} + mx' + \frac{3}{2}bmx'\right\}$, $\frac{1}{mv}\left\{\frac{9\pi}{2} + mx' + \frac{3}{2}bmx'\right\}$, &c. The phase of high water will be represented by giving to the sine its greatest negative value, or by making $mvt - mx' - \frac{3}{2}bmx'.\sin(mvt - mx') = \frac{3\pi}{2}$. The first approximation gives $mvt - mx' = \frac{3\pi}{2}$, $\sin mvt - mx = -1$. Substituting this in the second term, $mvt - mx' + \frac{3}{2}bmx' = \frac{3\pi}{2}$; therefore

$$\text{at high water,} \qquad t = \frac{1}{mv}\left\{\frac{3\pi}{2} + mx' - \frac{3}{2}bmx'\right\}.$$

The successive high waters will be obtained by putting $\frac{7\pi}{2}$, $\frac{11\pi}{2}$, &c., in place of $\frac{3\pi}{2}$. Thus we have

At low water $$t=\frac{1}{mv}\left\{\frac{\pi}{2}+mx'+\frac{3}{2}bmx'\right\}$$

At the next high water $$t=\frac{1}{mv}\left\{\frac{3\pi}{2}+mx'-\frac{3}{2}bmx'\right\}$$

At the next low water $$t=\frac{1}{mv}\left\{\frac{5\pi}{2}+mx'+\frac{3}{2}bmx'\right\}$$

At the next high water $$t=\frac{1}{mv}\left\{\frac{7\pi}{2}+mx'-\frac{3}{2}bmx'\right\}$$

&c.

Interval from low water to high water $$=\frac{1}{mv}\{\pi-3bmx'\}$$

Interval from high water to low water $$=\frac{1}{mv}\{\pi+3bmx'\}$$

Excess of the latter above the former $$=\frac{1}{mv}6bmx'=6b\frac{x'}{v}.$$

As $v=\frac{n}{m}$, this is $$6b\frac{mx'}{n}.$$

Rise of the water occupies less time than its fall.

(207.) The fraction $\frac{x'}{v}$ is the time occupied by the tide-wave in passing from the open sea to the station under consideration. Thus we find,

Excess of the time of water falling above the time of water rising $=6b\times$ time occupied by the tide-wave in passing from the open sea to the station under consideration.

$$\text{Where}\quad b=\frac{\text{rise of tide above the mean state}}{\text{mean depth of water}}.$$

Thus in any part of the canal far from the sea, the times of high water and of low water, and the interval between them, will on different days depend on the extent through which the surface of the water oscillates up and down, or upon the magnitude of the whole rise of tide. And in places on the canal at different distances from the sea, the inequality of the times of water rising and water falling will, on the same day, depend upon the distance of the places from the sea.

(208.) Since the high water occurs, in the place whose distance from the sea is x', at the time $\frac{1}{mv}\left\{\frac{3\pi}{2}+mx'-\frac{3}{2}bmx'\right\}$, it will occur, in the place whose distance is $x'+x_{,}$, at the time

$$\frac{1}{mv}\left\{\frac{3\pi}{2}+m\,.\,\overline{x'+x_{,}}-\frac{3}{2}bm.\overline{x'+x_{,}}\right\},$$

and therefore the time occupied by the phase of high water in describing the space $x_{,}$ will be $\frac{1}{mv}\left\{mx_{,}-\frac{3}{2}bmx_{,}\right\}$ $=\frac{x_{,}}{v}\left(1-\frac{3b}{2}\right)=\frac{x_{,}}{v\left(1+\frac{3b}{2}\right)}$ nearly; therefore the phase of high water has travelled along the canal with the velocity $v\left(1+\frac{3b}{2}\right)=\sqrt{gk}\times\left(1+\frac{3b}{2}\right)=\sqrt{\{gk(1+3b)\}}$ nearly. The velocity with which a shallow wave of great length would travel along the surface of water, whose depth $=$depth here at high water, would, by (172.), be $\sqrt{g\times\text{depth at high water}}=\sqrt{g\times k\,(1+b)}$. Consequently, the phase of high water travels along the canal with a velocity greater than that of a shallow wave on water of the same depth as the high water. In like manner, the phase of low water travels along the canal with the velocity $\sqrt{\{gk(1-3b)\}}$ nearly, which is less than that of a long shallow wave on water of the same depth as the low water. The following theorem will be easily remembered. If D_2 be the depth at low water, D_3 that at high water, and if D_1, D_2, D_3, D_4, are in arithmetical progression; then the phase of low water travels with the velocity due to the depth D_1, and the phase of high water with the velocity due to the depth D_4.

Simple rule for the velocity of the phases of high water and low water.

(209.) The actual velocity of the particles of the water, or the velocity of the tide-stream, is represented by $\frac{dX}{dt}$. Observing that $X=a.\cos(mvt-mx)-\frac{3}{8}a^2m^2x.\cos(2mvt-2mx)+\frac{5}{16}ma^2.\sin(2mvt-2mx)$, we have

$$\frac{dX}{dt}=amv\left\{-\sin(mvt-mx)+\frac{3}{4}am^2x.\sin(2mvt-2mx)+\frac{5}{8}ma.\cos(2mvt-2mx)\right\}.$$

This is the velocity of the particles whose original ordinate was x: if for x, as in (199.), we put $x'-a\cos(mvt-mx')$, the expression for the velocity of the particles of water which are passing the place whose ordinate measured on the canal-bank is x', will become

$$amv\left\{-\sin(mvt-mx')-\frac{ma}{2}+\frac{ma}{8}\cos(2mvt-2mx')+\frac{3}{4}am^2x'\sin(2mvt-2mx'\right\}.$$

At high water, $mvt-mx'=\frac{3\pi}{2}-\frac{3}{2}bmx'$, $\sin(mvt-mx')=-1$ nearly, $2mvt-2mx'=3\pi$ nearly, $\sin(2mvt-2mx')=0$, $\cos(2mvt-2mx')=-1$, and the expression for the velocity is

$$amv\left\{1-\frac{ma}{2}-\frac{ma}{8}\right\}=bv\left\{1-\frac{5b}{8}\right\}.$$

At low water, $mvt-mx'=\frac{\pi}{2}+\frac{3}{2}bmx'$, $\sin(mvt-mx')=1$, $\sin(2mvt-2mx')=0$, $\cos(2mvt-2mx')=-1$, and the expression for the velocity is $amv\left\{-1-\frac{ma}{2}-\frac{ma}{8}\right\}=$

Velocity of ebb-stream greater than that of flow-stream.

$$-amv\left\{1+\frac{ma}{2}+\frac{ma}{8}\right\}=-bv\left\{1+\frac{5b}{8}\right\}.$$

These are respectively the same (very nearly) as the greatest velocities of the water up and down the canal: hence the greatest velocity of the tide-stream downwards exceeds the greatest velocity of the tide-stream upwards.

(210.) If we wish to proceed with a third approximation to the value of X, we must use the value already found, namely,

$$X=a\cos\overline{mvt-mx}+\frac{5ma^2}{16}\sin\overline{2mvt-2mx}-\frac{3m^2a^2}{8}x\cos\overline{2mvt-2mx};$$

in which the two last terms are to be regarded as small quantities of the second order. We must then substitute this (having due regard to the orders of quantities) in the right-hand side of the equation,

$$4v^2.\frac{d^2X}{du\,dw}=\frac{d^2X}{dt^2}-v^2\frac{d^2X}{dx^2}=v^2\frac{d^2X}{dx^2}\times\left\{-3\frac{dX}{dx}+6\left(\frac{dX}{dx}\right)^2\right\};$$

in which the right-hand side of the equation is to be expressed in terms of u and w; and, solving it, we obtain a value of X, to which (as before) an arbitrary term $\phi(vt-x)$ is to be attached. This arbitrary term can only be of the third order, inasmuch as the last approximation comprehended every term of the second order. From this expression for X, that for V is obtained by substitution in the equation

$$V=k\left(1-\frac{dX}{dx}+\left(\frac{dX}{dx}\right)^2-\left(\frac{dX}{dx}\right)^3\right).$$

Then, to adapt our expression for the elevation to a fixed point on the canal-bank, we must solve the equation $x+X=x'$, including terms of the second order: this gives

$$x=x'-a.\cos\overline{mvt-mx'}+\frac{3}{16}ma^2.\sin\overline{2mvt-2mx'}+\frac{3}{8}m^2a^2x'.\cos\overline{2mvt-2mx'}.$$

This must be substituted in every part of the expression for V; and the arbitrary function $\phi(vt-x)$ must be so determined as to make the expression $=k\left(1-ma.\sin\overline{mvt-mx'}\right)$ at the mouth of the canal, or when $x'=0$. The expression finally obtained is this; putting b for ma,

Solution to third approximation.

$$V=k\left\{1-b.\sin\overline{mvt-mx'}+\left(\frac{3}{4}b^2.\sin\overline{2mvt-2mx'}+\frac{33}{32}b^3.\cos\overline{mvt-mx'}-\frac{21}{32}b^3.\cos\overline{3mvt-3mx'}\right)mx'\right.$$
$$\left.+\left(\frac{9}{32}b^3.\sin\overline{mvt-mx'}-\frac{27}{32}b^3.\sin\overline{3mvt-3mx'}\right)m^2x'^2\right\}.$$

In figures (14) and (15) we have represented the course of the waves up a channel on two suppositions: the first supposing $b=\frac{1}{3}$, the second supposing $b=\frac{1}{6}$. We shall leave to the reader the task of discussing the results of this approximation, or of proceeding with further approximations. That which we have obtained will be found very important in the explanation of some peculiar cases of tides.

(211.) On account of the great importance of this problem in applications to the tides, we shall examine another modification of it.

Problem.—To investigate the motion of the tide-wave under the same circumstances, when the water of the canal is supposed also to have a current-flow (independent of fluctuations of tide) towards the sea.

Investigation when there is a current-flow in the river.

(212.) The investigation of the equation of the last problem includes this case, and the differential equation, therefore, will be the same, namely,

$$\frac{d^2X}{dt^2}-v^2\frac{d^2X}{dx^2}=v^2\frac{d^2X}{dx^2}\times\left\{-3\frac{dX}{dx}+6\left(\frac{dX}{dx}\right)^2-\&c.\right\}.$$

For a first approximation, $\frac{d^2X}{dt^2}-v^2\frac{d^2X}{dx^2}=0$; the solution of which is $X=\phi(vt-x)+\psi(vt+x)$. Now it will not be sufficient to take $X=a\cos(mvt-mx)$, because this formula will not express the current-flow: but that circumstance will be expressed by the following assumption:—

Solution to first approximation.

$$X=a\cos(mvt-mx)-\frac{e}{2v}(vt+x)-\frac{e}{2v}(vt-x),$$

or

$$X=a\cos(mvt-mx)-et;$$

where e represents the mean velocity of the current-flow towards the sea.

Solution to second approximation.

(213.) For the second approximation we find the same equation as before (198.), and the general solution will be the same as that of (198.), namely,

$$X=\phi(vt-x)+\psi(vt+x)-\frac{3}{16}a^2m^2(vt+x)\cos(2mvt-2mx).$$

But for the arbitrary functions we cannot adopt the same form as in (198.): first, because, as we have already seen, the principal term of X must have the form $a\cos(mvt-mx)-et$: secondly, because, as in (199.), the term $x.\cos(2mvt-2mx)$ will introduce a term $x.\sin(2mvt-2mx)$ into the expression for V; and, as x is the original co-ordinate of the particles, and as these particles (from the flow of the current) will at length come to the mouth of the canal or to the sea, it would follow that in the sea, after the lapse of a certain time, we shall have an oscillation expressed by $\sin(2mvt-2mx)$, which is inadmissible. The latter fault, it is easily seen, will be remedied by introducing the proper multiple of $t.\cos(2mvt-2mx)$. This can be done by means of the arbitrary functions, since $f\times(vt-x)\times\cos(2mvt-2mx)$ is a function of $vt-x$. Adding, also, the arbitrary terms in the solution of (198.), we have

$$\begin{aligned}X=a\cos(mvt-mx)-et-\frac{3}{8}a^2m^2.x.\cos(2mvt-2mx)&\\+ca^2.\cos(2mvt-2mx)&\\+c'a^2.\sin(2mvt-2mx)&\\+f\times(vt-x)\times\cos(2mvt-2mx).&\end{aligned}$$

(214.) From this expression we obtain the following value of $\frac{dX}{dx}$:

$$\begin{aligned}am.\sin(mvt-mx)+\left\{2ca^2m+2fmvt-\left(2fm+\frac{3}{4}a^2m^3\right)x\right\}\sin(2mvt-2mx)&\\-\left\{2c'a^2m+f+\frac{3}{8}a^2m^2\right\}\cos(2mvt-2mx);&\end{aligned}$$

and substituting this in the expression $V=\frac{k}{1+\frac{dX}{dx}}$, we find

$$V=k\left\{\begin{aligned}&1-am\sin(mvt-mx)+\frac{a^2m^2}{2}\\&\qquad-\left\{2ca^2m+2fmvt-\left(2fm+\frac{3}{4}a^2m^3\right)x\right\}\sin(2mvt-2mx)\\&\qquad+\left\{2c'a^2m+f+\frac{3}{8}a^2m^2-\frac{a^2m^2}{2}\right\}\cos(2mvt-2mx).\end{aligned}\right\}$$

But this expression gives the height of the water at the place of those particles whose original ordinate was x. If we wish to find the height of the water at the place whose ordinate, measured along the bank of the canal, is x', we must ascertain the value of the original ordinate x of the particles whose ordinate at the instant in question is x', and must substitute that value of x in the expression. The first step, then, is to make $x+\mathrm{X}=x'$. Now X, by the expressions obtained, $=a\cos(mvt-mx)-et$ nearly: and the equation is

$$x+a.\cos(mvt-mx)-et=x'.$$

The first approximate solution is $x-et=x'$, whence $mx=mx'+met$. Substituting this in the small term, the equation becomes

$$x+a.\cos\left(\overline{mv-me}.t-mx'\right)-et=x',$$

or

$$x=x'+et-a.\cos\left(\overline{mv-me}.t-mx'\right).$$

This degree of approximation is sufficient. Substituting this expression for x in the second term of the formula for V, and substituting $x'+et$ for x in the remaining terms of the formula for V, we find

$$\mathrm{V}=k\left\{\begin{array}{l}1-am.\sin\left(\overline{v-e}.mt-mx'\right)+\left\{-2ca^2m+\left(-2fm.\overline{v-e}+\frac{3}{4}a^2m^3e\right)t+\left(2fm+\frac{3}{4}a^2m^3\right)x'\right\}\sin\left(2.\overline{v-e}.mt-2mx'\right)\\ \qquad\qquad+\left\{2c'a^2m+f-\frac{5}{8}a^2m^2\right\}\cos\left(2.\overline{v-e}.mt-2mx'\right).\end{array}\right\}$$

(215.) At the sea $x'=0$. Now we assume that at the sea the oscillation follows the simple law of $-kam.\sin nt$, or $-kam.\sin\left(\overline{v-e}.mt\right)$. Equating to zero the terms which remain in addition to that term when $x'=0$, we find: $-2ca^2m=0$; $-2fm.\overline{v-e}+\frac{3}{4}a^2m^3e=0$, or $f=\frac{3}{8}a^2m^2.\frac{e}{v-e}$; $2c'a^2m+f-\frac{5}{8}a^2m^2=0$.

Expression for tidal elevation.

Substituting the value of f, we now find

$$\mathrm{V}=k\left\{1-am.\sin\left(\overline{v-e}\,mt-mx'\right)+\frac{3}{4}a^2m^3\frac{v}{v-e}.x'.\sin\left(2.\overline{v-e}.mt-2mx'\right)\right\};$$

or, putting $am=b$, the expression for the elevation of the water above its mean height is,

$$\mathrm{K}=-bk.\sin\left(\overline{v-e}.mt-mx'\right)+\frac{3}{4}b^2k.\frac{v}{v-e}.mx'.\sin\left(2.\overline{v-e}.mt-2mx'\right).$$

(216.) We might proceed in the discussion of this expression and of that for X in the same manner as in (201.) &c. The only point, however, to which we shall attend here is the times of high water and low water. The expression just found may be put under the form

Investigation of the duration of rise and fall of the water.

$$-bk\left\{\sin\left(\overline{v-e}.mt-mx'\right)-\cos\left(\overline{v-e}.mt-mx'\right)\frac{3}{2}b.\frac{v}{v-e}mx'.\sin\left(\overline{v-e}.mt-mx'\right)\right\}$$

or

$$-bk\sin\left\{\overline{v-e}.mt-mx'-\frac{3}{2}b.\frac{v}{v-e}mx'.\sin\left(\overline{v-e}.mt-mx'\right)\right\}.$$

For low water we must give to the sine its greatest positive value, or must make the arc $=\frac{\pi}{2}$, or $\frac{5\pi}{2}$, or $\frac{9\pi}{2}$, &c. Adopting $\frac{\pi}{2}$, we have

$$\overline{v-e}.mt-mx'-\frac{3}{2}b.\frac{v}{v-e}.mx'.\sin\left(\overline{v-e}.mt-mx'\right)=\frac{\pi}{2}.$$

A first approximation is $\overline{v-e}.mt-mx'=\frac{\pi}{2}$, $\sin\left(\overline{v-e}.mt-mx'\right)=1$. Substituting in the second term,

$$\overline{v-e}.mt-mx'-\frac{3}{2}b.\frac{v}{v-e}.mx'=\frac{\pi}{2};$$

from which

$$t=\frac{1}{(v-e)\,m}\left\{\frac{\pi}{2}+mx'+\frac{3}{2}b.\frac{v}{v-e}.mx'\right\}.$$

Rise of tide occupies less time than the fall.

Proceeding in the same manner for high water, and taking different multiples of $\frac{\pi}{2}$, we find the following times.

At low water $$t=\frac{1}{(v-e)\,m}\left\{\frac{\pi}{2}+mx'+\frac{3}{2}b.\frac{v}{v-e}.mx'\right\}.$$

At the next high water $$t=\frac{1}{(v-e)\,m}\left\{\frac{3\pi}{2}+mx'-\frac{3}{2}b.\frac{v}{v-e}mx'\right\}.$$

At the next low water $$t=\frac{1}{(v-e)\,m}\left\{\frac{5\pi}{2}+mx'+\frac{3}{2}b.\frac{v}{v-e}mx'\right\}.$$

At the next high water $$t=\frac{1}{(v-e)\,m}\left\{\frac{7\pi}{2}+mx'-\frac{3}{2}b.\frac{v}{v-e}mx'\right\}.$$

&c.

Interval from low water to high water $$=\frac{1}{(v-e)\,m}\left\{\pi-3b\frac{v}{v-e}mx'\right\}$$

Interval from high water to low water $$=\frac{1}{(v-e)\,m}\left\{\pi+3b\frac{v}{v-e}mx'\right\}$$

Excess of the latter above the former $$=6bx'\frac{v}{(v-e)^2}.$$

As $v-e=\frac{n}{m}$, this is $$6b.\frac{m^2}{n^2}\left(\frac{n}{m}+e\right)x'=6bx'\left(\frac{m}{n}+e\frac{m^2}{n^2}\right).$$

This excess is greater than the excess when there is no current, in the proportion of $1+\frac{em}{n}$ to 1.

(217.) The preceding investigations suppose that the depth of the canal is uniform in its cross section. In the case of a wave of great length, as the tide-wave, we may, however, apply a satisfactory investigation to the motion of the wave along a canal of uniform section of any form.

Problem.—The section of the canal is defined by the equation $z=\phi\,(y)$, z being a horizontal co-ordinate: to find the equations for the motion of a long wave.

Investigation for long waves in a canal whose section is invariable but of any form.

(218.) Let $u=\psi\,(y)$ be the area of the section up to the height y, so that $\frac{du}{dy}=z$: let X be the horizontal displacement, in the direction of the canal's length, of the particles whose co-ordinate measured along the canal was x; V the present elevation of that part of the water. Then, as in (194.), the horizontal space which was h has now become $h\left(1+\frac{d\text{X}}{dx}\right)$, and the area of section which was $\psi\,(k)$ is now $\psi\,(\text{V})$, and therefore the volume which was expressed by $h.\psi\,(k)$ is now expressed by $h\left(1+\frac{d\text{X}}{dx}\right)\psi\,(\text{V})$. Making these equal,

$$\psi\,(\text{V})=\frac{\psi\,(k)}{1+\frac{d\text{X}}{dx}}.$$

This is the equation of continuity.

(219.) The pressure tending to push forward the water included between the two particles whose horizontal co-ordinates were x and $x+h'$, and whose horizontal co-ordinates are now $x+\text{X}$ and $x+\text{X}+\left(1+\frac{d\text{X}}{dx}\right)h'$, will, as in (195.) be the difference of the pressures caused by two vertical columns whose heights are $\text{V}-y$ and $\text{V}+\frac{d\text{V}}{dx}h'-y$ respectively. Hence, as in (195.), we find

$$\frac{d^2\text{X}}{dt^2}=-g.\frac{d\text{V}}{dx}.\frac{1}{1+\frac{d\text{X}}{dx}}.$$

To eliminate $\frac{d\text{V}}{dx}$, we must differentiate the equation of continuity found above, from which we obtain

$$\psi'\,(\text{V}).\frac{d\text{V}}{dx}=\frac{-\psi\,(k)}{\left(1+\frac{d\text{X}}{dx}\right)^2}.\frac{d^2\text{X}}{dx^2};$$

substituting from this, we have

$$\frac{d^2\text{X}}{dt^2}=\frac{g\,\psi\,(k)}{\psi'\,(\text{V})}.\frac{\frac{d^2\text{X}}{dx^2}}{\left(1+\frac{d\text{X}}{dx}\right)^3},$$

an equation which may be solved by successive substitution.

(220.) The first approximation will be made by considering V in the factor as equal to k, and neglecting

Solution to first approximation.

$\frac{dX}{dx}$. This gives $\frac{d^2X}{dt^2}=\frac{g\psi(k)}{\psi'(k)}\cdot\frac{d^2X}{dx^2}$. Let $k'=\frac{\psi(k)}{\psi'(k)}$, $v'^2=gk'$: then $\frac{d^2X}{dt^2}=v'^2\frac{d^2X}{dx^2}$; of which the solution is $X=\chi(v't-x)+\omega(v't+x)$. This expression denotes that the waves travel with the same velocity v' or $\sqrt{gk'}$, as if the depth of the canal were uniform and equal to $\frac{\psi(k)}{\psi'(k)}$ or $\frac{\psi(k)}{\phi(k)}$. For instance, if the section of the canal were a triangle, $\phi(y)=z=ay$: $\psi(y)=\int_y\phi(y)=\frac{ay^2}{2}$: therefore $\frac{\psi(k)}{\phi(k)}=\frac{\frac{ak^2}{2}}{ak}=\frac{k}{2}$, and the velocity would be that due to a canal of uniform depth equal to half the greatest depth of the triangular canal. If the section were a parabola, $\phi(y)=z=\sqrt{ay}$: $\psi(y)=\frac{2}{3}\sqrt{ay^3}$: $\frac{\psi(k)}{\phi(k)}=\frac{2}{3}\cdot\frac{\sqrt{ak^3}}{\sqrt{ak}}=\frac{2k}{3}$, and the velocity would be that due to a canal of uniform depth equal to two-thirds of the greatest depth of the parabolic canal.

Equation to second approximation.

(221.) For a second approximation, we must expand the equation $\psi(V)=\frac{\psi(k)}{1+\frac{dX}{dx}}$ to the first power of small quantities: this gives $\psi(k)+\psi'(k).(V-k)=\psi(k)-\psi(k)\frac{dX}{dx}$,

whence
$$V-k=\frac{-\psi(k)}{\psi'(k)}\cdot\frac{dX}{dx},$$

and $\psi'(V)$, or $\psi'(k+V-k)=\psi'(k)+\psi''(k).(V-k)=\psi'(k)-\frac{\psi(k).\psi''(k)}{\psi'(k)}\cdot\frac{dX}{dx}$:

therefore
$$\frac{g\psi(k)}{\psi'(V)}=\frac{g\psi(k)}{\psi'(k)}\cdot\left\{1+\frac{\psi(k).\psi''(k)}{\{\psi'(k)\}^2}\cdot\frac{dX}{dx}\right\};$$

and the equation becomes
$$\frac{d^2X}{dt^2}=\frac{g\psi(k)}{\psi'(k)}\cdot\left\{1-\frac{3\{\psi'(k)\}^2-\psi(k).\psi''(k)}{\{\psi'(k)\}^2}\cdot\frac{dX}{dx}\right\}\cdot\frac{d^2X}{dx^2},$$

which is to be solved in the same manner as in (198.), &c.

Limitations to the application of this solution.

(222.) The preceding solution, it is plain, depends on the assumption that the horizontal motion in the direction of z is insignificant in comparison with that in the direction of x. And this consideration points out the limitations within which the solution will apply. It will certainly apply when the sides of the canal at and near the surface of the water are so steep (say, for instance, inclined at an angle of 45° to the vertical) that the motion in the direction of z does not much exceed the vertical motion. It will certainly not apply when the shore is so flat that the motion in the direction of z greatly exceeds the vertical motion and is at all comparable with the motion in the direction of x.

Addition of solutions does not apply here.

(223.) Before closing this subsection, we must call the reader's attention to the circumstance that the addition of solutions, treated of in (186.), and which holds in all cases of waves where the motion of the particles is extremely small, does not hold here. For, if we have two functions X′ and X″, each of which separately satisfies one of the equations,

$$\frac{d^2X'}{dt^2}-v^2\frac{d^2X'}{dx^2}=v^2\frac{d^2X'}{dx^2}\times\left\{-3\frac{dX'}{dx}+,\text{ \&c.}\right\}$$
$$\frac{d^2X''}{dt^2}-v^2\frac{d^2X''}{dx^2}=v^2\frac{d^2X''}{dx^2}\times\left\{-3\frac{dX''}{dx}+,\text{ \&c.}\right\}$$

the sum of these functions will not satisfy the similar equation,

$$\frac{d^2(X'+X'')}{dt^2}-v^2\frac{d^2(X'+X'')}{dx^2}=v^2\frac{d^2(X'+X'')}{dx^2}\times\left\{-3\frac{d(X'+X'')}{dx}+,\text{ \&c.}\right\}$$

as will be evident on adding together the two equations.

Subsection 4.—Theory of Waves when the Water is under the action of Horizontal and Vertical Forces.

(224.) The general equations applicable to water under the action of forces (149.) are

$$Y=\Xi.\frac{d\eta}{dx}-\int_y\frac{dX}{dx}\ (\eta\text{ to }y)$$

$$\frac{d^2X}{dt^2}=F+\frac{d}{dx}\left\{-g'\,(k+K-y)-\int_y\frac{d^2Y}{dt^2}\,(y\text{ to }k)\right\},$$

General equations when there is external force.

where F is the extraneous force acting in an horizontal direction, and g' is gravity increased by the extraneous force acting in the vertical direction. If there be no extraneous vertical force, the second equation becomes

$$\frac{d^2X}{dt^2}=F+\frac{d}{dx}\left\{-gK-\int_y\frac{d^2Y}{dt^2}\,(y\text{ to }k)\right\}.$$

Before entering upon the legitimate objects of this subsection, we will apply these equations to the solution of two or three problems which would, at first sight, seem to be unconnected with them.

(225.) Hitherto we have assumed in every part of our investigation that there is upon the surface of the water an unlimited succession of waves. We shall now proceed to consider the case of a single wave.

Problem.—To ascertain the conditions under which it is possible that a single wave can travel along a canal.

Investigation for motion of a solitary wave.

(226.) By a moving wave, we mean in all cases (till we shall make new suppositions) a displacement of the particles, of such a nature that, upon examining the state of particles which are at a greater distance in the direction of x, and increasing the time by a quantity proportionate to that increase of distance, the displacement of the new particles at that later time will be found the same as the displacement of the old particles at the earlier time. It follows, therefore, that in all cases, whether there be a single wave or an indefinite succession of waves, the displacement must be represented by a function of $vt-x$. Yet any ordinary function of $vt-x$ will not fulfil our object, inasmuch as any ordinary function, though it may make the displacement $=0$ for particular values of $vt-x$, will not make the displacement $=0$ for all the values of $vt-x$ corresponding to the values of x preceding and following that part of the surface at which alone the wave is supposed to have a real existence at any instant of time t. We must adopt for our purpose a *discontinuous* function; that is (putting, for convenience, $vt-x=z$), we must suppose that for all values of z up to a certain limit, the value of X is 0; that from that limiting value of z up to another certain limit, the value of X will be expressed by a certain function of z, whose form is given: that from that second limiting value to a third, the value of X will be expressed by another certain function of z: and so on: and that for values of z exceeding that value which corresponds to the termination of the wave, the value of X will be constantly 0, or equal to some assigned quantity. This supposition corresponds exactly to the supposition that the surface of the water is perfectly level up to a certain point, and that then there exists a wave whose outline is determined by uniting portions of different curves, and that after that the water is again level. There is no objection whatever to our making such a supposition, provided that it be so framed as to satisfy the conditions which are evidently implied in forming our equations for the wave-motion.

Discontinuous function may be used for a single wave.

(227.) To simplify our language, we will suppose that only one function of z (which we will call the *wave-function*) is adopted to express the value of X between the values $z=0$, $z=a$: that for values of z less than 0, X is always $=0$: and that for values of z greater than a, X is always $=b$. (This will denote that the particles of water are, as it were, carried through the space b by the passage of the wave and then deposited.) Upon examining the equations (224.) it will be seen that Y depends on $\frac{dX}{dx}$, and (as the integration is with regard to y only) will be a multiple of $\frac{dX}{dx}$: that therefore $\frac{d^2Y}{dt^2}$, and its integral with respect to y which occurs in the second equation, will be a multiple of $\frac{d^3X}{dx\,dt^2}$: and, considering X as a function of $vt-x$, it is easily seen that $\frac{d^3X}{dx\,dt^2}=-v^2\frac{d^3X}{dz^3}$: and therefore its differential coefficient depends on $\frac{d^4X}{dz^4}$. Consequently our equations involve all the differential coefficients of X with regard to z, to the fourth inclusive. We will now examine the restrictions under which this number of differential coefficients places our assumptions for the form of X: they depend intimately on the way in which we conceive the differential coefficients to be formed.

(228.) In physical investigations there is but one definition of the differential coefficient which can be applied, namely, *the limiting value of the quotient formed by dividing the increment of the function by the increment of the independent variable.* If the reader will turn to the investigations of (144.), &c., he will see that this principle is adopted throughout; in estimating the sides of the small parallelogram by which the equation of continuity is formed; in estimating the pressure depending on the vertical column of fluid; and in estimating the motion which it will produce in the horizontal column of fluid. Taking this definition, then, it is clear, in the first place, that algebraic continuity in the form of the function is unnecessary: but, in the second place, it is clear that at these values of z where different forms of the function unite, there must be no *sudden* change in the value of X: for, if there were a sudden change, then at that point the value of $\frac{dX}{dz}$ (in consequence of the mode of forming the differential coefficient just described) would be infinite. In fact $\frac{dX}{dz}$ would be the limiting

The differential coefficients to a certain order must vanish at the extremities of the wave.

value of $\frac{\delta X}{\delta z}$, where δX had a finite value even when δz was indefinitely small: this would lead to suppositions of infinite forces, &c., such as were not contemplated at all in our investigation. There must, therefore, be no sudden change in the values of X at the beginning and the end of our wave-function. In like manner there must be no sudden change in the value of $\frac{dX}{dz}$ at the beginning and end of the wave-function, as such sudden change would make the values of $\frac{d^2X}{dz^2}$ infinite. In like manner there must be no sudden change in the values of $\frac{d^2X}{dz^2}$ and $\frac{d^3X}{dz^3}$. Then, as the equations of (224.) hold equally for still water, and for water in wave-motion, and as there is to be no sudden change in the values of the first and second and third differential coefficients, it follows also that there must be no sudden change in the value of the fourth. If, however, we are willing to suppose the sudden introduction of a finite force at a particular point of the wave, we may dispense with the last condition.

Forms of function which satisfy these conditions.

(229.) Now for the still water which precedes and follows the wave, all the differential coefficients are 0. Hence we must have,

At the beginning of the wave-function, or when $z=0$,

$$X=0, \quad \frac{dX}{dz}=0, \quad \frac{d^2X}{dz^2}=0, \quad \frac{d^3X}{dz^3}=0, \quad \frac{d^4X}{dz^4}=0.$$

At the end of the wave-function, or when $z=a$,

$$X \text{ must } =b, \quad \frac{dX}{dz}=0, \quad \frac{d^2X}{dz^2}=0, \quad \frac{d^3X}{dz^3}=0, \quad \frac{d^4X}{dz^4}=0.$$

The higher orders of differential coefficients may have any values whatever.

(230.) There is no difficulty in finding forms for X which will satisfy these conditions. For instance, if we take

$$X=\frac{630\,b}{a^9}\left\{\frac{1}{5}a^4z^5-\frac{2}{3}a^3z^6+\frac{6}{7}a^2z^7-\frac{1}{2}az^8+\frac{1}{9}z^9\right\};$$

then, on making $z=0$, X will $=0$: and, on making $z=a$, X will $=b$. The value of $\frac{dX}{dz}$ is $\frac{630\,b}{a^9}\,.\,z^4.(a-z)^4$; and this quantity, and its three next differential coefficients, vanish when $z=0$ or $z=a$.

(231.) Similarly, if we had chosen to assume that for values of z greater than a, X should $=0$, that condition, with all the others, would be satisfied by the assumption for the wave-function $X=c.z^5.(a-z)^5$.

(232.) It may, however, be more convenient to assume a form depending on sines and cosines. Thus, suppose the condition to be that all particles resume, after the passing of the wave, the same place which they had before its approach, or that X shall $=0$ for all values of $z<0$, and for all $>a$. It is easily seen that the condition that X, $\frac{dX}{dz}$, $\frac{d^2X}{dz^2}$, $\frac{d^3X}{dz^3}$, $\frac{d^4X}{dz^4}$ shall $=0$ for $z=0$ and $z=a$, will be satisfied by the assumption $X=\sin^5\frac{\pi z}{a}$; or by any higher power of the sine; or by $\sin^5\frac{\pi z}{a}$ increased by any number of such quantities as $\sin^5\frac{c\pi\,(z+e)}{a}$, where c may be any number (whole or fractional) greater than 1, and where the argument $\frac{c\pi\,(z+e)}{a}$ may be carried through as many whole multiples of π as we please, provided that the first value of z be >0, and the last $<a$. (This amounts to the same as supposing that any number of short waves, possessing the characteristic property in regard to the differential coefficients at their beginnings and ends, are piled upon the longer wave.) But if the condition is to be, that the particles are to be removed to the distance b and left there, it will be satisfied by the assumption $X=\frac{8b}{3a}\int_z \sin^4\frac{\pi z}{a}=\frac{2b}{3\pi}\left\{\frac{3}{2}\,.\,\frac{\pi z}{a}-\sin\frac{2\pi z}{a}+\frac{1}{8}\sin\frac{4\pi z}{a}\right\}$. For this makes $X=0$ when $z=0$, and $X=b$ when $z=a$; and it makes $\frac{dX}{dz}=0$, $\frac{d^2X}{dz^2}=0$, $\frac{d^3X}{dz^3}=0$, $\frac{d^4X}{dz^4}=0$, for $z=0$ and $z=a$. A higher power of the sine might have been taken; and the function might have been increased by any number of the supplementary functions mentioned above. Thus a single wave of any degree of complexity might be produced., We shall, however, for simplicity, confine ourselves to the simple form

$$X=\frac{8b}{3a}\int_z \sin^4\frac{\pi z}{a}.$$

(233.) But we have not yet examined whether these forms of wave are consistent in all respects with the equations of wave-motion. For this purpose we must substitute the assumed value of X in the equations

$$Y=-\int_y \frac{dX}{dx}\,(0 \text{ to } y)$$

$$\frac{d^2X}{dt^2}=F+\frac{d}{dx}\left\{-gK-\int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k)\right\};$$

and thereby ascertain whether any force F is necessary to maintain the particles of water in the assumed state of movement. Now taking the last assumption, and putting $vt-x$ for z; and putting $\phi''(y)$ for the factor depending on y, $\left(\text{it being understood that at the surface, or when } y=k,\ \phi''(k)=\frac{2b}{3\pi}\right)$, where $\phi''(y)$ is conceived to be the second derived function of some function $\phi(y)$, we have

$$X=\phi''(y).\left\{\frac{3\pi}{2a}(vt-x)-\sin\frac{2\pi}{a}(vt-x)+\frac{1}{8}\sin\frac{4\pi}{a}(vt-x)\right\}$$

$$-\frac{dX}{dx}=\phi''(y).\left\{\frac{3\pi}{2a}-\frac{2\pi}{a}.\cos\frac{2\pi}{a}(vt-x)+\frac{\pi}{2a}\cos\frac{4\pi}{a}(vt-x)\right\}$$

$$Y=\{\phi'(y)-\phi'(0)\}.\left\{\frac{3\pi}{2a}-\frac{2\pi}{a}.\cos\frac{2\pi}{a}(vt-x)+\frac{\pi}{2a}\cos\frac{4\pi}{a}(vt-x)\right\}$$

$$K=\{\phi'(k)-\phi'(0)\}.\left\{\frac{3\pi}{2a}-\frac{2\pi}{a}.\cos\frac{2\pi}{a}(vt-x)+\frac{\pi}{2a}\cos\frac{4\pi}{a}(vt-x)\right\}$$

$$\frac{d^2Y}{dt^2}=\{\phi'(y)-\phi'(0)\}.\left\{\frac{8\pi^3}{a^3}v^2.\cos\frac{2\pi}{a}(vt-x)-\frac{8\pi^3}{a^3}v^2.\cos\frac{4\pi}{a}(vt-x)\right\}$$

$$-\int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k)=\left\{\phi(y)-\phi(k)-\overline{y-k}.\phi'(0)\right\}.\left\{\frac{8\pi^3}{a^3}v^2.\cos\frac{2\pi}{a}(vt-x)-\frac{8\pi^3}{a^3}v^2.\cos\frac{4\pi}{a}(vt-x)\right\}$$

$$\frac{d}{dx}\left\{-gK-\int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k)\right\}=\left\{\frac{4\pi^2g}{a^2}.\overline{\phi'(k)-\phi'(0)}+\frac{16\pi^4v^2}{a^4}.\overline{\phi(y)-\varphi(k)-(y-k).\phi'(0)}\right\}\sin\frac{2\pi}{a}(vt-x)$$

$$+\left\{-\frac{2\pi^2g}{a^2}.\overline{\phi'(k)-\phi'(0)}-\frac{32\pi^4v^2}{a^4}\overline{\phi(y)-\phi(k)-(y-k).\phi'(0)}\right\}\sin\frac{4\pi}{a}(vt-x)$$

$$-\frac{d^2X}{dt^2}=-\phi''(y).\frac{4\pi^2v^2}{a^2}\sin\frac{2\pi}{a}(vt-x)+\phi''(y).\frac{2\pi^2v^2}{a^2}\sin\frac{4\pi}{a}(vt-x).$$

Expression for force which is necessary to maintain this wave.

The sum of the two last expressions, with sign changed, is the value of F, the horizontal force which must be applied to maintain this state of undulation.

(234.) It is evident that the value of F cannot generally be $=0$, since the different parts of the coefficients of $\sin\frac{2\pi}{a}(vt-x)$ and $\sin\frac{4\pi}{a}(vt-x)$ are not in the same proportion, and therefore those coefficients cannot vanish together. Therefore, in general, this discontinuous wave cannot exist without the application of force. But if the wave be long, so that the terms divided by a^4 may be neglected in comparison with those divided by a^2, F will $=0$ if $-\frac{2\pi^2g}{a^2}\overline{\phi'(k)-\phi'(0)}+\frac{2\pi^2v^2}{a^2}\phi''(y)=0$. This implies that $\phi''(y)$ is constant and $=\frac{2b}{3\pi}$; therefore, $\phi'(y)=\frac{2b}{3\pi}y$, and $\phi'(k)-\phi'(0)=\frac{2b}{3\pi}k$; and the equation becomes $-\frac{2\pi^2g}{a^2}k+\frac{2\pi^2v^2}{a^2}=0$, or $v^2=gk$. The same thing would have been found to be true if the wave-function had consisted of any number of sines. Thus it appears, that a single discontinuous wave of any degree of complexity may travel on water without any force to maintain it, provided, in the first place, that it satisfies the conditions laid down with regard to the differential coefficients at its terminations, and in the next place, that the wave is so long that a succession of simple waves, each of that length, would travel sensibly with the velocity due to waves of infinite length.

When the wave is very long, no force is necessary.

(235.) If the single wave is moderately long, a small force will maintain it as a discontinuous wave: but if it be short, the force must be (in proportion to the various pressures acting on the water) considerable. In fact, each of the different terms in the wave-function represents a wave of different length; and, when the waves are short, each of these would tend to travel on with its own peculiar velocity, which velocities are very different for the different waves. But when the waves are long, the peculiar velocities are very nearly the same for the different waves.

In a subsequent article (410.), we shall give the theory of a single wave, acted on by any force, and travelling with a velocity different from that mentioned above.

(236.) We may proceed in the same manner for the discussion of the motion of a single wave of considerable depth and of great length, observing that the equation for that case is $\frac{d^2X}{dt^2} = F + gk \cdot \frac{\frac{d^2X}{dx^2}}{\left(1+\frac{dX}{dx}\right)^3}$.

(237.) Hitherto we have supposed the depth in every part of the canal to be the same. We shall now suppose that the depth is different at different points of the canal; the variation, however, being supposed to be gradual. We have already seen (157.), that the equations cannot be satisfied in this case: and our investigation, therefore, cannot be quite so satisfactory in its character as the investigations undertaken where the equations can be satisfied. Still we conceive that the following will be found sufficiently certain and accurate to enable us to judge with confidence of the effect of the variation of depth upon the general circumstances of the waves.

Depth of canal variable.

Problem.—The depth being supposed variable: to find what alteration takes place in the magnitude, length, and velocity of the waves, in passing from one part of the canal to another.

(238.) In our former investigations, in which the horizontal bottom of the canal was taken as the axis of x, we found that the horizontal disturbance X might be represented by a collection of terms, each of which is of the form $A.(\varepsilon^{my}+\varepsilon^{-my}) \cos (nt-mx)$, where m and n are connected by the equation $n^2 = mg.\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}$. If, instead of taking the bottom of the canal, we had taken some other horizontal line for the axis of x, and if the ordinate of the bottom of the canal had then been η, the expression for X would have been $A\,(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)}) \cos (nt-mx)$; and the equation connecting m and n would have been $n^2 = mg.\frac{\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}}{\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)}}$. In this case, η is constant.

Assumption on which a plausible solution may be found.

Suppose now that in our canal of slowly varying depth there are, at different parts, portions of sensible length, whose depth is uniform through those lengths; then, through each of these lengths, the expression for X and the equation between n and m will have the same form as those above. It seems, then, not unreasonable to conjecture that the same form may apply to the parts of variable depth, or the parts where η is a function of x, and where, consequently, m will be a function of x, (for n, upon which the *period* of the waves depends, must be invariable through the whole extent of the disturbed water.) A must also be a function of x, whose form is yet to be determined. In regard to the term mx under the cosine, a necessity for change will be obvious. In a canal of uniform depth, mx represents the decrease of phase due to the space x, and, therefore, mh would represent the decrease of phase due to the small space h: if, then, (going upon the principle already announced,) we make the phase decrease for each small part of the canal of variable depth in the same manner as if that depth were continued uniform, we must not use mx for the decrease of the phase, but $\int_x m$. Let this integral $=$M: then our supposition will be

$$X = A.(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)}) \cos (nt - M)$$

$$n^2 = mg.\frac{\varepsilon^{m(k-\eta)}-\varepsilon^{m.(\eta-k)}}{\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)}};$$

where η is a given function of x, m in consequence is implicitly a given function of x, $\frac{dM}{dx}=m$, and A is an unknown function of x, whose form is to be determined so as to satisfy as nearly as possible the equations of waves.

(239.) The equation of continuity is

$$Y = \Xi.\frac{d\eta}{dx} - \int_y \frac{dX}{dx} \text{ (from } \eta \text{ to } y).$$

Now $$\frac{dX}{dx} = \frac{dA}{dx}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)}) \cos \overline{nt-M} + A\frac{dm}{dx}.(y-\eta)(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}) \cos \overline{nt-M}$$

$$-Am\frac{d\eta}{dx}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}) \cos \overline{nt-M} + Am\,(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)}) \sin \overline{nt-M}.$$

Therefore $-\int_y \frac{dX}{dx}$ from η to y, or $+\int_y \frac{dX}{dx}$ from y to η, is

$$-\frac{dA}{dx}\frac{1}{m}.(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}) \cos \overline{nt-M} - A\frac{1}{m}.\frac{dm}{dx}.(y-\eta)(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)}) \cos \overline{nt-M}$$

$$+\mathrm{A}.\frac{1}{m^2}.\frac{dm}{dx}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}+\mathrm{A}\frac{d\eta}{dx}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}$$

$$-\mathrm{A}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}).\sin\overline{nt-\mathrm{M}}-2\mathrm{A}\frac{d\eta}{dx}\cos\overline{nt-\mathrm{M}}.$$

And
$$\Xi.\frac{d\eta}{dx}=2\mathrm{A}\frac{d\eta}{dx}.\cos\overline{nt-\mathrm{M}}.$$

Therefore
$$\mathrm{Y}=-\frac{d\mathrm{A}}{dx}.\frac{1}{m}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}).\cos\overline{nt-\mathrm{M}}-\mathrm{A}.\frac{1}{m}.\frac{dm}{dx}(y-\eta)(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}$$

$$+\mathrm{A}\frac{1}{m^2}.\frac{dm}{dx}.(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}).\cos\overline{nt-\mathrm{M}}+\mathrm{A}.\frac{d\eta}{dx}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}$$

$$-\mathrm{A}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)}).\sin\overline{nt-\mathrm{M}};$$

or
$$\mathrm{Y}=-\frac{d}{dx}\left\{\frac{\mathrm{A}}{m}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}\right\}.$$

(240.) The equation of equal pressure is

$$\mathrm{F}=\frac{d^2\mathrm{X}}{dt^2}+\frac{d}{dx}\left\{g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}(y\text{ to }k)\right\}.$$

Now it is well known that, as x, y, and t, are independent of each other, the differentiation expressed by $\frac{d^2\mathrm{Y}}{dt^2}$ may be performed upon the quantity under the sign $\frac{d}{dx}$; and the integration with regard to y may also be performed on the quantity under the sign $\frac{d}{dx}$, provided that (as is the case here) the limits of the integral are independent of x. Thus we have

$$\frac{d^2\mathrm{Y}}{dt^2}=\frac{d}{dx}\left\{\frac{n^2\mathrm{A}}{m}(\varepsilon^{m(y-\eta)}-\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}\right\}$$

$$\int_y\frac{d^2\mathrm{Y}}{dt^2}(\text{generally})=\frac{d}{dx}\left\{\frac{n^2\mathrm{A}}{m^2}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}\right\}$$

$$\int_y\frac{d^2\mathrm{Y}}{dt^2}(y\text{ to }k)=-\frac{d}{dx}\left\{\frac{n^2\mathrm{A}}{m^2}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}\right\}$$
$$+\frac{d}{dx}\left\{\frac{n^2\mathrm{A}}{m^2}(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})\cos\overline{nt-\mathrm{M}}\right\}.$$

And
$$g\mathrm{K}=-\frac{d}{dx}\left\{\frac{g\mathrm{A}}{m}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})\cos\overline{nt-\mathrm{M}}\right\}.$$

Expression for force necessary to maintain this motion.

In adding these together, it will be observed, that the two last lines destroy each other, by virtue of the equation $n^2=mg\frac{\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}}{\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)}}$. And as $\frac{d^2\mathrm{X}}{dt^2}=-n^2\mathrm{A}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}$, the expression for F becomes

$$\mathrm{F}=-n^2\mathrm{A}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}-\frac{d^2}{dx^2}\left\{\frac{n^2\mathrm{A}}{m^2}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\cos\overline{nt-\mathrm{M}}\right\}.$$

(241.) To facilitate the differentiation of the last term, we will remark, that the alteration of depth is supposed to be gradual, so that $\frac{d\eta}{dx}$ is small, and therefore $\frac{dm}{dx}$ and $\frac{d\mathrm{A}}{dx}$ which depend on it are small, and therefore their powers and products and the second differential coefficients will be extremely small, and may be neglected. If then for the moment we make $\frac{n^2\mathrm{A}}{m^2}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})=\mathrm{P}$, in differentiating $\mathrm{P}.\cos\overline{nt-\mathrm{M}}$ we may omit $\frac{d^2\mathrm{P}}{dx^2}$. Thus we have

$$\frac{d}{dx}(\mathrm{P}.\cos\overline{nt-\mathrm{M}})=\frac{d\mathrm{P}}{dx}.\cos\overline{nt-\mathrm{M}}+\mathrm{P}.m.\sin\overline{nt-\mathrm{M}}$$

$$-\frac{d^2}{dx^2}(\mathrm{P}.\cos\overline{nt-\mathrm{M}})=-\left(2\frac{d\mathrm{P}}{dx}m+\mathrm{P}\frac{dm}{dx}\right)\sin\overline{nt-\mathrm{M}}+m^2\mathrm{P}.\cos\overline{nt-\mathrm{M}}.$$

Add to this the first term $= -m^2\text{P} . \cos \overline{nt-\text{M}}$

and the expression for F becomes $-\left(2\dfrac{d\text{P}}{dx}m + \text{P}\dfrac{dm}{dx}\right)\sin \overline{nt-\text{M}}$

$$= -2m^{\frac{1}{2}} . \frac{d}{dx}(\text{P}m^{\frac{1}{2}}) . \sin \overline{nt-\text{M}}$$

$$= -2n^2 . m^{\frac{1}{2}} . \frac{d}{dx}\left\{\text{A} . m^{-\frac{3}{2}}\left(\varepsilon^{m(y-\eta)} + \varepsilon^{m(\eta-y)}\right)\right\} . \sin \overline{nt-\text{M}} .$$

Expanding the differential coefficient, we should have one term multiplied by A and another by $\dfrac{d\text{A}}{dx}$; and by proper determination of $\dfrac{1}{\text{A}} . \dfrac{d\text{A}}{dx}$ we should be able to make the expression for $\text{F}=0$ for any one value of y; but it is not possible to make it $=0$ for all values of y; and thus it appears that some force, though perhaps extremely small, is necessary to maintain the sort of undulation which we have supposed.

Whole force, from the bottom to the surface, assumed to be 0.

(242) Among the different conditions on which we may fix for the determination of $\dfrac{d\text{A}}{dx}$, the following appears the most reasonable; that upon the whole, from the bottom to the surface of the water, the horizontal force necessary to maintain the assumed wave-motion shall be 0, or that $\int_y \text{F}$ shall $=0$. We cannot here perform the integration under the differential sign, because the limits of the integration will be η and k, the former of which is a function of x. Expanding the differential therefore, we find $\text{F}=$

$$\left\{\begin{array}{l} -2n^2\dfrac{d\text{A}}{dx} . \dfrac{1}{m} . \left(\varepsilon^{m(y-\eta)} + \varepsilon^{m(\eta-y)}\right) - 2n^2\text{A}\dfrac{1}{m} . \dfrac{dm}{dx}(y-\eta)\left(\varepsilon^{m(y-\eta)} - \varepsilon^{m(\eta-y)}\right) \\ +3n^2\text{A} . \dfrac{1}{m^2} . \dfrac{dm}{dx}\left(\varepsilon^{m(y-\eta)} + \varepsilon^{m(\eta-y)}\right) + 2n^2\text{A}\dfrac{d\eta}{dx}\left(\varepsilon^{m(y-\eta)} - \varepsilon^{m(\eta-y)}\right) \end{array}\right\} \sin \overline{nt-\text{M}}.$$

For $\int_y \text{F}$, the general integral is

$$\left\{\begin{array}{l} -2n^2\dfrac{d\text{A}}{dx} . \dfrac{1}{m^2} . \left(\varepsilon^{m(y-\eta)} - \varepsilon^{m(\eta-y)}\right) - 2n^2\text{A}\dfrac{1}{m^2} . \dfrac{dm}{dx}(y-\eta)\left(\varepsilon^{m(y-\eta)} + \varepsilon^{m(\eta-y)}\right) \\ +5n^2\text{A} . \dfrac{1}{m^3} . \dfrac{dm}{dx} . \left(\varepsilon^{m(y-\eta)} - \varepsilon^{m(\eta-y)}\right) + 2n^2\text{A}\dfrac{1}{m} . \dfrac{d\eta}{dx}\left(\varepsilon^{m(y-\eta)} + \varepsilon^{m(\eta-y)}\right) \end{array}\right\} \sin \overline{nt-\text{M}},$$

which from $y=\eta$ to $y=k$ is

$$\left\{\begin{array}{l} -2n^2\dfrac{d\text{A}}{dx} . \dfrac{1}{m^2} . \left(\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}\right) - 2n^2\text{A} . \dfrac{1}{m^2} . \dfrac{dm}{dx}(k-\eta)\left(\varepsilon^{m(k-\eta)} + \varepsilon^{m(\eta-k)}\right) \\ +5n^2\text{A} . \dfrac{1}{m^3} . \dfrac{dm}{dx}\left(\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}\right) + 2n^2\text{A} . \dfrac{1}{m} . \dfrac{d\eta}{dx}\left(\varepsilon^{m(k-\eta)} + \varepsilon^{m(\eta-k)}\right) - 4n^2\text{A} . \dfrac{1}{m} . \dfrac{d\eta}{dx} \end{array}\right\} \sin \overline{nt-\text{M}}$$

$$= -2n^2m^{\frac{1}{2}}\left\{\frac{d\text{A}}{dx} . m^{-\frac{5}{2}} . \left(\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}\right) + \text{A}\frac{d}{dx}\left\{m^{-\frac{5}{2}} . \left(\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}\right)\right\} + 2\text{A} . m^{-\frac{3}{2}} . \frac{d\eta}{dx}\right\} \sin \overline{nt-\text{M}}.$$

Making this $=0$, we find

$$\frac{1}{\text{A}} . \frac{d\text{A}}{dx} = \frac{-\dfrac{d}{dx}\left\{m^{-\frac{5}{2}} . \left(\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}\right)\right\} - 2m^{-\frac{3}{2}} . \dfrac{d\eta}{dx}}{m^{-\frac{5}{2}}\left(\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}\right)}$$

whence

$$\text{A} = \frac{\text{C}m^{\frac{5}{2}}}{\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}} \times \varepsilon^{\text{Q}},$$

where

$$\frac{d\text{Q}}{dx} = \frac{-2m\dfrac{d\eta}{dx}}{\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}}.$$

Coefficient of horizontal motion.

(243.) The coefficient of horizontal displacement of particles at the surface is

$$\text{A}\left(\varepsilon^{m(k-\eta)} + \varepsilon^{m(\eta-k)}\right) = \text{C} . m^{\frac{5}{2}} . \frac{\varepsilon^{m(k-\eta)} + \varepsilon^{m(\eta-k)}}{\varepsilon^{m(k-\eta)} - \varepsilon^{m(\eta-k)}} . \varepsilon^{\text{Q}} = \text{C} . m^{\frac{5}{2}} . \frac{mg}{n^2} . \varepsilon^{\text{Q}} = \frac{\text{C}g}{n^2}m^{\frac{7}{2}} . \varepsilon^{\text{Q}}.$$

(244.) The expression for the vertical displacement of particles at the surface, or K, consists of a large term and a small one, which may be represented by $B.\sin\overline{nt-M}+b.\cos\overline{nt-M}$. Making $\frac{b}{B}=\tan\alpha$, this expression $=\sqrt{B^2+b^2}.\sin\overline{nt-M+\alpha}$; but as b is small, $\sqrt{B^2+b^2}$ will not differ sensibly from B. The coefficient therefore of vertical displacement $=B$

Coefficient of vertical motion.

$$=A\left(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}\right)=C.m^{\frac{5}{2}}.\varepsilon^{Q}.$$

(245.) In order then fully to understand the changes in the magnitude of the waves, we have now only to estimate the exponential ε^{Q}. It does not appear that the requisite integration for the exponent can be effected, unless a value be assigned for η in terms of x. But the following limiting values will give a good idea of its general value.

Value of coefficient for short waves.

(246.) 1st. When the waves are short in proportion to the depth (as in common waves, or in the ordinary swell of the sea), m or $\frac{2\pi}{\lambda}$ is large; therefore $\varepsilon^{m(k-\eta)}$ is very large, and $\frac{m\frac{d\eta}{dx}}{\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}}$ is insignificantly small: therefore its integral $=0$; and the exponential term $=\varepsilon^{0}=1$. Thus we have this theorem. When the water is deep in proportion to the length of the waves, the coefficient of horizontal displacement varies as $m^{\frac{7}{2}}$ or inversely as $\lambda^{\frac{7}{2}}$; and the coefficient of vertical displacement varies as $m^{\frac{5}{2}}$ or inversely as $\lambda^{\frac{5}{2}}$. This theorem is not much in error if the depth be even as small as half the length of a wave.

Value of coefficient for long waves.

(247.) 2nd. When the waves are long in proportion to the depth (as in the tide-wave), m is small, and $\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}=2m(k-\eta)$ nearly; therefore the exponent $=-2\int_x\frac{m\frac{d\eta}{dx}}{2m(k-\eta)}=-\int\frac{\frac{d\eta}{dx}}{k-\eta}=\log(k-\eta)$; and the exponential term $=\varepsilon^{\log(k-\eta)}=k-\eta$. But when m is small, the equation $m=\frac{n^2}{g}.\frac{\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)}}{\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}}$ becomes $m=\frac{n^2}{g}.\frac{2}{2m(k-\eta)}$; therefore $m=\frac{n}{\sqrt{g}}.\frac{1}{(k-\eta)^{\frac{1}{2}}}$; $m^{\frac{7}{2}}=\frac{n^{\frac{7}{2}}}{g^{\frac{7}{4}}}.\frac{1}{(k-\eta)^{7}}$; $m^{\frac{5}{2}}=\frac{n^{\frac{5}{2}}}{g^{\frac{5}{4}}}.\frac{1}{(k-\eta)^{\frac{5}{4}}}$. Substituting these in the expressions for A and B we have this theorem. When the water is shallow in proportion to the length of the waves, the coefficient of horizontal displacement varies as $\overline{\text{the depth}}^{-\frac{3}{4}}$ or as $m^{\frac{3}{2}}$, or inversely as $\lambda^{\frac{3}{2}}$; and the coefficient of vertical displacement varies as $\overline{\text{the depth}}^{-\frac{1}{4}}$, or as $m^{\frac{1}{2}}$, or inversely as $\lambda^{\frac{1}{2}}$.

(248.) It may be worth while to examine into the value of F at the surface. We have already found for F the expression $-2n^2m^{\frac{1}{2}}.\frac{d}{dx}\{Am^{-\frac{3}{2}}(\varepsilon^{m(y-\eta)}+\varepsilon^{m(\eta-y)})\}.\sin\overline{nt-M}$; and putting k for y, the value of F at the surface is $-2n^2m^{\frac{1}{2}}.\frac{d}{dx}\{Am^{-\frac{3}{2}}(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})\}.\sin\overline{nt-M}$; which, on substituting for $A(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})$ the value already found, becomes

$$-2Cgm^{\frac{1}{2}}\frac{d}{dx}\{m^2.\varepsilon^{Q}\}\sin\overline{nt-M}.$$

$$=4Cgm^{\frac{3}{2}}.\varepsilon^{Q}\times\left\{\frac{m^2\frac{d\eta}{dx}}{\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}}-\frac{dm}{dx}\right\}.\sin\overline{nt-M}.$$

The sign of the coefficient of $\sin\overline{nt-M}$ will depend entirely on the sign of the quantity within the large brackets; the external multiplier being essentially positive when C is positive; that is, when A is positive. Now suppose the wave to be moderately long, or m to be moderately small. On differentiating the equation

$$\log n^2=\log mg+\log\left(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}\right)-\log\left(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)}\right)$$

and then expanding the exponentials to the third power of $m(k-\eta)$, we find

$$\frac{dm}{dx}=\frac{m\frac{d\eta}{dx}}{2(k-\eta)+\frac{2}{3}m^2(k-\eta)^3}.$$

Force necessary at the surface, to maintain the motion.

Expanding the exponentials in the term $\frac{m^2\frac{d\eta}{dx}}{\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)}}$ in the same manner, we find for the quantity within the large brackets

$$\frac{m\frac{d\eta}{dx}}{2(k-\eta)+\frac{1}{3}m^2(k-\eta)^3}-\frac{m\frac{d\eta}{dx}}{2(k-\eta)+\frac{2}{3}m^2(k-\eta)^3};$$

which, supposing the canal to become more shallow as the wave advances, that is, supposing $\frac{d\eta}{dx}$ to be positive, is evidently a very small positive quantity. The expression for F at the surface may therefore be represented by $+\text{E}.\sin\overline{nt-\text{M}}$. Now the elevation of the water, or K, has for its principal term $-\text{A}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})\sin\overline{nt-\text{M}}$; consequently at the crest of a wave, or where K is greatest and positive, $\sin\overline{nt-\text{M}}=-1$; therefore at the crest of the wave, $\text{F}=-\text{E}$. That is, the force, which must act at the crest of the wave to maintain the particles in the state of wave-motion, must be in the direction opposite to that in which the wave is advancing; or, supposing no such force to act, the particles at the crest of the wave will spring forwards from the wave, or will *break* towards the shallow side. This seems to be a complete explanation of the breaking of waves over the edge of a sunk shoal.

Breaking of waves over sunk shoals, &c. explained.

Cause of the surf.

(249.) The breaking of the surf upon a beach is not, perhaps, entirely to be ascribed to the cause which we have assigned for breakers over sunk banks; although that cause undoubtedly produces a great part of the effect. The excessive extent of disturbance of the particles of the water is probably one of the principal causes. As $k-\eta$ is diminished, m is increased or λ diminished; that is, on the water becoming shallower, the waves become shorter. And as we have seen, the elevation of the high parts and the depression of the low parts become greater, so that the general character of the waves is becoming steeper. Moreover, as the height of the wave bears a large proportion to the depth of the water, the front of the wave becomes, from that cause, very steep (203.). At the crest of the wave (nearly) $\sin\overline{nt-\text{M}}$ as we have found $=-1$, or $nt-\text{M}=\frac{3\pi}{2}$; therefore at the crest of the wave, X, which depends on $\cos\overline{nt-\text{M}}$, is 0; in front of the crest, M is greater, $nt-\text{M}$ is less than $\frac{3\pi}{2}$, and X therefore is negative, or the particles are thrown backwards towards the crest, and therefore (in addition to the cause already assigned) on account of the rapidly increasing coefficient of X as the breadth diminishes, the steepness of the front of the wave rapidly increases. At the same time, the particles on the crest are themselves moving in the direction of the wave-motion, and with a velocity which becomes greater and greater (for the particles which happen to be on the crest) as the wave approaches the shore. It is evident that the limit to these circumstances is, that the front of the wave becomes as steep as a wall, while the uppermost particles are moving towards the shore and the lowermost from the shore; that the former, therefore, will tumble over the latter; and this is the motion of surf.

Investigation when the breadth is not uniform.

(250.) Hitherto we have supposed the breadth of the canal uniform; or, rather, we have taken no account whatever of its breadth. We will now suppose the breadth of a narrow canal to be variable.

Problem.—To investigate the circumstances of the change of wave-motion in a narrow canal of uniform depth whose breadth at different points is different.

(251.) Let β, the breadth of the canal, be considered a function of x. It is easily seen that the equation of continuity must now be altered. For in figure 8, the small parallelogram ps is carried forward to the position PS, where the breadth of the canal is different; namely, is $\beta+\frac{d\beta}{dx}\text{X}$ instead of β: and, therefore, instead of making $hl=h\times\left(1+\frac{d\text{X}}{dx}\right)\times l\times\left(1+\frac{d\text{Y}}{dy}\right)$, as in (145.), we must make

$$\beta\times hl=\beta\left(1+\frac{1}{\beta}\cdot\frac{d\beta}{dx}\text{X}\right)\times h\left(1+\frac{d\text{X}}{dx}\right)\times l\left(1+\frac{d\text{Y}}{dy}\right):$$

from which we find

$$0=\frac{1}{\beta}\cdot\frac{d\beta}{dx}\text{X}+\frac{d\text{X}}{dx}+\frac{d\text{Y}}{dy};$$

and, integrating with respect to y,

$$\text{Y}=-\int_y\left(\frac{d\text{X}}{dx}+\frac{\text{X}}{\beta}\cdot\frac{d\beta}{dx}\right)\text{ (from 0 to }y\text{)}.$$

Sect. IV. Theory of Waves.

Subsec. 4. When the water is under the action of Horizontal and Vertical Forces.

The equation of equal pressure remains as before; namely,

$$\frac{d^2X}{dt^2}=F+\frac{d}{dx}\left\{-gK-\int_y \frac{d^2Y}{dt^2}\ (y \text{ to } k)\right\}.$$

(252.) We shall now proceed as in the case of variable depth, assuming that the expression for the wave-displacement may be represented by the same formula as if the breadth were uniform, with proper alteration of the constants. The assumption, however, is much simpler than where the depth is variable, inasmuch as we have no reason for assuming the expression for the phase to be different (the velocity of the wave, as depending on the depth, being everywhere the same). Thus, then, we have for X, $A(\varepsilon^{my}+\varepsilon^{-my})\cos\overline{nt-mx}$, where A is a function of x, and m is constant, connected with n by the equation $n^2=gm.\dfrac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}$.

(253.) The expression $\dfrac{dX}{dx}+\dfrac{X}{\beta}.\dfrac{d\beta}{dx}$ becomes

$$(\varepsilon^{my}+\varepsilon^{-my})\left\{\frac{dA}{dx}.\cos\overline{nt-mx}+mA.\sin\overline{nt-mx}+\frac{A}{\beta}.\frac{d\beta}{dx}.\cos\overline{nt-mx}\right\};$$

from which we obtain for Y,

$$-(\varepsilon^{my}-\varepsilon^{-my})\left\{\frac{1}{m}.\frac{dA}{dx}.\cos\overline{nt-mx}+A.\sin\overline{nt-mx}+\frac{1}{m}.\frac{A}{\beta}.\frac{d\beta}{dx}.\cos\overline{nt-mx}\right\};$$

therefore $$\frac{d^2Y}{dt^2}=(\varepsilon^{my}-\varepsilon^{-my})\left\{\frac{n^2}{m}.\frac{dA}{dx}.\cos\overline{nt-mx}+n^2A.\sin\overline{nt-mx}+\frac{n^2}{m}.\frac{A}{\beta}.\frac{d\beta}{dx}.\cos\overline{nt-mx}\right\},$$

and $$-\int_y \frac{d^2Y}{dt^2}\ (y \text{ to } k)=(\varepsilon^{my}+\varepsilon^{-my}-\varepsilon^{mk}-\varepsilon^{-mk})\left\{\frac{n^2}{m^2}.\frac{dA}{dx}.\cos\overline{nt-mx}+\frac{n^2}{m}A.\sin\overline{nt-mx}+\frac{n^2}{m^2}.\frac{A}{\beta}.\frac{d\beta}{dx}.\cos\overline{nt-mx}\right\}.$$

Also $$-gK=(\varepsilon^{mk}-\varepsilon^{-mk})\left\{\frac{g}{m}.\frac{dA}{dx}.\cos\overline{nt-mx}+gA.\sin\overline{nt-mx}+\frac{g}{m}.\frac{A}{\beta}.\frac{d\beta}{dx}.\cos\overline{nt-mx}\right\}.$$

Therefore, by virtue of the equation $n^2(\varepsilon^{mk}+\varepsilon^{-mk})=gm(\varepsilon^{mk}-\varepsilon^{-mk})$,

$$-gK-\int_y \frac{d^2Y}{dt^2}\ (y \text{ to } k)=(\varepsilon^{my}+\varepsilon^{-my})\left\{\frac{n^2}{m^2}.\frac{dA}{dx}.\cos\overline{nt-mx}+\frac{n^2}{m}A.\sin\overline{nt-mx}+\frac{n^2}{m^2}.\frac{A}{\beta}.\frac{d\beta}{dx}.\cos\overline{nt-mx}\right\}.$$

(254.) In differentiating this quantity with regard to x, we shall suppose the breadth to vary slowly, and shall, therefore, as in the similar differentiation of the last Problem, reject all products of differential coefficients, and all differential coefficients of the second order. Thus we obtain

$$\frac{d}{dx}\left\{-gK-\int_y \frac{d^2Y}{dt^2}\ (y \text{ to } k)\right\}=$$

$$(\varepsilon^{my}+\varepsilon^{-my})\left\{\frac{2n^2}{m}.\frac{dA}{dx}.\sin\overline{nt-mx}n^2-A.\cos\overline{nt-mx}+\frac{n^2}{m}.\frac{A}{\beta}.\frac{d\beta}{dx}.\sin\overline{nt-mx}\right\}.$$

Also $$\frac{d^2X}{dt^2}=(\varepsilon^{my}+\varepsilon^{-my})\,(-n^2A.\cos\overline{nt-mx}).$$

Subtracting the former from the latter,

$$F=-(\varepsilon^{my}+\varepsilon^{-my})\left\{\frac{2n^2}{m}.\frac{dA}{dx}+\frac{n^2}{m}.\frac{A}{\beta}.\frac{d\beta}{dx}\right\}\sin\overline{nt-mx}.$$

This expression vanishes, or no force is necessary to maintain the wave-motion, if $2\dfrac{dA}{dx}+\dfrac{A}{\beta}.\dfrac{d\beta}{dx}=0$, or if $\dfrac{2}{A}.\dfrac{dA}{dx}+\dfrac{1}{\beta}.\dfrac{d\beta}{dx}=0$, or if $\log(A^2\beta)=\log C^2$, or $A=\dfrac{C}{\sqrt{\beta}}$. That is, the coefficient of horizontal motion will be inversely as the square root of the breadth of the canal. The coefficient of vertical motion (in the same manner as in (244.)), will not sensibly differ from $-A(\varepsilon^{my}-\varepsilon^{-my})$, or, at the surface, $-A(\varepsilon^{mk}-\varepsilon^{-mk})$: and thus the height of the waves will be inversely as the square root of the breadth of the canal.

Coefficients of horizontal and vertical motion.

(255.) We might, without difficulty, have combined the investigations of the two last Problems; and we should have found the expression for the variation of the coefficient of either motion in a canal where both breadth and depth vary, to be represented by the product of those found separately in the two Problems.

(256.) Before quitting these important investigations, we shall make one remark. When waves travel along a canal of uniform breadth and depth, the motion of every particle, as we have seen (182.) and (183.), is in a circle, or an ellipse whose major axis is horizontal: if in a circle, its motion in that circle is uniform; if in an ellipse, the horizontal motion is, at any instant of time, in the same proportion to the whole horizontal motion as in a circle, and the vertical motion is also in the same proportion to the whole vertical motion as in a circle.

And (184.) the greatest horizontal motion forwards occurs when a particle is at the top of a wave: the greatest horizontal motion backwards occurs when a particle is at the bottom of a wave; and the horizontal motion is 0 when a particle is at its mean elevation. These assertions are not true if the depth or breadth is variable. Thus, in the case of variable depth; the horizontal displacement at the surface $=\mathrm{X}=\mathrm{A}(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})\cos\overline{nt-\mathrm{M}}$: the horizontal motion $=-n\mathrm{A}(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})\sin\overline{nt-\mathrm{M}}$: this is greatest when $nt-\mathrm{M}=\frac{3\pi}{2}$, $\sin\overline{nt-\mathrm{M}}=-1$, $\cos\overline{nt-\mathrm{M}}=0$; substituting these values in the expression for K, it becomes $\mathrm{A}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})$, which is not the greatest value of K; for the expression for K is of the form $\mathrm{B}\sin\overline{nt-\mathrm{M}}+b\cos\overline{nt-\mathrm{M}}$, the greatest value of which is $\sqrt{\mathrm{B}^2+b^2}$, where $\mathrm{B}=\mathrm{A}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})$. The horizontal motion is 0 when $\sin\overline{nt-\mathrm{M}}=0$; which (if we take the case of still water following the high water, implying that t has been increased above the value which made $\sin\overline{nt-\mathrm{M}}=-1$) gives $nt-\mathrm{M}=0$, $\cos nt-\mathrm{M}=1$; and the corresponding value of K is

$$-\frac{d\mathrm{A}}{dx}\cdot\frac{1}{m}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})-\mathrm{A}\frac{1}{m}\cdot\frac{dm}{dx}(k-\eta)(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})+\mathrm{A}\frac{1}{m^2}\cdot\frac{dm}{dx}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})+\mathrm{A}\frac{d\eta}{dx}(\varepsilon^{m(k-\eta)}+\varepsilon^{m(\eta-k)})$$

$$=-\frac{d}{dx}\left\{\frac{\mathrm{A}}{m}(\varepsilon^{m(k-\eta)}-\varepsilon^{m(\eta-k)})\right\}$$

$$=-\frac{d}{dx}\cdot\frac{\text{coefficient of vertical displacement}}{m}.$$

Cessation of flow occurs earlier than if the depth and breadth were uniform.

In the case of short waves in deep water, we have found the coefficient of vertical displacement to vary as $m^{\frac{5}{2}}$; therefore this value of K varies as $-\frac{d}{dx}m^{\frac{3}{2}}$: as the depth diminishes, m increases, and this value of K is negative. But in the case of long waves in shallow water (as the tide-wave), the coefficient of vertical displacement varies as $m^{\frac{1}{2}}$; therefore this value of K varies as $-\frac{d}{dx}\cdot\frac{1}{m^{\frac{1}{2}}}$: as the depth diminishes, m increases, and K is therefore positive. That is, in the case of long waves in shallow water, where the depth diminishes, the water is sensibly elevated above its mean height when the flow ceases; and in like manner it is sensibly depressed below its mean height when the ebb ceases.

(257.) In the case of variable width; the horizontal displacement at the surface $=\mathrm{A}(\varepsilon^{mk}+\varepsilon^{-mk})\cos\overline{nt-mx}$; the horizontal motion $=-n\mathrm{A}(\varepsilon^{mk}+\varepsilon^{-mk})\sin\overline{nt-mx}$: this is 0, for the still water following the high water, when $\sin\overline{nt-mx}=0$, $\cos\overline{nt-mx}=1$; the corresponding value of K is

$$-(\varepsilon^{mk}-\varepsilon^{-mk})\left\{\frac{1}{m}\cdot\frac{d\mathrm{A}}{dx}+\frac{1}{m}\cdot\frac{\mathrm{A}}{\beta}\cdot\frac{d\beta}{dx}\right\}=-\frac{1}{m\beta}(\varepsilon^{mk}-\varepsilon^{-mk})\frac{d}{dx}(\mathrm{A}\beta).$$

But we have found $\mathrm{A}=\frac{\mathrm{C}}{\beta^{\frac{1}{2}}}$, therefore $\mathrm{A}\beta=\mathrm{C}\beta^{\frac{1}{2}}$; and therefore $\mathrm{K}=-\frac{1}{m\beta}(\varepsilon^{mk}-\varepsilon^{-mk})\frac{d}{dx}(\mathrm{C}\beta^{\frac{1}{2}})$ If the breadth diminishes as x increases, the differential coefficient $\frac{d}{dx}(\mathrm{C}\beta^{\frac{1}{2}})$ is negative, and therefore the value of K is positive. That is, where the breadth diminishes, the water is sensibly elevated above its mean height when the flow ceases; and in like manner, the water is sensibly depressed below its mean height when the ebb ceases.

(258.) When (as usually happens) both these causes unite, that is when the channel along which the tide-wave is propagated becomes at the same time shallower and narrower, it may happen that the height of the water above its mean level is considerable when the flow ceases.

Investigation for a canal whose section is of any form and varies in any degree, when the wave is very long.

(259.) The change in the circumstances of the wave produced by the variation of the section of the channel in form as well as in breadth and in depth, when the wave is supposed to be very long, may be found more simply in the following manner.

(260.) Let the equation to the section, as in (218.), be $z=\phi(y,x)$, x being introduced into the function because the form and dimensions vary with x; and $u=\int_y z=\psi(y,x)$. For the same reason as in (238.), suppose X to be represented by $\mathrm{A}\cos\overline{nt-\mathrm{M}}$, where $\mathrm{M}=\int_x m$. Now we have found in (220.) that the function expressing a single wave on a uniform canal whose section is defined by the equation $u=\psi(y,x)$, is $\mathrm{X}=\chi(v't-x)=\chi\left(t\sqrt{\frac{g\psi_{\prime}(k)}{\phi(k)}}-x\right)=\chi_{\prime}\left(nt-nx\sqrt{\frac{\phi(k)}{g\psi(k)}}\right)$. Comparing this with our usual form for X in

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a uniform canal, namely, $\mathrm{A}\cos\overline{nt-mx}$, it appears that $m=n\sqrt{\dfrac{\phi(k)}{g\psi(k)}}$, and therefore in the variable canal

$$\mathrm{M}=n\int_x\sqrt{\frac{\phi(k,x)}{g\psi(k,x)}}.$$

(261.) Now the fluid, which when at rest was included between two vertical planes whose co-ordinates were x and $x+h$, is now included between two vertical planes whose co-ordinates are $x+\mathrm{X}$ and $x+\mathrm{X}+h+\dfrac{d\mathrm{X}}{dx}h$ nearly; and the area of the vertical section, which was $\psi(k,x)$, is now $\psi(\mathrm{V},x+\mathrm{X})$, or $\psi(k+\overline{\mathrm{V}-k},x+\mathrm{X})$, or $\left(\text{observing that } \dfrac{d\,.\,\psi(k,x)}{dk}=\phi(k,x)\right)$ $\psi(k,x)+\phi(k,x).\overline{\mathrm{V}-k}+\dfrac{d\,.\,\psi(k,x)}{dx}\mathrm{X}$. Thus the equation of continuity becomes

$$h\times\psi(k,x)=h\left(1+\frac{d\mathrm{X}}{dx}\right)\times\left\{\psi(k,x)+\phi(k,x).\overline{\mathrm{V}-k}+\frac{d.\psi(k,x)}{dx}\mathrm{X}\right\},$$

or
$$0=\frac{d\mathrm{X}}{dx}.\psi(k,x)+\phi(k,x).\overline{\mathrm{V}-k}+\frac{d.\psi(k,x)}{dx}\mathrm{X};$$

whence
$$\mathrm{V}-k=-\frac{1}{\phi(k,x)}\frac{d}{dx}\Big(\mathrm{X}.\psi(k,x)\Big).$$

(262.) The equation of pressure will be as before, $\dfrac{d^2\mathrm{X}}{dt^2}=-g\dfrac{d\mathrm{V}}{dx}.\dfrac{1}{1+\dfrac{d\mathrm{X}}{dx}}=-g\dfrac{d\mathrm{V}}{dx}$ nearly. Substituting for V, this becomes

$$\frac{d^2\mathrm{X}}{dt^2}=g\frac{d}{dx}\left\{\frac{1}{\phi(k,x)}\frac{d}{dx}\Big(\mathrm{X}.\psi(k,x)\Big)\right\}.$$

Performing the interior differentiation on the assumption $\mathrm{X}=\mathrm{A}\cos\overline{nt-\mathrm{M}}$, where A is a function of x, and $\dfrac{d\mathrm{M}}{dx}=n\sqrt{\dfrac{\phi(k,x)}{g\psi(k,x)}}$, this becomes

$$\frac{d^2\mathrm{X}}{dt^2},\text{ or }-n^2\mathrm{A}.\cos\overline{nt-\mathrm{M}}=g\frac{d}{dx}\left\{\frac{1}{\phi(k,x)}\frac{d}{dx}\Big(\mathrm{A}.\psi(k,x)\Big)\cos\overline{nt-\mathrm{M}}+n\mathrm{A}\sqrt{\frac{\psi(k,x)}{g\phi(k,x)}}\sin\overline{nt-\mathrm{M}}\right\}.$$

Performing the second differentiation, with the limitation that, as $\phi(k,x)$ and $\psi(k,x)$ vary very slowly, their second differentials and squares of first differentials may be rejected,

$$-n^2\,\mathrm{A}.\cos\overline{nt-\mathrm{M}}=n\sqrt{g}.\left\{\frac{1}{\sqrt{\phi(k,x).\psi(k,x)}}\frac{d}{dx}\Big(\mathrm{A}.\psi(k,x)\Big)+\frac{d}{dx}\left\{\frac{\mathrm{A}.\psi(k,x)}{\sqrt{\phi(k,x).\psi(k,x)}}\right\}\right\}\sin\overline{nt-\mathrm{M}}$$
$$-n^2\,\mathrm{A}.\cos\overline{nt-\mathrm{M}}.$$

Let $\mathrm{A}\psi(k,x)=\nu$, $\sqrt{\phi(k,x).\psi(k,x)}=\mu$: then this equation reduces itself to

$$\frac{1}{\mu}.\frac{d\nu}{dx}+\frac{d}{dx}\left\{\frac{\nu}{\mu}\right\}=0,$$

or
$$\frac{2}{\mu}.\frac{d\nu}{dx}-\frac{\nu}{\mu^2}.\frac{d\mu}{dx}=0,$$

or
$$\frac{2}{\nu}.\frac{d\nu}{dx}-\frac{1}{\mu}.\frac{d\mu}{dx}=0,$$

the integral of which is $\dfrac{\nu^2}{\mu}=\mathrm{C}^2$, or $\mathrm{A}^2.\{\psi(k,x)\}^{\frac{3}{2}}.\{\phi(k,x)\}^{-\frac{1}{2}}=\mathrm{C}^2$,

or
$$\mathrm{A}=\mathrm{C}.\{\psi(k,x)\}^{-\frac{3}{4}}.\{\phi(k,x)\}^{\frac{1}{4}}.$$

This is the coefficient of horizontal displacement of the particles.

Coefficient of horizontal motion.

(263.) For the vertical displacement,

$$\mathrm{K}=\mathrm{V}-k=-\frac{1}{\phi(k,x)}\frac{d}{dx}\Big(\mathrm{X}.\psi(k,x)\Big)=-\frac{1}{\phi(k,x)}.\frac{d}{dx}\Big(\mathrm{A}.\psi(k,x).\cos\overline{nt-\mathrm{M}}\Big)$$
$$=-\frac{1}{\phi(k,x)}\left\{\frac{d}{dx}\Big(\mathrm{A}.\psi(k,x)\Big).\cos\overline{nt-\mathrm{M}}+\mathrm{A}.\psi(k,x).\sin\overline{nt-\mathrm{M}}\,.\,n\sqrt{\frac{\phi(k,x)}{g\psi(k,x)}}\right\},$$

in which the first term may be neglected as much smaller than the second. Thus

$$K = -\frac{n}{\sqrt{g}} . A \sqrt{\frac{\psi(k,x)}{\phi(k,x)}} . \sin \overline{nt - M}.$$

Coefficient of vertical motion.

Putting for A its value found above, we have for the coefficient of vertical displacement

$$-\frac{n}{\sqrt{g}} C . \{\psi(k,x)\}^{-\frac{1}{4}} . \{\phi(k,x)\}^{-\frac{1}{4}}.$$

(264.) For instance: if the section of the canal be everywhere a rectangle, $\phi(k, x) =$ breadth, and $\psi(k, x) =$ breadth × depth: which gives for the coefficient of vertical displacement

$$-\frac{n}{\sqrt{g}} C . \{\text{breadth}\}^{-\frac{1}{2}} . \{\text{depth}\}^{-\frac{1}{4}},$$

a result which agrees with those found for long waves in shallow water in (247.) and (254.).

(265.) The two following problems regarding the motion of variable waves on a sea of uniform depth are interesting, as applying to some most important phænomena.

Elevations increase from wave to wave.

Problem.—The amplitude of each successive wave in a long series is the same, but the elevations of the waves increase progressively from wave to wave, the same elevation always corresponding to the same place: to find the forces which must act on the particles of water to maintain this state of undulation.

(266.) Here the value of Y at the surface, or K, must have a factor of $\sin \overline{nt - mx}$, which itself depends upon x. We may therefore suppose X to have such a factor: in other respects the form assumed for X may be the same as in the case of uniform waves. Let therefore

$$X = (\varepsilon^{my} + \varepsilon^{-my}) . \phi(x) . \cos \overline{nt - mx}:$$

where n and m are connected by the equation $n^2 = mg \frac{\varepsilon^{mk} - \varepsilon^{-mk}}{\varepsilon^{mk} + \varepsilon^{-mk}}$. Then

$$\frac{dY}{dy} = -\frac{dX}{dx} = -(\varepsilon^{my} + \varepsilon^{-my}) \frac{d}{dx}\left(\phi(x) \cos \overline{nt - mx}\right);$$

integrating from 0 to y,

$$Y = \frac{1}{m}(-\varepsilon^{my} + \varepsilon^{-my}) \frac{d}{dx}\left(\phi(x) . \cos \overline{nt - mx}\right);$$

$$K = \frac{1}{m}(-\varepsilon^{mk} + \varepsilon^{-mk}) \frac{d}{dx}\left(\phi(x) . \cos \overline{nt - mx}\right):$$

$$\frac{d^2Y}{dt^2} = \frac{n^2}{m}(\varepsilon^{my} - \varepsilon^{-my}) \frac{d}{dx}\left(\phi(x) . \cos \overline{nt - mx}\right)$$

$$\int_y \frac{d^2Y}{dt^2}(y \text{ to } k) = \frac{n^2}{m^2}(\varepsilon^{mk} + \varepsilon^{-mk} - \varepsilon^{my} - \varepsilon^{-my}) \frac{d}{dx}\left(\phi(x) . \cos \overline{nt - mr}\right):$$

therefore
$$gK + \int_y \frac{d^2Y}{dt^2}(y \text{ to } k) = -\frac{n^2}{m^2}(\varepsilon^{my} + \varepsilon^{-my}) \frac{d}{dx}\left(\phi(x) . \cos \overline{nt - mx}\right):$$

the terms $\frac{g}{m}(-\varepsilon^{mk} + \varepsilon^{-mk}) + \frac{n^2}{m^2}(\varepsilon^{mk} + \varepsilon^{-mk})$ destroying one another by virtue of the equation between n and m.

Also
$$\frac{d^2X}{dt^2} = -n^2(\varepsilon^{my} + \varepsilon^{-my}) . \phi(x) \cos \overline{nt - mx}.$$

Hence F or $\frac{d^2X}{dt^2} + \frac{d}{dx}\left\{gK + \int_y \frac{d^2Y}{dt^2}(y \text{ to } k)\right\}$ becomes

$$-n^2(\varepsilon^{my} + \varepsilon^{-my}) . \left\{\phi(x) . \cos \overline{nt - mx} + \frac{1}{m^2} . \frac{d^2}{dx^2}\left(\phi(x) . \cos \overline{nt - mx}\right)\right\}.$$

Performing the differentiation, this becomes

$$-n^2(\varepsilon^{my} + \varepsilon^{-my}) . \left\{\frac{2}{m} . \phi'(x) . \sin \overline{nt - mx} + \frac{1}{m^2} . \phi''(x) . \cos \overline{nt - mx}\right\}.$$

Force necessary to maintain this motion.

If $\phi(x) = Cx + D$, this expression for F becomes $-\frac{2Cn^2}{m}(\varepsilon^{my} + \varepsilon^{-my}) . \sin \overline{nt - mx}$. Now when the force is small, or the coefficient C small, the expression for Y will not sensibly differ from $-(\varepsilon^{my} - \varepsilon^{-my}) . (Cx + D) . \sin \overline{nt - mx}$. The term ε^{-my} is very small in comparison with ε^{my}, except very near the bottom. The result therefore is this: the force which, acting horizontally upon the particles of water, will maintain such a series of waves as we have supposed, must, for each particle, be nearly proportional to the elevation of that particle above its mean place: the proportion varying slowly, however, from one particle to another. Or we may state it thus: the force must be proportional to the horizontal velocity of each particle, and in the same direction as its motion.

The force of the wind corresponds closely with this expression.

(267.) Now we have nearly such a force in the action of the wind upon the waves. For the part of the wave which is principally or solely exposed to the action of the wind is its upper part. Thus the highest parts alone, or those near the crest of the wave, are urged forward: the lowest parts, or those in the hollow of the wave, either are not urged forward, or (more probably) by the eddies of the wind are actually urged backwards. Moreover, the action of the wind does not sensibly extend to a great depth below the surface: and in this respect it may be considered as fairly represented by ε^{my}. Thus then we have a mathematical explanation of the circumstance that the action of the wind will maintain the motion of a series of waves, whose elevation beginning at the windward shore is 0, and goes on increasing successively from wave to wave, without remarkable alteration from time to time (when they have once attained a certain magnitude). And as the proportion of the force of the wind, in regard to depth, is in pretty good agreement with that required by the theory for continuous unbroken waves, the theory explains the circumstance that, when the waves have attained this certain magnitude, their heads will scarcely be broken by the action of the wind.

In the open sea, the long waves only become important.

(268.) It may, however, be remarked that, in the open sea, the waves of large amplitude only are so much increased as to attract much attention. The reason appears to be that, when the action of the wind has in some degree increased all, the long waves protect the short ones from the continuance of its action upon them. And thus the long waves are conspicuous upon an open sea, not because the short waves are changed into long ones, but because the long ones only are conspicuously increased from the windward shore to the open sea.

The waves increase from time to time.

(269.) Problem.—The amplitudes of all the waves are equal, and the elevations are all equal at the same time, but increase constantly from time to time: to find the forces which must act on the particles of water to maintain this state of undulation.

(270.) Here we shall assume

$$X = (\varepsilon^{my}+\varepsilon^{-my})\,.\,\psi(t)\,.\,\cos\overline{nt-mx}.$$

Then

$$\frac{dY}{dy} = -\frac{dX}{dx} = -m\,(\varepsilon^{my}+\varepsilon^{-my})\,.\,\psi(t)\,.\,\sin\overline{nt-mx}:$$

integrating from 0 to y,

$$Y = (-\varepsilon^{my}+\varepsilon^{-my})\,.\,\psi(t)\,.\,\sin\overline{nt-mx}:$$

$$\frac{d^2Y}{dt^2} = (-\varepsilon^{my}+\varepsilon^{-my})\,\frac{d^2}{dt^2}\left(\psi(t)\,.\,\sin\overline{nt-mr}\right):$$

$$\int_y \frac{d^2Y}{dt^2}(y \text{ to } k) = \frac{1}{m}\,(-\varepsilon^{mk}-\varepsilon^{-mk}+\varepsilon^{my}+\varepsilon^{-my})\,\frac{d^2}{dt^2}\left(\psi(t)\,.\sin\overline{nt-mx}\right):$$

$$K = (-\varepsilon^{mk}+\varepsilon^{-mk})\,.\,\psi(t)\,.\,\sin\overline{nt-mx}:$$

therefore

$$\frac{d}{dx}\left\{gK+\int_y \frac{d^2Y}{dt^2}(y \text{ to } k)\right\} =$$

$$\frac{d}{dx}\left\{g\,(-\varepsilon^{mk}+\varepsilon^{-mk})\,.\,\psi(t)\,.\,\sin\overline{nt-mx}+\frac{1}{m}\,(-\varepsilon^{mk}-\varepsilon^{-mk}+\varepsilon^{my}+\varepsilon^{-my})\,\frac{d^2}{dt^2}\left(\psi(t)\,.\,\sin\overline{nt-mx}\right)\right\}$$

$$= gm\,(\varepsilon^{mk}-\varepsilon^{-mk})\,.\,\psi(t)\,.\,\cos\overline{nt-mx}+(\varepsilon^{mk}+\varepsilon^{-mk}-\varepsilon^{my}-\varepsilon^{-my})\,.\,\frac{d^2}{dt^2}\left(\psi(t)\,.\,\cos\overline{nt-mx}\right).$$

Also

$$\frac{d^2X}{dt^2} = (\varepsilon^{my}+\varepsilon^{-my})\,\frac{d^2}{dt^2}\left(\psi(t)\,.\,\cos\overline{nt-mx}\right).$$

Hence the expression for F or $\frac{d^2X}{dt^2}+\frac{d}{dx}\;gK+\int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k)\Big\}$ becomes

$$gm\,(\varepsilon^{mk}-\varepsilon^{-mk})\,.\,\psi(t)\,.\,\cos\overline{nt-mx}+(\varepsilon^{mk}+\varepsilon^{-mk})\,\frac{d^2}{dt^2}\left(\psi(t)\,.\,\cos\overline{nt-mx}\right).$$

Force necessary to maintain this motion.

Performing the differentiation, and observing that $gm\,(\varepsilon^{mk}-\varepsilon^{-mk})-n^2\,(\varepsilon^{mk}+\varepsilon^{-mk})=0$, we have

$$F = (\varepsilon^{mk}+\varepsilon^{-mk})\,\{\psi''(t)\,.\,\cos\overline{nt-mx}-2n\,.\,\psi'(\;)\,.\,\sin\overline{nt-mx}\}.$$

If $\psi(t) = Ct+D$, we must have

$$F = -2nC\,(\varepsilon^{mk}+\varepsilon^{-mk})\,.\,\sin\overline{nt-mx}.$$

That is: the elevations of the waves will be increased uniformly, provided a horizontal force acts upon every particle, which is equal for all the particles in the same vertical column, and which urges forward those particles that are above their mean places and urges backward those that are below their mean places, according to the law of the sine of the phase.

The force of the wind nearly represents this force, but not exactly.

(271.) Now the action of the wind produces a force strongly resembling this, but not so nearly as before. For the force (as in the former case) does act to urge forward those parts which are above their mean place, and in some degree to urge backwards those which are below their mean place. But it is not nearly equal for the particles at different elevations in the same vertical column: being effective only near the top and insensible for other parts. Hence it appears that there is too much horizontal force at the top of the waves for continuous undulation: and this excess of force is in the direction in which the wave is going, and therefore tends to push the top of the wave too rapidly. To this cause, as we conceive, is to be ascribed the breaking of the crests of the waves, which always takes place when the wind is raising the sea.

We may proceed in the same manner if $X = (\varepsilon^{my}+\varepsilon^{-my})\,(A+Bx+Ct)\cos\overline{ni-mx}$.

Manner in which the wind acts to raise the waves.

(272.) The effect of continued action of wind upon the sea may be conceived to show itself in the following manner. It is to be understood that, either from preceding disturbances, or from the trifling irregularities in the action of the wind while the water is smooth, there are very shallow undulations upon the water. When the wind begins to act, it will at first only increase the height of the waves in every part (271.), and during this time the heads of the waves will be broken. But after a time the waves, beginning with the windward shore, will be (for a short distance at least) so much increased, that the power of the wind will merely maintain them in that state without any increase (267.), but for all the sea in advance the wind will still be raising the waves. But as the waves successively attain that height which corresponds, according to the result of (266.), with the height which the wind can just maintain, those waves will no longer be increased, but the waves in advance will still be increased. Thus a wind of given intensity, however long it blows, can only raise the waves at a given point to a certain height: which height, however, will depend upon the distance of that point from the windward shore.

Limitation to the height of the waves raised by a wind of given force on a sea of given extent.

Motion of water under the action of forces similar to the tidal forces.

(273.) We shall now consider the motion of water in a canal of uniform breadth and depth under the action of a given force. The difficulties connected with a general solution of this problem would, in the present state of mathematics, be found insuperable; we shall therefore confine ourselves to a law of force which applies perfectly to the problem of tides.

(274.) Problem.—The water in a canal of uniform breadth and depth is acted on by a horizontal force represented at every point by $H.\sin\overline{it-mx}$ and by a vertical force represented by $G.\cos\overline{it-mx}$: to find the nature of the wave which will be produced.

(275.) The equations to be used are those of (224.), supposing the depth uniform or $\eta=0$. They become

$$Y = -\int_y \frac{dX}{dx}\,(0 \text{ to } y)$$

$$\frac{d^2X}{dt^2} = H.\sin\overline{it-mx} + \frac{d}{dx}\left\{-(g+G.\cos\overline{it-mx})\,(k-y+K) - \int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k)\right\}.$$

The forces are supposed to be small, and the term $\frac{d}{dx}(g.\overline{k-y})$ is evidently $=0$. These considerations reduce the second equation to this form:

$$\frac{d^2X}{dt^2} = H.\sin\overline{it-mx} + \frac{d}{dx}\left\{-gK - \overline{k-y}.G.\cos\overline{it-mx} - \int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k)\right\}.$$

Now the least trial would show that the expression for X must depend on $\sin\overline{it-mx}$. Assume therefore

$$X = \phi''(y).\sin\overline{it-mx};$$

$\phi''(y)$ being a function of y whose form is yet to be determined, and which is conceived as the second derived function from some function $\phi(y)$. Then

$$\frac{dX}{dx} = -m\phi''(y)\cos\overline{it-mx}; \qquad -\int_y \frac{dX}{dx} = m\phi'(y)\cos\overline{it-mx};$$

from which

$$Y = m\Big(\phi'(y)-\phi'(0)\Big)\cos\overline{it-mx}.$$

Therefore

$$\frac{d^2Y}{dt^2} = -i^2m\Big(\phi'(y)-\phi'(0)\Big)\cos\overline{it-mx};$$

$$-\int_y \frac{d^2Y}{dt^2}\,(y \text{ to } k) = +i^2m\Big(\phi(k)-\phi(y)-\phi'(0).\overline{k-y}\Big)\cos\overline{it-mx};$$

$$-gK = -gm\Big(\phi'(k)-\phi'(0)\Big)\cos\overline{it-mx}.$$

Adding to these terms $-\overline{k-y}.G.\cos\overline{it-mx}$, differentiating with regard to x, and putting for $\frac{d^2X}{dt^2}$ its value $-i^2.\phi''(y).\sin\overline{it-mx}$, our second equation becomes

$$\begin{aligned}-i^2.\phi''(y).\sin\overline{it-mx} = {} & H.\sin\overline{it-mx} \\ & -gm^2.\Big(\phi'(k)-\phi'(0)\Big).\sin\overline{it-mx} \\ & -m(k-y).G.\sin\overline{it-mx} \\ & +i^2m^2\Big(\phi(k)-\phi(y)-\phi'(0).\overline{k-y}\Big)\sin\overline{it-mx}.\end{aligned}$$

(276.) Equating to 0 the terms depending on y, we have

$$-i^2.\phi''(y) = -i^2m^2\phi(y); \quad 0 = -m(k-y)\,G - i^2m^2\phi'(0).\overline{k-y}.$$

The first gives $\phi(y)=\text{A}.\varepsilon^{my}+\text{B}.\varepsilon^{-my}$. From this we obtain $\phi'(y)=m\text{A}.\varepsilon^{my}-m\text{B}.\varepsilon^{-my}$; $\phi''(y)=m^2\text{A}.\varepsilon^{my}+m^2\text{B}.\varepsilon^{-my}$. The second equation gives $\text{G}+i^2m\phi'(0)=0$, or $\text{G}+i^2m^2\text{A}-i^2m^2\text{B}=0$, or $\text{B}-\text{A}=\frac{\text{G}}{i^2m^2}$; whence $\text{B}=\text{A}+\frac{\text{G}}{i^2m^2}$; $\phi(y)=\text{A}(\varepsilon^{my}+\varepsilon^{-my})+\frac{\text{G}}{i^2m^2}\varepsilon^{-my}$; $\phi'(y)=m\text{A}(\varepsilon^{my}-\varepsilon^{-my})-\frac{\text{G}}{i^2m}\varepsilon^{-my}$. Substituting in those parts of the equation which have not been already destroyed,

$$0=\text{H}-gm^2\left(m\text{A}(\varepsilon^{mk}-\varepsilon^{-mk})-\frac{\text{G}}{i^2m}\varepsilon^{-mk}+\frac{\text{G}}{i^2m}\right)+i^2m^2\left(\text{A}(\varepsilon^{mk}+\varepsilon^{-mk})+\frac{\text{G}}{i^2m^2}\varepsilon^{-mk}\right)$$

or
$$\text{A}\{-gm^3(\varepsilon^{mk}-\varepsilon^{-mk})+i^2m^2(\varepsilon^{mk}+\varepsilon^{-mk})\}=-\text{H}-\text{G}\left\{\frac{gm}{i^2}\varepsilon^{-mk}-\frac{gm}{i^2}+\varepsilon^{-mk}\right\}$$

whence
$$\text{A}=\frac{-\text{H}+\frac{gm}{i^2}\text{G}-\left(1+\frac{gm}{i^2}\right)\text{G}\varepsilon^{-mk}}{i^2m^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm^3(\varepsilon^{mk}-\varepsilon^{-mk})}$$

$$\text{B}=\text{A}+\frac{\text{G}}{i^2m^2}=\frac{-\text{H}+\frac{gm}{i^2}\text{G}+\left(1-\frac{gm}{i^2}\right)\text{G}\varepsilon^{mk}}{i^2m^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm^3(\varepsilon^{mk}-\varepsilon^{-mk})}$$

$$\phi''(y)=\frac{-\text{H}+\frac{gm}{i^2}\text{G}-\left(\frac{gm}{i^2}+1\right)\text{G}.\varepsilon^{-mk}}{i^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm(\varepsilon^{mk}-\varepsilon^{-mk})}\varepsilon^{my}+\frac{-\text{H}+\frac{gm}{i^2}\text{G}-\left(\frac{gm}{i^2}-1\right)\text{G}\varepsilon^{mk}}{i^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm(\varepsilon^{mk}-\varepsilon^{-mk})}\varepsilon^{-my}$$

$$m\phi'(y)=\frac{-\text{H}+\frac{gm}{i^2}\text{G}-\left(\frac{gm}{i^2}+1\right)\text{G}.\varepsilon^{-mk}}{i^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm(\varepsilon^{mk}-\varepsilon^{-mk})}\varepsilon^{my}-\frac{-\text{H}+\frac{gm}{i^2}\text{G}-\left(\frac{gm}{i^2}-1\right)\text{G}\varepsilon^{mk}}{i^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm(\varepsilon^{mk}-\varepsilon^{-mk})}\varepsilon^{-my}.$$

(277.) With these expressions forming the values of X and Y at the surface, we obtain the following:

Horizontal and vertical motions of the particles at the surface.

$$\text{X at the surface}=\frac{-\text{H}(\varepsilon^{mk}+\varepsilon^{-mk})+\text{G}.\frac{gm}{i^2}(\varepsilon^{mk}-2+\varepsilon^{-mk})}{i^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm(\varepsilon^{mk}-\varepsilon^{-mk})}\sin\overline{it-mx}$$

$$\text{Y at the surface, or K}=\frac{-\text{H}(\varepsilon^{mk}-\varepsilon^{-mk})+\text{G}(\varepsilon^{mk}-2+\varepsilon^{-mk})}{i^2(\varepsilon^{mk}+\varepsilon^{-mk})-gm(\varepsilon^{mk}-\varepsilon^{-mk})}\cos\overline{it-mx}.$$

Upon these expressions we may make the following remarks:

Distinction between *forced* wave and *free* wave.

(278.) 1st. The wave whose motions are defined by the values of $\text{X}=\phi''(y).\sin\overline{it-mx}$ and $\text{Y}=m\left(\phi'(y)-\phi'(0)\right)\cos\overline{it-mx}$, the particular values of which at the surface we have just exhibited, is that to which we have alluded (173.) by the name of the *forced* tide-wave. Its existence depends entirely on the continuance of the action of the external forces; if those forces ceased to act, the wave depending on $\sin\overline{it-mx}$ would immediately cease to exist. Other waves however would exist, as a consequence of these forces having been previously in action; and some (as we shall presently see) will at all times exist conjointly with the *forced* tide-wave. Thus, (as we shall see,) if the canal be interrupted, there will always exist a wave depending on $\sin\overline{it\pm m'x}$, where i and m' are connected by the equation $i^2=gm'\frac{\varepsilon^{m'k}-\varepsilon^{-m'k}}{\varepsilon^{m'k}+\varepsilon^{-m'k}}$; the magnitude and other circumstances of this wave will depend on the boundaries of the canal. This is what we have called the *free* tide-wave; its period is the same as that of the *forced* tide-wave, but its length is different; the continuance of the forces is not necessary for its existence. There may exist a wave depending on $\sin\overline{nt\pm mx}$, where n and m are connected by the equation $n^2=gm\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}$, and this would also be a *free* wave; its length is the same as that of the *forced* tide-wave, but its period is different, and for this reason we shall *not* call it a *free tide-wave;* it would not exist as a consequence of interruption of the canal, but as a consequence of sudden commencement or change or cessation of the forces, and its coefficient would depend on these circumstances (as we shall see hereafter).

(279.) 2nd. The expressions may be put into the following form:

$$\text{X at the surface}=\frac{-\text{H}+\text{G}.\frac{gm}{i^2}.\frac{\varepsilon^{mk}-2+\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}}{i^2-gm.\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}}\sin\overline{it-mx}$$

$$=\frac{-\text{H}+\text{G}.\frac{n^2}{i^2}.\frac{\varepsilon^{mk}-2+\varepsilon^{-mk}}{\varepsilon^{mk}-\varepsilon^{-mk}}}{i^2-n^2}\sin\overline{it-mx},$$

which, since $\varepsilon^{mk} - 2 + \varepsilon^{-mk} = \left(\varepsilon^{\frac{mk}{2}} - \varepsilon^{-\frac{mk}{2}}\right)^2$, and $\varepsilon^{mk} - \varepsilon^{-mk} = \left(\varepsilon^{\frac{mk}{2}} + \varepsilon^{-\frac{mk}{2}}\right) . \left(\varepsilon^{\frac{mk}{2}} - \varepsilon^{-\frac{mk}{2}}\right)$, becomes

$$\text{X at the surface} = \frac{-\text{H} + \text{G}\dfrac{n^2}{i^2} . \dfrac{\varepsilon^{\frac{mk}{2}} - \varepsilon^{-\frac{mk}{2}}}{\varepsilon^{\frac{mk}{2}} + \varepsilon^{-\frac{mk}{2}}}}{i^2 - n^2} \sin \overline{it - mx}.$$

Similarly

$$\text{K} = \frac{-\text{H} . \dfrac{n^2}{gm} + \text{G} . \dfrac{n^2}{gm} . \dfrac{\varepsilon^{mk} - 2 + \varepsilon^{-mk}}{\varepsilon^{mk} - \varepsilon^{-mk}}}{i^2 - n^2} \cos \overline{it - mx}$$

$$= \frac{-\text{H} . \dfrac{n^2}{gm} + \text{G}\dfrac{n^2}{gm} . \dfrac{\varepsilon^{\frac{mk}{2}} - \varepsilon^{-\frac{mk}{2}}}{\varepsilon^{\frac{mk}{2}} + \varepsilon^{-\frac{mk}{2}}}}{i^2 - n^2} \cos \overline{it - mx}.$$

Now mk is a small quantity. For, putting λ for the length of the forced tide-wave, (which at the equator $= 12500$ miles), $m = \frac{2\pi}{\lambda}$, and $mk = 2\pi\frac{k}{\lambda}$, so that, even if the depth of the sea were 25 miles, mk would be $= \frac{2\pi}{500} = \frac{6 \cdot 28}{500}$. Therefore $\varepsilon^{\frac{mk}{2}} - \varepsilon^{-\frac{mk}{2}} = mk$ very nearly, and $\varepsilon^{\frac{mk}{2}} + \varepsilon^{-\frac{mk}{2}} = 2$ very nearly, and therefore the expressions become

$$\text{X at the surface} = \frac{-\text{H} + \text{G}\dfrac{n^2}{i^2} . \dfrac{mk}{2}}{i^2 - n^2} \sin \overline{it - mx} ;$$

$$\text{K} = \frac{-\text{H}\dfrac{n^2}{gm} + \text{G}\dfrac{n^2}{gm} . \dfrac{mk}{2}}{i^2 - n^2} \cos \overline{it - mx}.$$

Effect of the vertical force is insignificant.

Consequently, if we consider G and H to be quantities not very dissimilar in magnitude, (which we shall find to be true,) the term depending on G in each of these expressions is wholly insignificant in comparison with that depending on H; and thus we arrive at this remarkable conclusion, that *the effect of the vertical disturbing force upon the phænomena of the tides is insignificant*, the whole of the sensible effect being due to the horizontal force. In (28.) and (69.) we have found the same result from different ways of treating the tides. There is no exception to this theorem as applied to the expression for K; if i were very small, it might not hold strictly with regard to the expression for X; but that is of small importance.

If the period of the forced wave were equal to the period of a free wave, the motion would be infinite.

(280.) 3rd. If the period of the *forced* tide-wave were the same as that of a *free* wave of the second class, that is, if i were equal to n, m being the same in the two waves, the denominator in the expressions for X and K would vanish, or X and K would become infinite. In this case then the expressions fail. It would seem that, if we suppose a canal surrounding the earth to be acted on by the forces of the sun and moon following the law assumed in this Problem, and if i were equal to n, the only interpretation that can be put on this failure would be,—that the motion of the water cannot be oscillatory in the manner of a wave, but that it must be that of a torrent of unequal depth passing round the earth so as to follow the apparent motion of the sun or moon. We shall find, however, that the introduction of friction prevents the expression from ever becoming infinite.

If the period of the forced wave is smaller, there is low water under the Moon.

(281.) 4th. If the period of the forced tide-wave be less than that of a free wave of the second class, (that is, if the wave be urged along more rapidly than it would go alone, or if the water be shallow,) or if i be greater than n, (i and n being, by (166.) inversely as the periods,) the denominator is positive; and the expression for K (omitting the term depending on G) is of the form $-\text{C} \cos \overline{it - mx}$. Now conceive the Moon to go round the Earth (apparently) from east to west; x must be measured in the same direction; because then, by proper determination of the value of m, the parts of the Earth which happen to be under the Moon will always have the same certain value of $\overline{it - mx}$. Conceive the horizontal force created by the Moon's action to be 0 for the parts under the Moon; to be positive at the parts, to a certain extent, which she has passed, and to be negative at the parts, to a certain extent, on the side to which she is approaching. (This is supposing, in other words, that to a certain extent the horizontal force caused by the Moon's action urges the water towards that part of the Earth which is under the Moon; a supposition which correctly represents the actual case.) Then $\overline{it - mx}$ must be 0 for the parts under the Moon. (For then, in the parts which the Moon has left, x is less than under the Moon, $\overline{it - mx}$ is positive, and therefore the force, which we have assumed to be H $\sin \overline{it - mx}$, is positive there; in the parts to which the Moon is approaching, x is greater than under the Moon, $\overline{it - mx}$ is negative, and therefore the force or H $\sin \overline{it - mx}$ is negative there.) Consequently the elevation of the water under the Moon is $-\text{C} . \cos 0 = -\text{C}$. That is, it is low water immediately under the Moon. The same conclusion will be found in (16.) and (109.).

(282.) 5th. In like manner, if the period of the forced tide-wave were greater than that of the free wave of the second class, (that is, if the water were so immensely deep that the wave would travel alone more rapidly than the disturbing forces urge it along,) or if i were less than n, the denominator would be negative, and there would be high water under the Moon.

If the period is very long, the form of the water is that of equilibrium nearly.

(283.) 6th. If the complete change of forces occupy a long time, that is if i be small, $\mathrm{K} = +\frac{\mathrm{H}}{gm}\cos\overline{it-mx}$ nearly. This is the same expression that would have been found from a theory of equilibrium, or by (152.); so that in this case the water assumes very nearly the form of equilibrium.

(284.) For shallow water, it will be remarked that $n^2 = gkm^2$ nearly, and the whole of the expressions may be altered by that substitution. And, as in (166.) and (165.), we may put $\frac{2\pi}{\tau'}$ for i, and $\frac{2\pi}{\lambda}$ for m: then τ' is the period of the changes of the *force*, which is the same as the period of the *forced wave*, and λ is the length of the wave. The expression for K, omitting the term depending on G, will thus be changed to

$$\frac{-\mathrm{H}.km}{i^2-gkm^2}\cos\overline{it-mx}, \text{ or } -\frac{\mathrm{H}k}{2\pi}.\frac{\lambda.\tau'^2}{\lambda^2-gk.\tau'^2}\cos\overline{it-mx}:$$

and, if the depth be *very* small,

$$\mathrm{K} = -\frac{\mathrm{H}k}{2\pi}.\frac{\tau'^2}{\lambda}.\cos\overline{it-mx}$$

In showing that in this case the elevation of the tide is proportional to the depth of the sea, we have obtained the same result as that deduced from Laplace's Theory of Tides, in (98.), (101.), and (107.).

(285.) The preceding conclusions are very important, as showing that the amount of elevation of the water under the action of forces depends in a most remarkable degree upon other circumstances than the magnitude of the forces. One is, the depth of the sea: another is, the periodic time of the forces. As depending upon the former, it appears that, if there were two parallel canals of different depths acted on by precisely the same forces, there might be high water in one when there was low water in the adjacent part of the other: or there might be elevations and depressions at the same time in both, but their magnitudes might have any proportion whatever. As depending upon the latter, it appears that, if there were two forces acting simultaneously upon the water in the same canal, the periods of those forces being different, (as, for instance, the forces depending upon the action of the Sun and the Moon,) the high water produced by one force might bear the same relation to the phases of that force which the low water produced by the other bears to the phases of that other force: (thus low water of the solar tide might accompany the transit of the Sun, and high water of the lunar tide might accompany the transit of the Moon, in the same canal.) Or the phases of the two tides might stand in the same relation to the phases of the two forces, but the proportion of their magnitudes to the magnitudes of the forces might differ in any degree whatever.

Coefficient of force supposed variable.

(286.) The conclusions of (93.), as to the effect of a variable coefficient of the force, apply here: but as we are able to give expressions here which could not be given in those investigations, we shall repeat a small part of the process. If the force, instead of being $\mathrm{H}.\sin\overline{it-mx}$, were $(\mathrm{H}+\mathrm{H}'\cos i't).\sin\overline{it-mx}$ or $\mathrm{H}.\sin\overline{it-mx} + \frac{\mathrm{H}'}{2}\sin\overline{(i+i')\,t-mx} + \frac{\mathrm{H}'}{2}\sin\overline{(i-i')\,t-mx}$, the expression for K would be (supposing the water shallow),

$$\frac{-\mathrm{H}km}{i^2-gkm^2}\cos\overline{it-mx} - \frac{1}{2}.\frac{\mathrm{H}'km}{(i+i')^2-gkm^2}\cos\overline{(i+i')\,t-mx} - \frac{1}{2}.\frac{\mathrm{H}'km}{(i-i')^2-gkm^2}\cos\overline{(i-i')\,t-mx};$$

and if i' were very small, that is, if the change of the variable coefficient were slow, we might expand to the first power of i', and it would become

$$\frac{-\mathrm{H}km}{i^2-gkm^2}\cos\overline{it-mx} - \frac{1}{2}\,\frac{\mathrm{H}'km}{i^2-gkm^2}\left\{\cos\overline{(i+i')\,t-mx} + \cos\overline{(i-i')\,t-mx}\right\}$$

$$+\frac{1}{2}.\frac{\mathrm{H}'km.2ii'}{(i^2-gkm^2)^2}\left\{\cos\overline{(i+i')\,t-mx} - \cos\overline{(i-i')\,t-mx}\right\}$$

$$= -\frac{km}{i^2-gkm^2}(\mathrm{H}+\mathrm{H}'\cos i't).\cos\overline{it-mx} - \frac{2kmii'}{(i^2-gkm^2)^2}\mathrm{H}'.\sin i't.\sin(it-mx).$$

(287.) The time of low water on a given day (supposing the variation of $i't$ in the day to be insensible) will be found by making

$$\tan(it-mx) = \frac{2ii'}{i^2-gkm^2}.\frac{\mathrm{H}'\sin i't}{\mathrm{H}+\mathrm{H}'\cos i't};$$

and, therefore, when $\sin i't$ is positive, it will be later than that determined by making $it-mx=0$, as in (281.).

(288.) The greatest amount of elevation or depression on that day, as in (93.), will be

$$\frac{km}{i^2-gkm^2}\sqrt{\left\{(\mathrm{H}+\mathrm{H}'\cos i't)^2 + \frac{4i^2i'^2}{(i^2-gkm^2)^2}\mathrm{H}'^2\sin^2 i't\right\}}.$$

If H′ be much smaller than H, this expression $=\frac{km}{i^2-gkm^2}(\mathrm{H}+\mathrm{H}'\cos i't)$ nearly.

(289.) If the term H do not exist, the whole force being $=\mathrm{H}'\cos i't\,.\sin\overline{it-mx}$, the expression for greatest elevation or depression on any day $=$

$$\frac{\mathrm{H}'km}{i^2-gkm^2}\sqrt{\left\{\cos^2 i't+\frac{4i^2i'^2}{(i^2-gkm^2)^2}\sin^2 i't\right\}}.$$

This quantity never vanishes, even on the days on which $\cos i't$ vanishes: but it is, on those days, either maximum or minimum. It is maximum if $2ii'$ be greater than i^2-gkm^2, and minimum if $2ii'$ be less than i^2-gkm^2. In the former case, however, it will be necessary to use for the coefficients of $\cos i't\,.\cos\overline{it-mx}$ and $\sin i't\,.\sin\overline{it-mx}$ their accurate values

$$\frac{-\mathrm{H}'km}{2}\left(\frac{1}{(i+i')^2-gkm^2}+\frac{1}{(i-i')^2-gkm^2}\right)\text{ and }\frac{-\mathrm{H}'km}{2}\left(\frac{1}{(i-i')^2-gkm^2}-\frac{1}{(i+i')^2-gkm^2}\right).$$

Subsection 5.—*On the Method of using the free Tide-Waves and forced Tide-Waves, and of introducing the limits of Canals in general.*

(290.) Before we exhibit the method of introducing the *free tide-wave* and other *free* waves, which we have mentioned as coexisting with the *forced tide-wave*, we will shortly remark on the general solution of the partial differential equation of waves. The same remarks apply to all other linear partial differential equations.

Solution which represents the forced tide-wave may be made more general by adding any number of expressions for free tide-waves.

(291.) We have found a solution X for the equation $\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\right\}$: where F is a given function of x and t. Now suppose that we know that each of the quantities X_1, X_2, X_3, &c. will satisfy the equation $\frac{d^2\mathrm{X}'}{dt^2}=0+\frac{d}{dx}\left\{-g\mathrm{K}'+\int_y\frac{d^2\mathrm{Y}'}{dt^2}\right\}$: the peculiarity of which, as distinguished from the former, is, that it contains no term independent of X′ $\left(\text{K}'\right.$ being a particular value of Y′, and Y′ being formed from X′ by integrating $\left.-\int_y\frac{d\mathrm{X}'}{dx}\right)$. Then the sum of all the quantities $\mathrm{X}+\mathrm{X}_1+\mathrm{X}_2+$&c. will satisfy the equation $\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\right\}$. For, upon substituting $\mathrm{X}+\mathrm{X}_1+\mathrm{X}_2+$&c., in this last equation, instead of X, we have

$$\frac{d^2(\mathrm{X}+\mathrm{X}_1+\mathrm{X}_2+\&\text{c.})}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g(\mathrm{K}+\mathrm{K}_1+\mathrm{K}_2+\&\text{c.})+\int_y\frac{d^2(\mathrm{Y}+\mathrm{Y}_1+\mathrm{Y}_2+\&\text{c.})}{dt^2}\right\};$$

which is necessarily true, because it is the same as the sum of all the equations

$$\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\right\};\quad\frac{d^2\mathrm{X}_1}{dt^2}=0+\frac{d}{dx}\left\{-g\mathrm{K}_1+\int_y\frac{d^2\mathrm{Y}_1}{dt^2}\right\},\ \&\text{c.}:$$

each of which is separately true. Therefore $\mathrm{X}+\mathrm{X}_1+\mathrm{X}_2+$&c. may be used as a solution of the equation $\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\right\}$: and it is more general than X merely, and therefore can better be adapted to the circumstances of the case.

(292.) In the problem of (274.), &c., we have found $\mathrm{C}\,.\cos\overline{it-mx}$* for a solution of the equation $\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\right\}$. But we know that $\mathrm{A}_1\cos(n_1t-m_1x+a_1)$, $\mathrm{B}_1\cos(n_1t+m_1x+b_1)$, $\mathrm{A}_2\cos(n_2t-m_2x+a_2)$, $\mathrm{B}_2\cos(n_2t+m_2x+b_2)$, &c. are solutions of the equation $\frac{d^2\mathrm{X}'}{dt^2}=\frac{d}{dx}\left\{-g\mathrm{K}'+\int_y\frac{d^2\mathrm{Y}'}{dt^2}\right\}$, provided only that $n_1^2=gm_1\frac{\varepsilon^{m_1k}-\varepsilon^{-m_1k}}{\varepsilon^{m_1k}+\varepsilon^{-m_1k}}$, &c. Therefore we may add any or all of these terms to $\mathrm{C}\,.\cos\overline{it-mx}$ for the solution of $\frac{d^2\mathrm{X}}{dt^2}=\mathrm{F}+\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\right\}$. We shall have occasion only to add terms in which $n_1=i$, or terms in which $m_1=m$. The nature of the conditions to be satisfied will determine our choice. If for some given instant of time a certain condition is to apply to all parts of the canal, the coefficient of x must be the same as in $\mathrm{C}\,.\cos\overline{it-mx}$; that is, m_1 must $=m$. But if at some certain part of the canal a certain

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* We use indifferently the sine or the cosine, as one is changed into the other by changing the origin of x or t.

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condition is to hold during all time, the coefficient of t must be the same as in $C.\cos\overline{it-mr}$, or n_1 must $=i$. The following instances will illustrate this.

(293.) Problem.—The disturbing force of the Moon, represented by the formula $H.\cos\overline{it-mx}$, begins to act suddenly when $t=0$: to find the nature of the waves produced on a canal: the water having been previously at rest.

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(294.) The value for X must be $C.\cos\overline{it-mx}+X_1+X_2+\&c.$: and the condition that the water was in its position of rest in every part of the fluid, and also was actually at rest, when $t=0$, requires that

$$C.\cos\overline{it-mx}+X_1+X_2+\&c.=0 \text{ when } t=0 \text{ for every value of } x;$$

and that
$$\frac{d}{dt}(C.\cos\overline{it-mx}+X_1+X_2+\&c.)=0 \text{ when } t=0 \text{ for every value of } x.$$

The only forms of X_1 and X_2 which can satisfy these conditions are

$$A.\cos(nt-mx+\alpha) \text{ and } B.\cos(nt+mx+\beta),$$

where $n^2=gm\dfrac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}$. The conditions now become

$$C\cos mx+A\cos(-mx+\alpha)+B\cos(mx+\beta)=0.$$
$$iC\sin mx-nA\sin(-mx+\alpha)-nB\sin(mx+\beta)=0.$$

Force begins to act at a certain instant.

Expanding the sines and cosines, we obtain $C+A\cos\alpha+B\cos\beta=0$; $A\sin\alpha-B\sin\beta=0$; $iC+nA\cos\alpha-nB\cos\beta=0$; $-nA\sin\alpha-nB\sin\beta=0$. The second and fourth equations give $A\sin\alpha=0$, $B\sin\beta=0$; the first and third give $A\cos\alpha=-\dfrac{(i+n)C}{2n}$, $B\cos\beta=\dfrac{(i-n)C}{2n}$. Hence $X_1=-\dfrac{(i+n)C}{2n}\cos\overline{nt-mx}$, $X_2=\dfrac{(i-n)C}{2n}\cos\overline{nt+mx}$: and the whole expression for X is

$$C.\cos\overline{it-mx}-\frac{(i+n)C}{2n}\cos\overline{nt-mx}+\frac{(i-n)C}{2n}\cos\overline{nt+mx}.$$

(295.) This instance is one of pure curiosity, given only as an example. The following, probably, is one of real application.

Canal bounded at both ends.

Problem.—A canal, the water of which is acted on by the forces of the moon's attraction, is bounded at both extremities: to find the nature of the waves produced on it.

(296.) The condition here is, that, at each extremity, X shall $=0$, and $\dfrac{dX}{dt}$ shall $=0$, whatever be the value of t. The second equation is evidently included in the first; so that, in fact, we have only one condition for each end of the canal. A moment's consideration will show that, the term for the forced tide-wave being $C.\cos\overline{it-mx}$, the other terms must have the same multiple of t. Let the value of X, therefore, at the surface be assumed $=$

$$C.\cos\overline{it-mx}\;+A.\cos\overline{it-lx+\alpha}\;+B.\cos\overline{it+lx+\beta},$$

where $i^2=gl\dfrac{\varepsilon^{lk}-\varepsilon^{-lk}}{\varepsilon^{lk}+\varepsilon^{-lk}}$. Let the values of x at the two extremities of the canal be a and b. Then, at one end of the canal, $X=C.\cos\overline{it-ma}+A.\cos\overline{it-la+\alpha}+B.\cos\overline{it+la+\beta}$; which is to be $=0$ for all values of t. At the other end, $X=C.\cos\overline{it-mb}+A.\cos\overline{it-lb+\alpha}+B.\cos\overline{it+lb+\beta}$, which also is to be $=0$ for all values of t. Expanding, and separating the coefficients of $\cos it$ and $\sin it$, we have these four equations:

$$C.\cos ma+A.\cos\overline{\alpha-la}+B.\cos\overline{\beta+la}=0;$$
$$C.\sin ma-A.\sin\overline{\alpha-la}-B.\sin\overline{\beta+la}=0;$$
$$C.\cos mb+A.\cos\overline{\alpha-lb}+B.\cos\overline{\beta+lb}=0;$$
$$C.\sin mb-A.\sin\overline{\alpha-lb}-B.\sin\overline{\beta+lb}=0.$$

Without here going through the details of the solution, it is plain that the conditions are satisfied by the following formula:

$$X=C.\cos\overline{it-mx}-\frac{C}{\sin l(b-a)}\left\{\cos\overline{it-mb}.\sin\overline{lx-la}-\cos\overline{it-ma}.\sin\overline{lx-lb}\right\};$$

which, upon multiplying the sines and cosines together, according to the rules of Trigonometry, may be put into the form $C.\cos\overline{it-mx}+A.\cos\overline{it-lx+\alpha}+B.\cos\overline{it+lx+\beta}$, and which becomes 0 when $x=a$ or $=b$. The first term of the expression is the *forced* tide-wave; the part within the brackets represents the *free* tide-waves, which may be conceived as the combination of the reflections of the forced tide-wave from the ends of the canal, and whose speed does not depend on the speed of the Moon's change of place.

(297.) It is proper to remark, that the condition $X=0$ cannot be strictly satisfied at all depths. For in the case of the forced tide-wave the value of X at different depths depends on $\varepsilon^{my}+\varepsilon^{-my}$, and in the case of the free tide-wave it depends on $\varepsilon^{ly}+\varepsilon^{-ly}$; and these are not exactly in the same proportion at different depths. It would seem, therefore, that there must, theoretically, be a breaking of water on the coast, in consequence of the oscillation of the tide. But when the wave is of very long period, (as is true of the tide-wave,) both $\varepsilon^{ly}+\varepsilon^{-ly}$ and $\varepsilon^{my}+\varepsilon^{-my}$ are at all depths sensibly $=2$, and, therefore, the condition $X=0$ may be considered as satisfied at all depths.

(298.) The corresponding value of K, or $-\int_y \frac{dX}{dx}(0 \text{ to } k)$, $\Big($observing that $\int_y (\varepsilon^{my}+\varepsilon^{-my}) = \frac{1}{m}(\varepsilon^{my}-\varepsilon^{-my}) = \frac{2my}{m}$ nearly $= y \times (\varepsilon^{my}+\varepsilon^{-my})$ nearly, and that the same holds when l is put for m,$\Big)$ is

$$-Cmk.\sin\overline{it-mx}+\frac{Clk}{\sin\,(lb-la)}\{\cos\overline{it-mb}.\cos\overline{lx-la}-\cos\overline{it-ma}.\cos\overline{lx-lb}\}.$$

In this case the disturbance may become infinite.

It may be interesting to examine the value of this expression in some particular cases.

(299.) 1st. If $\sin\overline{lb-la}=0$, that is, if $\overline{lb-la}=\pi$, or 2π, or 3π, &c., or if the length of the canal is any multiple of half the length of the free tide-wave, this expression becomes infinite. In reality the wave will become so large that the amount of friction, &c. will be so great as to neutralize the moon's force. This will be seen more clearly in the next section.

Tides in a canal of small extent.

(300.) 2d. If $\overline{lb-la}$ be small, that is, if the canal be short, we may proceed by approximation. Suppose a and b to be small quantities: x must also be a small quantity of the same order. Expanding $\sin\overline{it-mx}$, $\sin\overline{lx-la}$, &c., and substituting for the sines and cosines of the small arcs the series in terms of the arc, and preserving only the first power which remains in the resulting expressions, we find the following:

$$X=\cos it.C.\frac{m^2-l^2}{2}.\overline{x-a}.\overline{b-x}$$

$$K=\cos it.Ck(m^2-l^2).\left(x-\frac{a+b}{2}\right).$$

The first of these expressions shows that the horizontal motion will be greatest in the middle of the canal's length, and will diminish gradually both ways to the ends, where it is 0. The second shows that there is no variation of level at the middle of the canal's length, but that the variation of level in other parts is proportional to the distance from the middle, elevation taking place on one side of the middle at the same time as depression on the other side, so that the surface of the fluid remains sensibly plane, though inclined to the horizon. The law of motion as regards the time is the usual oscillatory law expressed by $\cos it$; but the motion of every particle differs in this respect from the motion of particles in an open sea affected by the tide: that here, the greatest horizontal displacement happens at the same time as the greatest vertical displacement; whereas, in an open sea, the greatest horizontal displacement happens when the vertical displacement is 0, and *vice versâ*. (182.) and (184.).

Consist simply of oscillations from one end to the other.

Solution in general case when one term is more important than the others.

(301.) 3d. In the general case, if we take the centre of the canal for the origin of our measures, or make $a=-b$, we have

$$X=C\cos it\left\{\cos mx-\frac{\cos mb}{\cos lb}\cos lx\right\}+C\sin it\left\{\sin mx-\frac{\sin mb}{\sin lb}\sin lx\right\}$$

$$K=Ck\cos it\left\{m.\sin mx-\frac{l.\cos mb}{\cos lb}\sin lx\right\}+Ck\sin it\left\{-m\cos mx+\frac{l\sin mb}{\sin lb}\cos lx\right\}.$$

In general, no simple meaning can be given to these expressions. In the particular case of $\cos lb$ being very small, the principal term of K is $-\cos it.\frac{Ckl\cos mb}{\cos lb}\sin lx$, which shows that, though there may be many oscillating elevations and depressions, yet all the waves will be of a stationary character in regard of space; that all the oscillations take place at the same time; and that an elevation on one side of the middle happens at the same instant as a depression at the same distance on the other side of the middle. If $\sin lb$ is very small, then the principal term in the expression for K is $\sin it.\frac{Ckl\sin mb}{\sin lb}\cos lx$: in this case the waves are stationary in place, and all the oscillations happen at the same time: but the elevations or depressions at equal distances from the middle towards both ends are similar.

(302.) Problem.—A canal, whose waters are acted on by the forces of the Moon, &c., communicates at one end with an open sea, whose waters have a tidal fluctuation, and is closed at the other end: to find the circumstances of tidal motion of the waters in the canal.

Tides in a deep gulf.

(303.) Let the mouth of the canal be the origin of the measures, its length being a; let the elevation of the sea-waters at the mouth of the canal be expressed by $A.\sin\overline{nt+B}$. Then the conditions to which our expressions for X and K must be subject, are these: that when $x=0$, K must $=A.\sin\overline{nt+B}$, and when $x=a$, X must

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$=0$ for all values of t. Suppose, then, that the value of X given by the solution of the differential equations is $\mathrm{C}.\cos\overline{nt-px}$. The complete value of X, as before (296.), will be

$$\mathrm{C}.\cos\overline{nt-px}+\mathrm{D}.\cos\overline{nt-mx+\mathrm{E}}+\mathrm{F}.\cos\overline{nt+mx+\mathrm{G}},$$

where $n^2=gm\dfrac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}$. And the complete value of K, as in (298.), will be

$$-\mathrm{C}pk.\sin\overline{nt-px}-\mathrm{D}mk.\sin\overline{nt-mx+\mathrm{E}}+\mathrm{F}mk.\sin\overline{nt+mx+\mathrm{G}}.$$

Our two conditions, therefore, become the following:

$$-\mathrm{C}pk.\sin nt-\mathrm{D}mk.\sin\overline{nt+\mathrm{E}}+\mathrm{F}mk.\sin\overline{nt+\mathrm{G}}=\mathrm{A}.\sin\overline{nt+\mathrm{B}}$$

$$\mathrm{C}.\cos\overline{nt-pa}+\mathrm{D}.\cos\overline{nt-ma+\mathrm{E}}+\mathrm{F}.\cos\overline{nt+ma+\mathrm{G}}=0.$$

Expanding the sines and cosines, and equating the coefficients of $\sin nt$ and $\cos nt$, we obtain the following equations for the determination of D sin E, D cos E, F sin G, and F cos G:

$$-\mathrm{C}pk-mk.\mathrm{D}\cos\mathrm{E}+mk.\mathrm{F}\cos\mathrm{G}=\mathrm{A}\cos\mathrm{B}$$

$$-mk.\mathrm{D}\sin\mathrm{E}+mk.\mathrm{F}\sin\mathrm{G}=\mathrm{A}\sin\mathrm{B}$$

$$\mathrm{C}\cos pa+\cos ma.\mathrm{D}\cos\mathrm{E}+\sin ma.\mathrm{D}\sin\mathrm{E}+\cos ma.\mathrm{F}\cos\mathrm{G}-\sin ma.\mathrm{F}\sin\mathrm{G}=0$$

$$\mathrm{C}\sin pa-\cos ma.\mathrm{D}\sin\mathrm{E}+\sin ma.\mathrm{D}\cos\mathrm{E}-\cos ma.\mathrm{F}\sin\mathrm{G}-\sin ma.\mathrm{F}\cos\mathrm{G}=0.$$

Solving these equations, and substituting the values so found in the expression for X, we find at length

$$\mathrm{X}=\frac{\mathrm{A}}{mk\cos ma}\sin\overline{nt+\mathrm{B}}.\sin\overline{ma-mx}+\mathrm{C}\cos\overline{nt-px}+\frac{\mathrm{C}p}{m\cos ma}\sin nt.\sin\overline{ma-mx}-\frac{\mathrm{C}}{\cos ma}\cos\overline{nt-pa}.\cos mx;$$

from which

$$\mathrm{K}=\frac{\mathrm{A}}{\cos ma}\sin\overline{nt+\mathrm{B}}.\cos\overline{ma-mx}-\mathrm{C}pk.\sin\overline{nt-px}+\frac{\mathrm{C}pk}{\cos ma}\sin nt.\cos\overline{ma-mx}-\frac{\mathrm{C}mk}{\cos ma}\cos\overline{nt-pa}.\sin mx;$$

which, it is easily seen, satisfy our equations of condition. The following considerations suggest themselves from the examination of this expression.

(304.) 1st. In a river or sea, such as we have supposed, the expression for the elevation of the tide is by no means a simple one: it may be represented by the combination of three stationary waves (of the nature of free waves) and one progressive wave (a forced tide-wave), whose compounded effects cannot easily be represented by those of a single wave, even though considered changeable.

(305.) 2nd. If we put the expression for K under the form $\mathrm{P}\sin nt-\mathrm{Q}\cos nt$, and make $\dfrac{\mathrm{Q}}{\mathrm{P}}=\tan\theta$, then $\mathrm{K}=\sqrt{\mathrm{P}^2+\mathrm{Q}^2}.\sin(nt-\theta)$, and the time of high water will be determined by making $nt=\theta+90°$, or $\tan nt=-\cot\theta=-\dfrac{\mathrm{P}}{\mathrm{Q}}$; and the whole rise of tide will be $2\sqrt{\mathrm{P}^2+\mathrm{Q}^2}$. In the case before us,

$$\mathrm{P}=\frac{\mathrm{A}.\cos\mathrm{B}.\cos\overline{ma-mx}}{\cos ma}-\mathrm{C}pk.\cos px+\frac{\mathrm{C}pk.\cos\overline{ma-mx}}{\cos ma}-\frac{\mathrm{C}mk.\sin pa.\sin mx}{\cos ma}$$

$$-\mathrm{Q}=\frac{\mathrm{A}.\sin\mathrm{B}.\cos\overline{ma-mx}}{\cos ma}+\mathrm{C}pk.\sin px-\frac{\mathrm{C}mk.\cos pa.\sin mx}{\cos ma}.$$

The expression, therefore, for nt at high water, as depending on x, will be one of great complexity; and that for $\dfrac{d\theta}{dx}$, on which the apparent velocity of the tide-wave will depend, will be complicated. It does not appear that in the general case we have any reason to think that the tide-wave will appear to travel always in the same direction.

(306.) 3rd. If the tidal oscillations of the sea be insensible, or $\mathrm{A}=0$, the remaining terms still have the same kind of complexity as in the last problem. But if $\cos ma$ be also very small, then the expression may be reduced to a single term multiplying $\sin mx$; and the tidal oscillation will consist of a stationary wave, or a series of stationary waves, in all which the oscillations take place simultaneously, and in which the horizontal motion of the particles and the vertical motion are greatest at the same time, or vanish at the same time.

Tide in a river stopped by a barrier.

(307.) 4th. If the effect of the moon's forces on the waters of the canal is insensible, then $\mathrm{C}=0$, and the value of K is reduced to $\dfrac{\mathrm{A}}{\cos ma}.\sin\overline{nt+\mathrm{B}}.\cos\overline{ma-mx}$. This shows that all the oscillations in different parts of the canal take place at the same time; or, if ma is less than 90°, that it is high water in every part of the canal at the same time. But the whole rise of the waters $=\dfrac{2\mathrm{A}}{\cos ma}\cos\overline{ma-mx}$ is different in different parts of the canal; and, if ma is less than 90°, it increases from the mouth of the canal to its termination. The corresponding value of X is $\frac{\mathrm{A}}{mk\cos ma}.\sin\overline{nt+\mathrm{B}}.\sin\overline{ma-mx}$: this shows that the greatest horizontal displacement of the particles occurs at the same time as the greatest vertical displacement; and the whole horizontal motion, or $\frac{2\mathrm{A}}{mk\cos ma}\sin\overline{ma-mx}$ diminisnes from the mouth of the canal to its termination.

(308.) As this investigation applies in some degree to the case of various gulfs and rivers, it may be proper to examine how far it is modified by taking into account the extent of the fluctuation.

Problem.—A canal is closed at one end and open at the other end to a tidal sea: to find the nature of the tide-wave in the canal, the elevation of the water being supposed to bear a sensible proportion to the whole depth.

Tide in a river stopped by a barrier, when the motion of the particles is large.

(309.) The equation, as in (196.), will be

$$\frac{d^2\mathrm{X}}{dt^2}-v^2\frac{d^2\mathrm{X}}{dx^2}=v^2\frac{d^2\mathrm{X}}{dx^2}\times\left\{-3\frac{d\mathrm{X}}{dx}+6\left(\frac{d\mathrm{X}}{dx}\right)^2-\&\mathrm{c.}\right\},$$

where $v^2=gk$. An approximate value of X is

$$\frac{\mathrm{A}}{mk\cos ma}.\sin\overline{mvt+\mathrm{B}}.\sin\overline{ma-mx}.$$

Substituting this value in the second side of the equation, as far as the term $\frac{d\mathrm{X}}{dx}$, and proceeding in the solution exactly in the same manner as in (198.) &c., we find the following expression for X:

$$\begin{aligned}\mathrm{X}=&\frac{\mathrm{A}}{mk\cos ma}.\sin\overline{ma-mx}.\sin\overline{mvt+\mathrm{B}}\\&+\frac{\mathrm{A}^2}{k^2\cos^2 ma}\left\{-\frac{3}{16m}\sin\overline{2ma-2mx}-\frac{3}{16}x.\cos\overline{2ma-2mx}.\cos\overline{2mvt+2\mathrm{B}}\right\}\\&+\phi(vt+x)+\psi(vt-x);\end{aligned}$$

from which we find the elevation of the water, at the place whose distance from the mouth of the canal, measured along the canal-bank, is $=x'$, to be represented by the following expression:

$$\begin{aligned}&\frac{\mathrm{A}}{\cos ma}.\cos\overline{ma-mx'}.\sin\overline{mvt+\mathrm{B}}\\&+\frac{1}{8}.\frac{\mathrm{A}^2}{k\cos^2 ma}\left(\cos\overline{2ma-2mx'}-\cos 2ma\right)+\frac{3}{8}.\frac{\mathrm{A}^2}{k\cos^2 ma}.mx'.\sin\overline{2ma-2mx'}.\cos\overline{2mvt+2\mathrm{B}}.\end{aligned}$$

In this case there are sometimes two high waters at every tide.

The law of the rise and fall of the water, at every part of the canal except its mouth, is now different from that which holds on the supposition that the oscillation is small in proportion to the depth of the canal. But the times of high water and of low water are still the same as before, and the high water and the low water are still simultaneous through the whole length of the canal. The second term of the expression above is constant for any given place; the first and third may be represented by $\mathrm{C}.\sin\overline{mvt+\mathrm{B}}+\mathrm{D}.\cos\overline{2mvt+2\mathrm{B}}$. In figures 16 and 17 we have represented these expressions by the ordinates of two curves: in fig. 16, $\frac{\mathrm{D}}{\mathrm{C}}=\frac{1}{6}$; in fig. 17, $\frac{\mathrm{D}}{\mathrm{C}}=\frac{1}{3}$. In the latter there are two high waters at each tide.

(310.) Problem.—A canal, on whose waters the action of the moon is insensible, communicates at both ends with tidal seas, in which the whole rise of the tide and the corresponding phases of the tide are different: to find the nature of the waves upon it.

Canal between two tidal seas.

(311.) The elevation of the water in the canal will evidently be represented by a combination of *free* tide-waves, of the form $\mathrm{P}.\sin\overline{nt-mx+p}+\mathrm{Q}.\sin\overline{nt+mx+q}$, or of a form equivalent to this; and the constants must be so determined as to make the elevations at the two extremities of the canal the same as those of the seas. Let a and b be the ordinates of the two extremities of the canal; $\mathrm{A}.\sin(nt+\alpha)$ the elevation of the water at one extremity, $\mathrm{B}.\sin(nt+\beta)$ that at the other extremity. Then, without going through the whole investigation, it is evident that all the conditions are satisfied by the following expression:

$$\mathrm{K}=\frac{\mathrm{A}}{\sin\overline{mb-ma}}.\sin\overline{mb-mx}.\sin\overline{nt+\alpha}+\frac{\mathrm{B}}{\sin\overline{mb-ma}}.\sin\overline{mx-ma}.\sin\overline{nt+\beta}.$$

The time of high water or low water at any part of the canal will be found by making $\frac{d\mathrm{Y}}{dt}=0$, from which we obtain

$$\tan nt'=\frac{\mathrm{A}\cos\alpha.\sin\overline{mb-mx}+\mathrm{B}\cos\beta.\sin\overline{mx-ma}}{\mathrm{A}\sin\alpha.\sin\overline{mb-mx}+\mathrm{B}\sin\beta.\sin\overline{mx-ma}};$$

and the velocity with which the phase of high water travels along the canal $=\frac{dt'}{dx}=$

$$\frac{m}{n}\cdot\frac{\mathrm{AB}\,.\sin\overline{\alpha-\beta}\,.\sin\overline{mb-ma}}{\mathrm{A}^2.\sin^2\overline{mb-mx}+\mathrm{B}^2.\sin^2\overline{mx-ma}+2\mathrm{AB}.\cos\overline{\alpha-\beta}.\sin\overline{mb-mx}.\sin\overline{mx-ma}}.$$

This expression has the same sign for all values of x; and therefore the high water travels in the same direction along all parts of the canal.

(312.) The expression for K at high water is

$$\frac{1}{\sin\overline{mb-ma}}\sqrt{\{\mathrm{A}^2.\sin^2\overline{mb-mx}+\mathrm{B}^2.\sin^2\overline{mx-ma}+2\mathrm{AB}.\cos\overline{\alpha-\beta}.\sin mb-mx.\sin\overline{mx-ma}\}}.$$

If $\overline{mb-ma}$ be nearly $=180°$ and $\overline{\alpha-\beta}$ small, then for all values of x which make $\overline{mb-mx}$ or $\overline{mx-ma}$ not very different from 90°, this value of K may much exceed the values when $x=a$ or $x=b$; that is, the rise of tide in the middle of the canal may be much greater than at its extremities. The velocity of the tide-wave at different points of the canal is inversely as the square of the rise of the tide at those points.

The time and height of high water depend upon the period of the waves.

It is to be remarked, that though in the theory of the tides, (439.) &c., α and β are found to be independent of n, yet m depends on n, and therefore $mb-mx$ and $mx-ma$ depend on n. If, then, the value of n alters, the time of high water determined from the expression for nt', and the height of high water, or K above, will be altered. This is the theorem used by Laplace, and to which reference is made in (121.).

(313.) The general expression for X, corresponding to the general expression for Y (remarking that $\mathrm{Y}=\frac{\mathrm{K}y}{k}$ nearly), is

$$-\frac{\mathrm{A}}{mk.\sin\overline{mb-ma}}\cos\overline{mb-mx}.\sin\overline{nt+\alpha}+\frac{\mathrm{B}}{mk.\sin\overline{mb-ma}}\cos\overline{mx-ma}.\sin\overline{nt+\beta}.$$

This expression vanishes when

$$\tan nt''=-\frac{-\mathrm{A}\sin\alpha.\cos\overline{mb-mx}+\mathrm{B}\sin\beta.\cos\overline{mx-ma}}{-\mathrm{A}\cos\alpha.\cos\overline{mb-mx}+\mathrm{B}\cos\beta.\cos\overline{mx-ma}}.$$

Thus it appears that the time at which the particles of water are in their mean horizontal position does not coincide with or bear any simple relation to the time at which they have the greatest elevation.

The greatest value of X for any given place (to which the greatest velocity of the particles is proportional) is

$$\frac{1}{mk.\sin\overline{mb-ma}}\sqrt{\{\mathrm{A}^2.\cos^2\overline{mb-mx}+\mathrm{B}^2.\cos^2\overline{mx-ma}-2\mathrm{AB}.\cos\overline{\alpha-\beta}.\cos\overline{mb-mx}.\cos\overline{mx-ma}\}}.$$

Canal between a tidal sea and a tideless sea.

(314.) If one of the seas have no tide, let $\mathrm{A}=0$. Then, for high water in every part, $\tan nt'=\cot\beta$, or the high water is simultaneous throughout the canal: the elevation is $\frac{\mathrm{B}.\sin\overline{mx-ma}}{\sin\overline{mb-ma}}$; the greatest value of X is $\frac{\mathrm{B}.\cos\overline{mx-ma}}{mk.\sin\overline{mb-ma}}$, or the water moves most rapidly where the canal joins the tideless sea; and the greatest horizontal displacement occurs at the same instant as the greatest vertical displacement.

Subsection 6.—Theory of Waves on Canals when Friction is taken into account.

(315.) It cannot be supposed that we can give a theory of friction among particles of water, and between water and land, which will pretend to any great degree of exactness. Nevertheless, we can give one of very plausible character, and one which, it is quite certain, will represent *generally* the effects of friction. We conceive that it will be found valuable, as assisting to explain some circumstances of the tides which hitherto have received no explanation.

Friction supposed to be proportional to the velocity.

(316.) It is well known that the resistance to bodies moving in fluids (to which the friction of fluids, running over an uneven surface, is entirely analogous) is nearly proportional to the square of the velocity, when that velocity is considerable. But all experiments have agreed in showing that, when the velocity is small, the resistance is greater than that which is given by this law. In fact, when the velocity is very small, the resistance may be represented as nearly, or more nearly, by supposing it proportional to the first power of the velocity. The velocity of tide-currents, for the most part, is so small that this law may be supposed to apply without great error to them. And it possesses the very great advantage of presenting equations which admit of solution with comparatively little trouble. We shall, therefore, suppose the friction proportional to the velocity, and shall measure it by $f\times$velocity. And as the velocity (using the same notation which we have throughout employed) is represented by $\frac{d\mathrm{X}}{dt}$, and as the friction tends always to diminish the velocity, and therefore acts backwards when the particles are moving forwards, and acts forwards when the particles are

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moving backwards, we shall represent the force of friction correctly, in conformity with our supposition, by the expression $-f\frac{dX}{dt}$.

(317.) Confining ourselves to the case of long waves, and supposing the rise and fall of the water to be small in proportion to its depth, we shall have, from the equation of (236.),

$$\frac{d^2X}{dt^2}=F+gk\frac{d^2X}{dx^2},$$

where k is the depth of the water, and F the whole force (independent of the pressure of the water) which acts upon any particle. If, then, the horizontal force produced by external causes is F′, and the force produced by friction is $-f\frac{dX}{dt}$, we must for F put $F'-f\frac{dX}{dt}$, and the equation becomes

$$\frac{d^2X}{dt^2}=F'-f\frac{dX}{dt}+gk\frac{d^2X}{dx^2}$$

or if $gk=v^2$,

$$\frac{d^2X}{dt^2}=F'-f\frac{dX}{dt}+v^2\frac{d^2X}{dx^2}.$$

Investigation when the external force is similar to tidal forces, and there is also friction.

(318.) Problem.—The external force, tending to urge the water of a canal forwards, is represented by $H.\sin\overline{it-mx}$: to find the motion of the water, taking friction into account.

(319.) As the motion for which we desire to obtain an expression is only that which is periodical, and in which the period of the motions is the same as the period of the forces, we must take the most general expression depending upon it which will satisfy the equation. Assume then $X=A.\cos it+B.\sin it$, A and B being functions of x to be discovered. Then $\frac{dX}{dt}=-iA.\sin it+iB.\cos it$; $\frac{d^2X}{dt^2}=-i^2A.\cos it-i^2B.\sin it$; $\frac{d^2X}{dx^2}=\frac{d^2A}{dx^2}.\cos it+\frac{d^2B}{dx^2}.\sin it$. Substituting these in the equation, and putting for F′ its value $H.\sin\overline{it-mx}$, and expanding the sine,

$$\begin{aligned}-i^2A.\cos it-i^2B.\sin it=H.\cos mx.\sin it&-H.\sin mx.\cos it\\ &+fiA.\sin it-fiB.\cos it\\ &+v^2\frac{d^2A}{dx^2}\cos it+v^2\frac{d^2B}{dx^2}\sin it.\end{aligned}$$

As this equation is to hold for all values of it, it will be equivalent to the two following:

$$-i^2A=-H.\sin mx-fiB+v^2\frac{d^2A}{dx^2} \qquad (1.)$$

$$-i^2B=H.\cos mx+fiA+v^2\frac{d^2B}{dx^2}. \qquad (2.)$$

From the former of these,

$$fiB=v^2\frac{d^2A}{dx^2}+i^2A-H.\sin mx \qquad (3.)$$

therefore

$$fi\frac{d^2B}{dx^2}=v^2\frac{d^4A}{dx^4}+i^2\frac{d^2A}{dx^2}+m^2H.\sin mx. \qquad (4.)$$

Multiplying (2.) by fi, and transposing its terms,

$$-fi^3B-fiv^2\frac{d^2B}{dx^2}=\quad f^2i^2A+fiH.\cos mx.$$

Multiplying (3.) by i^2

$$fi^3B=\quad i^2v^2\frac{d^2A}{dx^2}+i^4A-i^2H.\sin mx.$$

Multiplying (4.) by v^2

$$fiv^2\frac{d^2B}{dx^2}=\quad v^4\frac{d^4A}{dx^4}+v^2i^2\frac{d^2A}{dx^2}+v^2m^2H.\sin mx.$$

Adding together the three last equations,

$$0=v^4\frac{d^4A}{dx^4}+2v^2i^2\frac{d^2A}{dx^2}+(i^4+f^2i^2)A+(v^2m^2-i^2)H.\sin mx+fiH.\cos mx.$$

Differential equation for coefficient.

(320.) The equation just found is one of which the solution is well known. Its most general expression is the following. Let the four roots of the equation $0=v^4z^4+2v^2i^2z^2+(i^4+f^2i^2)$ be $\pm p\pm q\sqrt{-1}$: (it is easily found that $p^2=-\frac{i^2}{2v^2}+\sqrt{\left(\frac{i^4+f^2i^2}{4v^4}\right)}$, $q^2=\frac{i^2}{2v^2}+\sqrt{\left(\frac{i^4+f^2i^2}{4v^4}\right)}$:), and put M for $(v^2m^2-i^2)H.\sin mx+fiH.\cos mx$.

Then the expression for A is

$$-\frac{1}{v^4}\varepsilon^{px}.\cos qx\int_x\frac{1}{\cos^2 qx}\int_x\varepsilon^{-2px}.\cos^2 qx\int_x\frac{1}{\cos^2 qx}\int_x\varepsilon^{px}.\cos qx.\,\mathrm{M}.$$

the right hand integration being performed first, and arbitrary constants being added at each integration. But without going through the trouble of this integration, remarking that the part depending on the arbitrary constants will have the form $\mathrm{C}\varepsilon^{px}.\cos qx+\mathrm{C}'\varepsilon^{px}.\sin qx+\mathrm{C}''\varepsilon^{-px}.\cos qx+\mathrm{C}'''\varepsilon^{-px}.\sin qx$ (which may be proved more easily by substitution in the three first terms of the differential equation); and remarking that the substitution of $\frac{\sin mx}{\cos mx}$ in the three first terms will produce $(m^4v^4-2m^2v^2i^2+i^4+f^2i^2)\frac{\sin mx}{\cos mx}$; we easily find for the complete value of A

Solution of differential equation.

$$\frac{(i^2-v^2m^2)\mathrm{H}}{v^4m^4-2v^2m^2i^2+i^4+f^2i^2}\sin mx-\frac{fi\mathrm{H}}{v^4m^4-2v^2m^2i^2+i^4+f^2i^2}\cos mx$$
$$+\mathrm{C}\,\varepsilon^{px}.\cos qx+\mathrm{C}'\varepsilon^{px}.\sin qx+\mathrm{C}''.\varepsilon^{-px}.\cos qx+\mathrm{C}'''.\varepsilon^{-px}.\sin qx.$$

(321.) Substituting this value of A in equation (3.), and remarking that $v^2\frac{d^2}{dx^2}(\varepsilon^{px}.\cos qx)+i^2.\varepsilon^{px}.\cos qx=\varepsilon^{px}\{(p^2v^2-q^2v^2+i^2)\cos qx-2v^2pq.\sin qx\}$: which, on putting for p and q their values, is reduced to $-fi\,\varepsilon^{px}.\sin qx$; we obtain

$$fi\mathrm{B}=\frac{(i^2-v^2m^2)^2.\mathrm{H}}{v^4m^4-2v^2m^2i^2+i^4+f^2i^2}\sin mx-\mathrm{H}\sin mx-\frac{fi(i^2-v^2m^2)\mathrm{H}}{v^4m^4-2v^2m^2i^2+i^4+f^2i^2}\cos mx$$
$$+fi\{-\mathrm{C}.\varepsilon^{px}.\sin qx+\mathrm{C}'.\varepsilon^{px}.\cos qx+\mathrm{C}''.\varepsilon^{-px}\sin qx-\mathrm{C}'''.\varepsilon^{-px}.\cos qx\},$$

or

$$\mathrm{B}=\frac{-fi\mathrm{H}}{v^4m^4-2v^2m^2i^2+i^4+f^2i^2}\sin mx-\frac{(i^2-v^2m^2)\mathrm{H}}{v^4m^4-2v^2m^2i^2+i^4+f^2i^2}\cos mx$$
$$-\mathrm{C}.\varepsilon^{px}.\sin qx+\mathrm{C}'.\varepsilon^{px}.\cos qx+\mathrm{C}''.\varepsilon^{-px}.\sin qx-\mathrm{C}'''.\varepsilon^{-px}.\cos qx,$$

and finally $\mathrm{X}=\mathrm{A}\cos it+\mathrm{B}\sin it=$

Expression for displacement of particles.

$$\frac{-(i^2-v^2m^2)\mathrm{H}}{(i^2-v^2m^2)^2+f^2i^2}\sin\overline{it-mx}-\frac{fi\mathrm{H}}{(i^2-v^2m^2)^2+f^2i^2}\cos\overline{it-mx}+\varepsilon^{px}\{\mathrm{C}.\cos\overline{it+qx}+\mathrm{C}'.\sin\overline{it+qx}\}$$
$$+\varepsilon^{-px}\{\mathrm{C}''.\cos\overline{it-qx}-\mathrm{C}'''.\sin\overline{it-qx}\}.$$

We shall now proceed to apply this expression to some particular cases.

(322.) Problem.—To find the expression for the disturbance in a canal of indefinite length, or one returning into itself (as a canal surrounding the earth).

Canal supposed indefinite, or returning into itself.

Here it is plain that we cannot have any of the terms multiplying ε^{px} and ε^{-px}, because, in the case of a returning canal, these terms would have different values on returning by a different number of circuits to the same spot. Hence

$$\mathrm{X}=\frac{-\mathrm{H}}{(i^2-v^2m^2)^2+f^2i^2}\{(i^2-v^2m^2).\sin\overline{it-mx}+fi.\cos\overline{it-mx}\}.$$

If $\tan\mathrm{D}=\frac{fi}{i^2-v^2m^2}$,

$$\mathrm{X}=\frac{-\mathrm{H}}{\sqrt{\{(i^2-v^2m^2)^2+f^2i^2\}}}\sin\overline{it-mx+\mathrm{D}}.$$

Upon this we have to remark as follows:

(323.) 1st. This expression never becomes infinite. It is greatest nearly when $i^2-v^2m^2=0$, or $i^2-gkm^2=0$. Then $\mathrm{D}=90°$, and the expression for X is $\frac{-\mathrm{H}}{fi}\cos\overline{it-mx}$, which is very large if f is small.

(324.) 2nd. The expression for X is always less than that which is obtained by making f the coefficient of friction$=0$.

(325.) 3rd. The expression for V, the whole depth at any point in the disturbed state, is $\frac{k}{1+\frac{d\mathrm{X}}{dx}}=k-k\frac{d\mathrm{X}}{dx}$ nearly, and the fluctuation in height, or K, is therefore $-k\frac{d\mathrm{X}}{dx}$

$$=\frac{-km\mathrm{H}}{(i^2-v^2m^2)^2+f^2i^2}\{(i^2-v^2m^2).\cos\overline{it-mx}-fi.\sin\overline{it-mx}\}$$
$$=\frac{-km\mathrm{H}}{\sqrt{\{(i^2-v^2m^2)^2+f^2i^2\}}}\cos\overline{it-mx+\mathrm{D}}.$$

Tides and Waves.

Tides and Waves.

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Tide is accelerated by friction

This is maximum, or it is high water, when $it-mx+\mathrm{D}=\pi$, or when $t=\frac{\pi+mx-\mathrm{D}}{i}$. The force at the point in question vanishes, and the force on each side tends to draw the water towards the place, when $t=\frac{mx}{i}$. Consequently, high water follows the time of the force vanishing by $\frac{\pi-\mathrm{D}}{i}$. As the interval between high water and low water is $\frac{\pi}{i}$, it appears that high water is the next conspicuous phase which follows the evanescence of forces.

The acceleration is greater for a tide of longer period.

(326.) It appears here that high water is *accelerated* by the time $\frac{\mathrm{D}}{i}$, or $\frac{f}{i^2-v^2m^2}$ nearly. If then there were two forces acting at the same time, one depending upon $\overline{it-mx}$ and the other depending upon $\overline{(i-i'')\,t-mx}$, the tide depending upon the former would be accelerated by $\frac{f}{i^2-v^2m^2}$, and that depending upon the latter by $\frac{f}{(i-i'')^2-v^2m^2}$. The tide, therefore, depending upon $\overline{(i-i'')\,t-mx}$ would be more accelerated that that depending upon $\overline{it-mx}$ by $\frac{f}{(i-i'')^2-v^2m^2}-\frac{f}{i^2-v^2m^2}$, or (if i'' be small) $f\times\frac{2ii'}{(i^2-v^2m^2)^2}$. If $i^2-v^2m^2$ be small, this may be large enough to be very sensible. We shall hereafter find that this conclusion is most important.

(327.) 4th. If the water be so deep that v^2m^2 or gkm^2 is greater than i^2, the fluctuation in height or $-k\frac{d\mathrm{X}}{dx}=\frac{km\mathrm{H}}{(v^2m^2-i^2)^2+f^2i^2}\{(v^2m^2-i^2).\cos\overline{it-mx}+fi.\sin\overline{it-mx}\}$. Making $\tan\mathrm{D}'=\frac{fi}{v^2m^2-i^2}$, this becomes $\frac{km\mathrm{H}}{\sqrt{\{(v^2m^2-i^2)^2+f^2i^2\}}}\cos\overline{it-mx-\mathrm{D}'}$. This is maximum, or it is high water, when $it-mx-\mathrm{D}'=0$, or $t=\frac{mx+\mathrm{D}'}{i}$, which follows the time of evanescence of the forces by $\frac{\mathrm{D}'}{i}$. In all cases, therefore, the conspicuous phase which next follows the evanescence of forces is high water.

Coefficient of force supposed variable.

(328.) 5th. Suppose the coefficient of force to be variable and equal to $\mathrm{H}+\mathrm{H}'.\cos i't$, i' being small. Then the force $=(\mathrm{H}+\mathrm{H}'.\cos i't).\sin\overline{it-mx}=\mathrm{H}.\sin\overline{it-mx}+\frac{\mathrm{H}'}{2}\sin\overline{(i+i)\,t-mx}+\frac{\mathrm{H}'}{2}\sin\overline{(i-i')\,t-mx}$, and the expression for K is

$$\frac{-km\mathrm{H}}{(i^2-v^2m^2)^2+f^2i^2}\{(i^2-v^2m^2).\cos\overline{it-mx}-fi.\sin\overline{it-mx}\}$$
$$-\frac{km\mathrm{H}'}{\{(i+i')^2-v^2m^2\}^2+f^2.(i+i')^2}.\frac{1}{2}\left\{\{(i+i')^2-v^2m^2\}.\cos\overline{(i+i')\,t-mx}-f(i+i').\sin\overline{(i+i')\,t-mx}\right\}$$
$$-\frac{km\mathrm{H}'}{\{(i-i')^2-v^2m^2\}^2+f^2.(i-i')^2}.\frac{1}{2}\left\{\{(i-i')^2-v^2m^2\}.\cos\overline{(i-i')\,t-mx}-f(i-i')\sin.\overline{(i-i')\,t-mx}\right\}.$$

Suppose f so small that its square may be neglected. Then K becomes

$$-\frac{km\mathrm{H}}{i^2-v^2m^2}\cos\overline{it-mx}+\frac{kfim\mathrm{H}}{(i^2-v^2m^2)^2}\sin\overline{it-mx}$$
$$-\frac{1}{2}.\frac{km\mathrm{H}'}{\{(i+i')\}^2-v^2m^2}\cos\overline{(i+i')\,t-mx}+\frac{1}{2}.\frac{kf(i+i')\,m\mathrm{H}'}{\{(i+i')^2-v^2m^2\}^2}\sin\overline{(i+i')\,t-mx}$$
$$-\frac{1}{2}.\frac{km\mathrm{H}'}{\{(i-i')\}^2-v^2m^2}\cos\overline{(i-i')\,t-mx}+\frac{1}{2}.\frac{kf(i-i')\,m\mathrm{H}'}{\{(i-i')^2-v^2m^2\}^2}\sin\overline{(i-i')\,t-mx}.$$

Expanding these expressions to the first power of i', and omitting the general multiplier $\frac{km\mathrm{H}}{i^2-v^2m^2}$, the whole may be put in the following form :

$$\left\{-1-\frac{\mathrm{H}'}{\mathrm{H}}\cos i't-f\frac{\mathrm{H}'}{\mathrm{H}}.\frac{i'(3i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't\right\}\cos\overline{it-mx}$$
$$+\left\{-2\frac{\mathrm{H}'}{\mathrm{H}}.\frac{ii'}{i^2-v^2m^2}\sin i't+\frac{fi}{i^2-v^2m^2}+\frac{\mathrm{H}'}{\mathrm{H}}.\frac{fi}{i^2-v^2m^2}\cos i't\right\}\sin\overline{it-mx}.$$

(329.) On any given day (in treating of the tides) $i't$ may be considered nearly constant. Regarding the coefficients of cos $\overline{it-mx}$ and sin $\overline{it-mx}$ therefore as constant, the greatest elevation and depression of the water on that day will be expressed by the square root of the sums of their squares. Expanding the squares on the sup-

Subsec. 6. When friction is taken into account.

position that $\frac{H'}{H}$ is small, and taking only the first power of $\frac{H'}{H}$, the first power of f, the first power of i', and the combination of these first powers (a process which is legitimate for showing the combined effect of friction and variable coefficient, although the square of any one of these quantities is rejected), we have for the square of elevation or depression

$$1+\frac{2H'}{H}\cos i't+2f\frac{H'}{H}\cdot\frac{i'(3i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't-4f\frac{H'}{H}\cdot\frac{i^2i'}{(i^2-v^2m^2)^2}\sin i't$$
$$=1+\frac{2H'}{H}\cos i't+2f\frac{H'}{H}\cdot\frac{i'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't,$$

and the expression for the greatest elevation or depression, restoring the factor, is

$$-\frac{kmH}{i^2-v^2m^2}\left\{1+\frac{H'}{H}\cos i't+f\frac{H'}{H}\cdot\frac{i'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't\right\}.$$

The day on which the rise or fall is greatest is determined by making the quantity within the brackets maximum, or making $\cos i't+\frac{fi'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't$ maximum. This will be when $\tan i't$ or $i't=\frac{fi'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}$, or when $t=\frac{f(i^2+v^2m^2)}{(i^2-v^2m^2)^2}$. But the variable force is greatest when $t=0$. Therefore the greatest tide follows the greatest force by the time $\frac{f(i^2+v^2m^2)}{(i^2-v^2m^2)^2}$. This appears to us an important result, and one which no other theory has obtained. The equilibrium-theory of tides necessarily makes the tides to be greatest upon the same day on which the force is greatest. Laplace's theory (93.), and the theory of waves in canals without friction (288.), give the same result. But here we find a retardation accounted for by friction; and moreover this retardation is considerable. For the alteration in the time of any high water was found (326.) to be $\frac{D}{i}$, or $\frac{f}{i^2-v^2m^2}$: here the retardation in the time of the greatest tide is $\frac{f(i^2+v^2m^2)}{(i^2-v^2m^2)^2}$, or is equal to the alteration in the time of high water $\times\frac{i^2+v^2m^2}{i^2-v^2m^2}$. When i^2 is not much greater than v^2m^2 or gkm^2, or when the depth is not much less than that in which the free wave would move as rapidly as the forced wave, this multiplier may be considerable, and (for the tides) the retardation may amount to one or several days.

Greatest tide occurs some time later than the greatest force.

(330.) If the term $H=0$, or if the coefficient of force be merely $H'.\cos i't$, the expression for K is

$$\frac{kmH'}{2}\left\{\left(-\frac{1}{(i-i')^2-v^2m^2}-\frac{1}{(i+i')^2-v^2m^2}\right)\cos i't+\left(-\frac{f(i-i')}{\{(i-i')^2-v^2m^2\}^2}+\frac{f(i+i')}{\{(i+i')^2-v^2m^2\}^2}\right)\sin i't\right\}\cos\overline{it-mx}$$
$$+\frac{kmH'}{2}\left\{\left(-\frac{1}{(i-i')^2-v^2m^2}+\frac{1}{(i+i')^2-v^2m^2}\right)\sin i't+\left(\frac{f(i-i')}{\{(i-i')^2-v^2m^2\}^2}+\frac{f(i+i)}{\{(i+i')^2-v^2m^2\}^2}\right)\cos i't\right\}\sin\overline{it-mx}.$$

If i' be small, this is nearly

$$\frac{kmH'}{i^2-v^2m^2}\left\{-\cos i't-f.\frac{i'(3i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't\right\}\cos\overline{it-mx}$$
$$+\frac{kmH'}{i^2-v^2m^2}\left\{\frac{-2ii'}{i^2-v^2m^2}\sin i't+\frac{fi}{i^2-v^2m^2}\cos i't\right\}\sin\overline{it-mx}.$$

The square of the elevation or depression upon any day (regarding $i't$ as constant) is the sum of the squares of the coefficients of $\cos\overline{it-mx}$ and $\sin\overline{it-mx}$; which, omitting the factor $\frac{kmH'}{i^2-v^2m^2}$, and taking only the first power of f, is

$$\cos^2 i't+\frac{4i^2i'^2}{(i^2-v^2m^2)^2}\sin^2 i't+f\left\{\frac{2i'(3i^2+v^2m^2)}{(i^2-v^2m^2)^2}-\frac{4i^2i'}{(i^2-v^2m^2)^2}\right\}\sin i't.\cos i't$$

of which the second term, as depending on the square of i' is to be omitted. Thus the expression becomes

$$\cos^2 i't+f.\frac{2i'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin i't\ \cos i't$$

or

$$\tfrac{1}{2}+\tfrac{1}{2}\cos 2i't+\frac{fi'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}\sin 2i't;$$

whose maximum value occurs when $\tan 2i't$ or $2i't=\frac{2fi'(i^2+v^2m^2)}{(i^2-v^2m^2)^2}$, or $t=\frac{f(i^2+v^2m^3)}{(i^2-v^2m^2)^2}$, as before.

(331.) It is to be remarked that the slower the recurrence of the tide is in proportion to the space which it passes over (that is, the smaller i is in proportion to vm), the greater is the retardation. Thus the lunar tide is more retarded than the solar tide.

We shall now consider some cases in which the limits enter.

Tide propagated up a river, when there is friction.

(332.) Problem.—To find the nature of the tides propagated up a river of indefinite length communicating with a tidal sea, friction being considered, but no tidal forces being supposed to act on the river.

Here we must take the general solution of (321.) on the supposition that the force is 0, and we find for the general expression for X,

$$\varepsilon^{px}\{\mathrm{C}.\cos\overline{it+qx}+\mathrm{C}'.\sin\overline{it+qx}\}+\varepsilon^{-px}\{\mathrm{C}''.\cos\overline{it-qx}-\mathrm{C}'''.\sin\overline{it-qx}\}$$

where p^2 and q^2 have the values given in (320.).

As we suppose our wave to be propagated from the sea, and to meet with nothing which reflects it, we must take only those terms which depend on $it-qx$, and therefore X becomes

$$\varepsilon^{-px}\{\mathrm{C}''.\cos\overline{it-qx}-\mathrm{C}'''.\sin\overline{it-qx}\}.$$

From this,

$$\frac{d\mathrm{X}}{dx}=\varepsilon^{-px}\{(-p\mathrm{C}''+q\mathrm{C}''').\cos\overline{it-qx}+(q\mathrm{C}''+p\mathrm{C}''').\sin\overline{it-qx}\},$$

and

$$\mathrm{K}=-k\frac{d\mathrm{X}}{dx}\text{ nearly }=k\varepsilon^{-px}.\{(p\mathrm{C}''-q\mathrm{C}''').\cos\overline{it-qx}+(-q\mathrm{C}''-p\mathrm{C}''').\sin\overline{it-qx}\}.$$

At the sea, or where $x=0$, this becomes $k\{(p\mathrm{C}''-q\mathrm{C}''').\cos it+(-q\mathrm{C}''-p\mathrm{C}''').\sin it\}$. If we suppose the tidal elevation of the sea to be expressed by $\mathrm{A}.\sin nt$, we must have $i=n$, $k(p\mathrm{C}''-q\mathrm{C}''')=0$, $k(-q\mathrm{C}''-p\mathrm{C}''')=\mathrm{A}$. Substituting these in the general expression for K,

$$\mathrm{K}=\mathrm{A}\varepsilon^{-px}.\sin\overline{nt-qx},$$

p and q having the values obtained by putting n for i. And $\mathrm{C}''=\frac{-q\mathrm{A}}{k(p^2+q^2)}$, $\mathrm{C}'''=\frac{-p\mathrm{A}}{k(p^2+q^2)}$; therefore

$$\mathrm{X}=\frac{\mathrm{A}}{k(p^2+q^2)}\varepsilon^{-px}.\{-q.\cos\overline{nt-qx}+p.\sin\overline{nt-qx}\}.$$

Tidal disturbance gradually diminishes in going up the river.

These expressions indicate a wave whose vertical elevation and horizontal motion of particles diminish continually as it travels up the river. The value of $\frac{d\mathrm{X}}{dt}$ or the velocity of the water is,

$$\frac{n\mathrm{A}}{k(p^2+q^2)}.\varepsilon^{-px}.\{q.\sin\overline{nt-qx}+p.\cos\overline{nt-qx}\}.$$

The flow ceases before the water has dropped to mean height.

Now high water occurs when $\sin\overline{nt-qx}$ has its greatest positive value, or $nt-qx=90°$; the next mean elevation will occur then when $nt-qx=180°$; at this time the velocity of the water $=-\frac{n\mathrm{A}}{k(p^2+q^2)}\varepsilon^{-px}.p$; or the water will be flowing downwards; and therefore the direction of tide-current changes sooner after high water than if there were no friction.

Tide in a river stopped by a barrier, considering friction.

(333.) Problem.—To find the nature of the tides in the river when there is a barrier at a certain distance a from the mouth; other suppositions being as before.

In this case, taking the general solution, we cannot reject the first terms, because there may be a reflected wave. And, moreover, without taking these first terms, we shall not have a sufficient number of arbitrary constants for our conditions. These conditions are, that when $x=0$, K must $=\mathrm{A}\sin nt$, and when $x=a$, X must $=0$; for all values of t. Assume then

$$\mathrm{X}=\varepsilon^{px}\{\mathrm{C}.\cos\overline{it+qx}+\mathrm{C}'.\sin\overline{it+qx}\}+\varepsilon^{-px}.\{\mathrm{C}''.\cos\overline{it-qx}-\mathrm{C}'''.\sin\overline{it-qx}\}$$

$$\mathrm{K}=-k\frac{d\mathrm{X}}{dx}=-k\varepsilon^{px}\{(p\mathrm{C}+q\mathrm{C}').\cos\overline{it+qx}+(-q\mathrm{C}+p\mathrm{C}').\sin\overline{it+qx}\}$$

$$-k\varepsilon^{-px}\{(-p\mathrm{C}''+q\mathrm{C}''').\cos\overline{it-qx}+(q\mathrm{C}''+p\mathrm{C}''').\sin\overline{it-qx}\}.$$

When $x=0$, K becomes

$$-k\{(p\mathrm{C}+q\mathrm{C}').\cos it+(-q\mathrm{C}+p\mathrm{C}').\sin it\}-k\{(-p\mathrm{C}''+q\mathrm{C}''').\cos it+(q\mathrm{C}''+p\mathrm{C}''').\sin it\}.$$

The first condition therefore requires

$$i=n$$

$$p\mathrm{C}+q\mathrm{C}'-p\mathrm{C}''+q\mathrm{C}'''=0$$

$$-k\{-q\mathrm{C}+p\mathrm{C}'+q\mathrm{C}''+p\mathrm{C}''\}=\mathrm{A}.$$

The second condition requires

$$\varepsilon^{pa}.\{\mathrm{C}.\cos qa+\mathrm{C}'.\sin qa\}+\varepsilon^{-pa}.\{\mathrm{C}''.\cos qa+\mathrm{C}'''.\sin qa\}=0$$

$$\varepsilon^{pa}\{-\mathrm{C}.\sin qa+\mathrm{C}'.\cos qa\}+\varepsilon^{-pa}.\{\mathrm{C}''.\sin qa-\mathrm{C}'''.\cos qa\}=0.$$

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Determining from the last four equations the values of C, C′, C″, C‴, and substituting them in the expressions for X and K, we find

$$K=\frac{A}{\sqrt{\{\varepsilon^{2pa}+\varepsilon^{-2pa}+2\cos 2qa\}}}\times\{\varepsilon^{px-pa}.\sin\overline{it+qx-E}+\varepsilon^{pa-px}.\sin\overline{it-qx+F}\}$$

where $\tan E=\dfrac{\sin 2qa}{\varepsilon^{-2pa}+\cos 2qa}$, $\tan F=\dfrac{\sin 2qa}{\varepsilon^{2pa}+\cos 2qa}$, and therefore $E+F=2qa$.

Putting the quantity within the brackets under the form,

$$\cos it\times\{\varepsilon^{px-pa}.\sin\overline{qx-E}-\varepsilon^{pa-px}.\sin\overline{qx-F}\}+\sin it\times\{\varepsilon^{px-pa}.\cos\overline{qx-E}+\varepsilon^{pa-px}.\cos\overline{qx-F}\}$$

$$=P.\cos it+Q.\sin it=\sqrt{P^2+Q^2}.\left\{\frac{P}{\sqrt{(P^2+Q^2)}}\cos it+\frac{Q}{\sqrt{(P^2+Q^2)}}\sin it\right\},$$

and making $\dfrac{Q}{P}=\tan D$, this becomes

$$\sqrt{P^2+Q^2}.\cos\overline{it-D};$$

and, substituting for P and Q,

$$K=A\sqrt{\frac{\varepsilon^{2px-2pa}+\varepsilon^{2pa-2px}+2\cos\overline{2qa-2qx}}{\varepsilon^{2pa}+\varepsilon^{-2pa}+2\cos 2qa}}\times\cos\overline{it-D}.$$

(334.) At any given point of the river, the greatest elevation or depression of the water is the coefficient of $\cos\overline{it-D}$. This coefficient cannot, with any values of x and a, be infinite; and in this respect the result now found differs from that found in (307.) without friction. The numerator of K increases as x increases, only when $4q.\sin\overline{2qa-2qx}$ is greater than $2p\ (\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa})$; which differs from the result of (307.). The time t' of high water at any place will be determined by making $t'=\dfrac{D}{i}$, and the reciprocal of the velocity with which the phase of high water travels up the river will be $\dfrac{1}{i}.\dfrac{dD}{dx}$. Since $\tan D=\dfrac{Q}{P}$, $\dfrac{dD}{dx}=\dfrac{P\dfrac{dQ}{dx}-Q\dfrac{dP}{dx}}{P^2+Q^2}$. Putting for P and Q their values, we find for the reciprocal of the velocity of the phase of high water,

$$\frac{1}{i}.\frac{q(\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa})+2p.\sin\overline{2qa-2qx}}{\varepsilon^{2pa-2px}+\varepsilon^{2px-2pa}+2\cos\overline{2qa-2qx}}.$$

Wave travels up the river.

This quantity is always positive $\left(\right.$for even when $2qa-2qx=\dfrac{3\pi}{2}$, which gives the second term its greatest negative value $-2p$, the two first terms $=q\left(\varepsilon^{\frac{3\pi p}{2q}}-\varepsilon^{\frac{-3\pi p}{2q}}\right)=3\pi p+\&c.\left.\right)$. Thus, in all cases, there is a wave rolling up the river. This result is different from that of (307.), as found without supposing friction.

(335.) The value of X, found in the same manner, is

$$\frac{A}{k\sqrt{(p^2+q^2)}\sqrt{(\varepsilon^{2pa}+\varepsilon^{-2pa}+2\cos 2qa)}}+\{\varepsilon^{px-pa}.\cos\overline{it+qx-G}-\varepsilon^{pa-px}.\cos\overline{it-qx-H}\},$$

when $\tan G=\dfrac{-p-p\varepsilon^{2pa}.\cos 2qa+q\varepsilon^{2pa}.\sin 2qa}{q+p\varepsilon^{2pa}.\sin 2qa+q\varepsilon^{2pa}.\cos 2qa}$, $\tan H=\dfrac{p\varepsilon^{4pa}+p\varepsilon^{2pa}.\cos 2qa+q\varepsilon^{2pa}.\sin 2qa}{-q\varepsilon^{4pa}+p\varepsilon^{2pa}.\sin 2qa-q\varepsilon^{2pa}.\cos 2qa}$.

Putting the quantity within the brackets, or

$$\cos it\times(\varepsilon^{px-pa}.\cos\overline{qx-G}-\varepsilon^{pa-px}.\cos\overline{qx+H})-\sin it\times(\varepsilon^{px-pa}.\sin\overline{qx-G}+\varepsilon^{pa-px}.\sin\overline{qx+H}),$$

into the form

$$P'.\cos it-Q'.\sin it=\sqrt{P'^2+Q'^2}.\cos\overline{it+D'},\ \text{where}\ \tan D'=\frac{Q'}{P'},$$

and remarking that $H-G=-2qa$, the expression for X becomes

$$\frac{A}{k}\sqrt{\frac{\varepsilon^{2pa-2px}+\varepsilon^{2px-2pa}-2\cos\overline{2qa-2qx}}{(p^2+q^2)\ (\varepsilon^{2pa}+\varepsilon^{-2pa}+2\cos 2qa)}}\times\cos\overline{it+D'}.$$

The time at which, for any given place, the water has its greatest horizontal displacement up the river (or the time when the flow ceases and the ebb commences) is therefore $-\dfrac{D'}{i}$. But we have found for the time of high water $\dfrac{D}{i}$. Therefore the end of flow follows high water by $-\dfrac{1}{i}(D'+D)$.

Now $$\tan(\mathrm{D}'+\mathrm{D})=\frac{\mathrm{P}'\mathrm{Q}+\mathrm{P}\mathrm{Q}'}{\mathrm{P}\mathrm{P}'-\mathrm{Q}\mathrm{Q}'}=\frac{-(\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa})+2\sin\overline{2qa-2qx}.\tan\overline{\mathrm{E}-\mathrm{G}}}{(\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa})\tan\overline{\mathrm{E}-\mathrm{G}}+2\sin\overline{2qa-2qx}}.$$

Putting for tan E and tan G their values, it is easily found that $\tan\overline{\mathrm{E}-\mathrm{G}}=\frac{p}{q}$; therefore

$$\tan(-\mathrm{D}'-\mathrm{D})=\frac{q(\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa})-2p.\sin\overline{2qa-2qx}}{p(\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa})+2q.\sin\overline{2qa-2qx}}.$$

End of flow follows high water very soon.

Both numerator and denominator of this expression are positive, and therefore the end of flow *follows* high water. But near the mouth of the river $\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa}$ is reduced to $\varepsilon^{2pa}-\varepsilon^{-2pa}$ nearly, which is large, and the numerator is therefore large, or the end of flow follows high water by a considerable time. Near the barrier, $\varepsilon^{2pa-2px}-\varepsilon^{2px-2pa}$ becomes $4(pa-px)$ nearly, and $\sin\overline{2qa-2qx}$ becomes $2qa-2qx$ nearly, and the numerator is therefore very small, or the end of flow follows the high water very closely.

Tides in a canal bounded at both ends, considering friction.

(336.) Problem.—The external force $\mathrm{H}.\sin\overline{it-mx}$ acts upon the waters of a canal which is bounded at both ends: to find the motion of the water, taking friction into account.

Here we must use the general solution of (321). It gives

$$\mathrm{X}=\mathrm{L}.\cos\overline{it-mx}+\mathrm{M}.\sin\overline{it-mx}+\varepsilon^{px}\{\mathrm{C}.\cos\overline{it-qx}+\mathrm{C}'.\sin\overline{it+qx}\}+\varepsilon^{-px}\{\mathrm{C}''.\cos\overline{it-qx}-\mathrm{C}'''.\sin\overline{it-qx}\};$$

where L, M, p, q, are the constants whose values are to be found in (320.), and where C, C′, C″, C‴, are arbitrary. Let α and β be the two values of x at the boundaries of the canal. Then the conditions by which the values of C, C′, C″, C‴, are to be determined are that, for all values of t, $\mathrm{X}=0$ when $x=\alpha$, and when $x=\beta$.

(337.) The elimination presents no difficulty, and leads to the following expression. Let $\varepsilon^{2p\alpha-2p\beta}-2\cos\overline{2q\alpha-2q\beta}+\varepsilon^{2p\beta-2p\alpha}=\mathrm{B}$; then X=

$$\begin{aligned}
&\mathrm{L}\cos\overline{it-mx}+\mathrm{M}\sin\overline{it-mx}\\
&+\frac{\mathrm{L}}{\mathrm{B}}(-\varepsilon^{px+p\alpha-2p\beta}.\cos\overline{it+qx-m\alpha-q\alpha}+\varepsilon^{px-p\beta}.\cos\overline{it+qx-m\beta+q\beta-2q\alpha}+\varepsilon^{px-p\alpha}.\cos\overline{it+qx-m\alpha+q\alpha-2q\beta}\\
&\qquad\qquad-\varepsilon^{px-2p\alpha+p\beta}\cos\overline{it+qx-m\beta-q\beta})\\
&+\frac{\mathrm{M}}{\mathrm{B}}(-\varepsilon^{px+p\alpha-2p\beta}.\sin\overline{it+qx-m\alpha-q\alpha}+\varepsilon^{px-p\beta}.\sin\overline{it+qx-2q\alpha-m\beta+q\beta}+\varepsilon^{px-p\alpha}.\sin\overline{it+qx-m\alpha+q\alpha-2q\beta}\\
&\qquad\qquad-\varepsilon^{px-2p\alpha+p\beta}.\sin\overline{it+qx-m\beta-q\beta})\\
&+\frac{\mathrm{L}}{\mathrm{B}}(-\varepsilon^{-px-p\alpha+2p\beta}.\cos\overline{it-qx-m\alpha+q\alpha}+\varepsilon^{-px+p\beta}.\cos\overline{it-qx+2q\alpha-m\beta-q\beta}+\varepsilon^{-px+p\alpha}.\cos\overline{it-qx-m\alpha-q\alpha+2q\beta}\\
&\qquad\qquad-\varepsilon^{-px+2p\alpha-p\beta}.\cos\overline{it-qx-m\beta+q\beta})\\
&+\frac{\mathrm{M}}{\mathrm{B}}(-\varepsilon^{-px-p\alpha+2p\beta}.\sin\overline{it-qx-m\alpha+q\alpha}+\varepsilon^{-px+p\beta}.\sin\overline{it-qx+2q\alpha-m\beta-q\beta}+\varepsilon^{-px+p\alpha}.\sin\overline{it-qx-m\alpha-q\alpha+2q\beta}\\
&\qquad\qquad-\varepsilon^{-px+2p\alpha-p\beta}.\sin\overline{it-qx-m\beta+q\beta})
\end{aligned}$$

from which the expression for K will readily be inferred by observing that $\mathrm{K}=-k\frac{d\mathrm{X}}{dx}$; and that $-k\frac{d}{dx}(\varepsilon^{px+a}.\cos\overline{b+qx})=-k.\varepsilon^{px+a}.(p\cos\overline{b+qx}-q\sin\overline{b+qx})=-k\sqrt{p^2+q^2}.\varepsilon^{px+a}.\cos\overline{b+qx+c}$, where $\tan c=\frac{q}{p}$. The application of c is the same throughout (since $-p$ is accompanied by $-q$); but for $-p$, the sign of the multiplier $\sqrt{p^2+q^2}$ must be changed.

(338.) The purport of this expression will be best understood by examining the state of motion of the water at the middle of the canal and at one end, its length being supposed considerable. Suppose $\alpha>\beta$; the value of the divisor B will be nearly $\varepsilon^{2p\alpha-2p\beta}$. First suppose $x=\frac{\alpha+\beta}{2}$; then $\varepsilon^{px+p\alpha-2p\beta}=\varepsilon^{\frac{3p}{2}(\alpha-\beta)}$; when divided by B the quotient will be $\varepsilon^{-\frac{p}{2}(\alpha-\beta)}$; and as $\alpha-\beta$ is large, this term will be small. In like manner all the other terms will be small; and thus, near the middle of the canal, the expressions will be reduced to their first terms,

$$\mathrm{X}=\mathrm{L}.\cos\overline{it-mx}+\mathrm{M}.\sin\overline{it-mx},\qquad \mathrm{K}=-km\mathrm{L}.\sin\overline{it-mx}+km\mathrm{M}.\cos\overline{it-mx},$$

which are the same as if the canal were indefinitely long. Secondly, suppose $x=\alpha$; then $\varepsilon^{px+p\alpha-2p\beta}=\varepsilon^{2p\alpha-2p\beta}$; when divided by B the quotient is 1; this term therefore is sensible; but no other term is in the same predicament. Hence, near the end of the canal, we have only to consider the following terms in X;

The tidal motion is greater at the ends than in the middle.

$$\mathrm{L}.\cos\overline{it-mx}+\mathrm{M}.\sin\overline{it-mx}-\frac{\mathrm{L}}{\mathrm{B}}\varepsilon^{px+p\alpha-2p\beta}.\cos\overline{it+qx-m\alpha-q\alpha}-\frac{\mathrm{M}}{\mathrm{B}}\varepsilon^{px+p\alpha-2p\beta}.\sin\overline{it+qx-m\alpha-q\alpha};$$

from which (remarking that p is small and q is not small), Sect. IV. Theory of Waves. — Subsec. 6. When Friction is taken into account.

$$K = -k\frac{dX}{dx} = -kmL.\sin\overline{it-mx} + kmM.\cos\overline{it-mx} - \frac{kqL}{B}\varepsilon^{px+p\alpha-2p\beta}.\sin\overline{it+qx-m\alpha-q\alpha}$$
$$+\frac{kqM}{B}\varepsilon^{px+p\alpha-2p\beta}.\cos\overline{it+qx-m\alpha-q\alpha}.$$

Putting α for x and $\varepsilon^{2p\alpha-2p\beta}$ for B, this becomes

$$K = -kL\,(m+q).\sin\overline{it-m\alpha} + kM\,(m+q).\cos\overline{it-m\alpha},$$

or the vertical tide at the end of the canal is greater than that in the middle in the proportion of $m+q : m$.

A different form of arbitrary function sometimes necessary

(339.) The solution which applied to the several Problems which we have just terminated was founded essentially upon the assumption that the motions of the water recur with the same magnitude when it has changed by 2π. But it may happen that, even if there are oscillations, they may not recur with the same magnitude, but may continually decrease. This assumption will not affect that part of the solution in (321.) which depends on $\sin\overline{it-mx}$ and $\cos\overline{it-mx}$; but it will affect the terms which are multiplied by arbitrary constants. We shall now undertake this investigation. To find whether the equation $\frac{d^2X}{dt^2} = -f\frac{dX}{dt} + v^2\frac{d^2X}{dx^2}$ can be satisfied by the assumption $X = C.\varepsilon^{rt+sx}.\cos\overline{ut+wx+C'}$.

(340.) Differentiating, we have

$$\frac{dX}{dt} = C.\varepsilon^{rt+sx}\left(r.\cos\overline{ut+wx+C'} - u.\sin\overline{ut+wx+C'}\right)$$
$$\frac{d^2X}{dt^2} = C.\varepsilon^{rt+sx}\left((r^2-u^2).\cos\overline{ut+wx+C'} - 2ru.\sin\overline{ut+wx+C'}\right)$$
$$\frac{d^2X}{dx^2} = C.\varepsilon^{rt+sx}\left((s^2-w^2).\cos\overline{ut+wx+C'} - 2sw.\sin\overline{ut+wx+C'}\right)$$

Substituting in the equation,

$$(r^2-u^2).\cos\overline{ut+wx+C'} - 2ru.\sin\overline{ut+wx+C'} =$$
$$-fr.\cos\overline{ut+wx+C'} + fu.\sin\overline{ut+wx+C'} + v^2(s^2-w^2).\cos\overline{ut+wx+C'} - 2v^2sw.\sin\overline{ut+wx+C'}.$$

As this is to be true for all values of t and of x, it must hold for the separated coefficients of the sine and cosine; or

$$r^2-u^2 = -fr + v^2(s^2-w^2); \qquad -2ru = fu - 2v^2sw,$$

leaving C and C′ indeterminate. When the values of two of the quantities r, s, v, w, are fixed by the conditions of any Problem, the values of the other two can be found from these equations.

Tides on a current river, considering friction.

(341.) Problem.—A river of indefinite length runs on a declivity towards a tidal sea: to investigate the motion of its waters; its mean depth being uniform, and friction being taken into account.

Take for the axis of x a line drawn along the bottom of the river. It will be inclined to the horizon by a very small angle α. Resolve the force of gravity into one part perpendicular to the bottom of the river and one part parallel to it. The former will not sensibly differ from g; the latter will $= g \sin\alpha$. And (measuring x, as in all other cases, up the river) this force tends to diminish x, or is $= -g\sin\alpha$. Hence the general equation becomes

$$\frac{d^2X}{dt^2} = -g\sin\alpha - f\frac{dX}{dt} + v^2\frac{d^2X}{dx^2}.$$

A solution of this is, $X = -\frac{g\sin\alpha}{f}t$, which we will call $-bt$. (It is evident that this term alone denotes that the water will run towards the sea with a uniform velocity.) The most general solution then, which is adapted to our purposes, is

$$X = -bt + \text{ the general solution of the equation } \frac{d^2X}{dt^2} = -f\frac{dX}{dt} + v^2\frac{d^2X}{dx^2},$$

or $$X = -bt + C\varepsilon^{rt+sx}.\cos\,(ut+wx+C'),$$

where $$r^2-u^2 = -fr + v^2(s^2-w^2), \qquad -2ru = fu - 2v^2sw;$$

from which $$K = -k\frac{dX}{dx} = -k.C.\varepsilon^{rt+sx}.(s.\cos\overline{ut+wx+C'} - w.\sin\overline{ut+wx+C'}).$$

(342.) Let x' be the ordinate of any place upon the bank of the river, to which the situation of a particle of water corresponds at any instant. Then $x' = x + X$ nearly, or $x = x' - X$; and, as the only term of X which becomes large is $-bt$, we may take $x = x' + bt$. This makes $rt+sx = (r+bs)\,t + sx'$, $ut+wx = (u+bw)\,t + wx'$; and the expression for K corresponding to the place x' on the river bank is

$$K=-k.C.\varepsilon^{(r+bs)t+sx'}\left(s.\cos\overline{(u+bw)\,t+wx'+C'}-w.\sin\overline{(u+bw)\,t+wx'+C'}\right).$$

At the mouth of the river, $x'=0$, and this expression becomes

$$-kC.\varepsilon^{(r+bs)t}.\left(s.\cos\overline{(u+bw)\,t+C'}-w.\sin\overline{(u+bw)\,t+C'}\right).$$

But the law of the rise of the sea is assumed to be $A.\sin\overline{nt+B}$. Making these expressions coincide, we have

$$r+bs=0, \qquad u+bw=n,$$
$$-kCs.\cos(C'-B)+kwC.\sin(C'-B)=0, \qquad kCs.\sin(C'-B)+kCw.\cos(C'-B)=A.$$

The two last equations give only the values of C and C′. Combining the two preceding them with the two found in (341.) we have four equations for the four quantities r, s, u, w; and eliminating r, u, w, we obtain

$$-(v^2-b^2)s^2+(v^2-b^2)\left(\frac{fn-2bns}{2(v^2-b^2)s+fb}\right)^2-fbs+\frac{2bfn^2-4b^2n^2s}{2(v^2-b^2)s+fb}-n^2=0;$$

a biquadratic equation for s. Then $w=\frac{fn-2bns}{2(v^2-b^2)\,s+fb}$. Without going through the solution, we will only remark, that when b is small, a solution may be obtained which will allow a negative value $-s'=\frac{-f}{2v-2b}$ to be taken for s; that then w has a negative value $-w'=\frac{-1}{v-b}$; and that the elevation of the water is then expressed by

$$A.\varepsilon^{-s'x'}.\sin\overline{nt-w'x'+B}.$$

This elevation is above the mean level of the river at that point. But the mean level there is higher than that of the sea by $x'.\sin\alpha$. Therefore the surface of the river is higher than the mean level of the sea by

$$x'\sin\alpha+A.\varepsilon^{-s'x'}.\sin\overline{nt-w'x'+B}.$$

Low water in the river may be higher than high water in the sea.

(343.) At low water at any place, $\sin\overline{nt-w'x+B}=-1$, and therefore the elevation of low water at any place, above the mean level of the sea, is $x'\sin\alpha-A.\varepsilon^{-s'x'}$. The elevation of the high tide of the sea above its mean level is $+A$. The low water then at a point up the river will be higher than the high water of the sea if $x'\sin\alpha-A\varepsilon^{-s'x'}$ be greater than A. As, by increasing x', $x'\sin\alpha$ may be made as great as we please, and $A\varepsilon^{-s'x'}$ as little as we please, it is evident that a point may be found where this condition is satisfied. The circumstance that low water on a tidal river may be higher than high water on the sea, paradoxical as it may appear, is therefore a simple consequence of theory.

(344.) We shall conclude with the following Problem. The water being in the state of undulation represented by $X=L.\cos\overline{it-mx}$, the forces which have maintained it in that state suddenly cease when $t=\alpha$: to find the subsequent motion of the water.

Motion of water supposing the tidal forces to cease.

(345.) It is evident that there can be no such multiplier as ε^{sx} in the expression for X, since there is none such when $t=\alpha$. Let therefore

$$X=\Sigma.C.\varepsilon^{rt}.\cos(ut+wx)+\Sigma.C'.\varepsilon^{rt}.\sin(ut+wx).$$

It is plain that w must $=\pm m$. This restricts the assumption to

$$X=C.\varepsilon^{rt}.\cos(ut+mx)+E.\varepsilon^{rt}.\cos(ut-mx)+C'.\varepsilon^{rt}.\sin(ut+mx)+E'.\varepsilon^{rt}.\sin(ut-mx).$$

We have first to find r and u. The general equations become $r^2-u^2=-fr-v^2m^2$; $-2ru=fu$. From the second, $r=-\frac{f}{2}$. Substituting in the first, $\frac{f^2}{4}-\frac{f^2}{2}+v^2m^2=u^2$, or $u=\sqrt{\left(v^2m^2-\frac{f^2}{4}\right)}$. Then the special conditions to be satisfied are, that, when $t=\alpha$, X must $=L.\cos it.\cos mx+L.\sin it.\sin mx$, and $\frac{dX}{dt}$ must $=-iL.\sin it.\cos mx+iL.\cos it.\sin mx$; or X must then $=L\cos i\alpha.\cos mx+L\sin i\alpha.\sin mx$, and $\frac{dX}{dt}$ must then $=-iL.\sin i\alpha.\cos mx+iL\cos i\alpha.\sin mx$. Comparing these with the quantities deduced from the assumed expression, we have

$$\varepsilon^{r\alpha}\cos u\alpha\,(C+E)+\varepsilon^{r\alpha}\sin u\alpha\,(C'+E')=L\cos i\alpha$$
$$\varepsilon^{r\alpha}\sin u\alpha\,(E-C)+\varepsilon^{r\alpha}\cos u\alpha\,(C'-E')=L\sin i\alpha$$
$$\varepsilon^{r\alpha}(r\cos u\alpha-u\sin u\alpha).(C+E)+\varepsilon^{r\alpha}(r\sin u\alpha+u\cos u\alpha).(C'+E')=-iL\sin i\alpha$$
$$\varepsilon^{r\alpha}(r\sin u\alpha+u\cos u\alpha).(E-C)+\varepsilon^{r\alpha}(r\cos u\alpha-u\sin u\alpha).(C'-E')=iL\cos i\alpha.$$

From the first and third, C + E and C′ + E′ are found; from the second and fourth, E − C and C′ − E′ are found; and from these C, E, C′, E′, are found. Then the expression for X is

The motions will diminish rapidly.

$$\varepsilon^{-\frac{ft}{2}}.\left\{C.\cos\overline{ut+mx}+E.\cos\overline{ut-mx}+C'.\sin\overline{ut+mx}+E'.\sin\overline{ut-mx}\right\}.$$

The multiplier $\varepsilon^{-\frac{ft}{2}}$ shows that the oscillations will diminish rapidly and will therefore soon become insensible.

(346.) The functions $\varepsilon^{\pm px}.\genfrac{}{}{0pt}{}{\sin}{\cos}qr$, $\varepsilon^{rt}\genfrac{}{}{0pt}{}{\sin}{\cos}(ut\pm mx)$, $\varepsilon^{rt+sz}.\genfrac{}{}{0pt}{}{\cos}{\sin}(ut+wx)$, are in their meaning and application exactly analogous to the simpler expressions which we have used to represent the free tide-wave in (291.) and the articles which follow it.

Subsection 7.—Theory of Waves upon Open Seas.

Equations for water of three dimensions.

(347.) We shall now point out the form which the investigation assumes when the motion of water in space of three dimensions is considered.

Let z be the original horizontal co-ordinate of any particle measured at right angles to x, and Z the displacement of that particle in the direction of z at the time t, y being the vertical ordinate as before. Then, nearly as in (145.), we shall find the following equation of continuity:

$$\mathrm{Y}=\Xi\frac{d\eta}{dx}+\zeta\frac{d\eta}{dz}-\int_y\left(\frac{d\mathrm{X}}{dx}+\frac{d\mathrm{Z}}{dz}\right)(\text{from } \eta \text{ to } y),$$

where ζ is the value of Z corresponding to the bottom.

And, nearly as in (147.), we shall find the two following equations of equal pressure (no external force being supposed to act):

$$\frac{d^2\mathrm{X}}{dt^2}=\frac{d}{dx}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\ (\text{from } k \text{ to } y)\right\},$$

$$\frac{d^2\mathrm{Z}}{dt^2}=\frac{d}{dz}\left\{-g\mathrm{K}+\int_y\frac{d^2\mathrm{Y}}{dt^2}\ (\text{from } k \text{ to } y)\right\}.$$

We shall not attempt to solve these equations, except in the case where the depth is uniform, and where the oscillations are of uniform period (as in all the preceding investigations). And, even with these limitations, we shall find our solution exceedingly restricted.

(348.) Assuming, then, the same function of y as that which has occurred in the preceding investigations, (the propriety of which will be proved by its satisfying the equation so as to make the solution possible as regards the other co-ordinates, with the utmost generality,) and assuming the same relation between m and n, and making our expressions multiply $\cos nt$. (the same investigation applying in all respects to $\sin nt$,) let us suppose

$$\mathrm{X}=\mathrm{U}.(\varepsilon^{my}+\varepsilon^{-my}).\ \cos nt,$$
$$\mathrm{Z}=\mathrm{V}.(\varepsilon^{my}+\varepsilon^{-my}).\ \cos nt,$$

U and V being functions of x and z only.

Then $$\mathrm{Y}=-\int_y\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right).\ (\varepsilon^{my}+\varepsilon^{-my}).\ \cos nt=\frac{1}{m}.\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right).(-\varepsilon^{my}+\varepsilon^{-my}).\ \cos nt.$$

And $$\frac{d^2\mathrm{Y}}{dt^2}=\frac{n^2}{m}.\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right).\ (\varepsilon^{my}-\varepsilon^{-my}).\cos nt:$$

therefore $$-\int_y\frac{d^2\mathrm{Y}}{dt^2}\ (y \text{ to } k)=\frac{n^2}{m^2}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right).(-\varepsilon^{mk}-\varepsilon^{-mk}+\varepsilon^{my}+\varepsilon^{-my}).\cos nt.$$

Also $$-g\mathrm{K}=\frac{g}{m}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right)(\varepsilon^{mk}-\varepsilon^{-mk}).\cos nt.$$

Therefore $-g\mathrm{K}-\int_y\frac{d^2\mathrm{Y}}{dt^2}\ (y \text{ to } k)$, having regard to the equation $n^2(\varepsilon^{mk}+\varepsilon^{-mk})=gm\,(\varepsilon^{mk}-\varepsilon^{-mk})$, becomes

$$\frac{n^2}{m^2}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right)(\varepsilon\quad+\varepsilon^{-my}).\cos nt.$$

Also $$\frac{d^2\mathrm{X}}{dt^2}=-n^2\mathrm{U}.(\varepsilon^{my}+\varepsilon^{-my}).\cos nt,\qquad \frac{d^2\mathrm{Z}}{dt^2}=-n^2\mathrm{V}.(\varepsilon^{my}+\varepsilon^{-my}).\cos nt.$$

Hence the equations of equal pressure become

$$-n^2\mathrm{U}.(\varepsilon^{my}+\varepsilon^{-my}).\cos nt=\frac{n^2}{m^2}.\frac{d}{dx}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right).(\varepsilon^{my}+\varepsilon^{-my}).\cos nt,$$

$$-n^2\mathrm{V}.(\varepsilon^{my}+\varepsilon^{-my}).\cos nt=\frac{n^2}{m^2}.\frac{d}{dz}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right)(\varepsilon^{my}+\varepsilon^{-my}).\cos nt,$$

or

$$m^2\mathrm{U}+\frac{d}{dx}\left(\frac{d\mathrm{U}}{dx}+\frac{d\mathrm{V}}{dz}\right)=0$$

$$m^2 V + \frac{d}{dz}\left(\frac{dU}{dx} + \frac{dV}{dz}\right) = 0;$$

equations which are of the most general kind for the determination of U and V, and which are cleared of y and t.

Partial differential equation whose solution is necessary.

(349.) If we differentiate the first of these equations with regard to x, and the second with regard to z, and then add them together, and if we put W for $\frac{dU}{dx} + \frac{dV}{dz}$, we find

$$m^2 W + \frac{d^2 W}{dx^2} + \frac{d^2 W}{dz^2} = 0.$$

It will be remarked that W is proportional to the factor of cos nt in the expression for K, so that, if we could solve this equation, we should at once obtain the expression for the elevation of the wave (supposed stationary) at every point. And as $m^2 U = -\frac{dW}{dx}$, $m^2 V = -\frac{dW}{dz}$, all the circumstances of the motion of the water would be completely known.

Form of solution which would be desirable.

(350.) There are, however, great difficulties in the solution of this equation. The most convenient form for our purposes would be $W = P \cos Q$, P and Q being functions of x and z. If we could obtain this, we could also obtain another $W' = P \sin Q$; and combining the former of these as factor with cos nt and the latter with sin nt, we should have for the value of K

$$-\frac{1}{m}(\varepsilon^{mk} - \varepsilon^{-mk})\ P\,(\cos Q \,.\, \cos nt + \sin Q \,.\, \sin nt)$$

$$= -\frac{1}{m}\ (\varepsilon^{mk} - \varepsilon^{-mk}) \,.\, P \,.\, \cos \overline{nt - Q},$$

and the equation determining the position of the ridge of wave at any time t would be

$$Q = \text{constant}.$$

But the general solution of the equation in this form does not appear practicable.

(351.) There are two limited solutions (and perhaps others) which may be easily shown to satisfy the equation. The first may be interpreted partially; the second completely.

1st. Let $W = \phi(r)$, where $r = \sqrt{\{(x - a)^2 + (z - b)^2\}}$. Then $\frac{dW}{dx} = \frac{dW}{dr} \cdot \frac{dr}{dx} = \phi'(r) \,.\, \frac{x - a}{r}$: $\frac{d^2 W}{dx^2} = \frac{\phi'(r)}{r} + \overline{x - a} \,.\, \left(\frac{\phi''(r)}{r} - \frac{\phi'(r)}{r^2}\right) \cdot \frac{dr}{dx} = \frac{\phi'(r)}{r} + \left(\phi''(r) - \frac{\phi'(r)}{r}\right) \frac{(x - a)^2}{r^2}$. Similarly $\frac{d^2 W}{dz^2} = \frac{\phi'(r)}{r} + \left(\phi''(r) - \frac{\phi'(r)}{r}\right)\frac{(z - b)^2}{r^2}$. Substituting in the equation $m^2 W + \frac{d^2 W}{dx^2} + \frac{d^2 W}{dz^2} = 0$, we obtain

$$m^2 \phi(r) + \frac{\phi'(r)}{r} + \phi''(r) = 0.$$

Solution expressing annular waves.

The solution of this equation is the following, in which the letter S is put to denote the definite integral between the limits 0 and π:

$$\phi(r) = C \,.\, S_v \cos(mr \cos v) + C' \,.\, S_v \{\cos(mr \cos v) \,.\, \log(r \sin^2 v)\}$$

where v is a new variable, introduced solely for the purpose of forming a function which is to be integrated, and disappearing entirely from the result, which is the sum of two integrals between definite limits. But the values of the two definite integrals cannot be expressed by means of any usually tabulated quantities, and must be computed numerically. (A table of the values of the first integral, to a small extent, will be found in the *Philosophical Magazine* for January, 1841, page 7.) Putting S′ and S″ for the two integrals, corresponding to a given value of r, the most general form for W or $\phi(r)$ will be

$$E' \,.\, S' \,.\, \cos(nt + F') + E'' \,.\, S'' \,.\, \sin(nt + F''),$$

E′, E′, F′, and F″, being arbitrary constants. It is evident that this form of W expresses a series of circular waves converging to or diverging from the point whose co-ordinates are a, b.

(352.) The equation determining W will also be satisfied by the sum of any number of functions $\phi_{,}(r_{,})$, $\phi_{,,}(r_{,,})$, &c., where $r_{,} = \sqrt{\{(x - a_{,})^2 + (z - b_{,})^2\}}$, $r_{,,} = \sqrt{\{(r - a_{,,})^2 + (z - b_{,,})^2\}}$, &c., and where each of the functions $\phi_{,}$, $\phi_{,,}$, &c. satisfies the equation $m^2 \phi(r) + \frac{\phi'(r)}{r} + \phi''(r) = 0$. That is, there may be any number of systems of such circular waves, each system converging to or diverging from an arbitrary centre.

Solution expressing parallel waves.

(353.) 2d. Let $W = A \,.\, \cos(ax + bz)$: on substituting we obtain $m^2 - a^2 - b^2 = 0$ as the only condition. The same holds if we assume $W' = A \,.\, \sin(ax + bz)$: combining the former as factor with cos nt and the latter with sin nt, we find for the elevation of any part of the water

$$-\frac{1}{m}(\varepsilon^{mk}-\varepsilon^{-mk})\,\mathrm{A}.\cos(nt-ax-bz).$$

The positions of the ridges of waves at the time t are determined by making $ax+bz=nt\pm\pi$, or $=nt+3\pi$, or $=nt\pm5\pi$, &c. The ridges, therefore, are all parallel to the line whose equation is $ax+bz=0$. If from the origin of co-ordinates we draw a perpendicular upon one of the ridges, its length is found to be $\frac{nt\pm\pi}{\sqrt{a^2+b^2}}$, or $\frac{nt\pm3\pi}{\sqrt{a^2+b^2}}$, &c., that is $\frac{nt\pm\pi}{m}$, $\frac{nt\pm3\pi}{m}$, &c. The distance, therefore, from one ridge to the next is $\frac{2\pi}{m}$: and the velocity with which each ridge travels on is $\frac{n}{m}$.

(354.) The equation determining W will also be satisfied by the sum of any number of expressions $\mathrm{A}_{\prime}\cos(nt-a_{\prime}x-b_{\prime}z)$, $\mathrm{A}_{\prime}\cos(nt-a_{\prime\prime}x-b_{\prime\prime}z)$, &c., provided that $a_{\prime}^2+b_{\prime}^2=m^2$: $a_{\prime\prime}^2+b_{\prime\prime}^2=m^2$, &c. Each of these denotes a series of parallel waves with the interval $\frac{2\pi}{n}$ between one wave and the next, the waves being parallel to any arbitrary line. And the circumstance of the equation being satisfied by the algebraic sum of the different solutions indicates that the elevation of the water at the intersection of any ridges will be the algebraic sum of the elevations corresponding to each ridge. The same remark applies to the sum of the solutions representing circular waves, or to the sum of any number of solutions of both these kinds or of any other kinds.

Reflexion of parallel waves from a straight boundary.

(355.) Now suppose the water to be terminated on one side by a straight boundary: let the co-ordinates be so taken that the boundary may be parallel to z; let the corresponding value of x be c; then, whatever be the value of z while $x=c$, the motion of the particles of water in the direction of x must at all times be 0. For, all the particles which are once in contact with the boundary, that is, all those for which $x=c$, must remain in contact with the boundary; that is, they must always have $x=c$; and, therefore, X must $=0$. It is plain that this condition cannot be satisfied if we confine the expression for the elevation to the single term $-\frac{1}{m}(\varepsilon^{mk}-\varepsilon^{-mk})\,\mathrm{A}.\cos(nt-ax-bz)$. For then $\mathrm{W}=\mathrm{A}.\cos(ax+bz)$, $\mathrm{W}'=\mathrm{A}.\sin(ax+bz)$: $\mathrm{U}=-\frac{1}{m^2}.\frac{d\mathrm{W}}{dx}$ $=\frac{a}{m^2}\mathrm{A}.\sin(ax+bz)$, $\mathrm{U}'=\frac{-a}{m^2}\mathrm{A}.\cos(ax+bz)$; and the complete value of $\mathrm{X}=\mathrm{U}.\cos nt+\mathrm{U}'.\sin nt=$ $-\frac{a\mathrm{A}}{m^2}\sin(nt-ax-bz)$; which is not generally $=0$ when $x=c$. But it may be made to satisfy the required condition by adding another term which itself satisfies the original equation $m^2\mathrm{W}+\frac{d^2\mathrm{W}}{dx^2}+\frac{d^2\mathrm{W}}{dz^2}=0$. For let the values, as increased by the new terms, be

$$\mathrm{W}=\mathrm{A}\cos(ax+bz)+\mathrm{A}\cos(2ac-ax+bz),$$
$$\mathrm{W}'=\mathrm{A}\sin(ax+bz)+\mathrm{A}\sin(2ac-ax+bz);$$

then the new value of U or $-\frac{1}{m^2}\frac{d\mathrm{W}}{dx}$ is

$$\frac{a\mathrm{A}}{m^2}\sin(ax+bz)-\frac{a\mathrm{A}}{m^2}\sin(2ac-ax+bz);$$

and that of U′ is

$$-\frac{a\mathrm{A}}{m^2}\cos(ax+bz)+\frac{a\mathrm{A}}{m^2}\cos(2ac-ax+bz):$$

and, therefore, the new value of X or $\mathrm{U}.\cos nt+\mathrm{U}'.\sin nt$ is

$$-\frac{a\mathrm{A}}{m^2}\sin(nt-ax-bz)+\frac{a\mathrm{A}}{m^2}\sin(nt-2ac+ax-bz);$$

and upon making $x=c$ this expression becomes

$$-\frac{a\mathrm{A}}{m^2}\sin(nt-ac-bz)+\frac{a\mathrm{A}}{m^2}\sin(nt-ac-bz)$$

which $=0$ whatever be the values of x and t. Thus we find that the existence of one series of waves and the assumption of a rectilinear boundary imply the existence of another system of waves, whose elevation will be represented by substituting in the expression for K the additional terms of W and W′, and will therefore be

$$-\frac{1}{m}(\varepsilon^{mk}-\varepsilon^{-mk}).\mathrm{A}\,\{\cos(2ac-ax+bz)\cos nt+\sin(2ac-ax+bz)\sin nt\}$$
$$=-\frac{1}{m}(\varepsilon^{mk}-\varepsilon^{-mk}).\mathrm{A}.\cos(nt-2ac+ax-bz).$$

This expression, examined in the same manner as before, represents a series of parallel waves, in which the equation to the ridge of every one is $-ax+bz=nt-2ac\pm\pi$, &c., and which are all parallel to the line whose equation is $-ax+bz=0$. The ridges of the former waves were found to be parallel to the line whose equation is $ax+bz=0$. Hence the inclinations of the ridges of the two sets of waves to the boundary are equal, but they are inclined opposite ways. This is the mathematical explanation of the reflexion of waves from a straight boundary.

(356.) The whole elevation of the waves is

$$-\frac{1}{m}(\varepsilon^{mk}-\varepsilon^{-mk})\,\mathrm{A}\,(\cos\overline{nt-ax-bz}+\cos\overline{nt-2ac+ax-bz});$$

which, when $x=c$, becomes

$$-\frac{2}{m}(\varepsilon^{mk}-\varepsilon^{-mk})\,\mathrm{A}.\cos\overline{nt-ac-bz}.$$

The coefficient, therefore, for the undulation at contact with the boundary, is twice as great as that of an unreflected wave.

Reflexion of waves of any kind from a straight boundary.

(357.) In the same manner, if we take the expression for W in its most general state, putting it in the form $\phi(x,z)$, we may add to it another term $\phi(\overline{2c-x},z)$, which will satisfy the equation $m^2\mathrm{W}+\frac{d^2\mathrm{W}}{dx^2}+\frac{d^2\mathrm{W}}{dz^2}=0$ equally well, and which in the expression for X adds the new term $+\phi'(\overline{2c-x},z)$ to the former term $-\phi'(x,z)$, the sum of which is 0 when $x=c$. And, as above, the expression for K is

$$-\frac{1}{m}(\varepsilon^{mk}-\varepsilon^{-mk})\{\phi(x,z)+\phi(\overline{2c-x},z)\},$$

which, when $x=c$, becomes $-\frac{2}{m}(\varepsilon^{mk}-\varepsilon^{-mk}).\phi(c,z)$, or is double that at the same point in an unreflected wave. The additional term for W being $\phi(c+\overline{c-x},z)$, and the original term being $\phi(c+\overline{x-c},z)$, it is evident that the system of waves represented by one expression depends on $x-c$, in the same manner in which the other depends on $c-x$, and is, therefore, a reflected system whose form is exactly similar to the form in which the original system would have proceeded if not stopped by the boundary.

(358.) Leaving for the present the consideration of the motion of the waves as determined by the differential equations, we shall consider one case in which we seem to derive some assistance from general reasoning.

Form of the crest of the wave in broad channels which are shallow near the sides.

(359.) Suppose that a tide-wave is travelling along a canal of large dimensions, and of variable depth in its cross section, the depth diminishing gradually to both shores. (We may suppose the dimensions to be such as those of the English Channel, or any similar arm of the sea.) It is evident that the investigation of (218.) does not apply here: for, on account of the shallowness of the water at the sides, the velocity of flow towards both sides to produce the elevation of water there must be comparable with, perhaps equal to, the velocity of flow at mid-channel in the direction of the canal's length. Moreover, as the slope of the bottom is exceedingly small, the waves in every part of the channel will be travelling in nearly the same manner as if the extent of sea of the same depth were infinitely great, and will therefore travel with the velocity due to that depth: and, therefore, the ridge of wave cannot possibly stretch transversely to the channel, and travel along with uniform velocity lengthways of the channel. The state of things, then, will be this: the central part of the wave will advance rapidly (171.) along the middle of the channel; the lateral parts will not advance so rapidly; and the whole ridge will assume a curved shape, its convex side preceding. When this form is once acquired, it may perhaps proceed with little alteration; for if, as in figure 18, we suppose two such curves exactly similar, but one a little in advance of the other, the space which separates the wings of the two curves, measured perpendicularly to the curves, (the direction in which that part of the wave must really travel,) is much less than the space which separates the centres of the curves, and by proper inclination may be less in any proportion; and, therefore, may represent exactly the space travelled over by the wave at that depth while the wave at the greater depth travels over the greater space. That part of the ridge of the wave which is nearest to the coast will, therefore, assume a position nearly parallel to the line of coast.

(360.) Now the wave whose ridge is nearly parallel to the coast, or which advances almost directly towards the coast, will be a wave of the same character as that treated of in (307.). For the slope of the beach adds to the surface of the sea a very insignificant quantity, as compared with the breadth of the tide-wave, and the general effect is the same as if a perpendicular cliff terminated the sea on that side. Therefore, for those parts of the sea which are near to the coasts the law of (307.) holds; namely, the greatest horizontal displacement of the particles occurs at the same time as the greatest vertical displacement; and, therefore, when the sea is rising, the water is, for some distance from the coast, flowing towards the coast, and when it is falling, the water is flowing from the coast.

(361.) In mid-channel, the motion of the water will be such as is described in (184.), &c.; that is, the water will be flowing most rapidly up the channel at the time of high water, and its motion upwards will cease when the water has dropped to its mean height.

(362.) From this there follows a curious consequence with regard to the currents at an intermediate distance from the shore, where the effects of these two motions may be conceived to be combined.

At high water the water is not flowing to or from the shore, but is flowing up the channel.

When the water has dropped to its mean elevation, the water is ebbing from the shore, but is stationary with regard to motion up or down the channel.

At low water, the water is not flowing to or from the shore, but is running down the channel.

When the water has risen to its mean height, the water is flowing to the shore, but is stationary with regard to motion up or down the channel.

Revolving motion of the tide near the shore.

(363.) Consequently, in the course of one complete tide, the direction of the current will have changed through 360°, the water never having been stationary. And the direction of the change of current will be of such a kind that, if we suppose ourselves sailing up the mid-channel, the tide-current will turn, in those parts which are on the *left* hand, in the *same* direction as the hands of a watch; and in those parts which are on the *right* hand, in the direction *opposite* to that of the hands of a watch.

(364.) Beyond this we can add little to the Theory of Waves upon a sea extended in both dimensions. But the following remarks will be found important with reference to the method of determining from observations some of the phenomena of tides.

Cotidal lines on open seas.

(365.) In tracing the progress of the tide across an extended sea, we cannot observe the different waves as we can those upon a small piece of water. We can do nothing but make observations of the time of the rise and fall of the sea at many different points along the shores of the bounding continents, or at islands in different parts of the sea: and when we have thus ascertained the absolute time of high water at many different points, if they are sufficiently numerous, we may draw lines over the surface of the sea passing through all the points at which high water takes place at the same absolute instant. These lines (adopting the word introduced into general use by the highest authority on the discussion of tide-observations) we shall call *cotidal lines*. The tracing out the cotidal lines in different seas is the greatest advance that has yet been made in the discussion of the phenomena of the tides in open seas.

The relation between the cotidal lines and the waves from which they originate is very obscure.

(366.) Now when the series of waves is single, the cotidal lines correspond exactly with the lines marking the position of the ridge of the wave at different times. But when the series of waves is compound, it may happen that the form of the cotidal lines will not present to the eye the smallest analogy with the forms of the ridges of the mingled waves. This will be seen in a simple instance.

(367.) Suppose that there are two systems of parallel waves rolling across the sea at the same time, the ridges of all the waves of one series being parallel to x, and the ridges of all the waves of the other being parallel to z. The forms of the ridges, as they might be seen by an eye placed at a sufficient distance, would be the intersecting lines represented in figure 19. The elevation of water caused by the former at the time t will be represented by $b.\cos\overline{nt-mz}$: that caused by the latter will be represented by $a.\cos\overline{nt-px}$. We have taken the same coefficient for t in both these expressions, because the recurrence of tide-waves, whether forced waves, or free waves introduced in the integration for satisfying the limiting conditions, must be periodical as the exciting cause. But the coefficients of x and z may be different: thus, for instance, one of these waves may be a forced tide-wave, and the other may be a free tide-wave, in which case the coefficients (292.) will not necessarily be the same: this is, however, immaterial to the present investigation.

(368.) The whole elevation, then, of the water at any point will be

$$a.\cos\overline{nt-px}+b.\cos\overline{nt-mz}$$

$$=(a\cos px+b\cos mz)\cos nt+(a\sin px+b\sin mz)\sin nt,$$

which has its maximum value for that point when

$$\tan nt=\frac{a\sin px+b\sin mz}{a\cos px+b\cos mz}.$$

This expression for t determines the time of high water at that place; and, therefore, the line connecting all the points at which it is high water at the same instant will be determined by making $\frac{a\sin px+b\sin mz}{a\cos px+b\cos mz}$ equal to the same quantity $\tan nt$. Giving a definite value C to the quantity nt, we have for the equation to a certain cotidal line,

$$\frac{a\sin px+b\sin mz}{a\cos px+b\cos mz}=\tan \mathrm{C},$$

from which

$$\sin\overline{mz-\mathrm{C}}=-\frac{a}{b}\sin\overline{px-\mathrm{C}}.$$

(369.) If $a=b$, this equation becomes $\sin\overline{mz-\mathrm{C}}=-\sin\overline{px-\mathrm{C}}$; whence $mz-\mathrm{C}=-px+\mathrm{C}$, or $=-px+\mathrm{C}\pm 2\pi$, or $=-px+\mathrm{C}\pm 4\pi$, &c., or $mz-\mathrm{C}=px-\mathrm{C}\pm\pi$, or $=px-\mathrm{C}\pm 3\pi$, &c. These expressions evidently represent two series of straight lines, making equal angles with the co-ordinates: one of them is stationary, (as C or nt disappears from the equation,) and its deduction from the investigation above denotes that there is no sensible tide along those lines: the other is the real system of cotidal lines. The former is represented in figure 20 by the double lines, the latter by the single lines. In drawing the latter, we have supposed C to vary successively by $\frac{\pi}{2}$ at each step.

(370.) If $a<b$, the first expression for $mz-\mathrm{C}$ is included between the arc whose sine is $\frac{a}{b}$, and the arc

whose sine is $-\frac{a}{b}$, which values $mz - \mathrm{C}$ attains when $px - \mathrm{C}$ is $\frac{\pi}{2} \pm 2n\pi$ and $-\frac{\pi}{2} \pm 2n\pi$. The corresponding curves for values of C varying successively by $\frac{\pi}{2}$ are represented in figure 21.

Section V Experiments on Waves. —

(371.) If $a > b$, the curves will be such as those represented in figure 22.

(372.) It is evident that the contemplation of these curves (more especially if a small part only of each can be traced) will not easily enable us to discover the nature of the simple interfering systems of waves from which they originate. If either system were complicated, as in (303.), the difficulty would be still greater.

SECTION V.—ACCOUNT OF EXPERIMENTS ON WAVES.

(373.) Our theory of waves, as we have remarked in the Introduction, is not the most complete that could be devised. It embraces (as we believe) every case of general interest to which mathematics are at present applicable, but it does not comprehend those special cases which have been treated at so great length by Poisson (*Mémoires de l'Institut*) and Cauchy (*Savans Etrangers*). With respect to these we may express here an opinion, borrowed from other writers, but in which we join, that as regards their physical results these elaborate treatises are entirely uninteresting; although they rank among the leading works of the present century in regard to the improvement of pure mathematics. We shall not therefore trouble ourselves with detailing the few imperfect experiments of Biot (*Mémoires de l'Institut*) and Bidone (*Turin Memoirs*) which have been made in verification of these theories.

(374.) One of the most important works that has been published, in regard to experiments as well as to the theory of Waves, is that by the two brothers Weber, entitled "*Wellenlehre auf Experimente gegründet.*" This work contains an abstract of all the theories and all the principal experiments of preceding writers that the authors had been able to collect. The points however to which we shall allude here are the experiments made by the Webers themselves. These were made with an apparatus which they call *Wellenrinne.* It is a very narrow trough with glass sides. In one instance it was 5 ft. 4 in. long, (Paris measure,) about 8 inches deep, and about ½ an inch wide; in another instance it was 6 feet long, 2½ feet deep, and a little more than an inch wide. The glass sides were properly supported by pieces of wood connected with the bottom; in the smaller, the glass sides were continuous; in the larger, the glass only occupied 6 openings in different parts of the sides, the other parts being of wood. Some experiments were made with quicksilver, and some with brandy; but the principal part were made with water containing a great number of floating particles of the same specific gravity as the water; by observing the movements of these through the glass sides, sometimes with the naked eye and sometimes with a microscope, the motions of the particles of water, even to the bottom, were easily examined. The waves of experiment were generated by plunging a glass tube into the fluid, raising the fluid into the tube by suction, and then allowing it suddenly to drop.

Weber's *Wellenrinne.*

Motion of particles floating in the water observed.

Method of determining the form of the wave.

(375.) The form of the wave was determined by immerging into the liquid with its plane vertical a slate sprinkled with flour, and suddenly withdrawing it. In this way it appears that a satisfactory trace of the front of the wave was obtained; the liquid removing the flour from every part which it touched. In a nearly similar manner the form of the back of the wave was obtained; the slate being in this case suddenly plunged into the fluid; these determinations are however confessedly much less satisfactory than the former. The experimenters, however, were able to ascertain that, when the height of the wave was large in proportion to its depth, its front was much steeper than its back, as our theory of (203.) gives it.

(376.) The wave was sometimes observed when it had run to one end of the trough and was reflected there, (a method carried to great perfection in Mr. Russell's experiments, to be described hereafter.) But generally the observations were made soon after the wave was formed. We have no doubt that some irregularities in the results were entirely due to the mixture of waves of various lengths which always occurs at first, and that they would have been avoided if the actual observation had been deferred till the principal wave had cleared itself of the small waves.

(377.) By inspection of the motion of the particles, the Webers discovered the following general rules. When a wave-ridge is followed by an equal wave-hollow, every particle moves in an ellipse, (or a curve as near to an ellipse as the eye can judge,) whose major axis is horizontal; the motion of the particle when in the highest part of the ellipse being in the same direction as the motion of the wave, and in the opposite direction when at the lowest part of the ellipse. (Fig. 23 is copied from Weber's figure.) When a small wave-hollow follows a large wave-ridge, the motion is such as is represented in fig. 24; and when a large wave-hollow follows a small wave-ridge, the motion is such as is represented in fig. 25. These motions are all in general conformity with the results of our theory in (182.); it being remarked that, by the theory of (226.), &c., the same may (with certain combinations) apply to a single wave. At different depths the motion was different; the horizontal motion being diminished in some degree for the deeper particles, and the vertical motion being very much diminished, so that, on approaching the bottom, the ellipse became nearly a horizontal line, as shown in figure 26. These results agree with those of (177.), &c. It was also found that different particles in the same vertical line described corresponding parts of their courses at the same instant of time, as we have found in (162.).

Observed laws of motion of individual particles.

They agree well with theory.

(378.) From contemplation of these *experimental* circumstances, the Webers constructed figure 27 to represent the motion of particles at the surface of a progressive wave followed by other waves. We need scarcely point out to the reader that these motions coincide exactly with those which we have found in (182.).

Observed motion of particles in a succession of waves.

(379.) Some discordances were found in the results, depending on the manner in which the wave was produced, and which it would be extremely difficult to

Irregularities depending on the way of producing the wave.

Section V. Account of Experiments on Waves.

compare with theory. Thus when the suction-tube was plunged deep in the fluid, it was sometimes found that the horizontal motion of particles near the bottom was greater than that of particles at about half the depth. The form of the waves was varied by plunging the suction-tube to different depths. When it was very deep, the wave produced was long and flat; when it only touched the surface, the wave was short and high. In the latter case it was found that each particle performed its elliptical revolution in a shorter time than in the former; as the theory of (169.) gives.

(380.) In some experiments it was found that the time occupied by particles near the bottom in describing their elliptic courses was less than that occupied by particles near the surface. It is plain that some complicated system of waves was here produced by some peculiarity in the primary disturbance, of which we can give no further account.

(381.) Each particle described its second course in a shorter time than the first. This is evidently caused by a small wave following a large one.

(382.) In regard to the general velocity of the wave, the Webers found that it was increased by increasing the depth of the fluid in the trough, but they did not ascertain the law. They also found that it was independent of the specific gravity of the fluid. They found that a bulky wave travels more quickly than a small one, as appears from (208.).

Observed motion of particles when two waves meet.

(383.) Observations were also made of the motion of the particles when two equal waves meet each other. It was found here that the motion of each particle was backwards and forwards in a straight line, as is represented in fig. 28, which is copied from Weber's figure. We need scarcely to point out that this is precisely the same kind of motion as that which we have found from theory in (189.), &c.

(384.) Other observations were made by the Webers, but none which seem to bear closely upon our theory.

General value of Weber's experiments.

(385.) In regard to the experiments that we have abstracted, we may give our opinion as follows:—The contrivance of using a vessel with glass sides and observing the motions of floating particles is one so admirably adapted to overcome the greatest of all the difficulties attending the comparison of a wave-theory with experiment, namely, that of ascertaining the laws of movement of individual particles, that we think it gives these experiments a claim for superiority above all others. In other respects we think causes of uncertainty may be pointed out. The narrowness of the troughs used makes the effect of any irregularity of the sides great. The rapidity of the observation throws great doubt on the measures of time. However much the Webers might be inclined to trust to their "Tertien-Uhr," (a watch with which the $\frac{1}{60}$ part of a second of time could be observed,) we have little confidence in the use of it. The same cause—namely, the observation of the waves as soon as they were formed—has introduced great complexity into the facts of experiment, which would not have existed if the slower process used by Mr. Russell had been adopted. Although a *complete* theory ought to explain the *most complicated* experiments, yet, under all the difficulties of wave-mathematics, we must confine ourselves to simple cases if we wish to have valid comparisons of theory and observation. We would however point out to any future observer the use of Weber's *Wellenrinne* with several of Russell's methods of observation as likely to give better results than any yet obtained.

(386.) No allusion is made to theory, in the course of the Webers' experiments; and though they have stated the leading points of several theories, (in another part of their book,) they do not appear to have the power of familiarly applying them. We look upon their experiments therefore as quite free from theory, and for that reason we consider their coincidences with theory as peculiarly valuable.

Russell's experiments.

(387.) Mr. Russell's experiments on Waves are contained in the Report of the Seventh Meeting of the British Association, p. 417—496. They constitute, upon the whole, the most important body of experimental information in regard to the motion of Waves which we possess. We shall endeavour here to epitomize the principal contents of that paper, (omitting, for the present, all that relates to the tide-wave;) it will be necessary, however, to make some remarks upon Mr. Russell's references to theory, because we believe that any one who should derive his first knowledge of the nature of waves from that paper would receive from it a most erroneous notion of the extent of the Theory of Waves at the date of those experiments.

Apparatus for creating a wave.

(388.) We shall commence with the experiments made with apparatus arranged expressly for this purpose. A rectangular trough or cistern was constructed, 20 feet long, 1 foot broad, and more than 7 inches deep. At one end, an additional length of 7·3 inches was left, so that in fact the trough really was an uninterrupted trough, whose length was 20 feet 7·3 inches. Only 20 feet, however, was used in the experiments, the remaining part being used for the generation of a wave, in one of the following manners. A sluice being placed at the distance 7·3 inches from the end, water was poured into the small part of the trough behind the sluice, to a known height above the surface of the water in the trough; then, upon raising the sluice, that portion of this water which was higher than the general level (and whose volume therefore was known) rushed into the trough, forming a swell there which was immediately propagated as a wave along the surface of the water in the trough; and the sluice, being depressed, formed a smooth end to the trough in that part from which the wave began. Or, a vertical rectangular trunk, occupying the whole or a part of the small portion at the end of the horizontal trough was filled with water to a certain height, and, by lifting the trunk, that water was allowed to gush out below its lower edge. Or, the sluice of which we have spoken was used to form a wave by merely agitating it with the hand. And in some experiments the disturbance was given by pressing a solid into the water, and in others by withdrawing a solid from the water.

Mode of increasing the range of a wave.

(389.) The method used for measuring the velocities of the waves is extremely ingenious. The length of 20 feet was far too small to permit of any accurate determination of velocity. But Mr. Russell remarked that the wave, upon meeting one of the vertical ends of the trough, was reflected without alteration of form, and therefore could be observed in its reflected course as well as if the trough had been prolonged; and, as the same remark applied to every reflection at each end of the trough, the trough might be used as a channel of indefinite length. (The theory of (355.) and (357.) shows that the reflection from the plane end will in all cases produce a wave of exactly the same kind as that which comes in contact with the end, whatever that kind may be.) Thus the wave was sometimes observed after it had been reflected 60 times, or after it had really described a length of 1200 feet. Moreover, the progress of the wave was observed without difficulty at a great number of points in its course,—for instance, in the experiment just cited, at three points in each length of 20 feet, or in 180 points in the length of 1200 feet. The first observations were usually made after the wave had run the length of the trough once or twice; this allowed many small waves (such as apparently have injured Weber's experiments) to separate themselves and disappear.

Mode of observing the passage of a wave.

(390.) The method of observing the time at which the crest of a wave passed a given point was most happy. The flame of a candle, placed above the trough and at a small horizontal distance from it, was reflected by a mirror in an inclined position downwards to the water, then by the surface of the water it was reflected upwards, and being received upon another inclined mirror was reflected to the eye of an observer, who viewed it through an eye-tube, furnished with an internal wire and a more distant mark for directing the observer's eye. When the water was at rest, or when the horizontal surface at the top of the wave was passing under the mirror, the candle was seen in the centre of the eye-tube; when an inclined part of the wave (either the anterior or the posterior) was passing, the candle was seen on one or other side of the eye-tube. In this manner the passage of the highest part of a wave whose length was three feet, and whose height was only one-tenth of an inch, could be observed with accuracy.

Mode of observing the length and height of a wave.

(391.) The length of the wave was observed by adjusting two fine conical points, which nearly touched the quiescent surface, so that the anterior part of the wave would touch one and the posterior part would leave the other at the same instant. The height of the wave was observed by noting the elevation of the water in small pipes passing from the side of the trough and turning upwards at its outside. We doubt the accuracy of these determinations; they are, however, less important than the determination of velocity; yet we shall presently find that fuller information regarding them would have been valuable.

Species of wave observed by Russell.

(392.) Mr. Russell's researches, in these experiments, were directed entirely to the examination of what he denominates "The great primary wave," and which he describes as "differing in its origin, its phænomena, and its laws, from the undulatory and oscillatory waves which alone had been investigated previous to the researches of Mr. Russell." We are not disposed to recognize this wave as deserving the epithets "great" or "primary," (the wave being the solitary wave whose theory is discussed in (226.) &c.,) and we conceive that, ever since it was known that the theory of shallow waves of great length was contained in the equation $\frac{d^2X}{dt^2}=gk\frac{d^2X}{dx^2}$ (195.), with limitations similar to those in (226.), the theory of the solitary wave has been perfectly well known. Leaving this, however, we may state that Mr. Russell's experiments were all made upon a single wave of considerable length, similar to that discussed in (230.) and (232.), in which a particle is actually moved a certain distance by the wave and then remains at rest in a position differing from its original position. The result in (234.) shows that such a wave may travel, without any force to maintain its motion, provided it be long in proportion to the depth of the fluid; and provided that its velocity be $\sqrt{gk}$, k being the depth; or in other words, that it can, when moving freely, have no other velocity than $\sqrt{gk}$. We are not able to state fully how far Mr. Russell's experiments satisfy the former condition: for Mr. Russell was not aware of the influence of the length of the wave in any case, and therefore has not given it here. Supposing, however, (as his incidental remarks imply,) that the length was 3 feet, this length may correspond to a in the second or principal term in the expression of (232.), and as a has the same place there which λ has in (165.), &c., we may consider $\lambda=3$ feet. Therefore, when the depth was 6 inches, $\frac{\lambda}{k}=6$; when the depth was 4 inches, $\frac{\lambda}{k}=9$, &c. The velocity is therefore (see the table in (170.)) sensibly not the same as if the wave were infinitely long. The wave, therefore, (in conformity with the remarks in (235.),) would tend to split into several waves, each of which would move with its own velocity; and this appears to have happened in some instances (rejected in Mr. Russell's table of results). But the difference of velocities would be small, and probably the elevation observed would be that of the first part, which corresponds to $\lambda=3$ feet.

Influence of the length of the wave upon the velocity, neglected by Mr. Russell.

Influence of the height of the wave upon the velocity.

(393.) There is, however, another point to be considered, namely, that the height of the wave, in many of the experiments, bears a sensible proportion to the depth. According to the theory of (208.), supposing the succession of waves continuous, the top of the wave would travel with a velocity greater than that due to the undisturbed depth, and even greater than that due to the disturbed depth, and expressed by $\sqrt{gk}\times(1+3b)$, where $b=\frac{\text{height of wave}}{\text{depth of water}}$. But if, as in continuous waves, we refer our first calculation not to the undisturbed depth but to the mean depth; then instead of k we must put $k\left(1+\frac{b}{2}\right)$, the mean depth; and instead of $3bk$ we must put $\frac{3bk}{2}$, the greatest elevation above the mean being $\frac{bk}{2}$. Thus the last formula becomes

$$\sqrt{gk\left(1+\frac{b}{2}\right)\left(1+\frac{3b}{2}\right)}=\sqrt{gk}(1+2b).$$

(394.) To examine, then, the general coincidence of Mr. Russell's results with the theory, we have proceeded thus:—We have taken the abstract in pp. 440, 441, 442, of the *Report of the British Association*, having corrected a few errors in it, and have divided the experiments into groups in which the depth of the water and the height of the wave are nearly equal. We have assumed that the mean of the observed velocities corresponds to the mean of the depths, &c., an assumption which is not rigorously true, but probably much nearer to truth than any one experiment. We have then computed the theoretical velocity for the undisturbed depth by the formula of (169.), &c., supposing $\lambda=36$ inches; and in other columns we have altered this velocity in the proportion of $1:\sqrt{1+b}$, $1:\sqrt{1+3b}$, and $1:\sqrt{1+2b}$.

Tabular abstract of experiments.

Limits of Groups in the "Total Depth" of Mr. Russell's Table.	Undisturbed depth of Water in inches.	Height of Wave in inches.	Observed velocity in feet per Second.	Velocity computed for undisturbed Depth.	Computed Velocity $\times\sqrt{1+b}$.	Computed Velocity $\times\sqrt{1+3b}$.	Computed Velocity $\times\sqrt{1+2b}$.
EXPERIMENTS IN WHICH THE HEIGHT OF THE WAVE IS SMALL.							
1·05 and 1·10	1·000	0·075	1·670	1·629	1·689	1·803	1·747
1·30	1·150	·150	1·810	1·744	1·854	2·057	1·958
3·09 to 3·23	2·963	·207	2·860	2·702	2·795	2·972	2·885
3·32 and 3·40	3·080	·280	2·960	2·747	2·869	3·099	2·986
4·00 to 4·31	3·903	·256	3·310	3·016	3·114	3·300	3·208
5·20 to 5·50 (excluding 5·21)	5·088	·252	3·758	3·303	3·384	3·540	3·463
6·40 to 6·65	6·220	·304	4·094	3·495	3·579	3·742	3·662
7·42 to 7·70	7·040	·474	4·406	3·597	3·716	3·943	3·831
EXPERIMENTS IN WHICH THE HEIGHT OF THE WAVE IS LARGE.							
1·20	1·000	0·200	1·760	1·629	1·785	2·061	1·928
1·62	1·300	·320	2·060	1·858	2·072	2·446	2·267
2·19	1·900	·290	2·300	2·217	2·380	2·677	2·533
3·35 to 3·41	2·960	·420	3·010	2·701	2·887	3·225	3·061
3·50 to 3·61	3·020	·532	3·080	2·724	2·954	3·368	3·168
3·69 to 3·97	3·007	·830	3·252	2·719	3·072	3·677	3·388
4·40 to 4·75	3·910	0·625	3·505	3·018	3·250	3·671	3·467
5·21	3·870	1·340	3·820	3·007	3·488	4·293	3·911
5·61 to 5·82	5·070	0·692	3·970	3·300	3·518	3·917	3·723
6·15 to 6·40	5·080	1·160	4·170	3·302	3·659	4·286	3·985
6·69 to 7·20	6·034	0·823	4·262	3·468	3·697	4·117	3·912
7·74 to 8·00	6·946	0·884	4·497	3·586	3·808	4·216	4·017

The velocities agree generally with those computed from theory.

(395.) The experiments which are most favourable for determining the influence of the height of the wave are those of the second group. If we compare the column of "Velocity computed for undisturbed depth" with the column "Observed velocities," we find that *all* the computed velocities are too small. If we compare the "Computed velocity $\times \sqrt{1+b}$," which is the same as that due to the depth measured from the crest of the wave to the bottom of the trough, we find that 9 are too small and 3 too great. If we compare the "Computed velocity $\times \sqrt{1+2b}$," which is that deduced from our theory of (208.), we find that 5 are too small and 7 too great. If we compare the "Computed velocity $\times \sqrt{1+3b}$," we find that 3 are too small and 9 too great. The comparison of the first group leads to nearly the same result; the numbers in the corresponding columns being — all too small — 6 too small, 2 too great — 4 too small, 4 too great — 4 too small, 4 too great. On the whole, therefore, we think ourselves fully entitled to conclude from these experiments that the theory of (208.) is entirely supported; and that the velocity is correctly calculated by supposing it to be that due to the mean depth increased by three times the semi-oscillation in depth, or the whole depth from the crest of the wave increased by the whole oscillation in depth.

The small remaining discordance removed by probable supposition on the length of the waves.

(396.) The reader will, however, remark that the *excesses* of our computed quantities are for the most part in the *small* depths of water, and the *defects* in the *great* depths. We think it most likely that this is due to the difference in the lengths of the waves. It is not unlikely that λ was less than 36 inches in the small depths, and greater than 36 inches in the great depths. If we had calculated with such numbers, we should have found smaller computed velocities for the small depths, and greater for the great depths; and the agreement with the observed velocities would have been extremely close.

Experiments of a general character.

(397.) Other experiments of Mr. Russell's were directed to the inquiry, whether the mode of producing the wave (in other words, the form of the wave) influenced its velocity; it was found that no difference of velocity was perceptible with waves produced in different ways. This is in accordance with (234.). Experiments were also made, (of which no details are given,) which showed that the motion of the particles from the surface to the bottom of the channel is the same, and that particles once in a vertical plane continue in a vertical plane. These results agree with those of (180.).

Negative wave produced by the paddles of a steam-boat.

(398.) Some experiments were made by Mr. Russell on what he calls a *negative* wave—that is, a wave which is in reality a progressive hollow or depression. But (we know not why) he appears not to have been satisfied with these experiments, and has omitted them in his abstract. All the theories of our IVth Section, without exception, apply to these as well as to *positive* waves, the sign of the coefficient only being changed. We may remark, as a matter which may be observed (in some localities) in daily experience, that the phænomenon of a negative wave is given in great perfection by the paddles of a steam-boat: the first wave which passes away from it being a hollow of considerable depth. We were first made aware of this by observation of the traces made by Mr. Bunt's excellent self-registering tide-gauge on the banks of the Avon, at a short distance below Bristol; but we have since frequently remarked it in the waves caused by steamboats in other places, merely by watching the fall and rise of the water on the river banks.

Experiments in triangular and trapezoidal channels.

(399.) Experiments were also made by Mr. Russell as to the velocity of transmission of a wave in a channel whose section is a triangle. In some channels which he used the breadth was $\frac{3}{2}\times$ the depth, in others it was $3\times$ the depth; and one side in all cases was vertical. This construction, in conformity with the remark of (222.), could not be expected to give exactly the velocity determined by the formula of (220.). And it is expressly remarked by Mr. Russell that the wave did not stretch across the channel, but travelled in the manner described in (359.). We should therefore expect the experimental velocity of the leading part of the wave (or that corresponding to the deep side) to approach more nearly than the formula of (220.) gives to that of a rectangular channel of the same depth. This is what appears in some degree from the experiments, although their results are irregular. We select a few in which the height of the wave is small. With these we shall give some observations made in trapezoidal channels. In computing all these we shall use the following process. We shall take the undisturbed depth of the water from Mr. Russell's numbers. From this we shall compute the equivalent depth in a rectangular channel, by dividing the area of section by the breadth of the surface (220.). To this we shall add double the height of the wave, as in (395.), and with this sum we shall compute the velocity.

Tabular abstract of experiments.

Triangular Channel H, breadth $= \frac{3}{2}\times$ depth; both sides above the surface of the water being vertical.

Undisturbed Depth in Inches.	Depth of Equivalent Rectangular Channel.	Height of Wave.	Computed Velocity in Feet.	Observed Velocity.
6·04	3·02	0·14	2·80	2·85
7·04	3·52	0·08	2·96	3·03
7·04	3·52	0·11	2·96	3·05
7·04	3·52	0·12	2·97	3·04

Triangular Channel K, breadth $= 3\times$ depth; both sides above the surface of the water being vertical.

Undisturbed Depth in Inches.	Depth of Equivalent Rectangular Channel.	Height of Wave.	Computed Velocity in Feet.	Observed Velocity.
4·04	2·02	0·10	2·39	2·05
4·04	2·02	0·17	2·40	2·28
4·04	2·02	0·37	2·61	2·50

Trapezoidal Channel L, formed by pouring in one additional inch of water into K.

Undisturbed Depth in Inches.	Depth of Equivalent Rectangular Channel.	Height of Wave.	Computed Velocity in Feet.	Observed Velocity.
5·00	3·00	0·24	2·85	3·20
5·00	3·00	0·42	2·99	3·00
5·00	3·00	0·53	3·10	2·90
5·00	3·00	0·68	3·12	3·04

Trapezoidal Channel M, formed by pouring in one inch of water into L.

Undisturbed Depth in Inches.	Depth of Equivalent Rectangular Channel.	Height of Wave.	Computed Velocity in Feet.	Observed Velocity.
6·01	4·01	0·40	3·22	3·08
6·01	4·01	0·86	3·40	3·50

The general agreement with our theory is at least as close as the agreement of the observations among themselves. The mean of all the triangular results, and that of each of the trapezoidal results, agree closely with the theory.

Experiments in a channel of variable breadth.

(400.) A series of experiments was made in a channel of uniform depth, whose sides were vertical and met in a vertical line, the horizontal plan of the channel being a triangle with one acute angle. Observations of the height of the wave were made at three stations, at which the breadths of the channel were in the proportion of 4, 2, 1. According to the theory of (254.) and (263.), the heights at these points ought to be in the proportion of $\frac{1}{2}, \frac{1}{\sqrt{2}}, 1$. And this proportion very well represents those observed; the inequality being sometimes greater and sometimes less. Thus, comparing the first and third, which ought to be as $\frac{1}{2} : 1$, they are found to be (in the different observations)

1·5 : 3·5
2·0 : 3·3
2·0 : 3·6
1·25 : 2·5
1·5 : 3·25
2·0 : 3·3
1·0 : 2·0
0·25 : 0·4

The velocity cannot be easily calculated, the height of the wave being great in the narrow part of the channel; but it appears, on a general view, to agree well with that in a rectangular channel of the same depth.

Time of breaking of each wave.

(401.) It was always found that the wave broke when its elevation above the general level became equal or nearly so to the general depth. The application of mathematics to this circumstance is so difficult, that we confine ourselves to the mention of the observed fact.

Experiments in a channel of variable depth.

(402.) Observations were also made in a channel (we presume of uniform breadth) whose depth diminished gradually, the bottom being inclined at a slope of 1 in 51. The height of the wave here, in conformity with the theory of (247.) and (264.), ought to be inversely as the fourth root of the depth. The results are irregular, but it is easy to see that this proportion agrees well with the general mean, and that other proportions (for instance, inversely as the depth or as the square root of the depth) would not at all agree with it. Here too it was remarked that the wave broke when its height above the general level was equal to the depth of the water at that place.

Time at which the wave breaks.

Time at which the surf of the sea breaks.

(403.) In confirmation of the last remark, observations were made on the surf of the sea, a series of rods being fixed in the sand, by which the height of the wave at the instant of breaking could be observed. The general result was that the wave broke when its height above the antecedent hollow was equal to the depth of the water. Mr. Russell, however, has not stated whether this depth was measured from the mean level of the surface, or from the bottom of the hollow.

Peculiarity in canal navigation.

(404.) Before dismissing these experiments, however, we must allude to a very remarkable practical fact which led to the whole investigation. It had been accidentally discovered that the navigation of certain

Section V. Account of Experiments on Waves.

canals of small depth, on which boats are drawn by horses, could be greatly facilitated by increasing the speed of the boats; the resistance when the velocity was great being considerably less than when the velocity was small. The whole of the circumstances affecting this result are most ably analyzed by Mr. Russell in a paper in the *Edinburgh Transactions*, vol. xiv. The following general statement will put the reader in possession of the principal facts. There is one particular velocity (that defined by the equation $v^2=gk$) at which a free wave will travel along a canal of given depth k. Nevertheless a boat travelling along a canal, at whatever speed, will be accompanied by a wave travelling with the same speed as the boat, whose motion at that particular speed is maintained by the horizontal pressure of the boat upon the water. These two kinds of waves are in all respects analogous to the free-tide-wave and the forced-tide-wave, of which we have spoken frequently in Section IV., and may with propriety be called the *free-canal-wave* and the *forced-canal-wave*. Now the general fact is this. If the boat moves more slowly than the free wave, the forced wave precedes the middle of the boat, and the force necessary to keep up the speed of the boat is (in proportion to its velocity) considerable. If the boat moves more rapidly than the free wave, the forced wave follows the middle of the boat, and the force necessary to keep up the speed is (in proportion to its velocity) less than in the former case. But if the boat moves with a velocity equal to, or rather slightly exceeding, the velocity of the free wave, then the boat rides with its middle on the top of the wave, and is drawn with a force much less (in proportion to its velocity) than at lower speeds, and even absolutely less. And these are the facts to be explained by our theory.

Boat moving slowly is preceded by a wave.

Boat moving quickly is followed by a wave.

Resistance to a boat much diminished when it rides on the top of the great wave.

(405.) The theory of (229.), &c., is sufficient for this purpose. Suppose, as in (232.), the value of X to be 0 when $vt-x$ is <0; and to be $\frac{2b}{3\pi}\left\{\frac{3}{2}\cdot\frac{\pi(vt-x)}{a}-\sin\frac{2\pi(vt-x)}{a}+\frac{1}{8}\sin\frac{4\pi(vt-x)}{a}\right\}$ when $vt-x$ is $>0, <a$; and to be b when $vt-x$ is $>a$: (in which expressions v is not necessarily $=\sqrt{gk}$, but may be any quantity whatever). Since, by (194.), $k+\text{K}=\frac{k}{1+\frac{d\text{X}}{dx}}=k-k\frac{d\text{X}}{dx}$ nearly, we have $\text{K}=-k\frac{d\text{X}}{dx}=\frac{8kb}{3a}\sin^4\frac{\pi(vt-x)}{a}$. This is the elevation of the surface of the water; and, since it applies only from $vt-x=0$ to $vt-x=a$, it shows that through that extent the water will be elevated, its elevation decreasing very gradually at both extremities of the wave. Now in the expressions of (233.) and (234.), suppose a the length of the wave to be so large that the terms depending on $\frac{1}{a^4}$ may be neglected: then the expression for F, the force which is necessary to maintain the motion of the wave with the velocity v, will be found to be

$$\frac{32\,\pi b}{3a^2}(v^2-gk)\,.\,\sin^3\frac{\pi(vt-x)}{a}\,.\cos\frac{\pi(vt-x)}{a}.$$

(406.) Suppose now the velocity to be less than that of a free wave, or v^2 to be $<gk$. The form of the expression for the necessary force is then $-\text{C}\,.\sin^3\frac{\pi(vt-x)}{a}\,.\cos\frac{\pi(vt-x)}{a}$. The changes of sign of this expression depend only on the last term [$\sin\frac{\pi(vt-x)}{a}$ being always positive when $vt-x$ is $>0, <a$]. For the front of the wave, as far as its middle, $vt-x$ is $<\frac{a}{2}$, $\frac{\pi(vt-x)}{a}$ is $<\frac{\pi}{2}$, and F is negative. For the back of the wave, $vt-x$ is $>\frac{a}{2}$, $\frac{\pi(vt-x)}{a}$ is $>\frac{\pi}{2}$, and F is positive. It appears, then, that the wave may be retained at this slow speed if there be a force resisting its front and a force urging forward its back. And it will therefore easily be conceived that (with perhaps a trifling change in the wave's form) a portion of the effect will be produced by only a force urging forward its back. As the motion of a boat can produce no force except forwards, it is readily seen that the boat can produce the force requisite only by following the wave. Thus when the velocity is less than that of a free wave, the wave will precede the boat.

Position of the wave with regard to the boat, explained from theory.

(407.) If the velocity were greater than that of a free wave, or $v^2>gk$, the form of the expression for the force would be $+\text{C}\,.\sin^3\frac{\pi(vt-x)}{a}\,.\cos\frac{\pi(vt-x)}{a}$, which would be found to be positive for the front of the wave, and negative for its back. The force therefore to maintain a rapid wave ought to tend to accelerate its front and to retard its back. As before, the boat can produce only the accelerative part, and therefore must be upon the front of the wave, or must be followed by the crest of the wave.

(408.) If $v^2=gk$, no force is requisite. But as we have found in the preceding instances that an accelerating force on the front and a retarding force on the back produce the same effect, we may infer that the combination of an accelerating force in front and an accelerating force on the back will produce no effect, or will produce the same effect as no force, or will make $v^2=gk$. If, then, the head and the stern of the canal-boat equally urged the water, the boat must ride on the top of the wave to maintain this velocity.

(409.) But as the head of the boat urges the water more than the stern does, the velocity of the wave must be such as requires a greater accelerating pressure on the front than on the back, or such as requires a pressure similar to that treated of in (407.), or must be a little greater than the velocity of the free wave, in order that the boat may ride evenly on the crest of the wave. This is true in practice.

Theory of the long wave maintained by a force, in its general form.

(410.) There is no difficulty in determining the exact form of the wave which a given force exerted by the boat would produce, provided we suppose it to act equally on all the particles in the same vertical, the wave being long. The equation of (236.), omitting the product of $\frac{d\text{X}}{dx}$ and $\frac{d^2\text{X}}{dx^2}$, becomes $\frac{d^2\text{X}}{dt^2}-gk\frac{d^2\text{X}}{dx^2}=\text{F}$. If we make $gk=v'^2$, v' is the velocity of a free wave. And as F depends upon the position of the boat, and upon the position of any particle of water with respect to the boat, F may be expressed by $\chi''(vt-x)$. Thus the equation becomes

$$\frac{d^2X}{dt^2} - v'^2 \frac{d^2X}{dx^2} = \chi''(vt-x).$$

Making $v't-x=u$, $v't+x=w$, as in (197.), we have

$$4v'^2 \cdot \frac{d^2X}{du\,dw} = \chi''\left\{\frac{v+v'}{2v'}u + \frac{v-v'}{2v'}w\right\}.$$

Integrating this with respect to u and w, as in (198.), we have

$$4v'^2X = \phi(u) + \psi(w) + \frac{4v'^2}{v^2-v'^2}\chi\left\{\frac{v+v'}{2v'}u + \frac{v-v'}{2v'}w\right\}$$

or $4v'^2X = \phi(v't-x) + \psi(v't+x) + \dfrac{4v'^2}{v^2-v'^2}\chi(vt-x)$.

Omitting the two first terms, which express free waves,

$$X = \frac{1}{v^2-v'^2}\chi(vt-x).$$

And $K = -k\dfrac{dX}{dx} = \dfrac{k}{v^2-v'^2}\chi'(vt-x)$; or $F = \chi''(vt-x) = \dfrac{v'^2-v^2}{k} \cdot \dfrac{dK}{dx}$.

We leave the discussion of this equation to the reader, conceiving that the process which we have used in the last articles is better adapted to the general explanation of the observed facts. We may remark, however, that the breaking of the waves attending the boat, in certain cases, appears to be owing to this circumstance, that the form of the function χ' is such that the function χ does not satisfy the equations of (229.), and therefore at some point there is an infinite force on the particles of water.

Explanation from theory of the diminution of the resistance to the boat.

(411.) We have not yet, however, alluded to the algebraical circumstances on which the diminution of resistance to the boat depends. For this purpose we will consider the velocities of the individual particles of water. For any particle this is expressed by

$$\frac{dX}{dt} = \frac{2b}{3\pi}\left\{\frac{3\pi}{2a}v - \frac{2\pi v}{a}\cos\frac{2\pi(vt-x)}{a} + \frac{\pi v}{2a}\cos\frac{4\pi(vt-x)}{a}\right\}.$$

When $vt-x=a$ or 0, that is at the beginning or end of the wave, this velocity is

$$\frac{2b}{3\pi}\left\{\frac{3\pi}{2a}v - \frac{2\pi v}{a} + \frac{\pi v}{2a}\right\} = 0.$$

When $vt-x=\dfrac{a}{2}$, that is at the middle of the wave, this velocity is

$$\frac{2b}{3\pi}\left\{\frac{3\pi}{2a}v + \frac{2\pi v}{a} + \frac{\pi v}{2a}\right\} = \frac{8bv}{3a}.$$

The boat, riding on the top of the wave, has its head and stern almost free of the water, while its middle is immersed in water which is travelling in the same direction as itself with the velocity $\dfrac{8bv}{3a}$. Consequently the velocity through the water, upon which the resistance depends, is not v but $v-\dfrac{8bv}{3a}$; and the resistance is therefore diminished in the proportion of $1 : \left(1-\dfrac{8b}{3a}\right)^2$. To this is owing the great facility of drawing the boat at the high speed, provided it ride on the top of the wave. Perhaps it may be assisted by the circumstance that the head being raised above the water, or rather being left by the water, presents less of bluff resistance than usual. Neither of these circumstances (the rapid motion of the water in the same direction as the boat, and the elevation of the boat's head above the water) presents itself when the boat either precedes or follows the wave.

(412.) It was remarked that at low velocities the boat is followed by a breaking wave, but that at the advantageous velocity there is no broken wave. We cannot undertake to put the explanation of this fact in detail into a mathematical form.

Observations of waves on canals.

(413.) Observations were made by Mr. Russell on the velocity of waves in navigable canals. Their results agree generally with theory; but as the circumstances are never so perfectly known as in artificial canals, we do not think them worthy of the detailed notice which we have given to the others in (392.), &c.

Observations on waves of the sea.

(414.) Mr. Russell also made some observations on the motion of waves of the sea. These observations completely failed. Mr. Russell was not aware that the velocity of waves depends on their lengths, and, in deep water, depends sensibly on nothing else; and was therefore baffled by obtaining the most discordant results. His observations, too, are useless to any other person, as no attempt was made to measure the length of the waves. Observations were, however, made (by immersing a glass tube to a considerable depth in the water) to determine whether the agitation of the water produces a sensible effect at great depths. It was found that where the depth was 12 feet, waves 4 or 5 feet long produce no sensible motion near the bottom; waves 30 or 40 feet long produce a very inconsiderable effect. The reader will see that these results agree perfectly with those of (177.), &c.

(415.) We shall allude hereafter to some of Mr. Russell's observations of tides. Meantime we shall repeat our opinion of the great value of the experiments which we have abstracted, but we must warn the reader against attaching any importance to the theoretical expressions which are mingled with them in the original account.

General facts of sea-waves are notorious.

(416.) The general phænomena of waves of the sea, to which we have incidentally alluded in several parts of Section IV., are so notorious that it would be useless and indeed difficult to produce authorities. They are, the great velocity of large waves as compared with that of small ones (168.); the crossing of any number of systems of waves without obliterating each other, (352.) and (354.); the power of the wind to raise the waves, and their broken character during the raising, (271.); the power of the wind to maintain the waves in increasing height from the windward shore, and their smooth character, (267.); the reflection of waves, unbroken, from a vertical wall, (357.); the breaking of the waves on the edge of a shoal, (158.) and (248.); the breaking of the surf on a sloping beach, (249.); and some others.

Special treatises on waves of the sea.

We may, however, remark that some treatises have been written expressly on the motion of the waves of the sea, embodying the results of considerable thought and extensive observation, and in some respects worthy of the reader's attention. Among these we may particularly specify those of De la Coudraye, Bremontier, and Emy. The theories of these writers are throughout of very little value, (as will always hold with theories of waves which are not based upon the mathematical expressions of the general properties of fluids:) their observations are, in a great measure, directed to the prac-

Section V. Account of Experiments on Waves.

Breaking on the edge of deep-sunk banks.

tical effect of waves, upon sea beaches and erections in the sea. Among those which have a bearing on our theory, we may mention that the horizontal motion of the particles of water next to the bottom, produced in shallow water by long waves (180.), is proved to be sensible by the disturbance of the stones and sand at the bottom; and that the breaking over the edge of a shoal (248.) is stated as occurring on the edge of the bank of Newfoundland when the waves in general are high, although the depth on the shallow side is 500 feet, (that on the deep side being much greater.) We may mention, on the authority of the best charts, that a similar breaking is observed about the line of "no soundings," (that is, where the water suddenly becomes deeper than 600 feet,) which at some distance borders the British Isles. We defer the mention of the bore and other tidal phænomena (to which these authors allude) to our last Section.

Height of waves of the sea.

(417.) The estimations of the height of the greatest waves raised by the wind (272.), which are scattered through the works of different writers, are very unsatisfactory. We are, however, inclined to think that in no circumstances does the height of an unbroken wave exceed 30 or 40 feet.

SECTION VI.—INVESTIGATION OF THE MATHEMATICAL EXPRESSIONS FOR THE TIDES PRODUCED BY THE SUN AND MOON, CONSIDERED AS WAVES IN CANALS.

(418.) In our last section we quitted for a time the main object of this Essay, in order to show that the results of our Theory of Waves are supported by experiment, as far as experiments on the small scale can be found which admit of comparison with them. We now return to our principal object, (to which the remainder of the Essay will be devoted,) and shall employ the present Section in investigating the nature of the tides which would occur if the water were confined in a channel of uniform section and of simple form. The theorems of Section IV., as will appear from the following proposition, will be applicable to this inquiry, as soon as the magnitude of the disturbing forces shall be determined.

(419.) Proposition.—In investigating the motion of the sea in a narrow canal, it is unnecessary to consider the forces arising from the rotation of the Earth.

The general equations for the motion of the waters on the surface of the Earth, whether limited by fixed boundaries or not, are those of (84.), omitting the second for the reasons given in (87.). They are, therefore,

Laplace's equations, applicable to fluids in canals as well as to fluids covering the whole spheroid.

$$\frac{dp'''}{d\theta} = -r^2\frac{d^2u}{dt^2} + 2nr^2.\sin\theta.\cos\theta.\frac{dv}{dt}$$

$$\frac{dp'''}{d\varpi} = -2nr^2.\sin\theta.\cos\theta.\frac{du}{dt} - r^2\sin^2\theta.\frac{d^2v}{dt^2}$$

$$w = -\frac{d}{d\theta}(u\gamma) - u\gamma.\text{cotan}\,\theta - \gamma\frac{dv}{d\varpi}$$

$$= \text{I} + \frac{3\text{Sb}^2}{2\text{D}^3}\left\{(\sin\sigma.\cos\theta + \cos\sigma.\sin\theta.\cos\overline{nt+\varpi-s})^2 - \frac{1}{3}\right\} - gw + p''',$$

where ϖ is the terrestrial longitude (measured from a certain fixed meridian carried by the Earth) of the original or mean place of a particle of water, $\varpi+v$ its longitude as disturbed by tide at the time t; θ is the original angular distance from the north pole, $\theta+u$ the same angle as disturbed by tide at the time t; γ the depth of the water, w its tidal elevation at the time t; n the earth's angular velocity of rotation; p'' that part of the pressure of the particles which depends upon their motion; and the other quantities as in (22.) and (82.). Now it is our object to show that, when these equations are transformed into that shape which applies to the motion of water in a narrow canal, the symbol n will disappear. As this latter is found only in the two first, it will be sufficient for us to show that it disappears from the combination of the two first.

(420.) Now, in investigating the motion of water longitudinally in a narrow canal, putting l for the length of the canal as measured from a fixed point to any particle of water under consideration, we want to find $\frac{dp'''}{dl}$, to be used in the same manner as in (149.) or (195.) To do this by means of the equations above, we must put $\frac{dp'''}{dl} = \frac{dp'''}{d\theta}.\frac{d\theta}{dl} + \frac{dp'''}{d\varpi}.\frac{d\varpi}{dl}$. In figure 29, let $\text{BT}=l$, $\text{BT}'=l+\delta l$, T and T′ being two particles upon a line parallel to the axis of the canal; and let θ, ϖ, and $\theta+\delta\theta$, $\varpi+\delta\varpi$, correspond to the same two particles. Then it is evident that $\text{TU}=\text{b}\delta\theta$, $\text{UT}'=\text{b}\delta\varpi\times\sin\theta$. Putting T, therefore, for the acute angle at T, we have, $\sin\text{T} = \frac{\text{UT}'}{\text{TT}'}, = \frac{\text{b}\delta\varpi\times\sin\theta}{\delta l}$; $\cos\text{T} = \frac{\text{TU}}{\text{TT}'} = \frac{\text{b}\delta\theta}{\delta l}$: or $\frac{\delta\varpi}{\delta l} = \frac{\sin\text{T}}{\text{b}.\sin\theta}$, $\frac{\delta\theta}{\delta l} = \frac{\cos\text{T}}{\text{b}}$. Using these as the values of $\frac{d\varpi}{dl}$ and $\frac{d\theta}{dl}$ in the expression for $\frac{dp'''}{dl}$, we have

$$\frac{dp'''}{dl} = \frac{dp'''}{d\theta}.\frac{d\theta}{dl} + \frac{dp'''}{d\varpi}.\frac{d\varpi}{dl}$$

$$= -\frac{r^2}{\text{b}}\left(\cos\text{T}.\frac{d^2u}{dt^2} + \sin\text{T}.\sin\theta.\frac{d^2v}{dt^2}\right) + \frac{2nr^2}{\text{b}}\left(\cos\text{T}.\sin\theta.\cos\theta.\frac{dv}{dt} - \sin\text{T}.\cos\theta.\frac{du}{dt}\right).$$

Now considering the motion of the water produced by the tide to be only in the direction of the canal's length, Expressions for the Tides considered as Waves in Canals.

and putting L for the displacement of the particle in that direction, it is seen in figure 30 that bv.sin θ=L.sin T, or $v=\frac{\mathrm{L}.\sin \mathrm{T}}{\mathrm{b}.\sin\theta}$; and, therefore, $\frac{dv}{dt}=\frac{\sin \mathrm{T}}{\mathrm{b}.\sin\theta}.\frac{d\mathrm{L}}{dt}$, and $\frac{d^2v}{dt^2}=\frac{\sin \mathrm{T}}{\mathrm{b}.\sin\theta}.\frac{d^2\mathrm{L}}{dt^2}$ (omitting, in the differentiation, the terms depending on the differentials of θ and T, because they would introduce the squares of small quantities.) Similarly, bu=L.cos T, or $u=\frac{\mathrm{L}.\cos \mathrm{T}}{\mathrm{b}}$, and $\frac{du}{dt}=\frac{\cos \mathrm{T}}{\mathrm{b}}.\frac{d\mathrm{L}}{dt}$, and $\frac{d^2u}{dt^2}=\frac{\cos \mathrm{T}}{\mathrm{b}}.\frac{d^2\mathrm{L}}{dt^2}$. Substituting these quantities,

$$\frac{dp'''}{dl}=-\frac{r^2}{\mathrm{b}^2}.\frac{d^2\mathrm{L}}{dt^2}=-\frac{d^2\mathrm{L}}{dt^2}\text{ (since b and } r \text{ are sensibly equal).}$$

When the equations are applied to canals, the centrifugal force does not enter.

This equation does not contain n, and, therefore, is unaffected by the Earth's rotation: or, in other words, it is indifferent whether we consider the relative change of place between the point on the Earth and the disturbing luminary to arise, partly from the motion of the Earth and partly from the motion of the luminary, or entirely from the motion of the luminary. (This does not hold when we consider the Earth as covered, in whole or in part, by extensive sheets of water.) The equation, moreover, is exactly the same as if the canal were straight.

(421.) Having established this point, we shall desert this method of investigation, and shall proceed to ascertain the value and effects of the acting forces, as if the Earth were fixed, and the Sun and Moon revolved in diurnal movement round it.

(422.) Problem.—The form of a canal is a small circle upon the Earth, of any diameter, and in any position: the luminary being in any position, it is required to find its force upon the waters at each point, in the direction of the canal's length.

(423.) In figure 31, let He be the small circle, e the point at which the force is to be ascertained, S the place of the luminary: take KF the great circle parallel to the small circle, and therefore passing through E the centre of the sphere: draw SN perpendicular to its plane produced, meeting that plane in N; join EN by the line EFN; take G the centre of the small circle, and draw GH parallel to EF. Since GE is perpendicular to the planes of the circles, or parallel to SN, it is evident that G, E, H, F, S, N, are all in the same plane perpendicular to the plane of the circles. Draw es, en, parallel to ES, EN; join Ee, Se. Let the radius of the small circle, expressed by the angle corresponding to the distance in arc between its pole and its circumference, be α: let the linear radius of the sphere be b; then GH=b.sin α, EG=b.cos α. Also let ES=D, $\angle$GES=β; then EN=D.sin β, SN=D.cos β. And let $\angle$HGe=θ. Then, conceiving x to be measured parallel to EN, y perpendicular to it in the plane of the great circle, z parallel to EG, and E to be the origin of co-ordinates, the values of those co-ordinates respectively are as follows:

For S;	D.sin β,	0,	D.cos β.
For e;	b.sin α.cos θ,	b.sin α.sin θ,	b.cos α;

and, therefore,

$$(\mathrm{S}e)^2=(\mathrm{D}.\sin\beta-\mathrm{b}.\sin\alpha.\cos\theta)^2+(\mathrm{b}.\sin\alpha.\sin\theta)^2+(\mathrm{D}.\cos\beta-\mathrm{b}.\cos\alpha)^2$$
$$=\mathrm{D}^2-2\mathrm{Db}\,(\cos\beta.\cos\alpha+\sin\beta.\sin\alpha.\cos\theta)+\mathrm{b}^2.$$

(424.) Now the force which the luminary exerts upon the solid part of the sphere is the same as if the whole matter of the sphere were collected at E, and may, therefore, be represented by $\frac{\mathrm{S}}{(\mathrm{S\,E})^2}$ or $\frac{\mathrm{S}}{\mathrm{D}^2}$ in the direction ES. The force of the luminary upon the particle at e may be represented by $\frac{\mathrm{S}}{(\mathrm{S}e)^2}$ in the direction eS; or by $\frac{\mathrm{S}.e\mathrm{E}}{(\mathrm{S}e)^3}$ in the direction eE, and $\frac{\mathrm{S.ES}}{(\mathrm{S}e)^3}$ in the direction es parallel to ES. The former of these, by (279.), produces no sensible effect, and may, therefore, be put out of consideration. Hence we have

$$\text{Force upon the solid sphere parallel to ES}\quad=\frac{\mathrm{S}}{\mathrm{D}^2}.$$

$$\text{Force upon the particle at } e \text{ parallel to ES}\quad=\frac{\mathrm{S.D}}{(\mathrm{S}e)^3}.$$

Excess of the latter above the former, which is the true relative disturbing force upon the particle at e,

$$=\mathrm{S}\left(\frac{\mathrm{D}}{(\mathrm{S}e)^3}-\frac{1}{\mathrm{D}^2}\right).$$

(425.) This, however, is the disturbing force in the direction es. If we resolve it into two parts, one parallel to en and one perpendicular to it, or parallel to eq, the latter will be entirely transverse to the direction of the canal at that point, and will, therefore, in our investigation be rejected: the former will be found by multiplying the force in the direction es by cos sen or sin β. Thus we have

$$\text{Disturbing force in the direction } en\quad=\mathrm{S}\left(\frac{\mathrm{D}}{(\mathrm{S}e)^3}-\frac{1}{\mathrm{D}^2}\right)\sin\beta.$$

Sect. VI. Expressions for the Tides considered as Waves in Canals.

(426.) Finally, resolving this force into two, one parallel to G*e*, and the other perpendicular to it, or tangential to the small circle (and, therefore, urging the water along the canal), we find for the latter,

Expression for the disturbing force produced by the Sun or Moon on the water of a canal.

$$\text{Disturbing force in the direction of the canal at } e = \mathrm{S}\left(\frac{\mathrm{D}}{(\mathrm{S}e)^3} - \frac{1}{\mathrm{D}^2}\right)\sin\beta.\sin\theta.$$

Putting for $\frac{1}{(\mathrm{S}e)^3}$ the expression $\left\{\mathrm{D}^2 - 2\mathrm{Db}(\cos\beta.\cos\alpha + \sin\beta.\sin\alpha.\cos\theta) + \mathrm{b}^2\right\}^{-\frac{3}{2}}$, and expanding to the second power of b, the expression becomes

$$\mathrm{S}.\sin\beta.\sin\theta\left\{\frac{3\mathrm{b}}{\mathrm{D}^3}(\cos\beta.\cos\alpha + \sin\beta.\sin\alpha.\cos\theta) - \frac{3}{2}.\frac{\mathrm{b}^2}{\mathrm{D}^4} + \frac{15}{2}.\frac{\mathrm{b}^2}{\mathrm{D}^4}(\cos\beta.\cos\alpha + \sin\beta.\sin\alpha.\cos\theta)^2\right\}.$$

This force acts to urge the water in the direction *e*H. We shall, however, conceive the length of the canal to be measured positively in the same direction in which θ is measured positively, that is, from H towards *e*: and this gives for the proper expression for the disturbing force,

$$\mathrm{S}.\sin\beta.\sin\theta\left\{-\frac{3\mathrm{b}}{\mathrm{D}^3}(\cos\beta.\cos\alpha + \sin\beta.\sin\alpha.\cos\theta) + \frac{3}{2}.\frac{\mathrm{b}^2}{\mathrm{D}^4} - \frac{15}{2}.\frac{\mathrm{b}^2}{\mathrm{D}^4}(\cos\beta.\cos\alpha + \sin\beta.\sin\alpha.\cos\theta)^2\right\}.$$

(427.) Problem.—To find the expression for the force as varying with the time, the luminary being supposed to revolve uniformly in a small circle.

Let P*po*, figure 32, be a sphere whose centre is the centre of the earth, and whose circumference passes through the circle *o*S in which the luminary revolves: let P be the pole of the circle *o*S (it will, therefore, in fact be the point corresponding to the pole of the earth): and let *p* be the point corresponding to the pole of the canal; let P, *p*, *o*, O, E, G, be in the same plane; and let *p*, S, H, E, G, be in the same plane; let O be the origin of the angle or arc for the canal O*e*; and let *o* be the origin of the angle or arc for the small circle *o*S. The spherical angle *op*S is evidently equal to OGH. Let OG*e* $=\phi$: this angle is independent of the position of S, and may, therefore, be properly used for defining the point *e*. The angle θ in the investigation above is HG*e*, or $\phi -$ OGH, or $\phi - op$S, figures 32 and 33; β or GES is represented on the sphere by *p*S; α is here a constant. We will put $90° - \sigma$ for P*o* or PS (σ is then the declination of the luminary); and δ for P*p*; and *nt* for the angle *o*PS, which is proportional to the time.

(428.) Now our expressions above depend on $\sin\beta.\sin\theta$, $\cos\beta$, and $\sin\beta.\cos\theta$; and these we must endeavour to express so as to contain no variable but *nt*.

First,
$$\begin{aligned}\sin\beta.\sin\theta &= \sin p\mathrm{S}.\sin(\phi - op\mathrm{S})\\ &= \sin\phi.\sin p\mathrm{S}.\cos op\mathrm{S} - \cos\phi.\sin p\mathrm{S}.\sin op\mathrm{S}.\end{aligned}$$

Now
$$\cos op\mathrm{S} = -\cos \mathrm{P}p\mathrm{S} = \frac{\cos\beta.\cos\delta - \sin\sigma}{\sin\beta.\sin\delta}\ \text{(figure 33.)};$$

therefore
$$\begin{aligned}\sin p\mathrm{S}.\cos op\mathrm{S} &= \frac{\cos\beta.\cos\delta - \sin\sigma}{\sin\delta} = \frac{\cos\delta(\sin\sigma.\cos\delta + \cos\sigma.\sin\delta.\cos nt) - \sin\sigma}{\sin\delta}\\ &= \cos\sigma.\cos\delta.\cos nt - \sin\sigma.\sin\delta.\end{aligned}$$

And
$$\sin p\mathrm{S}.\sin op\mathrm{S} = \sin \mathrm{PS}.\sin o\mathrm{PS} = \cos\sigma.\sin nt;$$

therefore
$$\sin\beta.\sin\theta = -\sin\sigma.\sin\delta.\sin\phi + \cos\sigma.\cos\delta.\sin\phi.\cos nt - \cos\sigma.\cos\phi.\sin nt.$$

Second,
$$\cos\beta = \sin\sigma.\cos\delta + \cos\sigma.\sin\delta.\cos nt.$$

Third,
$$\begin{aligned}\sin\beta.\cos\theta &= \sin p\mathrm{S}.\cos(\phi - op\mathrm{S}) = \cos\phi.\sin p\mathrm{S}.\cos op\mathrm{S} + \sin\phi.\sin p\mathrm{S}.\sin op\mathrm{S}\\ &= -\sin\sigma.\sin\delta.\cos\phi + \cos\sigma.\cos\delta.\cos\phi.\cos nt + \cos\sigma.\sin\phi.\sin nt.\end{aligned}$$

Expressions which must be multiplied to give the disturbing force in terms of the time.

Fourth. By means of the two last we obtain $\cos\alpha.\cos\beta + \sin\alpha.\sin\beta.\cos\theta =$

$$\begin{aligned}&\cos\alpha.\sin\sigma.\cos\delta - \sin\alpha.\sin\sigma.\sin\delta.\cos\phi + \cos\alpha.\cos\sigma.\sin\delta.\cos nt + \sin\alpha.\cos\sigma.\cos\delta.\cos\phi.\cos nt\\ &\qquad + \sin\alpha.\cos\sigma.\sin\phi.\sin nt.\end{aligned}$$

And the first and fourth are now to be substituted in the expression at the end of (426.).

It is evident that this would lead to results of great complexity; and we shall content ourselves, therefore, with applying it to a few special cases.

Disturbing body supposed to be in the equator, the canal forming any small circle.

(429.) (I.) Suppose the luminary in the equator, or $\sigma = o$.

In this case
$$\sin\beta.\sin\theta = \cos\delta.\sin\phi.\cos nt - \cos\phi.\sin nt.$$

$$\cos\alpha.\cos\beta + \sin\alpha.\sin\beta.\cos\theta = \cos\alpha.\sin\delta.\cos nt + \sin\alpha.\cos\delta.\cos\phi.\cos nt + \sin\alpha.\sin\phi.\sin nt.$$

The product of these two expressions, with sign changed, which represents the principal part of the disturbing force, is

$$-\frac{1}{2}\cos\alpha.\sin\delta.\cos\delta.\sin\phi.(1 + \cos 2nt) - \frac{1}{2}\sin\alpha.\cos^2\delta.\sin\phi.\cos\phi(1 + \cos 2nt)$$

$$-\frac{1}{2}\sin\alpha.\cos\delta.\sin^2\phi.\sin 2nt+\frac{1}{2}\cos\alpha.\sin\delta.\cos\phi.\sin 2nt+\frac{1}{2}\sin\alpha.\cos\delta.\cos^2\phi.\sin 2nt$$
$$+\frac{1}{2}\sin\alpha.\sin\phi.\cos\phi.(1-\cos 2nt).$$

This expression consists of three distinct classes of terms:

Disturbance of mean level.

(430.) The first or $-\frac{1}{2}\cos\alpha.\sin\delta.\cos\delta.\sin\phi+\frac{1}{2}\sin\alpha.\sin^2\delta.\sin\phi.\cos\phi$, does not depend on nt, and therefore does not give rise to any oscillation, or any state of waters varying with the time. But as it depends on ϕ, it shows, (152.) and (283.), that the relative mean level in different parts of the canal is not the same as if the luminary did not exist.

(431.) The second, or $-\frac{1}{2}\cos\alpha.\sin\delta.\cos\delta.\sin\phi.\cos 2nt+\frac{1}{2}\cos\alpha.\sin\delta.\cos\phi.\sin 2nt$, depends on ϕ and $2nt$. Putting it in the form $\mathrm{A}.\sin\overline{2nt+\phi}+\mathrm{B}.\sin\overline{2nt-\phi}$, or $(\mathrm{A}+\mathrm{B})\sin 2nt.\cos\phi+(\mathrm{A}-\mathrm{B})\cos 2nt.\sin\phi$, we easily find for its equivalent

$$\cos\alpha.\sin^3\frac{\delta}{2}.\cos\frac{\delta}{2}.\sin\overline{2nt+\phi}+\cos\alpha.\sin\frac{\delta}{2}.\cos^3\frac{\delta}{2}.\sin\overline{2nt-\phi}.$$

If x be the length of the canal measured from the same origin as ϕ, $\phi=\frac{x}{\mathrm{b}\sin\alpha}$, and this expression becomes

$$\cos\alpha.\sin^3\frac{\delta}{2}.\cos\frac{\delta}{2}.\sin\overline{2nt+\frac{x}{\mathrm{b}\sin\alpha}}+\cos\alpha.\sin\frac{\delta}{2}.\cos^3\frac{\delta}{2}.\sin\overline{2nt-\frac{x}{\mathrm{b}\sin\alpha}}.$$

Each of these two terms is of the same form as those considered in (274.), putting H for the multiplier here, *it* for $2nt$, and mx for $\pm\frac{x}{\mathrm{b}\sin\alpha}$; and each will give rise to a wave, one of which will roll in one direction and the other in the opposite direction. If we examine the expression for Y at the surface, or K, in (279.), and (284.), and remark that for these two waves m has different signs, we shall see that the two coefficients above will be multiplied by equal factors with opposite signs; the absolute sign of either depending upon the depth of the water. The whole elevation of the water then, produced by these two waves, will be represented by

$$\frac{k\mathrm{b}\sin\alpha.\cos\alpha.\sin\delta}{8n^2\mathrm{b}^2\sin^2\alpha-2gk}\times\left\{\sin^2\frac{\delta}{2}.\cos\overline{2nt+\frac{x}{\mathrm{b}\sin\alpha}}-\cos^2\frac{\delta}{2}.\cos\overline{2nt-\frac{x}{\mathrm{b}\sin\alpha}}\right\},$$

or
$$\frac{k\mathrm{b}.\sin\alpha.\cos\alpha.\sin\delta}{8n^2\mathrm{b}^2\sin^2\alpha-2gk}\times\{-\cos\delta.\cos\phi.\cos 2nt-\sin\phi.\sin 2nt\}.$$

If we make $\tan\psi=\frac{\tan\phi}{\cos\delta}$, this becomes

$$-\frac{k\mathrm{b}.\sin\alpha.\cos\alpha.\sin\delta}{8n^2\mathrm{b}^2\sin^2\alpha-2gk}\times\sqrt{\cos^2\delta.\cos^2\phi+\sin^2\phi}.\cos\overline{2nt-\psi},$$

where it must oe remembered that k is the depth of the water, and that the multiplier $\frac{3\mathrm{Sb}}{\mathrm{D}^3}$ has been omitted.

First semidiurnal tide.

This expression shows, 1st, that *for the same place* the oscillation goes through all its phases while $2nt$ changes by 2π, and therefore the oscillation goes twice through all its phases while $2nt$ changes by 4π, or while nt changes by 2π, or while the luminary performs its revolution; that is, the tide is semidiurnal. 2nd, that *for the same time* the oscillation has all its different phases for values of ψ extending through 2π, that is for values of ϕ extending through 2π (for ϕ and ψ have the values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, together), that is, for the whole extent of the canal and for no smaller extent; therefore there is but one wave on the canal, which passes round it with an irregular motion twice in a tidal day. 3rd, that the magnitude of the fluctuation is different at different places.

But as the expression contains the factors $\cos\alpha.\sin\delta$, this wave has no existence unless the canal is a small circle whose pole does not coincide with the pole of the earth.

(432.) The third class of terms, or $-\frac{1}{4}\sin\alpha(1+\cos^2\delta)\sin 2\phi.\cos 2nt+\frac{1}{2}\sin\alpha.\cos\delta.\cos 2\phi.\sin 2nt$, depends upon 2ϕ and $2nt$. Putting it in the form $\mathrm{A}.\sin\overline{2nt+2\phi}+\mathrm{B}.\sin\overline{2nt-2\phi}$, it becomes

$$-\frac{1}{2}\sin\alpha.\sin^4\frac{\delta}{2}.\sin\overline{2nt+2\phi}+\frac{1}{2}\sin\alpha.\cos^4\frac{\delta}{2}.\sin\overline{2nt-2\phi};$$

which, as above, will produce an elevation of waters represented by

$$-\frac{1}{4}.\frac{k\mathrm{b}.\sin^2\alpha}{n^2\mathrm{b}^2\sin^2\alpha-gk}\left\{\sin^4\frac{\delta}{2}.\cos\overline{2nt+2\phi}+\cos^4\frac{\delta}{2}.\cos\overline{2nt-2\phi}\right\},$$

or
$$-\frac{1}{4}.\frac{k\mathrm{b}\sin^2\alpha}{n^2\mathrm{b}^2\sin^2\alpha-gk}\left\{\left(\frac{1}{2}+\frac{1}{2}\cos^2\delta\right)\cos 2\phi.\cos 2nt+\cos\delta.\sin 2\phi.\sin 2nt\right\}.$$

If $\tan\chi = \frac{2\cos\delta}{1+\cos^2\delta}\tan 2\phi$, this becomes

$$-\frac{1}{4}.\frac{k\mathrm{b}\sin^2\alpha}{n^2\mathrm{b}^2\sin^2\alpha - gk}\times\sqrt{\left(\frac{1}{2}+\frac{1}{2}\cos^2\delta\right)^2.\cos^2 2\phi + \cos^2\delta.\sin^2 2\phi}.\cos\overline{2nt-\chi},$$

to which the multiplier $\frac{3\mathrm{Sb}}{\mathrm{D}^3}$ is yet to be supplied.

Second semidiurnal tide.

This expression shows, 1st, that *for the same place* the oscillation goes through all its phases while $2nt$ changes by 2π, or is semidiurnal; 2nd, that *for the same time* the oscillation has all its different phases for values of χ extending through 2π, that is, for values of 2ϕ extending through 2π; or it has all its different phases twice repeated for values of χ extending through 4π, that is for values of ϕ extending through 2π; therefore there is a double wave on the canal which passes round it with an irregular motion once in a tidal day; 3rd, that the magnitude of the fluctuation is different in different places.

These two are compounded into one semidiurnal tide at every place.

(433.) The two waves which we have found will produce only one apparent semidiurnal wave at each place. For the expressions $\mathrm{L}.\cos\overline{2nt-\psi}+\mathrm{M}.\cos\overline{2nt-\chi}$ or $(\mathrm{L}\cos\psi+\mathrm{M}\cos\chi)\cos 2nt + (\mathrm{L}\sin\psi+\mathrm{M}\sin\chi)\sin 2nt$ may be represented by $\mathrm{N}.\cos\overline{2nt-\mathrm{O}}$ where $\mathrm{N}=\sqrt{\{(\mathrm{L}\cos\psi+\mathrm{M}\cos\chi)^2+(\mathrm{L}\sin\psi+\mathrm{M}\sin\chi)^2\}}$ and $\tan\mathrm{O} = \frac{\mathrm{L}\sin\psi+\mathrm{M}\sin\chi}{\mathrm{L}\cos\psi+\mathrm{M}\cos\chi}$; which, as estimated at the place, presents the appearance of a single semidiurnal tide-fluctuation. As the relation of the coefficients of the two waves, and even the similarity or opposition of their signs, will depend on the depth of the canal and other special circumstances, we cannot proceed further in the investigation.

(434.) There is no term depending on nt, and therefore no diurnal wave.

The canal supposed to be a great circle in any position, and the disturbing body in any position.

(435.) (II.) Suppose the canal to be a great circle, or $\alpha=90°$; the position of the great circle and the declination of the luminary being any whatever. (We shall, for convenience, assume this case in all our succeeding investigations.)

In this case $\sin\beta.\sin\theta = -\sin\sigma.\sin\delta.\sin\phi + \cos\sigma.\cos\delta.\sin\phi.\cos nt - \cos\sigma.\cos\phi.\sin nt$;

and $\cos\alpha.\cos\beta+\sin\alpha.\sin\beta.\cos\theta = -\sin\sigma.\sin\delta.\cos\phi+\cos\sigma.\cos\delta.\cos\phi.\cos nt+\cos\sigma.\sin\phi.\sin nt$.

If we multiply these terms with sign changed, we find in the result the following classes of terms.

Disturbance of mean level.

(436.) First, the terms $\left(-\sin^2\sigma+\frac{1}{2}\cos^2\sigma\right)\sin^2\delta.\sin\phi.\cos\phi$ independent of nt. These, as before, do not indicate an oscillation, but they show that the mean elevation of the water at each place is modified by the attraction of the luminary. And as the extent of this modification depends on σ, there will be a slow change in the mean elevation of the water depending on the slow changes of σ. The elevation will however be the same for equal values of σ with opposite signs. Introducing the factor $\mathrm{S}.\frac{3\mathrm{b}}{\mathrm{D}^3}$, putting $\left(\frac{1}{2}-\frac{3}{2}\sin^2\sigma\right)$ for $-\sin^2\sigma+\frac{1}{2}\cos^2\sigma$, and observing that $\phi=\frac{x}{\mathrm{b}}$, x being the length of the canal, the actual elevation, by (152.) is

$$+\frac{3\mathrm{Sb}}{2g\mathrm{D}^3}\left(\frac{1}{2}-\frac{3}{2}\sin^2\sigma\right)\sin^2\delta\int_x\sin 2\phi$$

or

$$+\frac{3\mathrm{Sb}^2}{2g\mathrm{D}^3}\left(\frac{1}{2}-\frac{3}{2}\sin^2\sigma\right)\sin^2\delta\int_\phi\sin 2\phi,$$

or

$$-\frac{3\mathrm{Sb}^2}{4g\mathrm{D}^3}\left(\frac{1}{2}-\frac{3}{2}\sin^2\sigma\right)\sin^2\delta.\cos 2\phi.$$

The part which depends on σ is $+\frac{9\mathrm{Sb}^2}{8g\mathrm{D}^3}\sin^2\sigma.\sin^2\delta.\cos 2\phi$. Now $\sin\sigma=$ sine of declination of luminary; and ϕ is 0 or 180°, or $\cos 2\phi=1$, for those parts of the canal which have the greatest geographical latitude; consequently at those parts the mean level of the water rises when the declination of the luminary is greatest; it falls by the same amount at the equator.

Computation in feet.

(437.) To estimate it in feet we may remark, that this coefficient for the Sun is $\frac{9}{4}$ of the coefficient computed in (31.), or $\frac{9}{4}\times 0{\cdot}2710$ foot, or $0{\cdot}61$ foot. Therefore when the Sun's declination is σ, the elevation of the water depending on these terms, produced by the Sun, is $0{\cdot}61$ foot $\times\sin^2\sigma.\sin^2\delta.\cos 2\phi$. If the canal pass through the pole, $\sin\delta=1$, and the expression is $0{\cdot}61$ foot $\times\sin^2\sigma.\cos 2\phi$. As ϕ in this case is measured from the pole, the expression shows that the water is raised at the poles by $0{\cdot}61$ foot $\times\sin^2\sigma$, and is lowered at the equator by the same quantity.

(438.) If we put M for the mass of the Moon, D′ for her distance, and μ for her declination, the corresponding quantity for the effect of the Moon's action will be $\frac{9\text{M}\text{b}^2}{8g\text{D}'^3}\sin^2\mu.\sin^2\delta.\cos 2\phi$. Observing that (if the Moon's mass be supposed $\frac{1}{80}$ of the Earth's) this coefficient is $\frac{9}{4}$ of that computed in (35.), the numerical coefficient is $\frac{9}{4}\times 0\cdot 5959$ foot or 1·34 foot; and the elevation of the water is 1·34 foot $\times \sin^2\mu.\sin^2\delta.\cos 2\phi$.

(439.) Secondly, there are the following terms depending on nt and 2ϕ

$$\sin\sigma.\cos\sigma.\sin\delta.(\cos\delta.\sin 2\phi.\cos nt-\cos 2\phi.\sin nt)$$

or

$$-\sin\sigma.\cos\sigma.\sin\delta.\left(\sin^2\frac{\delta}{2}.\sin\overline{nt+\frac{2x}{\text{b}}}+\cos^2\frac{\delta}{2}.\sin\overline{nt-\frac{2x}{\text{b}}}\right).$$

As before, the effects of these two terms will be represented by two waves, one moving in one direction and the other in the opposite direction; and the elevation produced by both will be represented by

$$\frac{2k\text{b}.\sin\sigma.\cos\sigma.\sin\delta}{n^2\text{b}^2-4gk}\left(-\sin^2\frac{\delta}{2}.\cos\overline{nt+\frac{2x}{\text{b}}}+\cos^2\frac{\delta}{2}.\cos\overline{nt-\frac{2x}{\text{b}}}\right)$$

or by

$$\frac{2k\text{b}.\sin\sigma.\cos\sigma.\sin\delta}{n^2\text{b}^2-4gk}.(\cos\delta.\cos nt.\cos 2\phi+\sin nt.\sin 2\phi).$$

Let $\tan\psi=\frac{\tan 2\phi}{\cos\delta}$; then the expression for the elevation becomes

$$\frac{2k\text{b}.\sin\sigma.\cos\sigma.\sin\delta}{n^2\text{b}^2-4gk}\times\sqrt{\cos^2\delta.\cos^2 2\phi+\sin^2 2\phi}\times\cos\overline{nt-\psi}$$

which is to be multiplied by $\frac{3\text{S}\text{b}}{\text{D}^3}$.

Diurnal tide; it depends on the declination of the disturbing body, and disappears if the body is in the equator.

From this expression it appears, 1st, that *for the same place* the oscillation of the water goes through all its phases while nt increases by 2π, or while the luminary goes round the earth; it is therefore a diurnal tide; 2nd, that *for the same time* the wave is in all its different stages for the extent through which ψ varies by 2π, or through which 2ϕ varies by 2π, and is therefore to be found twice in all its different stages for the extent through which ϕ varies by 2π; or there are two complete waves upon the canal, going round it once in two days, with an irregular motion; 3rd, that the amount of oscillation is different at different places; 4th, that, other circumstances being the same, the extent of the oscillation is proportional to the sine of twice the luminary's declination, and changes sign when the declination changes from north to south, and vanishes when the luminary is in the equator.

When the canal is equatorial there is no diurnal tide.

(440.) There are two cases that deserve distinct notice.

1st. If the canal is equatorial, $\delta=0$, and the whole expression vanishes; or, whatever be the declination of the luminary, there is no diurnal tide.

2nd. If the canal passes through the poles of the earth, $\delta=90°$, and the expression is reduced to

$$\frac{2k\text{b}.\sin\sigma.\cos\sigma}{n^2\text{b}^2-4gk}\sin nt.\sin 2\phi$$

When the canal passes through the poles, the tide-wave is a stationary wave.

or the wave is a stationary wave, such as is treated of in (187.). Its period is diurnal; there are two waves always existing at the same time. The origin of ϕ is, in this case, at the pole; and there will therefore be no vertical oscillation at the poles, (where $2\phi=0$ or 360°,) and none at the equator, (where $2\phi=180°$ or 540°,) but there will be a large oscillation at latitude $\pm 45°$ (where $2\phi=90°$, 270°, 450°, or 630°). An elevation in north latitude occurs at the same time as a depression in south latitude. The origin of nt is the plane which passes through the axis of the earth and the axis of the canal; and the water is therefore in its mean state, as depends on this diurnal wave, when the luminary is six hours from the meridian, and in its most elevated or depressed state when the luminary is on the meridian. The elevation or depression depends on the sign of $\sin\sigma$ or sine of declination; it depends also on the sign of $n^2\text{b}^2-4gk$, which will vary as the canal is deep or shallow.

(441.) Thirdly, there are terms depending on $2nt$ and 2ϕ, namely,

$$\frac{1}{2}\cos^2\sigma.\cos\delta.\cos 2\phi.\sin 2nt-\frac{1}{4}\cos^2\sigma.(1+\cos^2\delta).\sin 2\phi.\cos 2nt,$$

or

$$-\frac{1}{2}\cos^2\sigma.\sin^4\frac{\delta}{2}.\sin\overline{2nt+2\phi}+\frac{1}{2}\cos^2\sigma.\cos^4\frac{\delta}{2}.\sin\overline{2nt-2\phi},$$

Semi-diurnal tide, different in different parts of the canal.

which produce two waves travelling in opposite directions, the sum of the corresponding elevations fo which will be represented by

$$-\frac{1}{4}.\frac{k\text{b}.\cos^2\sigma}{n^2\text{b}^2-gk}\left\{\sin^4\frac{\delta}{2}.\cos\overline{2nt+2\phi}+\cos^4\frac{\delta}{2}.\cos\overline{2nt-2\phi}\right\}$$

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or by

$$-\frac{1}{4}\cdot\frac{k\mathrm{b}.\cos^2\sigma}{n^2\mathrm{b}^2-gk}\left\{\left(\frac{1}{2}+\frac{1}{2}\cos^2\delta\right).\cos 2nt.\cos 2\phi+\cos\delta.\sin 2nt.\sin 2\phi\right\}.$$

Making $\tan\chi=\frac{2\cos\delta}{1+\cos^2\delta}\tan 2\phi$, this becomes

$$-\frac{1}{4}\cdot\frac{k\mathrm{b}.\cos^2\sigma}{n^2\mathrm{b}^2-gk}\cdot\sqrt{\left(\frac{1}{2}+\frac{1}{2}\cos^2\delta\right)^2.\cos^2 2\phi+\cos^2\delta.\sin^2 2\phi}.\cos\overline{2nt-\chi}$$

which is to be multiplied by $\frac{3\mathrm{Sb}}{\mathrm{D}^3}$.

From this it is readily seen; 1st, that the tide is semidiurnal; 2nd, that there are two waves on the canal at the same time, revolving irregularly in a day; 3rd, that the extent of oscillation is different in different parts of the canal; 4th, that it is in all proportional to the square of the cosine of the luminary's declination.

If the canal is equatorial, the tide is equal in all parts.

(442.) If the canal is equatorial, $\delta=0$, $\cos\delta=1$, and the expression becomes

$$-\frac{3\mathrm{Sb}}{4\mathrm{D}^3}\cdot\frac{k\mathrm{b}\cos^2\sigma}{n^2\mathrm{b}^2-gk}\cos\overline{2nt-2\phi},$$

or the extent of oscillation is everywhere the same, and the water is high or low on the meridian under the luminary according as gk is greater or less than $n^2\mathrm{b}^2$.

If the canal passes through the poles, the wave is a stationary wave.

(443.) If the canal pass through the pole, $\delta=90°$, $\cos\delta=0$, and the expression is reduced to

$$-\frac{3\mathrm{Sb}}{8\mathrm{D}^3}\cdot\frac{k\mathrm{b}\cos^2\sigma}{n^2\mathrm{b}^2-gk}\cos 2\phi.\cos 2nt,$$

which shows that there is a stationary wave at each pole and at each intersection with the equator, the high water at the pole corresponding to the low water at the equator. The extreme phases of the oscillation occur when the luminary is on the meridian of the canal, and when it is in the meridian six hours from that of the canal; if $n^2\mathrm{b}^2$ be greater than gk, it is high water at the equator and low water at the poles in the former of the cases, and low water at the equator and high water at the poles in the latter case.

Relation of the sign of the expressions to the depth of the sea.

(444.) The reader will remark that gk is less than $n^2\mathrm{b}^2$ if $\frac{k}{\mathrm{b}}$ is less than $\frac{n^2\mathrm{b}}{g}$, or if $\frac{\text{depth of sea}}{\text{earth's radius}}$ is less than $\frac{\text{equatorial centrifugal force}}{\text{gravity}}$, or less than $\frac{1}{289}$; or if the depth of the sea is less than 14 miles. And $4gk$ is less than $n^2\mathrm{b}^2$ if $\frac{k}{\mathrm{b}}$ is less than $\frac{n^2\mathrm{b}}{4g}$, or if the depth of the sea is less than $3\frac{1}{2}$ miles. In this calculation we suppose n the apparent angular velocity of the luminary to be not sensibly different from the real angular velocity of the earth; this applies very nearly to the sun and nearly enough to the moon.

(445.) In the expression for the force urging the water along the canal, we have preserved the terms depending on $\frac{\mathrm{b}^2}{\mathrm{D}^4}$. We do not however think that the examination of the effects of substituting in these expressions the values found in (428.) or (429.) or (435.) would repay us for the trouble. The additional factor $\frac{\mathrm{b}}{\mathrm{D}}$ is about $\frac{1}{60}$, and the terms multiplied by this factor are of the same order as those already discussed, so that the new terms may be considered as of magnitude corresponding to $\frac{1}{60}$ part of those already found (more or less). The terms depending on nt and $2nt$ are nearly similar to those already found; the only novelty is, that there are terms depending on $3nt$. But as these terms are evidently small, and as observation has not yet given reason to suppose that there is a sensible sea-tide occurring three times every day, we shall not further consider these terms.

(446.) Now it must be remarked, that the coefficient of each of the terms which we have discussed is slowly variable. The coefficient of the term examined in (439.) depends on $\frac{\sin\sigma.\cos\sigma}{\mathrm{D}^3}$; that of the term in (441.) depends on $\frac{\cos^2\sigma}{\mathrm{D}^3}$; where σ is the declination of the sun (or moon), and D its distance; two elements which are continually varying. The elements, however, and the combinations of them which we have just set down, may be expanded (by the usual astronomical developments) in such series as $\mathrm{H}+\mathrm{H}'\cos i't+$ &c., where, for $\frac{\sin\sigma\ .\ \cos\sigma}{\mathrm{D}^3}$, H is 0; and, for $\frac{\cos^2\sigma}{\mathrm{D}^3}$, H is much larger than H′. In the latter of these cases, which applies to the semidiurnal tide, supposing the force only to vary, the coefficient of elevation or depression, by (288.), Variation of the disturbing body's distance and of its angular velocity considered.

would not sensibly differ from $\frac{km}{i^2-gkm^2}(\mathrm{H}+\mathrm{H}'\cos i't)$, which is the same as that corresponding to the position of the luminary at the time of the tide, supposing the elevation or depression computed from that position as if it remained constant; but the time of high or low water would be sensibly different. But in reality, when the force varies from change of the attracting body's distance from perigee and of declination, its angular motion varies at the same time: it will be convenient, then, to consider the effect of these two causes together. Now the reader will perceive that in (427.) and all the articles following it, we have used nt merely to denote the hour-angle of the attracting body; and, therefore, upon supposing that angle to increase irregularly, we must, in (427.) &c., put a new symbol. Thus we shall put $-\frac{3}{2}\cdot\frac{\mathrm{Sb}\cos^2\sigma}{\mathrm{D}^3}\cdot\sin^4\frac{\delta}{2}\cdot\sin\overline{2\text{ hour-angle}+2\phi}$ for the first term in (441.); or $-\frac{3}{2}\cdot\frac{\mathrm{Sb}\cos^2\sigma}{\mathrm{D}^3}\sin^4\frac{\delta}{2}\cdot\sin\overline{2\text{ hour-angle}+2\frac{x}{\mathrm{b}}}$. Now let e be the eccentricity of the orbit in which the body moves round the earth, ht its mean anomaly, ω the inclination of its orbit to the earth's equator, lt its mean distance from the intersection of its orbit with the equator. Then, as its true right ascension may be expressed very nearly by the formula,

$$\text{mean right ascension}+2e\sin ht-\frac{\omega^2}{4}\sin 2lt,$$

its true hour-angle may be expressed very nearly by

$$nt-2e\sin ht+\frac{\omega^2}{4}\sin 2lt;$$

and the expression above becomes

$$-\frac{3}{2}\cdot\mathrm{Sb}\cdot\sin^4\frac{\delta}{2}\times\frac{\cos^2\sigma}{\mathrm{D}^3}\cdot\sin\overline{2nt+\frac{2x}{\mathrm{b}}-4e\sin ht+\frac{\omega^2}{2}\sin 2lt}.$$

Expanding the sine, and remarking that, when z is small, $\sin\overline{2nt+\frac{2x}{\mathrm{b}}+z}$ is expressed with sufficient accuracy by

$$\sin\left(2nt+\frac{2x}{\mathrm{b}}\right)+z\cdot\cos\left(2nt+\frac{2x}{\mathrm{b}}\right),$$

the expression becomes

$$-\frac{3}{2}\mathrm{Sb}\sin^4\frac{\delta}{2}\times\frac{\cos^2\sigma}{\mathrm{D}^3}\times\left\{\sin\left(2nt+\frac{2x}{\mathrm{b}}\right)-2e\cdot\sin\left(2nt+ht+\frac{2x}{\mathrm{b}}\right)+2e\cdot\sin\left(2nt-ht+\frac{2x}{\mathrm{b}}\right)\right.$$
$$\left.+\frac{\omega^2}{4}\sin\left(2nt+2lt+\frac{2x}{\mathrm{b}}\right)-\frac{\omega^2}{4}\sin\left(2nt-2lt+\frac{2x}{\mathrm{b}}\right)\right\}.$$

(447.) Now $\cos^2\sigma=1-\sin^2\sigma=1-\omega^2\cdot\sin^2 lt$ nearly $=1-\frac{\omega^2}{2}+\frac{\omega^2}{2}\cos 2lt$. And, putting D_m for the mean distance, as in (22.), $\mathrm{D}=\mathrm{D}_m(1-c\cdot\cos ht)$ nearly, or $\frac{1}{\mathrm{D}^3}=\frac{1}{\mathrm{D}_m{}^3}(1+3e\cdot\cos ht)$ nearly; therefore $\frac{\cos^2\sigma}{\mathrm{D}^3}=\frac{1-\frac{\omega^2}{2}}{\mathrm{D}_m{}^3}\times\left(1+3e\cdot\cos ht+\frac{\omega^2}{2}\cos 2lt\right)$ nearly. Substituting this, the expression above becomes

$$-\frac{3}{2}\left(1-\frac{\omega^2}{2}\right)\cdot\frac{\mathrm{Sb}}{\mathrm{D}_m{}^3}\cdot\sin^4\frac{\delta}{2}\times\left\{\sin\overline{2nt+\frac{2x}{\mathrm{b}}}-\frac{e}{2}\sin\overline{2nt+ht+\frac{2x}{\mathrm{b}}}+\frac{7e}{2}\sin\overline{2nt-ht+\frac{2x}{\mathrm{b}}}+\frac{\omega^2}{2}\sin\overline{2nt+2lt+\frac{2x}{\mathrm{b}}}\right\};$$

and the corresponding expression for the elevation of the water, omitting the constant factor $-3\left(1-\frac{\omega^2}{2}\right)\cdot\frac{\mathrm{Sb}^2k}{\mathrm{D}_m{}^3}\sin^4\frac{\delta}{2}$, is

$$\frac{1}{4n^2\mathrm{b}^2-4gk}\cos\overline{2nt+\frac{2x}{\mathrm{b}}}-\frac{1}{(2n+h)^2\mathrm{b}^2-4gk}\cdot\frac{e}{2}\cos\overline{2nt+ht+\frac{2x}{\mathrm{b}}}+\frac{1}{(2n-h)^2\mathrm{b}^2-4gk}\cdot\frac{7e}{2}\cdot\cos\overline{2nt-ht+\frac{2x}{\mathrm{b}}}$$
$$+\frac{1}{(2n+2l)^2\mathrm{b}^2-4gk}\cdot\frac{\omega^2}{2}\cdot\cos\overline{2nt+2lt+\frac{2x}{\mathrm{b}}}.$$

Expression for the elevation of the tide, with force and angular velocity variable.

(448.) Expanding this expression to the first power of h and l, and omitting the constant factor $\frac{1}{4n^2\mathrm{b}^2-4gk}$, it may be put in the form $\mathrm{P}\cdot\cos\overline{2nt+\frac{2x}{\mathrm{b}}}+\mathrm{Q}\cdot\sin\overline{2nt+\frac{2x}{\mathrm{b}}}$,

where $$\mathrm{P}=1+\left(3+\frac{16n\mathrm{b}^2h}{4n^2\mathrm{b}^2-4gk}\right)e\cdot\cos ht+\left(\frac{1}{2}-\frac{4n\mathrm{b}^2l}{4n^2\mathrm{b}^2-4gk}\right)\omega^2\cdot\cos 2lt$$

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$$Q=\left(4+\frac{12nb^2h}{4n^2b^2-4gk}\right)e.\sin ht+\left(-\frac{1}{2}+\frac{4nb^2l}{4n^2b^2-4gk}\right)\omega^2.\sin 2lt\,;$$

and this may be put in the form $R.\cos\overline{2nt+\frac{2x}{b}-T}$, or $R.\cos\overline{2nt+2\phi-T}$, wher $R=\sqrt{P^2+Q^2}=P$ nearly, and $T=4e.\sin ht-\frac{\omega^2}{2}\sin 2lt$ nearly.

Modifications to be made in the computation proceeding on the supposition of constant force.

(449.) If the magnitude of the wave had been computed on the supposition that the force was constant and equal to that given by the values of D and σ for the time of computation, the expression would have been merely $\left(1+3e\cos ht+\frac{\omega^2}{2}\cos 2lt\right)\times\cos\overline{2nt+\frac{2x}{b}}$. It appears, therefore, that one effect of introducing the motions which we have considered is, to increase the term (in the coefficient of elevation of tides) which depends on the eccentricity of the orbit, or to make the tide proportional to a higher power than the cube of the parallax (supposing the depth of the sea less than 14 miles); another effect is, to diminish the term depending on the declination, or to make the tide proportional to a lower power than the square of the cosine of declination (on the same supposition): a third effect is, to make the phase of tide depend on $\overline{2nt-4e\sin ht+\frac{\omega^2}{2}\sin 2lt+2\phi}$ nearly, or on $\overline{2\text{ true hour-angle }+2\phi}$. The hour-angle, however, is to be computed with an increased value of ellipticity of the moon's orbit, and a diminished obliquity. The same remarks apply, in all respects, to the term depending on $\overline{2nt-2\phi}$, and therefore to their combination.

The terms in the moon's longitude and distance depending on the evection (which is but a slowly varying eccentricity of the moon's orbit) follow the same law as those depending on the eccentricity, so nearly that the same investigations may be held to apply to them, without farther examination. The law of the inequality called *Variation* is a little different. If the inequality in the moon's distance depending on variation be called $-D_m.p.\cos qt$, that in longitude will be $+\frac{11}{8}p.\sin qt$, or that in hour-angle will be $-\frac{11}{8}.p.\sin qt$. Treating these terms in the same manner as those above, it will be found that there is added to P the term $\left(3+\frac{11nb^2q}{4n^2b^2-4gk}\right)p.\cos qt$, and to Q the term $\left(\frac{11}{4}+\frac{12nb^2q}{4n^2b^2-4gk}\right)p.\sin qt$. The second term within each of these brackets bears to the first a proportion which is not exactly the same as that for the terms depending on e, but is not very different from it; and, as the numerical value of Variation is small, there will be no sensible error in assuming that it is exactly the same. Thus, the remarks which we have made regarding the term depending on the eccentricity apply also to the evection and variation: the term depending on the obliquity following a different law. There is no other inequality in the moon's motion worthy of notice.

Friction taken into account.

(450.) To examine the effect of friction, we will neglect the square of f in the expression of (325.); and putting i successively $=2n$, $2n+h$, $2n-h$, and $2n+2l$; $v^2=gk$, $m=\frac{2}{b}$, and omitting the same constant factor as in (447.), we have the following terms to add to those at the end of (447.):

$$-\frac{fb^2 2n}{(4n^2b^2-4gk)^2}\sin\overline{2nt+\frac{2x}{b}}+\frac{fb^2(2n+h)}{((2n+h)^2.b^2-4gk)^2}.\frac{e}{2}\sin\overline{2nt+ht+\frac{2x}{b}}$$
$$-\frac{fb^2(2n-h)}{((2n-h)^2b^2-4gk)^2}.\frac{7e}{2}.\sin\overline{2nt-ht+\frac{2x}{b}}-\frac{fb^2(2n+2l)}{((2n+2l)^2b^2-4gk)^2}.\frac{\omega^2}{2}.\sin\overline{2nt+2lt+\frac{2x}{b}}\,;$$

which, if expanded to the first power of h and l, may be put in the form $P'.\sin\overline{2nt+\frac{2x}{b}}+Q'.\cos\overline{2nt+\frac{2x}{b}}$, where

$$P'=\frac{fb^2}{4n^2b^2-4gk}\left\{-2n-\left(6n+4h\frac{12n^2b^2+4gk}{4n^2b^2-4gk}\right)e.\cos ht-\left(n-2l\frac{6n^2b^2+2gk}{4n^2b^2-4gk}\right)\omega^2.\cos 2lt\right\}$$
$$Q'=\frac{fb^2}{4n^2b^2-4gk}\left\{\left(8n+h\frac{36n^2b^2+12gk}{4n^2b^2-4gk}\right)e.\sin ht+\left(-n+l\frac{12n^2b^2+4gk}{4n^2b^2-4gk}\right)\omega^2.\sin 2lt\right\};$$

the additional factor $\frac{1}{4n^2b^2-4gk}$ being omitted, as in (448.). It is easily seen that

$$P'=-P\times\frac{2nfb^2}{4n^2b^2-4gk}-fb^2h.\frac{16n^2b^2+16gk}{(4n^2b^2-4gk)^2}e.\cos ht+fb^2l\frac{4n^2b^2+4gk}{(4n^2b^2-4gk)^2}\omega^2.\cos 2lt$$
$$Q'=Q\times\frac{2nfb^2}{4n^2b^2-4gk}+fb^2h.\frac{12n^2b^2+12gk}{(4n^2b^2-4gk)^2}e.\sin ht+fb^2l\frac{4n^2b^2+4gk}{(4n^2b^2-4gk)^2}\omega^2.\sin 2lt.$$

(451.) If we combine these terms with those of (448.), introducing some terms of the order f^2eh and $f^2\omega^2l$ (which are perfectly insensible), and if we make $\frac{f\mathrm{b}^2}{4n^2\mathrm{b}^2-4gk}=\rho$, $\frac{f\mathrm{b}^2(4n^2\mathrm{b}^2+4gk)}{(4n^2\mathrm{b}^2-4gk)^2}=\rho_{,}$ it will be found that the complete expression for the height of the water at any instant, omitting the factors $-3\left(1-\frac{\omega^2}{2}\right)\frac{\mathrm{Sb}^2k}{\mathrm{D}_m^{\ 3}}\sin^4\frac{\delta}{2}\frac{1}{4n^2\mathrm{b}^2-4gh}$, is

Alteration which it produces in the last expression.

$$\begin{aligned}&\{\cos(2nt+2\phi)-2n\rho\,.\,\sin(2nt+2\phi)\}\times\left\{1+\left(3+\frac{16n\mathrm{b}^2h}{4n^2\mathrm{b}^2-4gk}\right)e\,(\cos ht+h\rho_{,}.\sin ht)\right.\\&\qquad\qquad\left.+\left(\frac{1}{2}-\frac{4n\mathrm{b}^2l}{4n^2\mathrm{b}^2-4gk}\right)\omega^2(\cos 2lt+2l\rho_{,}.\sin 2lt)\right\}\\&+\{\sin(2nt+2\phi)+2n\rho\,.\,\cos(2nt+2\phi)\}\times\left\{\left(4+\frac{12n\mathrm{b}^2h}{4n^2\mathrm{b}^2-4gk}\right)e\,(\sin ht-h\rho_{,}.\cos ht)\right.\\&\qquad\qquad\left.+\left(-\frac{1}{2}+\frac{4n\mathrm{b}^2l}{4n^2\mathrm{b}^2-4gk}\right)\omega^2(\sin 2lt-2l\rho_{,}.\cos 2lt)\right\};\end{aligned}$$

or

$$\begin{aligned}&\cos\overline{2n(t+\rho)+2\phi}\times\left\{1+\left(3+\frac{16n\mathrm{b}^2h}{4n^2\mathrm{b}^2-4gk}\right)e.\cos\overline{h(t-\rho_{,})}+\left(\frac{1}{2}-\frac{4n\mathrm{b}^2l}{4n^2\mathrm{b}^2-4gk}\right)\omega^2.\cos\overline{2l(t-\rho_{,})}\right\}\\&+\sin\overline{2n(t+\rho)+2\phi}\times\left\{\left(4+\frac{12n\mathrm{b}^2h}{4n^2\mathrm{b}^2-4gk}\right)e.\sin\overline{h(t-\rho_{,})}+\left(-\frac{1}{2}+\frac{4n\mathrm{b}^2l}{4n^2\mathrm{b}^2-4gk}\right)\omega^2\,.\,\sin\overline{2l(t-\rho_{,})}\right\}.\end{aligned}$$

May be represented by using the elements of an earlier epoch, and then adding a constant to the phase of the tide.

(452.) The interpretation of this expression is as follows:

The tides, as affected by friction, may still be computed by the formula of (449.), provided that first we take the co-ordinates of the attracting body's place, not for the time for which the calculation is made, but for a time anterior to it by $\rho_{,}$; and, secondly, that, having thus computed the time of any phase of the tide (high water, for instance), we adopt a time earlier than the time so found, by ρ.

(453.) In the case of the diurnal tide, for which the coefficient has the form $\mathrm{H}'.\cos i't$, and in which the phases of tide depend on $it\pm mx$ where $i=n$ and $m=\frac{2}{\mathrm{b}}$ (439.), we think it is desirable to call the reader's attention to the circumstance that the approximate expressions in (286.), (287.), and (289.), may not apply. For, these expressions suppose that i^2-gkm^2 or $n^2-\frac{4gk}{\mathrm{b}^2}$ is large, or that $\frac{n^2\mathrm{b}}{g}-\frac{4k}{\mathrm{b}}$ is large. Now we have seen that this quantity vanishes if the depth of the sea is $3\frac{1}{2}$ miles, and changes sign if the depth is still greater. Our knowledge of the depth of the sea is extremely imperfect, but, such as it is, it entitles us to suppose that the depth may equal or exceed $3\frac{1}{2}$ miles. In this case it will be necessary to resort to the expressions at the end of (289.), and it will be found that it may happen that the greatest diurnal tide will occur on the day when the force which causes it is smallest. We think it unnecessary to remark on the time at which the high or low diurnal tide occurs, because that time has not been a subject of accurate observation.

Effects of two bodies considered.

(454.) We shall now consider the tides in a canal caused by the simultaneous action of two bodies, as the sun and the moon. We shall consider all the symbols of the present Section, up to this point, as applying to the sun, and (as in Section II.) shall put M for the mass of the moon, μ for its declination, and D' for its distance. And, n being the apparent angular motion of the sun round the earth in its diurnal motion, we shall find n' for the apparent angular motion of the moon round the earth, where $\frac{n'}{n}=\frac{29}{30}$ nearly, and $\left(\frac{n'}{n}\right)^2=\frac{14}{15}$ nearly. It will be convenient to compare our conclusions with those of the equilibrium-theory.

(455.) First, in regard to the ordinary semidiurnal tide. The coefficient of solar tide is $-\frac{3}{4}.\frac{\mathrm{S}k\mathrm{b}^2\cos^2\sigma}{\mathrm{D}^3(n^2\mathrm{b}^2-gk)}$, (441.), omitting those factors which depend only on the position of the canal and of the place of observation upon it; the coefficient of lunar tide is $-\frac{3}{4}.\frac{\mathrm{M}k\mathrm{b}^2\cos^2\mu}{\mathrm{D}'^3(n'^2\mathrm{b}^2-gk)}$. Hence,

$$\text{the lunar coefficient} = \text{solar coefficient}\times\frac{\mathrm{M}\cos^2\mu}{\mathrm{D}'^3}\times\frac{\mathrm{D}^3}{\mathrm{S}\cos^2\sigma}\times\frac{n^2\mathrm{b}^2-gk}{n'^2\mathrm{b}^2-gk}.$$

But by the equilibrium-theory (44.), the lunar coefficient = solar coefficient $\times\frac{\mathrm{M}'}{\mathrm{S}'}\cdot\left(\frac{\mathrm{P}'}{\mathrm{P}'_m}\right)^3\cdot\left(\frac{\mathrm{P}_m}{\mathrm{P}}\right)^3\cdot\frac{\cos^2\mu}{\cos^2\sigma}$, which upon substituting from the articles preceding (44.) becomes

$$\text{lunar coefficient} = \text{solar coefficient}\times\frac{\mathrm{M}\cos^2\mu}{\mathrm{D}'^3}\times\frac{\mathrm{D}^3}{\mathrm{S}\cos^2\sigma}:$$

The solar and lunar tides are not proportional to to the solar and lunar force: the lunar tide being too great.

Consequently the proportion of the lunar coefficient to the solar coefficient given by the wave-theory is greater than that given by the equilibrium-theory in the ratio of $n^2 b^2 - gk$ to $n'^2 b^2 - gk$, or in the ratio of $\frac{n^2 b}{g} - \frac{k}{b}$ to $\left(\frac{n'}{n}\right)^2 \cdot \frac{n^2 b}{g} - \frac{k}{b}$, or in the ratio of $\cdot 00346 - \frac{k}{b}$ to $\cdot 00323 - \frac{k}{b}$. Any probable supposition that we may make in regard to the depth of the sea will give for this ratio a value sensibly different from that of equality. Suppose, for instance, the sea were 4 miles deep; then $\frac{k}{b} = \cdot 001$ nearly, and the ratio is $\cdot 00246 : \cdot 00223$, or $11 : 10$ nearly. The mass of the moon, therefore, as inferred from the tides with a sea 4 miles deep would be too great by $\frac{1}{10}$ part. If the sea were 8 miles deep, the ratio would be $\cdot 00146 : \cdot 00123$, or $13 : 11$ nearly, and the inferred mass of the moon would be too great by $\frac{1}{6}$ part. Thus we find,

1st. If the depth of the sea is less than 14 miles, the mass of the moon inferred from the tides is inevitably too great.

2d. The error will be different (or the moon's mass will appear different) in canals of different depths.

The excess different in canals of different depth.

(456.) In regard to the variations of these coefficients as produced by the variations of distance and declination, the equilibrium-theory requires that those coefficients be used which correspond to the actual distance, &c., of the sun and moon at the moment: the wave-theory with friction requires that coefficients be used which correspond to an earlier time, preceding the tide by $\frac{f}{4} \cdot \frac{\frac{n^2 b^2}{g^2} + \frac{k}{g}}{\left(\frac{n^2 b}{g} - \frac{k}{b}\right)^2}$. This applies at present to the coefficients only.

(457.) In regard to the mode of combining these coefficients, the equilibrium-theory, (44.) and (49.), putting M_2 and S_2 for the coefficients, gives for the height of tide at any instant $M_2 . \cos 2 . \overline{l - m} + S_2 . \cos 2 . \overline{l - s}$, or

$$\sqrt{\{M_2^2 + 2 M_2 S_2 . \cos 2 . \overline{m - s} + S_2^2\}} \times \cos 2\,\overline{(l - m) + F},$$

$$\text{where } \tan F = \frac{S_2 . \sin 2 . \overline{m - s}}{M_2 + S_2 . \cos 2 . \overline{m - s}},$$

$l - m$ being the moon's hour angle, $l - s$ the sun's hour angle, and therefore $m - s$ being the excess of the sun's hour angle above the moon's at the instant of computation for tide. The wave-theory gives in the first instance (441.) for the sun $S_3 . \cos \overline{2\,nt - \chi}$, for the moon $M_3 . \cos \overline{2\,n't - \chi}$: but these receive some modifications. Both forces are variable, but (on account of the extreme slowness of the variations for the sun) it will be sufficient to consider those of the moon. Both tides are affected by friction, and this must be taken into account for both. Referring then to the expressions at the end of (451.), to the explanation in (452.), and to the formula of (449.), it will be seen that we have to combine two such expressions as the following:

For the Sun	$S_3 . \cos \overline{2nt + 2n\rho - \chi}$.
For the Moon	$M_3 . \cos \overline{2n't + 2n'\rho' - T - \chi}$,

S_3, M_3, and T, being certain functions of the bodies' co-ordinates at a time anterior by $\rho_{,} = \frac{f b^2 (4n^2 b^2 + 4gk)}{(4n^2 b^2 - 4gk)^2}$; and ρ being $= \frac{f b^2}{4 n^2 b^2 - 4gk}$, and $\rho' = \frac{f b^2}{4n'^2 b^2 - 4gk}$.

Combining these as in (49.), an expression will be found exactly similar to that of (49.) cited above, but in which the angle entering in the second term of the coefficient, and in the expression for F, instead of $2 . \overline{m - s}$, is

$$(2n - 2n')\,t + \frac{2nf}{4n^2 - gk\frac{4}{b^2}} - \frac{2n'f}{4n'^2 - gk\frac{4}{b^2}} + T.$$

(458.) If we expand the third term by putting $n' = n + (n' - n)$, and retain only the first power of $(n' - n)$, the expression for the angle becomes

$$(2n - 2n')\,t - \frac{f}{4} \cdot \frac{\frac{n^2 b^2}{g^2} + \frac{k}{g}}{\left(\frac{n^2 b}{g} - \frac{k}{b}\right)^2} (2n - 2n') + T,$$

or

$$(2n - 2n') \times \left\{ t - \frac{f}{4} \cdot \frac{\frac{n^2 b^2}{g^2} + \frac{k}{g}}{\left(\frac{n^2 b}{g} - \frac{k}{b}\right)^2} + \frac{T}{2n - 2n'} \right\}.$$

Since $2nt$ is the sun's mean hour angle from a certain plane at the time t, and $2n't$ is the moon's mean hour angle from the same plane, $(2n-2n')\,t$ is the difference of their mean hour angles or the difference of their mean right ascensions at the time t. Consequently the angle above is the difference of mean right ascensions of the sun and moon, not at the time t, but at the time

$$t-\frac{f}{4}\cdot\frac{\dfrac{n^2\mathrm{b}^2}{g^2}+\dfrac{k}{g}}{\left(\dfrac{n^2\mathrm{b}}{g}-\dfrac{k}{\mathrm{b}}\right)^2}+\frac{\mathrm{T}}{2n-2n'},$$

or it is the difference of *true* right ascensions at the time $t'=t-\dfrac{f}{4}\cdot\dfrac{\dfrac{n^2\mathrm{b}^2}{g^2}+\dfrac{k}{g}}{\left(\dfrac{n^2\mathrm{b}}{g}-\dfrac{k}{\mathrm{b}}\right)^2}$.

The relative positions of the two bodies at an anterior time are to be used in the calculation of tide as affected by friction.

(459.) The second term shows as the effect of friction that we may calculate the height and time of high water on the equilibrium-theory, (the constant χ excepted,) provided we consider the Sun and Moon to have the relative position which they really had at the time preceding the time of tide by $\dfrac{f}{4}\cdot\dfrac{\dfrac{n^2\mathrm{b}^2}{g^2}+\dfrac{k}{g}}{\left(\dfrac{n^2\mathrm{b}}{g}-\dfrac{k}{\mathrm{b}}\right)^2}$, or $\rho_{\prime}$. It is singular that this quantity (although found by a very different process) should be exactly the same as the quantity found in (452.) for the retroposition of time for which the coefficients are to be computed. The combination of results shows that both the coefficients and the relative positions are to be used which correspond to a time anterior by a certain quantity.

(460.) With regard to the diurnal tide, we leave the reader to make similar remarks. We will only observe that as the coefficients of the diurnal and of the semidiurnal tide depend in very different ways on the depth of the sea, it is impossible, without a precise knowledge of the depth, to assign any proportion between the coefficients as depending on the proportion of the forces.

Effect of propagation of tide up a river considered.

(461.) Suppose now that a shallow river, or even a shoaly sea of considerable length, communicates with such a canal as we have supposed. The insignificance of such a river will prevent it from altering the tides of the sea in the smallest sensible degree; and we may consider the sole effect of their communication to be, that the sea will always maintain at the mouth of the river the height given by the preceding theory, and that it will always supply or receive the water necessary for the propagation up the river of such waves as are consistent with the circumstances of the river. The only points to which we shall now allude are, the time and height of the tides during the different parts of a lunation.

The relation of the height of the river-tide to the height of the sea-tide always the same.

(462.) With regard to the height of the tide: the highest tide at the station on the river will always be that which is propagated from the highest tide of the sea. The tide occupies always the same time in passing from the sea to the river station (with a small inequality, perhaps of a few minutes, which, although important in the next article, is quite unimportant for our present purpose). The effect of this is only to require a greater retroposition of the places of the sun and moon by which the magnitude of the tide is computed. Thus, suppose that for the sea-tide it is necessary to compute for places of the sun and moon earlier by forty-four hours, and suppose that the tide-wave occupies three hours in passing up the river: then the height of high water at the inland station must be computed from the positions of the sun and moon forty-seven hours earlier than the time of high water at the inland station; because that time is forty-four hours earlier than the time of high water in the sea-tide by which the river-tide is produced.

The relation of the time of the river-tide to the time of the sea-tide not always the same.

(463.) But for the calculation of the time of high water a different rule is necessary. It will be seen by (206.) that the time of high water at the inland station, as measured from a certain fixed phase of the tide at the shore, is $\dfrac{3\pi}{2mv}+\dfrac{x'}{v}-\dfrac{3x'}{2v}b$, or $\mathrm{C}-\dfrac{3x'}{2v}b$, where b is the proportion of the rise of tide above the mean state to the mean depth of water in the river. The quantity b therefore will be proportional to $\sqrt{\{\mathrm{M}_3^2+2\mathrm{M}_3\mathrm{S}_3\cos\overline{2\,(\mathrm{A}'-\mathrm{A})}+\mathrm{S}_3^2\}}$, where A′ is the moon's right ascension and A the sun's, computed for t', the time whose expression is at the end of (458.); or supposing S_3 much smaller than M_3, b will be proportional to $\mathrm{M}_3+\mathrm{S}_3\cos\overline{2\,(\mathrm{A}'-\mathrm{A})}$. Substituting this in $\mathrm{C}-\dfrac{3x'}{2v}b$, it takes the shape $\mathrm{E}-\mathrm{G}x'.\cos\overline{2\,(\mathrm{A}'-\mathrm{A})}$. Now the time of high water on the sea-coast is determined by making $\cos\overline{2n't}\,\overline{-\chi'+\mathrm{F}}$ maximum, where $\mathrm{F}=\dfrac{\mathrm{S}_3}{\mathrm{M}_3}\sin\overline{2\,(\mathrm{A}'-\mathrm{A})}$ nearly; and where $\chi'=\chi+\mathrm{T}-\dfrac{2n'.f}{4n'^2-gk\dfrac{4}{\mathrm{b}}}$. As this requires that $2n't-\chi'+\mathrm{F}=2m\pi$, m being a whole number, we have the time of high water on the coast $=\dfrac{2m\pi+\chi'}{2n'}-\dfrac{\mathrm{F}}{2n'}=\dfrac{2m\pi+\chi'}{2n'}-\dfrac{\mathrm{S}_3}{2n'\mathrm{M}_3}\sin\overline{2\,(\mathrm{A}'-\mathrm{A})}$, and therefore the time of high water at the inland station =

$$\frac{2m\pi+\chi'}{2n'}+\mathrm{E}-\frac{\mathrm{S}_3}{2n'\mathrm{M}_3}\sin\overline{2(\mathrm{A}'-\mathrm{A})}-\mathrm{G}x'\cos\overline{2(\mathrm{A}'-\mathrm{A})}.$$

Let $\frac{2m\pi+\chi'}{2n'}+\mathrm{E}=\mathrm{K}$, $\sqrt{\left(\frac{\mathrm{S}_3}{2n'\mathrm{M}_3}\right)^2+\mathrm{G}^2x'^2}=\mathrm{L}$, and $\frac{2n'\mathrm{M}_3\mathrm{G}x'}{\mathrm{S}_3}=\tan\mathrm{N}$: then the time of high water at the inland station is

The time for a river-tide is to be computed with the places of the Sun and Moon corresponding to a later epoch than that used for the heights.

$$\mathrm{K}-\mathrm{L}\sin\overline{2(\mathrm{A}'-\mathrm{A})+\mathrm{N}}.$$

Now $\overline{2(nt-n't)+\mathrm{N}}$, or $(2n-2n')\left(t'+\frac{\mathrm{N}}{2n-2n'}\right)$ is the difference of mean right ascension of the sun and the moon at the time $t'+\frac{\mathrm{N}}{2n-2n'}$; therefore $\overline{2(\mathrm{A}'-\mathrm{A})+\mathrm{N}}$ is very nearly the difference of true right ascension of the sun and moon at that time. Thus it appears that for computing the time of high water it is necessary to use, not the positions of the sun and moon at the true time of the tide, nor the positions at that anterior time which is employed in computing the height of high water, but a time later than that which is used for computing the height, and therefore a time which is nearer to the true time of high water.

The mass of the Moon inferred from these times of tide will be a little too large.

(464.) It appears also that the effect of the passage of the tide along the shallow river, as shown by the variation in the time of high water as referred to the moon's transit, if computed by the formulæ of the equilibrium-theory, is to give a mass of the moon which is a little smaller than that corresponding to the variation of heights. For we adopt the quantity L as the representative of $\frac{\mathrm{S}_3}{2n'\mathrm{M}_3}$; but L really is $\sqrt{\left(\frac{\mathrm{S}_3}{2n'\mathrm{M}_3}\right)^2+\mathrm{G}^2x'^2}$: therefore we adopt for $\frac{\mathrm{S}_3}{2n'\mathrm{M}_3}$, a quantity which is too great, or for M_3, a quantity which is too small.

Modification in the statement for the times of low water.

(465.) If we investigated the effect of the passage up the shallow river upon the time of low water, we should find that the positions of the sun and moon corresponding to an earlier time than that used for the height of the high water must be employed; but we should still find that the mass of the moon inferred from the variations of the time of low water as referred to the moon's transit is too small.

Advantages and disadvantages of the theory of waves as applied to tides.

(466.) We shall here close our exposition of the Wave-Theory as applied to the tides. As nearly the whole of this theory is published for the first time in the present treatise, we shall not remark upon it at great length. We think it right, however, to point out to the reader its great and important defect as applied to the explanation of tides upon the earth, namely, that in the case of nature the water is not distributed over the surface of the globe in canals of uniform breadth and depth, or in any form very nearly resembling them. In this regard its fundamental suppositions are probably as much, or nearly as much, in error as those of Laplace's theory. But we also think it right to point out that in regard to the completeness of detail with which the principles can be followed out, there is no comparison between the two theories. This will be seen by the reader who has remarked the facility with which the results of "difference between the angular velocities of the sun and moon," "variable coefficients of force," and "friction," are obtained in finite form. For these, Laplace's theory is quite useless. And though (as we have stated) the fundamental suppositions differ much from the real state of the seas, yet no one can hesitate to admit that the same general conclusions will appply:—for instance, that the moon's mass inferred from the height of the tides is too great, and by different degrees in different places: that the effect of friction will be a retroposition of tides in reference to the places of the sun and moon, &c. The peculiarities of river-tides, which no other theory has touched upon, are almost completely mastered by this.

(467.) With these remarks we terminate our Theory of Tides. Any mathematical deduction which may be required in reference to any special phænomenon of observation will be given in the place where such phænomenon is mentioned. The remainder of our Essay will be devoted to the Observations of Tides and their Comparison with the Theory.

SECTION VII.—METHODS USED FOR OBSERVING THE TIDES, AND FOR REDUCING THE OBSERVATIONS.

Observations of Tides are for the most part confined to high water. Much more valuable when low water also is observed.

(468.) The greater part of the observations of tides hitherto made have been observations of time and height of high water only, made at the entrances of the docks of our principal commercial towns. These observations, as regards *time*, are all affected by the circumstance treated in (208.) and (463.), namely, that the time of transmission of the tide from the sea is less for a large tide than for a small one; and, as regards *height*, they do not even give the coefficient of vertical oscillation. Both these defects are removed by adding observations of time and height of low water: and such a system of observation, if made under favourable circumstances, and where the water is quiet, will usually give results of considerable accuracy for the times of high and low water on the coast.

Observations of time of greatest height and greatest depression are inaccurate.

(469.) But this supposes that the rise of the water is continuous and its fall continuous, and that both follow simple laws; a thing which is by no means to be assumed, and which we shall find incorrect in application to various instances. Moreover, as regards time, it is inaccurate, because it is impossible to fix precisely on the time when the surface of the water, having risen with a decreasing velocity that at last is imperceptible, begins to fall with a velocity which at first is insensibly

Mr. Whewell's proposed method of observation.

Objections to that method.

small. To obviate the latter inconvenience, it was proposed by Mr. Whewell that the time should be noted at which the surface of the water passes any fixed points on a wall in its ascent and in its descent. If the rise and fall followed the same simple law as in the sea, (the elevation above a mean point being expressed by sin nt,) the time intermediate to these observed times would give the time of high or low water with precision, because the instant at which the water passes the line, when in rapid rise or fall, can be very accurately observed. But in fact, the laws of rise and fall are so different that this deduction would be erroneous for high water, and very greatly in error for low water. The author of this paper endeavoured to use, for the time of low water, observations of height made at equal intervals of time for an hour and half in the neighbourhood of and including low water, but even in that time the fall and rise occurred with such different velocities that it was necessary to abandon the use of the mean of any of the times, and to judge as well as could be done, under all circumstances, from the general course of the fall and rise during the whole interval.

It is highly important to observe the whole course of the tide.

(470.) There is no method, in fact, which will give satisfactory results as to low and high water, and none which gives any knowledge whatever of the general course of the tide, except the observation—by the senses of an observer stationed for the purpose, or by the indications of a self-registering tide-gauge—of the height of the water at every instant of time, or at least of the height at very small intervals. When a self-registering instrument is established, it is as easy to keep it in action constantly as occasionally, and thus the register of every tide may be preserved. When the observations of a special observer are used, it is necessary to limit the observations, either to two or three days which exhibit the principal changes of circumstance, (as a day near spring tides and a day near neap tides,) or to a period which embraces those changes (as a half-lunation). We shall mention the cautions with which both kinds of observations ought to be made, and shall notice the principles of construction of a self-registering tide-gauge.

Caution necessary for accurate observation.

(471.) The first thing is, to obtain a surface of water, communicating so freely with the sea as to assume the same mean level, and yet unaffected by the agitations of ordinary waves. A sheltered situation ought therefore to be chosen; but, in parts opening immediately to the sea, it is necessary to use other precautions. The most effectual is, to place in the water a large vertical trunk or trough, communicating with the water only by a small hole at or near to its bottom. Suppose, for instance, this hole to be 10 feet below the surface; the agitation of the water by waves 10 feet long is, by (178.), diminished to $\frac{1}{535}$ part of that at the surface, and therefore would not sensibly disturb the water in the trunk even if there were no limitation of the communicating aperture. But if a wave be very long—as for instance any of the modifications of the tide wave—then the agitation caused by it is as great at 10 feet depth as at the surface (180.), and its time is so great that it will be able always to maintain the water in the trunk sensibly at the same level as the external water, even though the communicating aperture be small. If then a float be placed in the trunk, carrying an index above, this index will rise or fall with the general mean level of the water as affected by the tides only. We have had great pleasure in watching the steady motion of the index on the admirable tide-gauge erected at the Royal Dockyard of Sheerness, while the swell on the outside was so heavy that to judge of a mean level within the accuracy of many inches appeared quite impracticable. The instrument, however, is more likely to fail for observations near low water. For suppose the surface to drop till the hole is only one foot below the surface, then the motion of the water produced by waves 10 feet long is nearly as great as that at the surface, and therefore, however much the aperture be limited, irregularity in the height of the surface will be produced by the waves. The only simple way of preventing this is to let the bottom of the trunk and its communicating aperture be carried as low as possible below low water. It might perhaps be advisable to have the trunk divided into two chambers by a vertical partition, the first chamber receiving the water from the sea by a small hole in its bottom, the second receiving the water from the first by a small hole in the partition; the motion of the water in the second would be extremely steady. It is almost needless to observe that a fixed vertical scale of feet and inches, or other measures of length, is indispensable; in simple observations it may be traced upon the quay wall or post at which the elevation of the surface is noted; where a float carries a vertical rod, the rod may be marked as a scale of feet and inches, and the indication opposite to a fixed index may be noted, or the rod may carry an index which in rising or falling will point to different divisions on a fixed scale.

(472.) In some of these ways, observations of the height and time of high water (and for the most part of low water also) are regularly made by direction of the Board of Admiralty at Portsmouth, Plymouth, Ramsgate, Liverpool, Dublin, and occasionally at Harwich and other ports when maritime surveys are in progress. Observations of the same kind are also usually made at the principal docks in the Thames, the Mersey, and the Clyde, as well as at other ports. The principal data for determinations of the time, &c., of tides in ports all over the world have been obtained by some of these methods, either from the loose observations made by harbour-masters and merchant-sailors, or from the more accurate observations conducted by the officers of government surveying-ships.

General principle of self-registering tide-gauge.

(473.) The principle of the self-registering tide-gauge is in all cases the following. By means of a pendulum-clock urged by a sufficient weight, a sheet of paper either spread upon a flat surface, or rolled as required upon a large cylinder, or fixed in a tubular form upon a solid cylinder, is made to travel uniformly. The first of these methods is used for Osler's anemometer, in which the self-registering principle is similar to that of a tide-gauge; the second in Palmer's tide-gauge, (*Phil. Trans.* 1831;) the third in Bunt's tide-gauge. A pencil, carried by mechanism connected with the float, is made to move through a space proportional to the vertical motion of the float, and in a direction perpendicular to the direction of the paper's motion (where the paper is on a cylinder, the motion of the pencil is parallel to the cylinder's axis). A curve is thus traced, whose abscissa represents time, and whose ordinate represents rise of the surface of the water. Occasional examination is always necessary to verify the correctness of action of the machinery. We cannot perhaps do better than copy (with some alterations) from the *Phil. Trans.* 1838, the description of Mr. Bunt's tide-gauge, erected on the bank of the river Avon, nearly a mile below Bristol.

Description of Mr. Bunt's tide-gauge.

(474.) The principal parts of Mr. Bunt's tide-gauge (figures 34 to 39) are A, an eight-day clock, which turns a vertical cylinder B once in 24 hours; the upper end of B carrying on its circumference a toothed contrate-wheel, in which works a pinion *p* carried by the axis of one of the clock-wheels. C is a wheel with grooved circumference; to which is attached the wire E, which passes over the upper part of C, (lying in its groove,) and over a large pully F attached to the outside of the quay-wall, and depends in the water-trunk, supporting the float D at its lower end. A counterpoise G is suspended by a wire attached to a smaller barrel carried by the same axis as C, and thus keeps the wire E constantly stretched, and moves the wheel C when, by the rising of the water, the float D ceases to pull upon E. H is a still smaller drum carried by the same axis as C; a finer wire is attached to it and wrapped round it, and on this wire is suspended the bar I which carries the pencil K. The diameters of the drum H and the wheel C are so adjusted that the vertical motion of the pencil is one-eighteenth part of the vertical motion of the surface of the water. a is a strong oak frame attached to the quay-wall; from it project arms cc whose ends are supported by pillars dd; there also projects an arm e whose end is supported by the pillar f. cc carry the mahogany frame b, to which the clock, the bush for the upper pivot of B, and one bearing of the spindle of C, are attached, (the other bearing of the spindle of C being in another upright S;) e carries the bush for the lower pivot of B. The nature of its bearing may be seen in figure 38; it is in a plug *b*, which may be lowered by a screw and winch, either for adjustment, or for taking the cylinder out, (for mounting papers, &c., upon it;) as by lowering it, first the upper pivot is disengaged, then the lower, and then the end of the cylinder may be made to slide upon the pieces *aa*.

(475.) The upper end of the bar I slides between two guides P and N, and the lower end between two guides QQ. It carries the small brass bar *d* (fig. 37), which has movement upon an axis whose projection is at *e*, (the movement being in the direction to or from the surface of the cylinder, and the axis being sufficiently long to permit no other sensible movement.) The bar *d* carries the pencil-holder K, which is pressed towards the great barrel by the crooked lever carrying the small weight L. *h* is a screw for adjusting the height of the pencil-frame in reference to its suspending wire. The great barrel B is 24 inches in length and 48 in circumference; it is made of mahogany staves, screwed upon mahogany ends and diaphragm, and covered with white enamel.

(476.) The action of this machinery will be understood from the remarks in (473.). Referring to the *Philosophical Transactions*, 1838, for some parts which we have omitted, we think the following points important. The pallets (in the escapement of the clock) may be detached from the train, by lifting a latch behind the clock, and drawing them backward; this arrangement is required from its being necessary to fix the hands so that they cannot, as in other clocks, be made to slide without carrying the train along with them. The pinion which drives the cylinder may be detached by pulling it forwards, so as to allow the cylinder to be turned freely. From inequalities in the teeth of the wheels the cylinder is not moved through exactly equal spaces in equal times; to prevent error from this cause, the places of the hours and minutes marked on the top of the large cylinder were thus determined: the pallets were detached from the clock, (the large cylinder remaining connected with it,) and the hands were moved to indicate 0 hours 0 minutes, 0 hours 20 minutes, 0 hours 40 minutes, &c., and at each of these positions a pencil line was drawn on the cylinder by sliding the pencil-frame between its guides. The scale of height was determined by marking feet and inches on the outside of the float-trunk, and noting the internal indication of the gauge as the tide rose to each successive foot on the trunk. A small scale M, carried by the frame b, may be pushed in contact with the barrel, and thus the heights and the times may be read; or lines of feet may be *turned* on the barrel, by causing it to revolve while the pencil is held to it. The float is of pine, well saturated with oil; the aperture by which the water enters is about $\frac{1}{200}$th of the sectional area of the trunk.

(477.) A sheet of paper is wrapped round the cylinder and expanded by moisture; its ends are then pasted together, and it may (if necessary) be fixed with pins or bands. The curves are then well traced by the mechanism upon the paper, and it may be removed at pleasure.

The same sheet may be retained on the tide-gauge during a fortnight.

(478.) If it were desired to take the record of each tide so as to exhibit the course of a great number of tides in sequence, as in fig. 40, it would be necessary to apply a fresh paper every day to the barrel. But remarking that the tide comes later on each successive day by nearly an hour, and that the same hour of high water recurs only in 15 days, it is evident that the same paper may be kept upon the cylinder for a fortnight without risk of the curves interfering, although there may be a great complication of lines on the paper. In figure 41 we give a copy of the curves delineated by the self-registering tide-gauge at Sheerness, (extracted by permission of Captain Beaufort, R. N., Hydrographer,) from 1840, December 23, (new moon,) to 1841, January 7, (full moon.) The reader, in tracing the curves, must conceive the two ends of the drawing to be united, and must begin at the point marked with the word "Change." We have selected this period, because it exhibits one of the greatest irregularities that we have ever known in the tides, namely, that of 3d January, 1841, morning tide, when the water was five feet lower than was expected. This is fully confirmed by observations made at the same time at Woolwich, at Deptford, and at the London and St. Katherine Docks. It followed a very heavy gale which had blown partly from S.W., and partly from N.W. or N. In figure 42 we give a copy of some of the curves traced by the Bristol tide-gauge at the same time, (with which we have been favoured by Mr. Bunt.) In figure 43 are represented a few curves, (corresponding to another period,) traced by the self-registering tide-gauge at Swansea, (for which we are indebted to the kindness of J. W. G. Gutch, Esq.) It must be remarked that the floats of the tide-gauges at Bristol and Swansea do not descend sufficiently low to record the phænomena of low-water; that at Sheerness fully records the circumstances of low water. The reader will at once see that the information furnished by this instrument is infinitely more valuable than could have been obtained by any system of personal observation.

Specimens of the curves traced by self-registering tide-gauges.

Methods of reducing tidal observations

(479.) We shall now proceed with the methods of reducing tide-observations. And first, in regard to the law of rise during each tide at any given locality.

applicable to the course of a single tide at any place.

Although times of high water and of low water have been observed, and their difference of velocities up the same river have been remarked, and although curves exhibiting to the eye the laws of rise and fall have been traced, (by some of the means above described,) we are not aware that they have been reduced to algebraical form except by the writer of this paper in the *Philosophical Transactions*, 1842. The elevation of the water had been observed at Deptford at every quarter of an hour during half a lunation. The spring-tides and those near them were classed together as one group, and the neap-tides and those near them as another group. In order to combine those of each group, it was assumed that the predicted time of high water in the Nautical Almanac was correct, (a constant difference excepted,) and the interval from one predicted high water to the next was conceived to correspond to 360° of *phase*, and the time of every intermediate observation was converted into *phase* by that proportion. (In a subsequent discussion of tides observed every five minutes at Southampton, we have, instead of using any predicted time, fixed upon the estimated times of low water as the origin of phase.) To bring all the observed heights to a comparable state, the range from high water to low water in every half-tide was supposed to correspond to 2·000, and the depression below the nearest high water at every observation was converted into number by that proportion. The various tides in each group were thus made entirely comparable. The means of all the phases and all the converted depressions within every 10° of phase were taken, and thus a series of mean phases very near to 5°, 15°, 25°, &c., and the corresponding converted depressions, were obtained. By observation of the progress of the numbers, it was easy to alter the latter so as to obtain converted depressions corresponding exactly to 5°, 15°, 25°, &c. Then it was assumed that these could be represented by the following formula:—

$$\text{Converted depression} = A_0 + A_1 \cos.\,\text{phase} + A_2 \cos.\,2\,\text{phase} + \&c., + B_1 \sin.\,\text{phase} + B_2 \sin.\,2\,\text{phase} + \&c.,$$

which, it is well known, is sufficient for the representation of a function which is periodical for 360° of phase. Then the values of A_0, A_1, B_1, &c., are determined with comparative facility in the following manner. 1st. A_0 is the mean of all the converted depressions. 2nd. Multiply every converted depression by cos. phase, and take their sum; then, (since $\cos 5° + \cos 15° + \&c.$ to $355° = 0$; $\cos 5°.\sin 5° + \cos 15°.\sin 15° + \&c. = 0$; $\cos 5°.\cos 10° + \cos 15°.\cos 30° + \&c. = 0$; and so for every one except the multiplier of A_1, where $\cos^2 5° + \cos^2 15° + \&c.$ to $355° = 18$), that sum $= 18\,A_1$. 3rd. Multiply every converted depression by sin. phase, and take their sum; it will be found to be $= 18\,B_1$. 4th. Multiply every converted depression by cos. 2 phase and take the sum; it $= 18\,A_2$, and so on. Thus the values of all the coefficients are obtained.

Methods of reduction applicable to the laws of height and time at any place in different positions of the sun and moon.

(480.) Secondly, in regard to the laws of time and height of tide at the same place in different positions of the sun and moon. Every examination referring to this object proceeds on the supposition that the times and heights may be represented generally by the forms given by the equilibrium-theory, (45.), (46.), (53.), and (54.), though perhaps with altered proportion of coefficients, arguments altered by addition or subtraction of constants, and (in some cases) altered form of function.

Laplace's methods of treating the heights of the tides.

Laplace is (so far as we are aware) the first person who combined observations in considerable groups for comparison with theory. His labours will be found in the second and fifth volumes of the *Mécanique Céleste*. In the second volume he treats the observations made at Brest from 1711 to 1716. He commences with the height of syzygial tides. These he divides into four classes, corresponding to the two equinoxes and the two solstices, and uses at least two syzygies (one new and one full moon) for each equinox or solstice, the whole number being 24 syzygies of each class. For each of these he takes the heights for several neighbouring days; and uses, as the whole range of tide on any day, the difference between the mean of two high waters on that day and the intervening low water.

His methods for the age of the tide.

In order to determine the interval by which the greatest tide follows the syzygy, he does not remark which tide appears to be the greatest, but he remarks that the tides on the second day before syzygy, and the fifth day after it, are very nearly equal; then, observing the extent of change of tide in one day, (which at those points of the lunation is considerable,) he is able to correct this approximation, and thus to find very exactly the times at which the tides are equal; the time intermediate to these is that at which the total tide is highest. It is found to be almost exactly 36 hours after the syzygy.

For the alteration of the moon's effect, depending on her motion in right ascension.

(481.) Laplace has assumed (without assigning any reason for it except the possibility of a communication between the port of observation and two tidal seas, as in (121.) and (312.)) a result similar to that which we have found in (455.), namely, that the rapidity of the moon's motion in right ascension increases her effect on the tides, as compared with the sun's effect. To discover the value of the coefficient, he proceeds in this manner. Assuming that in the mean of his syzygial tides the moon's declination is sensibly the same as the sun's, the syzygial solstitial tides ought to bear to the syzygial equinoctial tides the proportion of the mean value of the square of the cosine of declination at the solstices to the similar quantity at the equinoxes. Now they are found to be somewhat greater than this proportion gives. The difference is attributed to the moon's quicker motion in right ascension, or slower motion in hour-angle, at the solstitial syzygies than at the equinoxial syzygies; and, assuming the moon's mean effect to be to that of the sun as 3 : 1, or to be $\frac{3}{4}$ of the whole mean effect, the proportional change in the moon's effect caused by that difference of velocities in right ascension is found, and from this the proportional change caused by the whole mean velocity in right ascension is inferred, (supposed to be always proportional to that velocity in right ascension.) From this Laplace finds the moon's effect to be increased $\frac{1}{10}$: this, as we have seen (455.), implies that the sea is 4 miles deep.

For changes of parallax

(482.) For the effect of the sun's variation of distance, Laplace compares the heights at the winter solstices with those at the summer solstices. For the effect of the moon's variation of distance, Laplace compares 12 apogeal tides with 12 perigeal tides (in syzygies), and finds that their magnitudes are precisely in the proportion given by theory, without any allowance for the difference of the moon's movement in right ascension at perigee and at apogee. (The effect of this, in our theory, will be found in (448.).)

For diurnal tide.

(483.) The effect of diurnal tide is found on comparing the morning syzygial tides with the evening syzygial tides, at the solstices. In the summer, the evening tides are the greater at Brest; in the winter the morning tides. See (48.) and (63.). But Laplace remarks that observations of low water are necessary for a complete determination, (we shall hereafter find an instance of such a determination.)

For the proportion of the sun's mean effect to the moon's mean effect.

(484.) For the quadratures, Laplace first determines the time of smallest tide by a process nearly similar to that for the time of the largest tide near syzygies, and finds exactly the same value for its retardation. In order to determine the proportion of the sun's effect to the moon's effect, by comparing the neap tides with the spring tides, (see (51.) and (52.),) it may be assumed in the mass of observations that the declinations of the two bodies will be equal; but a correction is required, because the inequality in the moon's motion called the *variation* always diminishes her distance and increases her velocity in right ascension at syzygies and produces the opposite effect at quadratures. Making due allowance for this, it is found that a corrected mean spring tide is double a corrected mean neap tide, or that the moon's effect is three times as great as the sun's. On comparing those in which the declinations are remarkably different, the coefficient $\frac{1}{10}$ is again found for the increase of the moon's effect depending on her motion in right ascension. The effect of change of the moon's distance is said to be as distinct in the quadratures as in the syzygies.

(485.) The diurnal tide in the equinoxial quadratures follows the same law as in solstitial syzygies, the moon's declination being the same; but as in the former the sun's declination vanishes, the amount of diurnal tide is only $\frac{3}{4}$ of its amount in the latter. This is verified by the observation.

The same from the times of high water.

(486.) To determine the proportions of the effects of the sun and moon from the times of high water, Laplace has taken observations nearly 7 days apart, (nearly 3½ days before and nearly 3½ days after the greatest tides.) For the first of these $2(m-s)$ in (49.) is nearly $-90°$, and for the second it is nearly $+90°$, and therefore these two values give nearly the two greatest possible values for F with opposite signs, and their difference is particularly well adapted to determine the proportion of S' and M'. The effect of the moon is thus also found to be three times that of the sun, or a little greater.

(487.) The hours at which the greatest tides and the least tides occur are determined by interpolating among the times of the observed tides following the syzygies and quadratures. It is found thus that the hour of smallest tide is not quite six hours later than the hour of greatest tide. There are many other deductions compared with theory, but those which we have mentioned are the most important, and they will serve to give the reader an idea of the process which Laplace has followed, using no great number of tides, and not comparing the general laws of particular phænomena, so much as the special values which the expressions assume in extreme cases.

(488.) In the fifth volume of the *Mécanique Céleste*, Laplace has discussed the observations made at Brest during sixteen years, from 1807 to 1822. The theory which he has used is precisely the same as that of his second volume, with this addition, that he has pursued to great length the consequences of his assumption that the effect of each of the attracting bodies, both as regards the coefficient of tide and as regards the constant in the argument, will contain a multiple of the body's angular movement round the earth. In this respect his theory is greatly superior to that of the English philosophers whom we shall mention shortly. The method of expansion which he has adopted is exactly similar to that of (447.) and (450.), though its principles are absolutely arbitrary. We shall defer to the next Section the statement of his results; and shall only mention here that his method of discussion is exactly the same as in his second volume, not using all the observations made at all times, but only comparing those which are made at or near to the times at which the irregularity which he is seeking has its extreme values.

(489.) The method of discussing the observations which we have above described as Laplace's (independently of the advances, however arbitrary their foundations may be considered, in the theory) was undoubtedly a great improvement upon that used by Lalande in his *Traité du Flux et Reflux de la Mer*, who had contented himself with picking out, from the same collection as that first used by Laplace, a single observation here and there, and thus (taking them as affected by winds and other accidents) had sometimes arrived at conclusions opposite to those which Laplace established. But the great principle of employing masses of observations was first used in its greatest extent by Mr. (now Sir J. W.) Lubbock, in the discussion of the observations of high water made at the London Docks during nineteen years (from 1808 to 1826).—See the *Philosophical Transactions*, 1831. The process used by Mr. Lubbock was nearly as follows. The quantity treated in regard to times was, the interval between the moon's passage over the meridian and the time of high water: the quantity treated in regard to heights was, the height of high water above a certain fixed mark on the dock-wall. Then, first, the observations were divided by months, (the observations of every month of January during all the nineteen years being collected into one group, those of every month of February into another group, and so on.) As the moon's node has performed one revolution, and the moon's perigee two revolutions, almost exactly, in the nineteen years, and as the inequalities in the moon's motion not connected with the time of year depend only upon these elements, and have gone through all their changes in the nineteen years, it is plain that the means inferred from these groups will give an accurate representation of all the phænomena which depend solely upon the time of year. The groups were then subdivided into parcels, each parcel including all the observations at which the moon's passage over the meridian occurred in each half hour of apparent solar time. Thus for the month of September, the first parcel included all the observations corresponding to the moon's passage over the meridian within half an hour after the sun for every September through the nineteen years; the second included all in which the moon passed more than half an hour and less than an hour after the sun, and so on. Thus the first table of rough results was obtained, which, when its irregularities were smoothed down, gave a table of results adapted to further use. Secondly, the observations were divided into groups corresponding to minutes of the moon's horizontal parallax, (the first group including all in which the moon's parallax was greater than 54′,

Laplace's method much superior to methods employed before him.

Mr. Lubbock's methods of treating times and heights of high water.

For mean values depending only on the time of the year, and the time of the moon's transit.

For the effect of variation of parallax.

and less than 55′, and so on,) and each group was subdivided into parcels corresponding to each hour of apparent solar time at which the moon passed the meridian; the observations made in the afternoon were, however, alone employed here. Thirdly, the observations were divided into groups corresponding to every three degrees of declination of the moon, (the middle group containing all in which the declination was between $1\frac{1}{2}^\circ$ north and $1\frac{1}{2}^\circ$ south,) and each group was subdivided into parcels corresponding to each hour of apparent time in which the moon passed the meridian. The mean of each parcel was in all cases taken, as well for the times as for the heights. Thus three sets of tables were obtained, the leading division of the first being the month, that of the second being the parallax, that of the third being the declination; and all being subdivided according to time of moon's transit. In the first table, the means of all the results for the different months corresponding to the same half hour of moon's transit were taken. These means were subtracted from all the quantities of the three sets of tables, and thus new tables were formed. Then it was intended by the author that a tide (either in time or in height) should be computed as follows:—1st. The quantity depending on the moon's transit only should be taken. 2nd. The correction to this depending on month and moon's transit should be added. 3rd. The correction depending on parallax and moon's transit should be added. 4th. The correction depending on declination and moon's transit should be added. It was, however, pointed out by Mr. Whewell and acknowledged by Mr. Lubbock (*Philosophical Transactions*, 1834) that the last correction is nearly included in the second, because for a given month and given hour of moon's transit the moon's declination is given, excepting that part which depends on the inclination of her orbit to the ecliptic.

For the effect of variation of declination.

(490.) Observations were also selected in classes, to show that there is no sensible diurnal tide at London; and others showing that the direction of the wind produces no sensible effect. Although little of mathematical deduction accompanied this work, we must allow that it was far more complete as a classified discussion of observations than any that had preceded it.

(491.) In the *Philosophical Transactions*, 1833, Mr. Lubbock applied the same method (merely so far as the process for the first table mentioned above) to observations made at St. Helena, Brest, Plymouth, Portsmouth, and Sheerness. In the volume for 1834 he has modified the tables of 1831, so as to correct the error which we have mentioned. In the volume for 1835 he has discussed an immense number of observations, made at Liverpool during nineteen years from 1774 to 1792. They are treated in exactly the same manner as the London tides, except that the observations when the moon's declination is north are separated from those in which it is south, the difference between these showing the existence of a diurnal tide. In the volume for 1836, the same observations are discussed specially for the discovery of the diurnal tide; the process is simply to divide by months and to subdivide by half hours of transit, (as before,) which defines in every case the moon's declination nearly, and then to divide each parcel once more into two parts, one part including upper transits of the moon with north declination, and lower transits with equal south declination, (the tides corresponding to which ought to be similar,) and the other part including upper transits of the moon with south declination, and lower transits with north declination.

Mr. Whewell's methods of treating Mr. Lubbock's results.

(492.) Thus far, however, we have merely numerical values, embodying the result of all the observations in a convenient form, and adapted to prediction, but not giving mathematical laws. The first step to this object was made by Mr. Whewell in the *Philosophical Transactions*, 1834. The following is his method, for the times of high water; it being premised that the parallaxes and declinations spoken of are the same as those used by Mr. Lubbock, and therefore are those which correspond to the time of high water, or a time near it, and not those which correspond to the anterior time indicated by theory in (452.). First, he remarks that (as pointed out by Mr. Lubbock) the mean of all the intervals between the moon's transit and the high water contains a term corresponding very exactly in value for different hours of moon's transit to the first term in (54.) increased by a constant (for London 1 hour 26 minutes), provided we diminish $\overline{m-s}$ by another constant (for London 2 hours); the value of $\frac{S}{M}$ being properly determined $\left(\text{for London it is } \frac{1}{2\cdot 99}\right)$. Next, he takes the tables of difference for each half hour of moon's transit between the mean interval and interval corresponding to different values of the moon's parallax; and, by inspection, he sees that in regard to the changes produced by change of parallax, the numbers in the table change sign when they pass 57′ (the mean parallax); and, in regard to changes of hour of transit, they may be represented by $\sin^2(\text{hour of transit} - 4^h)$, and therefore that the formula is something like a multiple of $p'.\sin^2(\text{hour of transit} - 4^h)$. Determining the multiplier so that the sum of all the numbers in the table will be represented by the formula, he obtains finally $3^m \times p' \times \{1 + \sin(2.\overline{m-s} - 2^h)\}$. This does not correspond exactly to the second term in (54.). For the declination, he changes all Mr. Lubbock's numbers expressing the difference from the mean state depending on declination, to the difference from the numbers when the declination is 0, and shows that in regard to change of declination the sum for all the different hours of transit may be expressed nearly enough by a multiple of the square of the sine of declination, and that when the mean for declination 0 is restored, the mean correction may be expressed by $11 - 132 \times \sin^2$ declination. He then applies this quantity with sign changed to Mr. Lubbock's numbers, and finds that there remains another set of numbers whose law is to be investigated. From inspection, nearly as before, this is found to be represented nearly by

Determines the mathematical expressions for various corrections by inspection of Mr. Lubbock's numbers.

$$84\left(\frac{1}{12} - \sin^2 \text{declination}\right) \sin\left(2.\overline{m-s} - 8^h\right);$$

and thus the whole correction for declination is

$$\left(\frac{1}{12} - \sin^2 \text{declination}\right)\left\{132 + 84 \sin(2.\overline{m-s} - 8^h)\right\}.$$

The observations of height are discussed in a similar manner; the general principle being, to take the principal term of the expression in (53.), and to examine whether the principal part of the height agrees with it, (the correction to $\overline{m-s}$ being somewhat different from that used for the times of high water.) Then the dif-

Sect. VII. Methods used for observing the Tides, and for reducing the Observations.

ferences depending on parallax and on declination are resolved into parts nearly in the same manner as above. We shall not delay further on this, but shall remark that, however much the conclusions were modified by later suppositions or by choice of expressions analogous to those of (457.), &c., we confidently refer the reader to this investigation as one of the best specimens of the arrangement of numbers given by observation under a mathematical form. In the *Philosophical Transactions*, 1836, Mr. Whewell has used the same method with some small additions, for reducing to law the numbers given by Mr. Lubbock's discussion of the Liverpool tides.

Curves used by Mr. Lubbock.

(493.) In the *Philosophical Transactions*, 1833, Mr. Lubbock exhibited the inequalities of the time of tide, (neglecting those depending on parallax and declination,) by graphical construction, for six different places. The method is, to construct a curve, in which the time of the moon's transit is the abscissa, and the interval from transit to high water is the ordinate. In the same volume, page 232, Mr. Whewell recommended the adoption of this as a general method of obtaining the numbers useful for prediction of tides. In discussing a series of observations made by the persons employed on the Preventive Service, (*Philosophical Transactions*, 1835,) he appears to have used projections extensively. In examining the effect of the sun's declination in the Liverpool tides, graphical projection was used by Mr. Whewell, to exhibit the numbers remaining after the application of other known corrections; and, a curve being drawn with a free hand among the points laid down from the actual numerical data, this curve was treated as the proper representation of those remaining numbers, and the values at different times were measured from the curve, instead of adopting the numbers themselves. Since that time, the use of curves has been commonly adopted by Mr. Whewell and Mr. Lubbock, (see the *Philosophical Transactions* for 1837, and especially the investigation of the progress of the diurnal wave by Mr. Whewell;) and in the *Philosophical Transactions*, 1838, Mr. Whewell (adopting Mr. Bunt's methods) has given rules for the most advantageous employment of the method of curves. They are as follows:

Mr. Whewell's rules for the employment of curves instead of numerical reductions.

(494.) "Upon a series of parallel ordinates corresponding to the times of moon's transit, I lay down the successive tides, that is, the heights or the lunitidal intervals, as the one or the other are the subject of examination. This curve is more or less irregular, but for most places the leading feature is the zigzag form which arises from the diurnal inequality. A curve is drawn by the eye so as to cut off this inequality, leaving equal differences above and below. We may then proceed as follows to find the other inequalities.

"Having laid down the observed intervals and heights, referring both to the *apparent* time of the moon's transit, and having drawn through each series of points the dotted line which cuts off the diurnal inequality only, but retains every other, I trace off on a piece of transparent paper, having an axis drawn on it extending from 0 hours 0 minutes to 12 hours 0 minutes transit, the successive portions of the dotted line of observation just mentioned, which are included between those hours during the first three months of the year; fitting the tracing paper in its place every time by means of the two extreme points of the axis. I thus obtain six irregular curves, [the same curves, nearly, occurring twice in each lunation,] the mean of which is found by drawing across them, at equal distances, twenty-four vertical lines, and finding by my scale a point in each which is the exact mean of the six intersections. In this manner I get four mean curves [one for each time of three months] on separate pieces of paper, which by repeated combinations are reduced into one, being the mean semimenstrual curve for the year.

"The next step is to reduce this curve to a mean parallax (57′·2) at each hour of transit. For this purpose an arrangement must be made, showing the mean parallax for that year at each of the twelve hours, which will be found to vary from about 56′·9 to 57′·5. [That is, for each of the days throughout the year, at which the moon's transit occurs between 0 hour and 1 hour, the parallax must be taken from the Nautical Almanac, and the mean of all these must be supposed to apply to 0 hour 30 minutes; and so for other hours.] The parallax table of the preceding year, if already discussed, will be sufficiently near for making the requisite small alteration of the curve to the mean parallax; otherwise an approximate parallax table for the current year must be first made.

"I then calculate [in the same manner] the mean declination, [for each hour of the moon's apparent transit,] which varies, not only as the parallax, slightly from hour to hour, but also considerably from year to year. The hourly differences (being only about half a degree from the mean) I have disregarded, and I prefer marking on each annual curve the mean declination of that year, to any attempt to reduce the different annual curves to one common declination.

"Having [as before mentioned] very carefully obtained the mean semimenstrual curve, I cut it out nicely on a piece of thick drawing-paper, and laying the intersections of the vertical hour lines of 0 hour and 12 hours with the axis, on the corresponding points on my sheets, [viz. those mentioned in the first paragraph,] I pencil-in the mean curves, and then ink them. The residue, or space between the mean curve and that of observation, is next transferred to a straight line below [as an ordinate to a new curve.]

"In examining this residue, the first step is to lay on an *approximate* line of parallax. For this I make an arrangement for every hour of transit, and for 54′, 55′, 56′, and 59′, 60′, 61′ of parallax [that is, two groups for each hour] (omitting 57′ and 58′) of all the vertical distances, at the successive hours of transit, of the curve from the straight line, adding to each a constant to avoid negative quantities. I thus obtain twenty-four parcels, the means of which give an approximate correction for 55′ and 60′ at every hour of transit. From this a first line of parallax is laid down, preparatory to the obtaining of the declination-corrections.

"These are obtained by collecting the measured distances from the parallax line to the curve, into parcels of 0° decl. to 13° decl., and 21° decl. to 28° decl. for every hour of transit, omitting those of 14° to 20° decl., and taking the axis, with the mean declination of the year, as a better representative of the mean declinations. The means of these twenty-four parcels give me a declination-correction very near the truth. From this I lay down, on the parallax line, the effect of declination, and thus get an approximate curve of declination combined with parallax. Both this and the former curve of parallax are drawn in with pencil only, being merely used as approximations, whence *corrections* of the first parallax and declination-*corrections* are obtained. This is done by an arrangement (as before) of the spaces still remaining between the pencil curve of declination combined with parallax, for hours of transit and for 54′, 55′, 56′, and 59′, 60′, 61′ parallax, which gives a small additional correction of the parallax corrections first obtained. From this corrected parallax correction, I draw in, with ink, the true parallax line, making the requisite alteration in the line of declination combined with parallax, which is then also inked in. Finally, a *correction* of the declination-corrections being made from this latter line, is considered as giving the true effect of the declinations."

The accuracy of result depends on the assumption of a correct law.

(495.) We have only to remark further on this, that much will depend upon the operator's mental assumption as to the *law* of the inequality as depending on either of the elements of correction. If, for instance, he assumes (as required by theory) that the correction for declination is nearly proportional to the square of the declination, it will be better to use the square of declination throughout. It is evident that there is nothing in the precepts above to prevent the operator from referring the parallax, declination, &c., to any time previous to the transit which immediately precedes the tide; and this, in fact, is done by Mr. Whewell in some discussions of tides.

Comparison of Laplace's methods with Mr. Lubbock's and Mr. Whewell's.

(496.) In remarking on these methods we cannot fail to observe that Mr. Lubbock's method of using *all* the observations for his first results, and the methods founded on this by Mr. Whewell, are in some respects greatly superior to Laplace's, which used only observations made at particular times. Thus Laplace *assumed* that the principal corrections for parallax *must* occur on the days when parallax was greatest or least; Mr. Lubbock's method, if properly used, would show whether that is true or not. On the other hand, Laplace's method is much superior in the facility which it gives for introducing such considerations as that of the variation of effect produced by variation of velocity in right ascension, (a real quantity, as shown by the second terms in our expression of (448.), and by the remarks in (455.), though for a reason very different from Laplace's.) But, viewing the two independent methods introduced by Mr. Whewell, of reducing the tabular numbers to law by a process of mathematical calculation, and of exhibiting the law to the eye without any mathematical operation by the use of curves, we must characterize them as the best specimens of reduction of new observations that we have ever seen; although, with a more accurate knowledge of theory, they might have been much improved.

Methods applying to the progress of the tide-wave across the ocean.

(497.) Thirdly, in regard to the progress of the tide over the different parts of the sea covering the greater portion of the earth. As the tide is itself a changeable state of the water, and as its elements vary from day to day, it is necessary to fix upon some definite phænomenon at some definite time in order to be able to compare the times of tide at different places. By common consent of mariners, the time adopted is "the time of high water on days of new moon and full moon," or rather the interval from the moon's transit at which high water occurs on those days. This is called the *establishment* of a port. It is to be observed that the interval taken on the day of new or full moon is not the same as the mean of all the intervals taken under all circumstances; this latter is termed by Mr. Whewell the *corrected establishment*, and it is by far the more important element in scientific discussion. The former however is most easily understood by practical men, and it is therefore the element most usually adopted in such comparisons as those of which we are speaking.

Establishment; its meaning.

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(498.) Now the establishment is commonly given in the time of the place. But in order to make the establishments of different places comparable, with the view of tracing the general course of the tide, we must express them all in the time of one standard place, Greenwich for instance. For this purpose we have only to subtract the east longitude or add the west longitude, (expressed in time,) and we have then a set of numbers which are strictly comparable, and which express the Greenwich time of high water at each of the places on the day of new or full moon.

(499.) Suppose now that these last numbers are marked on a chart, and that, on inspecting the chart, we see at two points on opposite sides of a sea and at several small islands between them the same indication III^h. This denotes that it was high water at all those places at the same time, which on a day of new or full moon was 3 o'clock. We may then consider the tide as a great wave whose ridge passed through those places at 3 o'clock on that day. Suppose that, on looking at another part of the same sea, we find a series of points marked IV^h. From this we infer that the ridge of the wave passed through this series of places at 4 o'clock on the same day. We conclude therefore that the ridge of the wave has travelled from one of these lines to the other in one hour; not by the motion of the particles of water through that space, but by the change in the relative motion described in (135.), and other parts of Section IV. If now we draw a line through the first series of points that we have mentioned, it is called the *cotidal line* of III^h; a line drawn through the second series of points is called the *cotidal line* of IV^h; and so on.

Cotidal lines.

(500.) Cotidal lines of this kind had been drawn by Dr. Young and other writers, but to a small extent. The first instance (so far as we are aware) in which they were traced on a large scale, was in the charts inserted by Mr. Lubbock in the *Philosophical Transactions*, 1831. But greater extent, as well as greater accuracy, were given by Mr. Whewell's investigations in the *Philosophical Transactions*, 1833, 1835, and 1836.

The precepts assume that the diurnal tide is unimportant, and fail where it is very large.

(501.) It is proper to remark that the whole of our directions regarding this matter go on the supposition that the semidiurnal tide is the only one of any importance; at least that the diurnal tide, though perhaps unsensible, does not materially disturb the times and general order of the semidiurnal tide. We shall however find instances in which the diurnal tide is far greater than the semidiurnal tide. In these cases the word *establishment* ceases to be applicable. And though it would not be difficult to extract, from a long series of observations, a fundamental number bearing the same relation to the semidiurnal tide which the ordinary establishment bears to the semidiurnal tide of other places, yet it is not easy so to state a rule that nautical persons would be able speedily to ascertain its value for such a place. In regard to these localities therefore cotidal lines are yet wanting.

(502.) We cannot close this part of our work without remarking on the confusion, and the consequent

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Confusion produced by the ignorance of mariners.

difficulty of drawing cotidal lines, produced in many places by the inaccurate habits of seamen and all who minister to them, (in England at least,) in not distinguishing between high water and the termination of flood or slack water. On this point see (5.) and (184.). In French works these are carefully distinguished: thus in Romme (*Tableau des Vents et des Marées*) the words *flot* and *jusan* are appropriated to the flow and ebb, while *montant* and *perdant* are used only for the rise and fall; and the want of synchronism between them is clearly explained. But in some of the best English charts, from ignorance of this circumstance, the times of high water on the coast and in the sea at a few miles distance are marked as differing 2½ hours or 3 hours; and even in the table of tides in the *Nautical Almanac*, the establishments at Portland Road and Portland Race are marked as differing 3 hours. We trust that this stain on the scientific character of our mariners will soon be removed.

Section VIII.—Comparison of the preceding Theories of the Tides, with Observations.

(503.) In the present Section we propose to bring together the principal results obtained by the methods described in the last Section, and to confront them with the theories of the preceding Sections. For convenience we shall divide this Section into the following Subsections:

Subsection 1.—On the individual tides in rivers, and in bays and estuaries in which the character of the tides is nearly similar to that of rivers.

Subsection 2.—On the individual tides in some small seas.

Subsection 3.—On the laws of the tides for varying positions of the Sun and Moon, at several different places.

Subsection 4.—On the progress of the tide over different parts of the ocean.

Subsection 1.—*On the individual Tides in Rivers, and in Bays and Estuaries in which the character of the Tides is nearly similar to that of Rivers.*

(504.) The common property which unites the various classes of canals considered here is, that the elevation and depression of the surface of the water bear a sensible proportion to the depth of the water. The theories, therefore, which are to be considered as specially applying to them are, the whole of the Subsection commencing at (192.) and illustrated by figures 9, 10, 11, 12, 13, 14, 15; and the theory from (307.) to (309.), illustrated by figures 16 and 17. Besides these, however, we shall find it necessary to apply the theories relating to variation of breadth and depth (238.) to (264.), because in far the greater number of rivers, &c., the breadth and depth diminish in proceeding from the sea. And in all cases the laws of friction must be supposed to apply, and the theorems from (332.) to (335.) must therefore be used. We shall now advert to special instances.

(505.) The theories from (238.) to (264.) lead us to expect that, where a river contracts rapidly and shoals rapidly, the range of tide will certainly increase. We are not prepared to say that the law which we have laid down in (264.) would apply with extreme accuracy, because the assumptions of very slow contraction and very slow variation of depth on which the process of (262.) is founded might not be safe except the contraction, &c., extended through the length of several waves. Still there is no doubt of the general correctness of the result. But the theorem of (332.) shows, as a consequence of friction, that if the channel be uniform the range will decrease. Combining these, we see that there is a certain rate of contraction with which the range of tide will be stationary: if the river contracts more rapidly, the range will increase from the preponderating effect of contraction: if it contracts less rapidly (and *à fortiori* if it expands), the range will diminish from the preponderating effect of friction. We can easily supply the reader with instances.

Instances of the change of magnitude of tide in passing up a river; first increasing, afterwards decreasing.

In the Thames, the mean range at Sheerness is about 13 feet, at Deptford about 17 feet, at London Bridge about 15 feet, and from this point it diminishes gradually to the weir drawn across the river at Teddington, (a few miles above Richmond,) where it is about 2 feet. A longitudinal section of the river, given by Messrs. G. and J. Rennie in the *Fourth Report of the British Association*, is copied in figure 44, (as far as relates to the tides,) and exhibits clearly to the eye this change of extent of oscillation.

The ranges in different parts of the Severn we have mentioned in (7.).

On approaching the Seine, the mean range is about 13 feet; at Havre it is 20 feet; at Quillebœuf it is about 13 feet; and from this point it decreases rapidly.

At the Cumbray islands, in the Firth of Clyde, the mean range is about 6 feet; at Greenock about 7; at Port Glasgow about 9; at Glasgow about 7; after which it decreases.

In the river St. Lawrence, the range increases from 4 or 5 feet at its mouth to 14 feet at Quebec; after which it dies away.

These instances might be multiplied to any extent, the alteration of range of tide in all contracting rivers following nearly the same laws.

(506.) There is, however, another cause which sometimes operates to increase the tides considerably without any convergence; it is the interruption of the canal, supposed uniform, by a barrier. This case is treated theoretically in (307.) and (333.). From (307.) it appears that, if there is no friction, the high water will occur at the same time in every part of the canal, and its range may become very great at the upper end: the horizontal motion, or tide-current, will be very great at the mouth. In (333.) it appears that, from the introduction of friction, the height will not increase so much; and the time of high water will be later as we ascend the canal. Perhaps the best instance of this case is in the Bay of Fundy. At its entrance the currents are very rapid, but the range is only about 8 feet. But at the head of the bay (where it branches into two shoaly arms) the range of tide sometimes amounts to 60 or 70 feet. The high water is later here than at the entrance of the bay by one hour nearly. In the Gulf of California the tide rises to a great height, though small at its entrance. In like manner, the range of tide in the Wash is about 26 feet, while it is only 6 feet at Yarmouth; and at St. Malo, Avranches, &c., it is nearly 40 feet, while at Cherbourg it is only 15. We consider this last instance analogous to the others, because we imagine

Instances of large tides in interrupted estuaries.

that the islands of Jersey, &c., almost effectually prevent the tide from flowing on the eastern side of the bay (in any important degree as regards the supply of water for the bay); and, therefore, the tidal entrance to Avranches is, in reality, one of much less breadth than it at first appears.

Instance of small interval between high or low water and stationary water.

(507.) The next circumstance in river-tides which we shall notice is, the relation of the times of slack water (or the turn of the current) to the times of high and low water. In (184.) we have seen that if the river-tide were a wave propagated without friction along a uniform channel of indefinite length, the times of turn of the tide would be exactly intermediate to the times of high and low water, or the cessation of flow would be three hours after high water, and the cessation of ebb would be three hours after low water. But every modifying circumstance which occurs in common rivers brings the cessation of flow nearer to the time of high water, and the cessation of ebb nearer to the time of low water. Thus, in (256.) we find that the shoaling of the bottom produces this effect; in (257.) the contraction of its breadth does the same; in (332.) friction produces a similar effect; and in (335.) it does so in a greater degree if the river is stopped by a barrier: this last effect, however, is small at the mouth of the river. From all these causes combined, the turn of current ought to follow the high and low water at a small interval, especially on advancing near to anything like a barrier in the river. It is notorious among all persons acquainted with river-navigation that this is strictly correct in fact; but we are unable to refer to any numerical values, except for Deptford on the Thames. The mean of a great number of observations (discussed by the author of this Essay in a paper in the *Philosophical Transactions*, 1842) gave, for the interval between high or low water and slack water, 37 to 40 minutes of time.

Instance of low water in river higher than high water in the sea.

(508.) The theoretical circumstance to which we shall next allude is, that, in a tidal river affected by a general current towards the sea, and subject to friction, the low water in the upper part may be higher than the high water near the sea (343.). We cannot refer to a better exhibition of the observed fact than that in figure 44; the same thing, however, (we believe,) has been ascertained from accurate levelling along the river Forth.

(509.) The next theoretical point, and the most difficult, (from the imperfection of limited solutions where the convergence of the series employed is very slow,) is the general law of rise and fall of the water at some distance from the sea, the extent of vertical oscillation of the surface bearing a sensible proportion to the depth of the water. Where there is no barrier across the river, (at least within a great distance of the sea,) and where the oscillation is not *very* great, the theory of (192.) to (221.) will apply. From this we find as follows.

Instances of the change in the form of the tidal curve in passing up a river.

(510.) First, supposing the place not very far from the sea, the principal effect will be, that the rise of the water will occupy a shorter time than the fall (see figure 11); and the inequality will be greater far from the sea than near the sea, and greater at spring-tides than at neap-tides. In illustration of this, we are able to place before the reader figure 45, representing the registered tidal curve at Sheerness on the spring-tide of 1841, February 24, afternoon; figure 46, representing the tidal curve formed from observations of the same tide at Deptford; and figure 47, representing the tide of 1821, July 29, (in nearly similar position of the moon, &c.,) near London Bridge (*Fourth Report of British Association*, page 495). In figure 48 is represented the course of the neap-tide of 1841, March 2, afternoon, at Sheerness; and, in figure 49, that of the same tide at Deptford. All these curves are drawn to the same scale, and illustrate fully the points that we have mentioned. In figures 45, 46, 47, the times of rise and fall are, respectively, at Sheerness 6 h. 5 m. and 5 h. 55 m.; at Deptford 5 h. 0 m. and 7 h. 0 m.; and at London Bridge, 4 h. 48 m. and 7 h. 59 m. (the whole time being accidentally too small in the former and too large in the last). In figures 48 and 49 the times of rise and fall are, at Sheerness, 6 h. 30 m. and 6 h. 30 m.; and at Deptford, 6 h. 0 m. and 6 h. 30 m. At Teddington the times are about 1 h. 15 m. and 11 h. We may also state from Lalande, that the tide at Havre occupies six hours in its rise, but that at Quillebœuf and places above it, it occupies not more than two hours. A better instance is that of the St. Lawrence: at 40 leagues below Quebec, the rise and fall occupy equal times; at six leagues below Quebec, the rise occupies five hours, and the fall seven hours; at 20 leagues above Quebec, the rise occupies three hours, and the fall nine hours.

Instances of diminished duration of rise in passing up a river.

Instance of double tide in a river.

(511.) Secondly, supposing the place somewhat further from the sea, there will be a sensible check to the descent of the water; there may even be a small rise near the middle of the descent. That such a check does occur in fact will appear from figure 50, which is copied from a diagram given by Mr. Russell in the *Seventh Report of the British Association*, representing the tidal curve on the river Dee, in Cheshire.

Instance of triple tide.

(512.) Thirdly, supposing the place still further from the sea, and the tide somewhat larger, so as to make another term in the approximation necessary, it appears, from figure 14, (remarking that the tidal curve for any station may be formed nearly enough by reversing the form of the wave near that station,) that there may be a treble elevation in the course of one tide; and comparing this with figure 15, it appears that this phænomenon may be very conspicuous for spring-tides at the same station at which it cannot be perceived at neap-tides. Now this treble tide does occur in the Forth, where it is known by the name of the Leaky. (Something like it occurs also in the Tay, and probably in the Thames and other long rivers.) It is understood that Mr. Russell has in preparation an elaborate account of observations of the Leaky; as it is not yet published, (a short notice only being given in the *Tenth Report of the British Association*,) we must draw our information from Wright's account in the *Philosophical Transactions* for 1750, page 412. The principal points mentioned are, that the Leaky begins at Queen's Ferry, or a few miles above it, and in neap-tides extends 25 miles up the river above Queen's Ferry, and in spring-tides 19 miles further (three miles above Stirling); that the highest water is not that from the first rise, but that from the second rise. (In this respect it appears to differ from the tide represented in figure 14, in which the first rise is the highest; but it is extremely probable that the consideration of friction and of other impediments might sensibly alter this form.) It is stated, also, that there are no Leakies at full moon, though there are at new moon. We doubt the accuracy of the writer in regard to this point, because the circumstances of sea-tide, which alone could affect the river-tide, (namely, semi-diurnal and diurnal tide,) are the same at new and full moon. On the whole, while we see beyond doubt that the Leaky is a legitimate instance of the application of our theory, (having been observed 90 years before the theory explaining it was formed,) we think it has not yet been described with the accuracy that a phænomenon so remarkable deserves.

Nature of river-tide when the rise is very great.

(513.) But suppose the tide to be so very large in proportion to the depth of the water that our series of terms in (210.) ceases to be convergent; what shall we conclude to be the form of the tide-wave? Our mathematics here are at fault; and (till the analytical difficulty has been overcome) we must refer to observation. We may state this, then, as the result. As the very large tide progresses, its time of rise becomes shorter, and its time of fall longer; but the descent is not uniform, the greater part of the descent occupying not much more time than the rise, and the last part of the descent being almost imperceptibly slow, which again is succeeded by a very sudden and rapid rise of the next tide. The course up the Severn may be gathered from figures 51, 52, 53. At the lower part of the Bristol Channel the rise and fall, we believe, occupy nearly equal times, (although we are not aware of any careful observations,) and figure 51 may be taken as the representation of the tidal curve there. Figure 52 is a copy of a curve traced by a self-registering tide-gauge erected at Hung Road, (near the confluence of the Avon and the Severn,) for which we are indebted to the kindness of Mr. Bunt. The whole time of rise appears to be four hours at the least. Figure 53 represents the tidal curve at Newnham, drawn generally from our own rough observations. The whole time occupied by the rise ($1\frac{1}{2}$ hour) is accurate. We believe that this may be taken as a type of the tide-wave in rivers where the entering tide is remarkably large.

Causes and appearance of the Bore.

(514.) Connected with this rapid rise of the tide, in the case of the last article, is the Bore; a phænomenon on which, we believe, some misconception prevails. We believe that the following description of its cause and appearance will be found correct. It is necessary for its formation that there be a very large tide rising with great rapidity (thus, at Newnham, where the water rises 18 feet in an hour and half, the bore is considerable). It is necessary, also, that the channel of the river be bordered with a great extent of flat sands, near to the level of low water. These circumstances hold in the Severn, the Seine, the Amazons, the bays at the head of the Bay of Fundy, (Chignecto Bay and the Bay of Mines,) and other places where the bore is remarkable: the second does not hold in the Thames; and, in consequence, that river has no bore. When the rise of the tide begins, the surface of the water is disturbed in mid-channel, so as to distort reflexion, but the water is not *broken*; it is merely like a common wave. This point (which we state from our own observation) is fully confirmed by De la Condamine's account of the bore in the Amazons, and by the practice, in the Hoogly river, (near Calcutta,) and other places subject to a bore, of rowing boats, &c., into the middle of the channel on the approach of the bore, in order to place them out of danger. But as this rapid rise elevates the surface suddenly above the level of the flat sands, the water immediately rushes over them with great velocity and with a broken front, making a great noise. And this is the whole of the Bore. It is, however, a majestic phænomenon, especially when witnessed from a station which commands the river for several miles above and below, (we may particularly mention Newnham churchyard, on the Severn,) and at an hour when other sounds are stilled. The rise of the water continues, after the Bore has passed, with unabated rapidity, as far as can be seen by general observation, and the tidal current flows rapidly up the river, which is now quite full. At last its rapidity of rise diminishes, and at 90 minutes after the bore, at the place which we have mentioned, it begins to drop, the current still flowing up. About 15 minutes after the beginning of the drop, a singular line of ripple (of which we can give no further explanation) is seen, stretching across the whole river, and moving very slowly downwards. This appears to be the place at which the ascending current and the descending current meet; for, as soon as it has passed, the water is seen to be running slowly downwards, and in a very short time it is running with a speed which is scarcely to be seen under any other circumstances of any river, except, perhaps, some of the largest rapids. The remarkable point attending this last phænomenon is, that the phase of change of current moves *down* the river. In every other instance, the phases, whether of high water, of low water, or of change of current, move upwards; although that of low water (208.) moves upwards much more slowly than that of high water. It would seem here that one of the velocities has actuallly changed its sign.

Instance of Deptford tide expressed by a mathematical law.

(515.) The only instance in which the phænomena of tides in a continuous river have been reduced to mathematical law by such a process as that of (479.) is in the tides of Deptford (*Phil. Trans.* 1842). Referring to (479.) for the process used, it appears that the result expressed the depression of the water for every value of *phase;* the *phase* being an angle increasing by 360° from high water to high water, and the unit in terms of which the depression is expressed being the half oscillation. Thus it was found that at Deptford, when the range of tide was 15 feet 3 inches, putting $p=$ phase $-77°.50'$, the depression of the water below a fixed mark might be represented by

$$13\text{ ft. }10\text{ in.}+7\text{ ft. }7\cdot5\text{ in.}\times\{0\cdot939\sin p-0\cdot066\sin(2p-38°.22')-0\cdot048\sin(3p-48°.40')\\+0\cdot012\sin(4p-97°.2')\}$$

and that when the range of tide was 19 feet 2 inches, putting $p=$ phase $-86°.23'$, the depression of the water below the same mark might be represented by

$$13\text{ ft. }3\text{ in.}+9\text{ ft. }7\text{ in.}\times\{0\cdot902\sin p-0\cdot106\sin(2p-19°.11')-0\cdot069\sin(3p-62°.19')\\+0\cdot033\sin(4p-75°.35')\}.$$

Difficulties in reconciling the observed law with the theoretical law.

The first thing to be remarked is, that the mean level (or the term independent of sines) is higher in the large tides than in the small ones. Probably a part of this arises from the influence of the second term in the expression of (309.); but a part also is produced by the difference of the same kind at the mouth of the river, as it appears in Mr. Lloyd's paper (*Phil. Trans.* 1831) that the mean level at Sheerness is higher in spring tides than in neap tides by 7 inches nearly. The next thing which deserves notice is, that the coefficient of sin $(2p-19^\circ.11')$ is larger than that of sin $(2p-38^\circ.22')$ in a greater proportion than that of the range of tides; whereas the expressions of (210.) and (309.), supposing them to be put under brackets with the coefficient of the first term as a general multiplier, would give for the coefficient of the term depending on $2mvt$ a quantity proportional to the range. This seems to show merely that the approximation is not carried far enough for such a tide as that of the Thames. The third point to be noticed is, that the argument of the second term (or the angle whose sine enters into the second term) is not double the argument of the first term, as it ought to be by the expression of (200.) or (210.), but is double the argument of the first term diminished by a constant. This may arise from the same imperfection of the mathematics; or it may be explained thus. The bridges and other impediments may produce in some degree the effect of a barrier, and therefore the true expression ought to be intermediate to those of (200.) and (309.). The former with changed sign, and putting C for bk, gives for the variable part of depression

Explained in some degree by the impediments to the free motion of the water.

$$C\left\{\sin(mvt-mx')-\frac{3}{4}b.mx'.\sin(2mvt-2mx')\right\}.$$

The latter, with changed sign, making $\frac{A}{\cos ma}.\cos\overline{ma-mx'}=C=bk$, and putting $B=180^\circ$, gives for the variable part of depression

$$C\left\{\sin mvt-\frac{3}{4}b.mx'.\tan(ma-mx').\cos 2mvt\right\}.$$

If then the true expression were the mean of these two, it would be

$$C\left\{\cos\frac{mx'}{2}.\sin\overline{mvt-\frac{mx'}{2}}-\frac{3}{8}b.mx'.(\cos 2mx'.\sin 2mvt-\overline{\sin 2mx'-\tan(ma-mx')}\cos 2mvt)\right\}.$$

If $\frac{\sin 2mx'-\tan(ma-mx')}{\cos 2mx'}=\tan\overline{2mx'-D}$, where D will be small if the place of observation is near to the bridges, &c., this will become

$$C\left\{\cos\frac{mx}{2}.\sin\overline{mvt-\frac{mx'}{2}}-E.\sin\overline{2mvt-2mx'+D}\right\};$$

and if we put $\overline{mvt-\frac{mx'}{2}}=p$, it becomes

$$C\left\{\cos\frac{mx'}{2}.\sin p-E.\sin\overline{2p-mx'+D}\right\}$$

or

$$C\left\{\cos\frac{mx'}{2}.\sin p-E.\sin\overline{2p-F}\right\},$$

an expression possessing the peculiarity in the form of the argument of the second term to which we have adverted. On the whole, we regard the phænomena of the Deptford tides as agreeing pretty well with the laws given by our mathematical investigations.

Instance of a river (the Amazons) with several high and low tides existing upon it at once.

(516.) Before quitting the subject of tides in continuous rivers we may remark, that it is easy to conceive a river so long that there may be at the same instant several tides of high and low water alternately at different points along the river, its course being so long that the wave of high water occupies several days in passing up. We are aware of only one instance in which this is recorded as actually having been observed, namely the river Amazons. De la Condamine, who descended this river in 1743, (see the French *Memoirs* for 1745,) has stated that the tide is observable at Pauxis, 200 leagues from the mouth of the river, and that between this place and the mouth there are at any instant a score of places, more or less, [*une vingtaine de parages, plus ou moins,*] at which it is high water at that instant; the places intermediate to these having low water at those times. He conceives the tide-wave to advance there at the rate of one league per hour nearly.

Instance of estuary tide (Southampton) reduced to mathematical law. Difficulty of explaining it.

(517.) When the river is stopped by a sudden barrier, (a circumstance which holds nearly in the Southampton Water, the Orwell, the Stour, several arms of the sea on the coast of Cornwall, many of the sea-lochs of Scotland, &c., and many other estuaries,) the theory of (309.) illustrated by figures 16 and 17 ought to apply. We are not aware that sufficient observations have been made upon any estuary-tide except that of Southampton, and there they give the very singular curve represented in figure 54. Here is a double high water, as in figure 17, and it appears therefore that there is some agreement between the theoretical and the observed forms. But this is not so obvious when the height is expressed algebraically by the process of (479.). It appears, then, that the variable part of the depression below a fixed point (as applying to seven tides between 24th February and 27th February, 1842) may be expressed by

$$7\text{ ft. }3\text{ in.}\times\{0\cdot898.\sin p+0\cdot194\sin\overline{2p-1^\circ}+0\cdot191\sin\overline{3p-68^\circ.8'}+0\cdot034\sin\overline{4p+5^\circ.43'}\}$$

where p is an angle increasing uniformly with the time, and changing through 360° in the course of one tide. The theoretical expression (309.), as modified in (515.) is

$$C \times \left\{ \sin mvt - \frac{3}{4} b . mx' . \tan (ma - mx') . \cos 2mvt \right\}.$$

This matter is open to further investigation.

(518.) The phænomenon of long continued high water or double high water in estuaries is, we believe, not uncommon. Thus at Havre, as described by Dicquemarc, (*Journal de Physique*, 1779,) there is a long continued elevation called *la tenue du plein.* The tidal curve, as far as we have been able to construct it from Dicquemare's description, is given in figure 55. On the coast of Holland, (which possesses all the properties of an estuary in this respect, that the tidal water runs for a great distance over a shoaly bottom and then is suddenly stopped by the coast,) there is a double high water (see Moll, *Nieuwe Verhandelingen der eerste classe ... van het instituut ... te Amsterdam, zevende deel*). The first high water appears to be considered by the Dutch, in common language, as the real high water, and the second is called the *agger*, but so little difference is there between them, that, in making the extensive series of simultaneous tide-observations proposed by Mr. Whewell, the agger was sometimes observed for high water.

Instances of double high water on sea coasts.

(519.) We may here properly advert to the anomaly which has been observed in the tides at Christchurch, Poole, and Weymouth, and which by some writers (and by nautical persons on the coast in general) is ascribed to the effect of a tidal current from the Solent or western channel behind the Isle of Wight. We shall first state, that we conceive this explanation to be untenable. Without laying stress on the algebraic circumstance, that the union of two separate tides, one of the form $a . \sin \overline{nt + A}$, and the other of the form $b . \sin \overline{nt + B}$, will produce a simple tide of the form $c . \sin \overline{nt + C}$, it will be seen from figure 56 that the line of the mainland is continued by a long projecting shoal from the visible mouth of the Solent, so as in reality to prolong that channel to a length which, by discharging its water at once into the open sea, must effectually prevent it from affecting a long line of coast. Now in figures 57, 58, 59, we give the tidal curves of these places; in which the part marked with a strong line has been carefully observed by ourselves, (by noting the height of the water every five or ten minutes,) and the rest has been supplied from the information of persons on the spot. It is quite evident, on comparison with figures 12, 52, 53, that these are simply the tides produced by running over a shallow bottom; that at Weymouth corresponding to a case in which the tide is greater than at Christchurch and Poole; which is correct, the tide at Weymouth being double that at Christchurch and Poole. The two latter tides were observed at the mouth of the Avon and at the quay at Poole, and in each of these places the tide has flowed over a shoal, (Christchurch Bay being very shallow,) and has still a considerable distance to flow over it; so that they may be regarded as tides in continued shallow channels. With regard to the Weymouth tide, we cannot explain fully the length of the shoal; but it is remarkable that the cotidal lines follow in close sequence near to Weymouth, which implies that the tide-wave travels slowly *in the same manner* as if the water were shallow, whether that be or be not the cause of it. It would be a curious circumstance if it should be found, theoretically or experimentally, that the alteration of the cotidal lines by the form of the coast produces the same modification of the tides as a shoal which retards the velocity to the same degree. In all three instances also the form of the coast compels the tide to turn backward to these parts, (by passing round Portland Bill to Weymouth, round St. Alban's Head to Poole, and round the reef projecting from Hengstbury Head to Christchurch;) it would be a curious circumstance if it should be found that this assists to produce that modification.

Apparent anomalies in tides at Christchurch, Poole, and Weymouth.

Appear to be merely the effect of running over a shallow bottom.

(520.) In the *Philosophical Transactions*, 1840, Mr. Whewell has given tidal curves for Liverpool, determined by observations every half hour and also by observing the exact time at which the surface of the water passed two fixed marks; and tidal curves determined from observations of high water and the time of passing two fixed marks at Plymouth. He remarks that, at Liverpool, the rise occupies less time than the fall; but his tables and his curves imply the contrary for Plymouth. This however is an error, arising from an error in the tabular arrangement of his figures (see page 264, January 4, where the figures 3.28.0 in the last column are improperly raised). On correcting this, it is found that the rise occupies less time than the fall, as, theoretically, it ought to do (204.). And this is the general rule. The tide-observations published by the Admiralty seem to show that at Portsmouth the rise is the longer; but we have good authority for saying that the observations of low water there are very uncertain. It is not impossible, however, that Portsmouth may share in some degree in the peculiarities of Southampton.

Tidal curves at Liverpool, Portsmouth, Plymouth.

Subsection 2.—On the Individual Tides in some Small Seas.

(521.) The small information that we possess with regard to the tides of the Mediterranean Sea may be first mentioned. It is notorious that the tides are so small that they cannot usually be recognized; or cannot be distinguished from the effects of wind, (except in the neighbourhood of the Strait of Gibraltar, where they are affected by the tide of the Atlantic Ocean.) This agrees with the result of (300.), in which the greatest value of Y for a closed canal contains as a factor the length of the canal, which for the Mediterranean (considered with reference to tidal phænomena) is small. Yet that there is a tide in every part is shown by the observations of Toaldo at Venice (*Phil. Trans.* 1777). The spring tide there is about four feet, although at the mouth of the Adriatic it can hardly be recognized. It is plain that this case is similar to that of (307.), or to those of the Bay of Fundy, &c., mentioned in (506.), and that a tide, insensible at the mouth, has by virtue of the factor $\frac{1}{\cos ma}$ become sensible at the head.

Small tides in the Mediterranean.

Tides in the Adriatic Sea.

(522.) The tides in the English Channel claim notice, Tides in the English Channel. False notions on the meeting of tides.

as having been the subject of careful examination by many persons, English and French. A paper in the *Phil. Trans.* 1819, by Captain Anderson, deserves to be mentioned. This writer conceives, that if there is any where a *meeting* of two tides, as from the English Channel and from the North Sea, there must be a terrible disturbance of waters, as if two streams ran to abut each against the other. The reader who has entered into the Theory of Waves will at once perceive that no such consequence follows. While calling to mind the analogy of two ordinary waves on water running in opposite directions and crossing each other without any shock, he may also refer to the result of the investigation in (310.), which is precisely that of two tides from two tidal seas meeting in the same canal, and which shows that at every point the motion of the water is similar to that in ordinary tides, except that there is no simple relation between the times of high water and slack water. If the consideration of friction were introduced, the general expression of (321.), making $H=0$, must be applied, and the same limiting conditions must be used as in (310.), which will show that at both entrances the tide-waves travel towards the centre (except one is much greater than the other). But, dismissing this writer's speculations, we find his information valuable. It appears that in the upper part of the Channel (to which his detailed remarks are confined) the water flows up the Channel nearly three hours after high water, and runs down nearly three hours after low water. He mentions this as a most singular and astonishing fact, although it is merely the law of (184.), which is that of the simplest case of waves. We may mention here, that this continuance of the current after high water, &c., if it last three hours, is called by sailors *tide-and-half-tide ;* if it last one hour and a half, it is called *tide-and-quarter-tide*, &c.

Reversion and revolution of currents in the English Channel.

(523.) The results of far more extensive observations are collected by Monnier, in his *Mémoire sur les Courants de la Manche, &c.* From these it appears, as a universal rule throughout the English Channel, that at any great distance from either shore the current runs up the Channel nearly three hours after high water, and down the Channel nearly three hours after low water; and that on the English side of the Channel, especially opposite the entrances of bays, the directions of the currents turn in 12 hours in the same direction as the hands of a watch; and that on the French side they turn in the opposite direction. This is entirely in conformity with the theories of (184.) and (363.). The same laws are recognized as holding in the British Channel, and in the German or North Sea near the Scotch and English coasts. Two instances are mentioned, (off the Start Point and off Ushant,) in which it seems that the direction of the tide, after having turned through a certain angle, turns back through the same directions; but they do not appear to be well established.

Different height of tides in different parts of the Irish Channel.

(524.) With regard to the Irish Channel, we have little to remark, except that there is a very great difference in the height of the tide on the different sides, the tide on the east side being considerably the greater. They are also greater in the northern part (north of Wicklow, on one side, and of Bardsey Island on the other side) than in the southern part. Between Wexford and Wicklow they are very small. For these facts, and for information regarding the height of the tides along a considerable extent of coast, we would refer to fig. 60, (*Chart of British Isles and North Sea,*) in which the small figures along different parts of the coast denote the extreme range of the tide in yards (Whewell, *Phil. Trans.* 1836). The theory of the motion of waves on a large surface of water bounded by an irregular figure is so imperfect that we cannot explain this mathematically ; although it seems likely enough that the great wave coming from the Atlantic and passing the south-eastern coast of Ireland should not produce a great rise on its eastern coast. The greater rise in the north is undoubtedly due to the similarity to estuary tides, for which we have had to refer so frequently to (307.) and (333.).

Peculiarity of the tides of the German Sea.

(525.) The tides of the German Sea present a very remarkable peculiarity. Along the eastern coast of England, as far as the mouth of the Thames, the tide-wave, coming from the Atlantic round the Orkney Islands, flows towards the south. Thus, on a certain day, it is high water in the Murray Firth at 11 o'clock, at Berwick at 2 o'clock, at Flamborough Head at 5 o'clock, and so on (see the cotidal lines in figure 60) to the entrance of the Thames. But on the Belgian and Dutch coasts immediately opposite, the tide-wave flows from the south towards the north. Thus, on the day that we have supposed, it will be high water off the Thames at 11 o'clock, (the tide having travelled in 12 hours from the Murray Firth,) and at Calais nearly at the same time : but at Ostend it will be at 12, off the Hague at 2, off the Helder at 6, and so on. This circumstance is so strange, that Mr. Whewell, in order to explain it, has had recourse to the supposition of a revolving tide in the German Ocean, in which the tide-wave would run as on the circumference of a wheel, the line of high water at any instant being in the position of a spoke of the wheel. Although our mathematical acquaintance with the motion of extended waters is small, we have little hesitation in pronouncing this to be impossible. The only conceivable case in which it could hold would be, when there was a shoal in the middle nearly touching the surface of the water : a supposition which does not apply here, the sea between Suffolk and Belgium being somewhat deepest in the middle.

Explanation from the position of the shoals.

(526.) We believe that a complete explanation may be found in the arrangement of the great shoals of the North Sea. It must be remarked that (except within a very small distance of Norway) the North Sea is considerably deeper on the English side than on the German side ; so much so that the tide-wave coming from the north runs into a deep bay of deep water, bounded on the west side by the Scotch and English coasts as far as Newcastle, and on the east side by the great Dogger Bank. As far as the latitude of Hull, the English side is still the deep one : and though a species of channel through the shoal there allows an opening to the east, yet immediately on the south of it is the Wells Bank, which again contracts the deep channel to the English side. After this (that is, in the latitude of Yarmouth) the deep channel expands equally to both sides. It seems reasonable to conclude from this that the great set of north tide is on the English side of the North Sea, both between the Dogger Bank and England and between the Wells Bank and England, (a branch stream of tide having been given off to the east between these two banks :) and that any passage of tide-wave over these banks may be neglected. Now this view is supported in a remarkable degree by the tidal observations on two dangerous shoals called the

Ower and Leman, lying between Cromer and the Wells Bank, but nearer to the latter. It appears that, on these shoals, the direction of the tide-current revolves in the same manner as the hands of a watch. (See Norie's or any other charts, or Monnier, p. 23.) Now, in conformity with the remarks in (363.), this proves beyond doubt that the Ower and Leman are on the left hand of the main stream of tide, (supposing the face turned in the direction in which the tide proceeds,) or are on its eastern border; and therefore that the central stream is still nearer to the coast of Norfolk. From a point not far south of this we may suppose the tide to diverge in a fan-shaped form over the uniformly-deep Belgian sea. Along the English coast, the wave will flow to the south: but it will reach the whole of the Belgian and Dutch coast at the same instant: and, if this tide alone existed, we doubt not that the time of high water would be sensibly the same along the whole of that coast.

(527.) But there is another tide of great magnitude, namely, that which comes from the English Channel through the Straits of Dover. This also diverges, we conceive, in a fan-form, affecting the whole Belgian sea: the western part turns into the estuary of the Thames: the eastern part runs along the Dutch coast, producing, at successive times, high water (even as combined with the North Sea tide) along successive points of that coast from Calais towards the Helder. And this we believe to be the complete explanation of the apparently opposite tide currents. The branch tide of the North Sea running between the Dogger Bank and the Wells Bank will assist in propagating the tide along the German coast from the Helder towards the mouth of the Elbe. We have gone into some detail in this explanation for the purpose of showing the importance of considering the form of the bottom in explanations of specific tides.

Captain Hewett's observations on a point in the North Sea where there is no rise of tide.

(528.) A remarkable set of observations was made by the late Captain Hewett, R.N., on the depth and motion of the water at a strictly definite point (a boat being moored over a small hillock discovered in the bottom of the sea) in latitude 52° 27′ 30″ N., longitude 3° 14′ 30″ E., or a few miles south of the Wells Bank, near the middle of the Belgian sea. They have since been communicated to the British Association. The result was, that the change of elevation of the surface was insensible: but that there was a considerable stream of tide alternately N.E. and S.W. (magnetic). The point in question corresponds pretty well to the intersection of the cotidal lines of 9 o'clock of the North Sea tide, and 3 o'clock of the English Channel tide (ordinary establishment), and these tides would therefore wholly or partially destroy each other as regards elevation. As regards the compound tide-stream, the greatest positive current from one tide will be combined with the greatest negative current of the other, and this will produce a stream whose direction agrees well with Captain Hewett's. At 3^h on the day of new moon, the North Sea tide would be running north (magnetic), and the English Channel tide would be running east, and therefore the compound current would be running north-east; at 9^h it would be running south-west. Both currents, and consequently the compound current, would cease at about 0^h, 6^h, &c., on the day of new moon: and as Captain Hewett's observations were made rather more than a day before new moon, the slack water would occur an hour or more before noon. The whole of this also agrees well with the observation.

Races—Portland Race.

(529.) The phænomenon known by the term *Race* is peculiar to promontories projecting into narrow channels. Thus nearly every headland of the English and Irish channels has more or less of a race. The most remarkable of these is Portland Race. When the rising tide in the neighbourhood is nearly at its mean level, the water begins to run eastward past Portland Bill and over the sunk ledge which projects from the Bill: when the tide has reached its greatest height, the Race is raging most furiously, the velocity of the current to the east being 6 or 7 miles per hour. When the water has dropped to its mean level nearly, the current turns, and at the time of low water it is raging towards the west. It is plain that this current is only the ordinary tide-current treated in (184.), but it necessarily runs past the promontory with great rapidity in order to transfer the proper quantity of water from the bay on one side of the promontory to the bay on the other side.

Rapid currents from bays communicating with tidal seas by two channels.

(530.) It is scarcely necessary to mention at length the rapid currents which are sometimes produced in the channels communicating from a surface of water to two tidal seas, or to two parts of the same sea in which the hours of high water are different. The more rapid current is always that which passes through the narrower channel. Thus, the mouth of the Southampton water communicates with the English Channel by two courses, namely, by the east and by the west of the Isle of Wight: the former, being wide and deep, allows the water to take nearly the same level as that at the east end of the island, and this at some states of the tide is different from that at the west end of the island, and therefore the water runs with considerable velocity in the western channel (the Solent). A more remarkable instance is that of Loch Alsh, on the west coast of Scotland, opposite the Isle of Skye: this island very much interrupts the great tidal channel between Scotland and the Long Island (Lewis, Harris, &c.), and the times of high water on the north and south coasts of Skye are therefore different: Loch Alsh is principally supplied from the north side by the broad entrance called Kyle Haken, and its water therefore nearly takes the level of the northern sea: its height is therefore at some stages of the tide considerably different from that of the southern sea, with which it communicates by the narrow strait of Kyle Rhea; and the water consequently rushes through Kyle Rhea with a current of fearful rapidity, changing its direction twice in every tide.

Mean level of the sea nearly independent of the range of tide.

(531.) We shall close this subsection by stating that, at the expense of the British Association (see their 8th *Report*), a series of levels was carried across the land from Portishead on the Bristol Channel to Axmouth on the English Channel, by Mr. Bunt, and simultaneous observations of the tides were made at these stations, and were referred to the marks whose difference of level was obtained. Although the whole tide at Portishead was 35½ feet, and that at Axmouth only 10 feet, the mean level at Portishead was only 9 inches higher than that at Axmouth: its high water being 13 feet 7 inches higher, and its low water being 12 feet 2 inches lower. Thus it appears that the Theory of Waves, in the manner in which we have throughout used it, considering the greatest elevation above the mean level, and the greatest depression below it, as very nearly equal, is properly applicable to the tides. We have also been informed that the levellings in the Ordnance Survey have shown that the high water in the upper part of the Bristol Channel is higher than at its mouth, and the low water in the upper part lower than at its mouth, by nearly equal quantities: but we possess no documentary information on this point.

Subsection 3.—On the Laws of the Tides for varying Positions of the Sun and Moon.

(532.) The Equilibrium-Theory (44.), Laplace's Theory (98.), (101.), and (107.), and the Theory of Waves (436.) and (283.), (439.) and (441), modified in (448.) and (452.), agree in the general form of the expressions for the elevation of the sea, supposing the coefficients altered and the arguments of the angles altered. We shall therefore (for convenience), in the first instance, use the formulæ of the Equilibrium-Theory in our comparisons, introducing the modifications given by the other theories as may appear necessary.

Dependence of the mean level of the sea on the Moon's declination.

(533.) The first line in the expression of (44.), with which (436.) agrees, gives us the slowly varying effect of the two attracting bodies, in feet,

$$(1\cdot 34\times\sin^2\mu+0\cdot 61\times\sin^2\sigma)\times(\cos 2.\ \text{terrestrial latitude}+\mathrm{C}),$$

μ and σ being the declinations of the Moon and the Sun.

In the expression of (44.) C is $\frac{1}{3}$: in that of (436) it is 0. The constant, in both cases, depends on the condition of giving a certain external form to a certain quantity of fluid; the form is the same in both, but the different proportions of the polar to the equatorial surface in the two suppositions (one, of the earth completely covered with water, the other, of a canal passing through the poles) give different absolute elevations. In either case, however, the elevation at mean latitudes is small; it will be most conspicuous near the equator or near the poles. The quantity of information which we have bearing upon this point is small. In the *Phil. Trans.*, 1839, p. 157, Mr. Whewell shows that the observations of high and low water at Plymouth give a mean height of water increasing as the Moon's declination increases, and amounting to three inches when the Moon's declination is 25°. This is in the same direction as that corresponding in the expressions above to a high latitude. The effect of the Sun's declination is not investigated from the observations. In the *Phil. Trans.*, 1840, p. 163, Mr. Whewell has given the observations of some most extraordinary tides at Petropaulofsk in Kamschatka, and at Novo-Arkhangelsk in the island of Sitkhi on the west coast of North America. From the curves in the *Phil. Trans.*, as well as from the remaining curves relating to the same places (which, by Mr. Whewell's kindness, we have inspected), there appears to be no doubt that the mean level of the water at Petropaulofsk and Novo-Arkhangelsk rises as the Moon's declination increases. We have no further information on this point.

(534.) The third line of (44.), which we shall next treat, gives the expression for the ordinary semidiurnal tide, on the Equilibrium-Theory: the corresponding term for one body on Laplace's Theory will be found in (108.); and that on the Wave-Theory for the general combined effect of two bodies, in (457.) and the preceding articles. It appears that the forms of (53.) and (54.), with extensive changes of angles, &c., may be adopted as representing any theory. We shall consider the different parts of these formulæ separately.

Theoretical formulæ for semimenstrual inequalities.

(535.) *Semimenstrual inequality of time and of height.*—This is the name given by Mr. Whewell to the variation of that term, in the expression for the time of high water, or for the height of high water, which in (489.) is called the quantity depending on the Moon's time of transit only; and which, for the time, putting θ for the Moon's hour angle to the west of the meridian at high water, is represented in the first line of the last expression of (54.) by

$$2\theta=-\frac{\overset{\prime}{720}{}^{m}}{\pi}\cdot\frac{\mathrm{S}'.\ \sin 2.\overline{m-s}}{\mathrm{M}'+\mathrm{S}'.\ \cos 2.\overline{m-s}},$$

or more correctly by

$$\tan 2\theta=-\frac{\mathrm{S}'.\ \sin 2.\overline{m-s}}{\mathrm{M}'+\mathrm{S}'.\ \cos 2.\overline{m-s}}=-\frac{\mathrm{S}_\triangle.\ \sin 2.\overline{m-s}}{\mathrm{M}_\triangle+\mathrm{S}_\triangle.\ \cos 2.\overline{m-s}},$$

(see (56.)): and for the height, by the first term in the expression of (53.), which is proportional to

$$\sqrt{\{\mathrm{M}_\triangle{}^2+2\mathrm{M}_\triangle\,\mathrm{S}_\triangle.\cos 2.\overline{m-s}+\mathrm{S}_\triangle{}^2\}}.$$

The expressions of (457.), and the articles preceding it, show that, on the wave-theory, these expressions will be changed to

$$\tan(2\theta-2\lambda)=-\frac{\mathrm{S}''.\ \sin(2.\overline{m-s}-2\alpha)}{\mathrm{M}''+\mathrm{S}''.\ \cos(2.\overline{m-s}-2\alpha)}.$$

$$\text{Height}\qquad\propto\sqrt{\{\mathrm{M}'''^2+2\mathrm{M}'''\,\mathrm{S}'''.\cos(2.\overline{m-s}-2\alpha)+\mathrm{S}'''^2\}}$$

where the proportion of S″ to M″, or S‴ to M″ is not the same as on the equilibrium-theory, in consequence of the relation of the magnitude of the effect to the periodic time of the forces (455.). In considering the mean of many observations in all relative positions of the Sun, the Moon, the perigee, and the equinox, the variable term

T in (457.) may be neglected. Now the points which we shall examine are, 1st, whether consistent values of $\frac{S''}{M''}$ can be found; 2nd, whether consistent values of α can be found; 3rd, whether, with adopted values of $\frac{S''}{M''}$ and α, the general forms of the observed results agree with those of the formulæ.

Methods of determining the proportion of solar and lunar effects.

(536.) 1st. On the value of $\frac{S''}{M''}$. For the times this may be found by choosing one mean of observations in which $m-s-\alpha$ is as near as possible to 3^h (say 3^h+x) and choosing another mean of observations in which it differs 6^h from the former (or is 9^h+x). These will be respectively the times when the Moon's transit (in solar time) is later by 3^h and by 9^h (nearly) than the time when the tide is greatest. For the first of these, $\tan(2\theta_1-2\lambda)=-\frac{S''.\cos 2x}{M''-S''.\sin 2x}$: for the second, $\tan(2\theta_2-2\lambda)=+\frac{S'.\cos 2x}{M''+S''.\sin 2x}$: and half the difference of these, when x is small, will not differ sensibly from $\tan(\theta_2-\theta_1)=\frac{S''}{M''}$. Or it may be found by taking the least and greatest values of $\theta-\lambda$: for the first of these, $\sin(2\theta_3-2\lambda)=-\frac{S''}{M''}$: for the second, $\sin(2\theta_4-2\lambda)=+\frac{S''}{M''}$: taking half the difference, $\sin(\theta_4-\theta_3)=\frac{S''}{M''}$.

For the heights, the easiest way is to take the greatest and least ranges, which correspond to $M'''+S'''$ and $M'''-S'''$ in the formula, from which the value of $\frac{S'''}{M'''}$ is found.

Different determinations of this proportion

(537.) Thus the following results have been obtained:

At Brest, by all the observations near to 128 syzygies and 128 quadratures (*Méc. Cél.* liv. xiii. p. 204.), from heights, $\frac{S'''}{M'''}=\frac{1\cdot 643}{4\cdot 755}=0\cdot 3456$.* There is no deduction from the times, except that made by Mr. Lubbock (*Phil. Trans.*, 1832) from a year's observations, which gives $\frac{S''}{M''}=0\cdot 33945$.

At London, by 19 years' observations, from times (Lubbock, *Phil. Trans.*, 1831), $\frac{S''}{M''}=0\cdot 37887$,† or from another discussion (Lubbock, *Phil. Trans.*, 1836) 0·38386. There is no satisfactory determination from heights, the whole range not having been observed: but supposing the mean range to be 15 feet, and the greatest difference in the height of high water at spring tides and neap tides to be 3·4 feet as shown by Mr. Lubbock's observations, and supposing that (as in the open sea) the variations between spring tides and neap tides are the same for low water as for high water (which does not agree with the observations at Plymouth, where the variation for low water is the greater in the proportion of 3 : 2), the value of $\frac{S'''}{M'''}$ would be $\frac{1\cdot 7}{7\cdot 5}=0\cdot 2267$. This result is worthless.

At Liverpool, by 19 years' observations, from times (Whewell, *Phil. Trans.*, 1836), $\frac{S''}{M''}=0\cdot 37866$, or (Lubbock, *Phil. Trans.*, 1837) $=0\cdot 37124$. From heights, (Whewell, *Phil. Trans*, 1836,) $\frac{S'''}{M'''}=0\cdot 39875$; but as there were no observations of low water, this result is valueless. If the mean range were considered $=24$ feet, and the difference of high water for springs and neaps $=5\cdot 48$, $\frac{S'''}{M'''}$ computed as for London $=0\cdot 228$. This is worthless.

These determinations (omitting those to which we have taken exceptions) may be considered as first-rate. The following are greatly inferior:

From times only (Lubbock, *Phil. Trans.*, 1833, page 20; and Whewell, *Phil. Trans.*, 1834, page 20).

At Sheerness, from 12 months' observations . . . . $\frac{S''}{M''}$	$=0\cdot 37865$
At Portsmouth, 12 months'	$=0\cdot 34611$
At Plymouth, 9 months'	$=0\cdot 40673$

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* Mr. Whewell (*Phil. Trans.*, 1834, p. 20) has quoted as Laplace's value $\frac{1}{2\cdot 6157}=0\cdot 38231$. We know not on what authority this is given: it is certainly not on that of the *Mécanique Céleste.*

† Mr. Whewell has given the number $\frac{1}{2\cdot 9884}=0\cdot 3346$: we know not on what authority.

From times (Whewell, *Phil. Trans.*, 1838, page 245).

At Plymouth, from 4 years' observations . . . . . . $\frac{S''}{M''} = 0\cdot4025$

At Bristol, 4 years' . . . . $= 0\cdot394$

At Leith, Portsmouth, Pembroke, and Ramsgate, (periods not mentioned), $\frac{S''}{M''}$ appears a little less than at Liverpool.

From heights, at Portsmouth $\frac{S''}{M'''} = 0\cdot2$; at Plymouth $= 0\cdot294$; at Bristol $= 0\cdot303$.

From heights at Plymouth (Whewell, *Phil. Trans.*, 1839, page 153, &c.), by 6 years' observations of high and low water . . . . $\frac{S'''}{M'''} = \frac{43\cdot5}{144} = 0\cdot3021$

At Dundee (page 157), by one year's observation of high and low water, from times . . . . $\frac{S''}{M''} = 0\cdot3584$

From heights . . . . $\frac{S''}{M'''} = 0\cdot2774$

At Dundee, Mr. Mackie (*British Association*, *7th Report*) has found by 7 months' observations from times . . $\frac{S''}{M''} = 0\cdot3420.$

In the *Philosophical Transactions*, 1836, page 305, Mr. Whewell has insisted on the difference of the extreme values of the semimenstrual inequalities $\left(\text{and the consequent difference of deduced values of } \frac{S''}{M''}\right)$ at different places, as ascertained by three weeks' observations. We have not the smallest confidence in such a result, having in our own observations, for four days only, encountered a discordance of two hours between the times occupied by two successive tides.

The proportions deduced from different places do not agree.

(538.) Confining ourselves now to the values of $\frac{S''}{M''}$, deduced from the times only, which are incomparably more exact than those deduced from the heights (the Brest and Plymouth observations excepted), the following remarks suggest themselves. The two places (London and Liverpool), at which the greatest number of observations have been made, agree precisely in the value which they give for $\frac{S''}{M''}$. Yet it seems impossible to deny that at some places (we particularly cite Plymouth and Bristol) the value of $\frac{S''}{M''}$ deduced from the times is certainly greater by $\frac{1}{20}$th part, and that at other places (as Portsmouth and Dundee) it is less by nearly $\frac{1}{10}$th part. And these we consider to be real and certain differences in the comparative effects of the Sun and Moon in the seas near to those places. They cannot be explained by any difference in the speed of spring-tide-waves and neap-tide-waves, as in (463.), because, by the rules of (536.), the value of $\frac{S''}{M''}$ is obtained by comparing two times of tide when (the values of $m-s-\alpha$ being the same with opposite signs) the magnitude of the tide, which depends on $\cos 2.\overline{m-s-\alpha}$, was the same. Indeed we consider it perfectly certain that the value of $\frac{S''}{M''}$ deduced from the times is an accurate representation of the proportion of the magnitudes of the solar and lunar tide-waves which come to the port. And the question now is, whether such a difference for different ports, as we consider to be established, is consistent with our theory? We answer that it is, and that it may be ascribed to one or more of the following causes.

This disagreement is consistent with theory.

(α.) If tides were formed independently (or nearly so) in different channels of different depths, the proportion of the solar and lunar waves, which by (455.) contains the factor $\frac{n'^2 b^2 - gk}{n^2 b^2 - gk}$, would be different in the different channels. This can scarcely apply to the different ports of England, but it may well apply in comparing the tides of the Atlantic with those of the Pacific.

(β.) If tides are communicated by different channels to the same port, the proportion of the solar and lunar waves, (121.) and (312.), will depend on the length, &c., of those channels, and may be different at different places. This applies, probably, to every port in England.

(γ.) The harbour-tides are, without exception, tides which have acquired a greater magnitude than the ocean-tides, in consequence of the tide having passed over a sloping bottom. Now though we cannot give accurately the values of the expressions used in (246.) and (247.), still it is certain that they will depend on n and m as well as on the depth and slope, and, therefore, that the solar and lunar tides will be altered in different degrees.

A correct mass of the Moon ought to be inferred from any of these observations.

(539.) The difference in the values of $\frac{S''}{M''}$, therefore, however difficult to be explained from the actual circumstances of each port, presents no difficulty in

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the general theory. One point, however, we must specially remark. Our explanations of the difference depend entirely on the difference of the periodic times of the forces, and, therefore, the consideration introduced by Laplace (481.) for the varying alteration of the effect of *each* body as connected with its varying velocity in right ascension, applies to all of them. This consideration is embodied in our formulæ of (448.). And if the observations applying to this were properly treated, in a manner equivalent to that of Laplace, (481.), and if the observations were sufficiently numerous and good, we ought to obtain the same value of the Moon's mass from every port. Unfortunately, the philosophers who have reduced the English observations, do not appear to have been aware of the principle on which the true value of the mass may (theoretically, at least,) be deduced from $\frac{S''}{M''}$; Mr. Whewell has even expressly said (*Phil. Trans.*, 1838, page 245): "We see, therefore, how different the mass of the Moon would be found to be by calculations from the tide-observations of different places." This would be true for an equilibrium-theory; but, as that contemptible theory is entirely inapplicable, except for some forms which require such extensive changes that they are now useless, it is scarcely worth while to refer to it. For a wave-theory it is not correct. We shall have occasion to revert to this subject.

Proportions deduced from heights of tide are different from those deduced from times of tide. This is not contradictory to theory.

(540.) Now, considering the values of $\frac{S'''}{M'''}$ deduced from the vertical range of tide, we find that there are but three accurate determinations (those from Brest, Plymouth, and Bristol), to which that of Dundee may be added as somewhat inferior. They agree in giving a smaller value at each place than that given by the times. We cannot explain this, but we assert distinctly that there is nothing in it contradictory to theory in its present state. We believe it to depend upon circumstances purely local, upon the depth and breadth of the channels through which the tide passes, and the like. For here, unlike the case of (538.), the determination depends *entirely* upon the difference of spring tides and neap tides. The most singular circumstance is, that, apparently at London and Liverpool, and certainly at Plymouth (Whewell, *Phil. Trans.*, 1839, page 151, &c.), the semimenstrual inequality is greater for low water than for high water. That is, upon increasing the general magnitude of the sea-wave, it produces a greater effect on shore-low-water than on shore-high-water. From these observed facts, as well as from those of (513.), &c., we are induced to fix upon the subject partially discussed in the third subsection of Section IV., as more strongly deserving attention at the present time than any other point in the Theory of Tides.

Method of determining the retard, or the age of the tide.

(541.) 2nd. On the value of 2α. This is determined from the times by taking that value of $2.\overline{m-s}$ which is found from observation to give to θ at high water its mean value. For, if we take pairs of equal values of $\overline{m-s}-\alpha$ with opposite signs, we obtain pairs of equal values of $2\theta-2\lambda$ with opposite signs: the mean of all is $2\theta-2\lambda=0$, which corresponds to $2.\overline{m-s}-2\alpha=0$ or gives $2\alpha=$ the corresponding value of $2.\overline{m-s}$. We may here observe that the quantity λ, which is the value of θ corresponding to $2.\overline{m-s}-2\alpha=0$, or which is the mean value of θ, is that upon which Mr. Whewell has with great propriety fixed as the characteristic of the time of high water at any place. He has sometimes called it "the mean lunitidal interval," (or the mean hour-angle of the moon at high water,) or sometimes "the corrected establishment," for a reason to which we shall advert shortly.

The value of 2α is determined from the heights by taking that value of $2.\overline{m-s}$ which gives to the height is greatest value. For, the expression for the height is evidently greatest when $2.\overline{m-s}-2\alpha=0$.

Since $m-s$ is the angle in right ascension by which the Moon has separated from the Sun, and since the mean separation in a solar day is 49 minutes nearly, the number of days after syzygy corresponding to the separation $\overline{m-s}$ (supposed to be expressed in minutes of time) is $\frac{m-s}{49}$ nearly, or $m-s=$number of days × 49. Therefore α, if expressed in minutes of time $=49$ × number of days after syzygy, or α, if expressed in degrees, $=12°\ 12'$ × number of days after syzygy, at which the lunitidal interval = mean lunitidal interval, or at which the highest tide occurs. In the same manner, $\alpha=49^m$, or $12°\ 12'$ × number of days after quadrature at which the lunitidal interval has its mean value, or at which the lowest tide occurs. At solstices (separately considered) these numbers are to be increased $\frac{1}{12}$th, and at equinoxes to be diminished $\frac{1}{12}$th.

Determinations from different places.

(542.) Thus the following determinations have been obtained:—

At Brest (*Méc. Cél.*, vol. v., pages 187 to 200), from observations of heights at equinoxial syzygies, the highest tide follows syzygy by $1^d\cdot480$, or .	$\alpha=16°\ 36'$
At solstitial syzygies by $1^d\cdot547$, or .	$\alpha=20°\ 29'$
From observations of heights at equinoxial quadratures, the lowest tide follows quadrature by $1\cdot5097$, or .	$\alpha=19°\ 56'$
At solstitial quadratures by $1\cdot5127$, or	$\alpha=16°\ 25'$
Laplace has not given the details of computation applying to the times, but he remarks that the hour of low water (expressed in fractions of a day) corresponding to the greatest tide, supposed to occur at $1^d\cdot5$ after syzygy, is $0\cdot441170$, and the hour of high water corresponding to the least tide, supposed to occur at $1^d\cdot5$ after quadrature, is $0\cdot431600$. If a later epoch were used, the first of these numbers would be increased by a less quantity than the Moon's daily motion ($\theta-\lambda$ being then negative, and increasing in its negative value), and the second would be increased by a greater quantity than the Moon's daily motion: and thus they might be made exactly equal. The value of $\overline{m-s}$ at this time would determine α. Thus it appears that α, as found from the times, is greater than as found from the heights. Mr. Lubbock, from one year's observations of times (*Phil. Trans.*, 1832), found .	$\alpha=22°\ 0'$

At London, from times (Lubbock, *Phil. Trans.*, 1831, page 400) . . . . $\alpha=32^\circ$
From heights (page 403) . . . $\alpha=26^\circ$
At Liverpool, from times (Lubbock, *Phil. Trans.*, 1835, page 283; and Whewell, 1836, page 8, observing that the tide occurs 11 hours after the moon's transit) . . . . . . $\alpha=24^\circ\ 15'$
From heights . . . . $\alpha=20^\circ\ 30'$
At Bristol, from times (Whewell, *Phil. Trans.*, 1838, page 237), the value of α is the Moon's motion during 56 hours, or . . . . $\alpha=28^\circ\ 30'$
From heights, the value of α is the Moon's motion during 40 hours, or . . . . $\alpha=20^\circ\ 20'$
At Dundee, from times (Whewell, *Phil. Trans.*, 1839, page 157) . . . . $\alpha=23^\circ$
From heights . . . . $\alpha=20^\circ$
At Portsmouth, from times (Lubbock, *Phil. Trans.*, 1833, page 21) . . $\alpha=26^\circ\ 30'$
At Plymouth, from times (same reference) . . . . $\alpha=28^\circ\ 45'$
At Sheerness, from times (same reference) . . . . $\alpha=30^\circ$

(543.) We have now to consider how the existence and variety of magnitude of this quantity α can be explained. And first we must remark that a part of it depends on the circumstance that we have not observed the tides in the ocean in which they are actually formed by the Sun and Moon, but on rivers and narrow seas up which they have been propagated as waves without external forces. Now the tide at London (the mean effect of two waves at 12 hours' interval, as we shall hereafter explain) has travelled from Brest in 16 solar hours nearly, and, therefore, to refer the London tide to its origin in that part of the Atlantic Ocean we must subtract from α the Moon's motion in 16 solar hours, or 8° 15′. In this manner we find the following table of values* of α referred to that part of the Atlantic ocean which is near Brest:—

Reduction of all the values of retard to one locality.

Place.	Number of hours occupied by passage of tide from Brest.	Corresponding correction to α.	Corrected value of α from times.	Corrected value of α from heights.
Brest . . .	0h. 0m.	0° 0′	22° 0′	18° 20′
London . . .	16 10	−8 15	23 45	17 45
Liverpool . .	7 40	−4 0	20 15	16 30
Bristol . . .	3 30	−1 50	26 40	18 30
Dundee . . .	10 50	−5 30	17 10	14 30
Portsmouth. .	7 52	−4 0	22 30	
Plymouth . .	1 45	−1 0	24 45	
Sheerness . .	14 51	−7 30	26 0	

The values of α from the heights agree as well as we can expect. The only sensible discordance is that of Dundee, where the result is deduced from the observations of a single year. The values of α deduced from the times cannot be made to agree.

The retard cannot be explained by the transmission of the tide from the Southern Ocean. It is explained by friction.

(544.) Now a little consideration will show that the amount of α cannot possibly be explained by delay in the transmission of the tide. The smallest value of α implies a delay of 29 hours; while it will appear that the tide at the Cape of Good Hope is earlier by only 15 hours, and that at Cape Horn by a still smaller quantity, than that at Brest; and thus it would seem that, even in the southern sea, where no reason for delay of tide can be imagined, it has already been delayed at least 14 hours. Putting this aside, we shall at once state our conviction that the apparent delay arises entirely from friction, as explained from theory in (459.). We may remark that the expression there found has the same sign whether the depth of the sea be greater or less than 14 miles. We do not deny that it is possible to explain the same thing by the interference of two tides with nearly opposite phases, of which one has suffered no retardation, and the other a small retardation; and that, in the instance of Brest, two such mingling tides may be produced, one from S. to N., and the other from E. to W. But the other explanation is so much more natural that we shall adhere to it.

The value of retard to be adopted for the sea is that determined by the heights of the tides.

(545.) But, as we have one system of values of α given by the heights of tide, and another given by the times, it may be asked which is the true one? We answer, without doubt, that given by the heights. We have shown in (462.) that local circumstances do not alter, except by a constant quantity, the times of high water for a given range of tide, and therefore do not alter the day of highest tide (upon which α, as deduced from the heights, depends); but in (463.) it appears that local circumstances do alter the day when the tide occurs at mean lunitidal interval, or do alter the time for which the relative positions of the sun and moon are to be used in computation of the time of tide. They may be expected, therefore, to alter it differently in different ports, and thus the discordance of values of α in the first column is explained. It is remarkable that the difference between the time-values and the height-values of α is less at the two most open ports (Dundee and Brest) than at any of the others, as might have been expected.

Nature of the difference between the retards determined from heights and from times, and probable explanation of it.

(546.) It is, however, worthy of remark, that all the time-values of α are greater than the height-values. This implies that the effect of the local circumstances is opposite to that assumed in (463.), and therefore that the great tides travel more slowly from the sea than the small tides. We cannot venture positively to explain this, but we may suggest the following consideration for the reader's judgment. Before the old London Bridge was removed, the higher passage of small tides was materially obstructed by the sterlings of the bridge; that of great tides much less. The small tides were therefore nearly similar to tides in an interrupted canal, in which the time of high water is the same as that at the sea (307.); while the large tides partook of the nature of tides in an uninterrupted canal, and the time of high water up the river was progressive. Similar considerations may apply to the other ports. The quantity α is sometimes called the *retard*, and the time in which the Moon moves through that angle from the Sun is called the *age of the tide*.

(547.) 3rd. On the agreement of the laws of the observed semimenstrual irregularities with those given by the for-

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* We have been anxious to combine with these the value of α, deduced from the observations of height at Leith, which are tabulated in the Report of the 11th meeting of the British Association, page 36. But in the want of explanation of the table, and the apparent inference from it as it stands, that the highest tide *precedes* syzygy (contrary to universal experience), we have not ventured to use it.

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mulæ. With regard to the times we may state, that for those places for which the comparison has been made with the values of $\frac{S''}{M''}$ and α peculiar to those places, the agreement is perfect. We subjoin a Table similar to Mr. Lubbock's, of observed and computed intervals between the Moon's transit over the meridian at London and the time of high water (altered only so as to adapt it to the value of $\alpha=32°$ as above, instead of 30° which Mr. Lubbock has used,) and we copy from the *Philosophical Transactions*, 1836, Mr. Whewell's table of the observed and computed intervals for Liverpool.

Comparison of theoretical law and observed law of semimenstrual inequalities of time.

Observed and computed values of θ for London; the computed values being found by the second formula of (535.), supposing $\frac{S''}{M''} = 0{\cdot}37887$, $\alpha=32°$, λ (in time)$=1^h.25^m.35^s$.

Solar Time of Moon's Transit.		$m-s$ at Moon's Transit.		$m-s$ at High Water.		$m-s-\alpha$.		Computed Value of $\theta-\lambda$ in Time.		Computed Value of θ in Time.			Observed Value of θ in Time.	
h	m	°	′	°	′	°	′	m	s	h	m	s	h	m
0	0	0	0	1	0	329	0	+31	42	1	57	17	1	57
0	30	7	30	8	30	336	30	+24	50	1	50	25	1	50
1	0	15	0	16	0	344	0	+17	16	1	42	51	1	42
1	30	22	30	23	20	351	20	+ 9	20	1	34	55	1	35
2	0	30	0	30	50	358	50	+ 1	18	1	26	53	1	26
2	30	37	30	38	20	6	20	− 6	59	1	18	36	1	18
3	0	45	0	45	40	13	40	−14	50	1	10	45	1	11
3	30	52	30	53	10	21	10	−22	32	1	3	3	1	3
4	0	60	0	60	30	28	30	−29	30	0	56	5	0	56
4	30	67	30	68	0	36	0	−35	44	0	49	51	0	51
5	0	75	0	75	30	43	30	−40	42	0	44	53	0	45
5	30	82	30	83	0	51	0	−43	50	0	41	45	0	43
6	0	90	0	90	30	58	30	−44	18	0	41	17	0	42
6	30	97	30	98	0	66	0	−41	18	0	44	17	0	44
7	0	105	0	105	30	73	30	−33	38	0	51	57	0	52
7	30	112	30	113	10	81	10	−20	20	1	5	15	1	5
8	0	120	0	120	50	88	50	− 2	50	1	22	45	1	23
8	30	127	30	128	20	96	20	+15	7	1	40	42	1	39
9	0	135	0	136	0	104	0	+29	54	1	55	29	1	56
9	30	142	30	143	30	111	30	+39	18	2	4	53	2	5
10	0	150	0	151	0	119	0	+43	46	2	9	21	2	10
10	30	157	30	158	30	126	30	+44	20	2	9	55	2	10
11	0	165	0	166	0	134	0	+42	0	2	7	35	2	8
11	30	172	30	173	30	141	30	+37	34	2	3	9	2	3

Observed and computed values of $\theta-\lambda$ for Liverpool, supposing $\frac{S''}{M''} = 0{\cdot}37866$, and $\alpha = 24°.15'$ nearly. (*Note.*—Our α is not the same quantity as Mr. Whewell's α, *Philosophical Transactions*, 1836, page 8. Mr. Whewell uses α as the correction to the Moon's distance from the Sun at the time of transit preceding the tide; ours is the correction to the Moon's distance from the Sun at the time of high water.) λ for Liverpool is $11^h\ 6^m$.

Solar Time of Moon's Transit.		Computed Value of $\theta-\lambda$.		Observed Value of $\theta-\lambda$.		Solar Time of Moon's Transit.		Computed Value of $\theta-\lambda$.		Observed Value of $\theta-\lambda$.	
h	m	m	s	m	s	h	m	m	s	m	s
0	30	+12	16	+12	12	6	30	−25	8	−25	0
1	30	− 4	7	− 4	36	7	30	+ 9	2	+ 9	6
2	30	−20	6	−20	0	8	30	+36	28	+36	36
3	30	−34	0	−33	54	9	30	+44	20	+45	36
4	30	−43	6	−42	48	10	30	+39	40	+39	48
5	30	−42	40	−43	12	11	30	+27	36	+26	6

It would scarcely be possible to obtain closer agreement. In figure 61 we have represented the values of $\theta-\lambda$ for London by the ordinates of a curve, the abscissa representing the solar time of the Moon's transit. The dark line is the curve representing the values of $\theta-\lambda$ computed from theory, and the dotted line is that representing the observed values. The same form precisely is given by the observations at other places; we leave to the reader the task of laying down that for Liverpool, and we refer him to the *Philosophical Transactions*, 1833, page 19, for the curves of six different places.

Difference between Vulgar Establishment and Corrected Establishment.

(548.) We may now point out the difference between the vulgar *Establishment* of which we have spoken in (2.) and (497.), and Mr. Whewell's *Corrected Establishment*. At London, for instance, the vulgar Establishment, taken from the table above, would be $1^h\ 57^m$. But the mean interval between the Moon's transit and the time of high water is $1^h\ 25^m\ 35^s$; and this is the *Corrected Establishment*, differing 31^m from the other. It is better for adoption as a mathematical element, because it expresses a simple quantity (namely, the delay of high water after the Moon's transit, supposing no Sun to exist); and it is better for practical use, because it differs less, upon the whole, from the real interval on any day, than the vulgar Establishment does. Thus at London, if we roughly computed the time of high water with the interval $1^h\ 57^m$ from Moon's transit, we might sometimes be $1^h\ 15^m$ in error; if we used $1^h\ 26^m$, we should never be more than 44^m in error.

Comparison of theoretical and observed law of semi-menstrual inequalities of heights.

(549.) With regard to the heights, it is possible to use such constants in the second formula of (535.) as will give a very close approximation to the observed heights. Thus, for the representation of the heights of high water at Liverpool, Mr. Whewell has used the values $M'' = 6\cdot872$ feet, $S' = 2\cdot74$ feet, $\alpha = 15°$ (applicable to the Moon's distance from the Sun at the preceding transit), and has subtracted from the result of the computation, the constant $7\cdot19$ (the mean of all the computed numbers). This he compares with the difference between each of the heights, and the mean of all the heights, as measured from a certain fixed point. The result is as follows:—

Solar Time of Moon's Transit.		Difference of computed height from mean.	Difference of observed height from mean.
h	m	Feet.	Feet.
0	30	+2·35	+2·35
1	30	+2·35	+2·39
2	30	+1·83	+1·88
3	30	+0·84	+0·90
4	30	—0·48	—0·38
5	30	—1·89	—1·76
6	30	—2·90	—2·91
7	30	—2·90	—2·94
8	30	—1·89	—1·85
9	30	—0·48	—0·38
10	30	+0·84	+1·04
11	30	+1·83	+1·81

The agreement is good, but it has been obtained only by taking for M'' a quantity which shall make a good agreement, and not by taking it from the range of the tide. To make an equally good agreement for the heights in the London tides (*Phil. Trans.*, 1831, page 390), it is necessary to use a value of M'' much smaller than that used by Mr. Lubbock, so that the value of $\frac{S''}{M''}$ is unreasonably large. But, for the reasons mentioned in (540.), we do not think this important. In figure 62 we give the curve representing the semimenstrual inequality of height for London; the abscissa is the Moon's solar time of transit, and the ordinate is the height of high water above a fixed point.

(550.) We now come to the corrections in these semimenstrual formulæ depending on the varying declinations and varying distances of the Moon and Sun. And first, the correction for the Moon's declination. The term which the equilibrium-theory gives for correction of time, (54.) and (56.), is

$$-\frac{360^m}{\pi}.\frac{M_\Delta S_\Delta . \sin 2.\overline{m-s}}{(M_\Delta + S_\Delta . \cos 2.\overline{m-s})^2 \cos^2\Delta}(\sin^2\mu - \sin^2\Delta)$$

and that for correction of height, (53.) and (56.),

$$-3.\frac{M_\Delta^2 + M_\Delta S_\Delta . \cos 2.\overline{m-s}}{\sqrt{\{M_\Delta^2 + 2.M_\Delta S_\Delta . \cos 2.\overline{m-s} + S_\Delta^2\}}\cos^2\Delta}(\sin^2\mu - \sin^2\Delta)$$

where the whole height is represented by $3\sqrt{\{M_\Delta^2 + 2M_\Delta S_\Delta . \cos 2.\overline{m-s} + S_\Delta^2\}}$. Here μ is the Moon's declination, and μ and $\overline{m-s}$ are to be taken for the time of the tide. The theory of waves, (448.), (449.), and (452.), makes several important modifications in this expression. First μ and $\overline{m-s}$ are to be taken for a time anterior by a *constant* quantity $\rho_{\prime}$ (452.) and (459.), (not by a quantity which makes α constant). This quantity $\rho_{\prime}$ for Brest is about 36 hours, for London 52 hours, for Liverpool 44 hours, and for Bristol 40 hours. Secondly, in computing the retardation or its tangent, or $\theta - \lambda$ (the expression which in (535.) we have found it necessary to use) from the formula

Formulæ for lunar declination corrections of times.

$$-\frac{360^m}{\pi}\times\frac{(S_\Delta + T) . \sin 2.\overline{m-s}}{M_\Delta + N + (S_\Delta + T)\cos 2.\overline{m-s}}$$

and in computing the height from the formula

$$\sqrt{\{M_\Delta^2 + 2M_\Delta S_\Delta . \cos 2.\overline{m-s} + S_\Delta^2 + (2M_\Delta + 2S_\Delta . \cos 2.\overline{m-s})N + (2S_\Delta + 2M_\Delta . \cos 2.\overline{m-s})T\}},$$

we must increase $2m$ further by $\frac{n'\mathrm{b}^2 l}{n'^2\mathrm{b}^2 - gk}\omega^2 . \sin 2lt$ (ω = obliquity of Moon's orbit to equator, lt = Moon's longitude or right ascension); and for M + N, so far as depends on declination, we must put

$$M\left\{1 - \frac{\omega^2}{2} + \left(\frac{1}{2} - \frac{n'\mathrm{b}^2 l}{n'^2\mathrm{b}^2 - gk}\right)\omega^2 \cos 2lt\right\}.$$

The mean of all the values of this expression is $M.\cos^2\Delta$, or $M_\Delta = M\left(1-\frac{\omega^2}{2}\right)$, and this expression is therefore

$$M_\Delta\left\{1+\frac{1}{\cos^2\Delta}\left(\frac{1}{2}-\frac{n'b^2l}{n'^2b-gk}\right).\left(\omega^2-\omega^2.\overline{1-\cos 2lt}\right)\right\}$$

$$=M_\Delta\left\{1+\frac{1}{\cos^2\Delta}\left(1-\frac{2n'b^2l}{n'^2b^2-gk}\right)\left(\frac{\omega^2}{2}-\omega^2.\sin^2 lt\right)\right\}$$

$$=M_\Delta\left\{1-\left(1-\frac{2n'b^2l}{n'^2b^2-gk}\right).\frac{\sin^2\mu-\sin^2\Delta}{\cos^2\Delta}\right\}.$$

$$\text{Therefore } N=-M_\Delta\left(1-\frac{2n'b^2l}{n'^2b^2-gk}\right)\frac{\sin^2\mu-\sin^2\Delta}{\cos^2\Delta}.$$

Also, putting $2m'$ for $\frac{n'b^2l}{n'^2b^2-gk}\,\omega^2.\sin 2lt$, $\sin 2.\overline{m-s}$ is to be changed to $\sin\overline{2m-2s+2m'}=\sin 2.\overline{m-s}+2m'.\cos 2.\overline{m-s}$, and $\cos 2.\overline{m-s}$ is to be changed to $\cos\overline{2m-2s+2m'}=\cos 2.\overline{m-s}-2m'.\sin 2\overline{m-s}$. Omitting therefore all variations depending on the varying distance and declination of the sun, and all except those of declination of the moon, we have to compute the time of tide after transit, or $\theta-\lambda$, by the formula

$$-\frac{360^m}{\pi}\times\frac{S_\Delta\{\sin 2.\overline{m-s}+2m'.\cos 2.\overline{m-s}\}}{M_\Delta+N+S_\Delta\{\cos 2.\overline{m-s}-2m'.\sin 2.\overline{m-s}\}}$$

or (expanding to the first power of N and $2m'$), by the formula

$$-\frac{360^m}{\pi}.\frac{S_\Delta.\sin 2.\overline{m-s}}{M_\Delta+S_\Delta.\cos 2.\overline{m-s}}-\frac{360^m}{\pi}.2m'\frac{M_\Delta S_\Delta.\cos 2.\overline{m-s}+S_\Delta{}^2}{(M_\Delta+S_\Delta.\cos 2.\overline{m-s})^2}+\frac{360^m}{\pi}N.\frac{S_\Delta.\sin 2.\overline{m-s}}{(M_\Delta+S_\Delta.\cos 2.\overline{m-s})^2};$$

and putting for N and m' their values, we have for the second and third term, which constitute the declination-correction of time of high water,

$$-\frac{360^m}{\pi}\left\{\frac{n'b^2l}{n'^2b^2-gk}\,\omega^2.\sin 2lt.\frac{M_\Delta S_\Delta.\cos 2.\overline{m-s}+S_\Delta{}^2}{(M_\Delta+S_\Delta.\cos 2.\overline{m-s})^2}\right.$$

$$\left.+\left(1-\frac{2n'b^2l}{n'^2b^2-gk}\right)\frac{\sin^2\mu-\sin^2\Delta}{\cos^2\Delta}.\frac{M_\Delta S_\Delta.\sin 2.\overline{m-s}}{(M_\Delta+S_\Delta.\cos 2.\overline{m-s})^2}\right\}.$$

We proceed now to advert to the bearing of observations, as far as they have yet been discussed, upon these terms.

One term has been omitted in comparisons with observations.

(551.) The first term, depending upon $\omega^2\sin 2lt$, has not been elicited from observations at all, for very good reasons. The square of the sine of declination is $\omega^2\sin^2 lt$, and therefore $\omega^2\sin 2lt$ is proportional to the increase of the square of the sine of declination. Now Laplace gave very little attention to the times of tides; and moreover, in the discussion of solstitial and equinoxial observations, he confined himself to the times bordering very closely on solstices and equinoxes, on both sides of them (480.), &c., and therefore the increase of the square of declination could not appear. Mr. Lubbock and Mr. Whewell, though they have used observations made with all values of declination, have unfortunately (from inattention to this theory) classified the tides, as regards declination, only by the amount of declination, grouping together the observations at which the declination was increasing and those at which it was diminishing: in the mean of these, therefore, the term depending on the increase or decrease of declination has been entirely lost. We regret this much, because that term would be of singular use in giving the value of the important constant $\frac{n'^2b^2l}{n'^2b^2-gk}$; of which we shall shortly speak further.

Explanation of a difficulty produced by using a wrong place of the moon.

(552.) For the second term, the observations were arranged with reference to the declination, (though somewhat complicated at first by the introduction of the calendar month,) and they are therefore so far in a fit state to compare with theory. It must be remarked that m, s, and μ, are to be taken for the moon's place at a period anterior by that constant which we have called the *age of the tide* or $\rho_{\prime}$; and therefore that the interval of the time of tide from the moon's transit, or $\theta-\lambda$, ought not to be reckoned from the last transit of the moon, but from that transit of the moon over some meridian at which it occurred at the time $\rho_{\prime}$ nearly before the tide. Mr. Whewell, in discussing the intervals of tides from the nearest transit of the moon at London and Liverpool, (*Phil. Trans.* 1834 and 1836,) found, besides terms multiplying $\sin 2.\overline{m-s}$, terms of the form $132^m(\sin^2\Delta-\sin^2\mu)$ for London, and $84^m.(\sin^2\Delta-\sin^2\mu)$ for Liverpool. These terms, not at first understood, are a singular proof of the theory as regards the necessity of referring the interval to a transit earlier by the age of the tide. For the moon's velocity in right ascension $=$ vel. in long. $\times\frac{\cos\text{ obliquity}}{\cos^2\text{ declination}}=$ vel. in long. $\times\frac{\cos^2\Delta}{\cos^2\mu}=$ vel. in long. $\times(1-\sin^2\Delta+\sin^2\mu)$, therefore, supposing the theory correct, the moon's right ascension or time of transit would be increased, from the true anterior epoch to the transit near to the time of tide, by vel. in long. $\times$ age of tide $\times(1-\sin^2\Delta+\sin^2\mu)$ or by $\alpha\times(1-\sin^2\Delta+\sin^2\mu)$; of which the part that depends on declination is $\alpha\times(\sin^2\mu-\sin^2\Delta)$; and therefore the interval from that transit to the tide would be increased by $\alpha(\sin^2\Delta-\sin^2\mu)$; which, with the values above for α, gives for London $128^m(\sin^2\Delta-\sin^2\mu)$, and for Liverpool about $80^m(\sin^2\Delta-\sin^2\mu)$, (referring the end of α, in Mr. Whewell's manner, to the place of the moon at the transit preceding the tide;) the agreement of which with the quantity deduced from observation is very close.

Difficulty of comparing theory with existing results of observation.

(553.) The general comparison, however, of the results for declination-correction of times deduced by Mr. Whewell (*Philosophical Transactions*, 1834, 1836, 1838) with the term given by theory, is almost impracticable. Where the tide ought to be compared with a transit of the moon earlier by two days, it is evidently unsafe to attempt to use results deduced from comparison with a transit on the same day, especially for an element which changes so rapidly as declination, and more particularly as one term whose influence would probably then be sensible is entirely neglected. This consideration excludes all but the investigation in the *Philosophical Transactions*, 1838. In that paper Mr. Whewell has gone through the labour of reducing two years' observations at Bristol by reference to three different transits of the moon, (a process which, with proper management, would give the results depending on the change of the moon's declination and change of the moon's elliptic inequality, which we are seeking;) and he finds that the same transit (B), to which it is necessary to refer for semi-menstrual inequality of height, will give a declination-correction agreeing *generally* with theory. But how nearly this agreement holds is not explained. Our theoretical term is

$$-\frac{360^m}{\pi}\left(1-\frac{2n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}\right)\frac{\sin^2\mu-\sin^2\Delta}{\cos^2\Delta}\cdot\frac{\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\sin.2.\overline{m-s}}{(\mathrm{M}_{\Delta}+\mathrm{S}_{\Delta}.\cos 2.\overline{m-s})^2};$$

Mr. Whewell's has the form + a factor × $s\,(2\phi-12^h)$, s being a periodical function which is positive when its argument is less than 12^h and negative when its argument is between 12^h and 24^h, and has for maximum and minimum values +1 and −1, these values occurring, the first before 6^h in the argument, and the second after 18^h. (This appears clearly, from the instance p. 239, to be the nature of the curve.) Now this law agrees well with that of the theory. The factor is $+6^m$ for declination 24° and -7^m for declination 7°; or the difference for decl. 24° and decl. 7° is 13^m. But if from the theoretical term we compute, with $\frac{\mathrm{S}_{\Delta}}{\mathrm{M}_{\Delta}}=0{\cdot}394$, as found from the times at Bristol (537.), the maximum value of that term (occurring when $2.\overline{m-s}=129°$), we find the difference of its values for declination 7° and 24° to be $10^m{\cdot}1\times\left(1-\frac{2n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}\right)$. Are we to make this equal to 13^m, and to infer from it that gk is greater than $n'^2\mathrm{b}^2$, or that (444.) the depth of the sea is greater than 14 miles? We think not. Calculations on the equilibrium-theory, assuming the received mass of the moon (33.) and (35.), give for the proportion of the moon's effect to the sun's effect 0·596 : 0·271, or 1 : 0·45 nearly; and the only way in which we can explain the raising of that ratio to the proportion 1 : 0·394 is by supposing $n'^2\mathrm{b}^2-gk$ to be positive. (455.) A contradiction of the same kind will be found if we use any other supposition, as Laplace's: but will the reduced observations justify us in drawing any theoretical deduction? We scarcely think that they will. Independently of the general omission of steps of the reduction, we may remark that, as appears from Mr. Whewell's description of the mode of reducing, (extracted in 494.) the means of corrections were supposed to apply to the means of declinations, instead of to the means of the squares of declinations. This error cannot fail to injure the whole deduction.

(554.) The declination-correction for heights will be thus found. In the expression

$$\sqrt{\{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}+\mathrm{S}_{\Delta}{}^2+(2\mathrm{M}_{\Delta}+2\mathrm{S}_{\Delta}.\cos 2.\overline{m-s})\mathrm{N}\}},$$

(53.), (56.), and (535.), omitting T, we are to put $\cos 2.\overline{m-s}-2m'.\sin 2.\overline{m-s}$ for $\cos 2.\overline{m-s}$; and then to expand to the first power of m' and N. This gives for the small terms

Formulæ for lunar declination-correction of heights.

$$-2m'\frac{\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\sin 2.\overline{m-s}}{\sqrt{\{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}+\mathrm{S}_{\Delta}{}^2\}}}+\mathrm{N}\frac{\mathrm{M}_{\Delta}+\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}}{\sqrt{\{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}+\mathrm{S}_{\Delta}{}^2\}}};$$

or, restoring for m' and N their values, the small terms are

$$-\frac{n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}\omega^2\sin 2lt.\frac{\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\sin 2.\overline{m-s}}{\sqrt{\{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}+\mathrm{S}_{\Delta}{}^2\}}}$$

$$-\frac{\mathrm{M}_{\Delta}{}^2+\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}}{\sqrt{\{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}+\mathrm{S}_{\Delta}{}^2\}}}\left(1-\frac{2n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}\right).\frac{\sin^2\mu-\sin^2\Delta}{\cos^2\Delta}.$$

Now the first of these terms has been neglected, for the same reasons which we have assigned in (551.). The treatment of the second term by English mathematicians has been unsatisfactory, in the same manner as that for the declination-correction of times. Moreover, as the whole range has not been observed, except at Plymouth, it can generally lead to no result as to the value of $1-\frac{2n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}$; and at Plymouth the two phases (high and low water) appear to depend upon such different laws that we cannot use the variations for high water only in conjunction with the whole range.

Laplace's deduction of the mass of the moon from de-

(555.) Laplace's deductions for the whole ranges of tide at Brest are so important that we must devote a few words to them. The proportion of the small term above to the whole mean range is

$$\frac{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}}{\mathrm{M}_{\Delta}{}^2+2\mathrm{M}_{\Delta}\mathrm{S}_{\Delta}.\cos 2.\overline{m-s}+\mathrm{S}^{\ 2}}\left(1-\frac{2n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}\right).\frac{\sin^2\mu-\sin^2\Delta}{\cos^2\Delta}.$$

Sect. VIII. Comparison of Theory and Observations of Tides.

Subsec. 3. On the Laws of the Tides for varying positions of the Sun and Moon.

clination-corrections.

Now Laplace had found $\frac{S_\vartriangle}{M_\vartriangle}$, as we have explained (536.); and using this value with the values of $\overline{m-s}$ and μ in the observations which he discussed, and comparing the difference of the observed ranges when $\mu=0$ and $\mu=23°$, and the ratio of that difference to the whole range, with the formula, he obtained a value for $1-\frac{2n'\mathrm{b}^2 l}{n'^2\mathrm{b}^2-gk}$, and therefore for $\frac{n'\mathrm{b}^2 l}{n'^2\mathrm{b}^2-gk}$, or for $\frac{l}{n'}\cdot\frac{\frac{n'^2\mathrm{b}}{g}}{\frac{n'^2\mathrm{b}}{g}-\frac{k}{\mathrm{b}}}$. It is evident that $\frac{n'}{l}=$ number of lunar days in moon's sidereal period; that $\frac{n'^2\mathrm{b}}{g}=\frac{n'^2}{n^2}\cdot\frac{n^2\mathrm{b}}{g}=\left(\frac{\text{solar day}}{\text{lunar day}}\right)^2\times$ the proportion of centrifugal force at the equator to gravity; and that $\frac{k}{\mathrm{b}}$ is a constant which in our theory represents the proportion of the depth of the sea to the earth's radius. Thus a constant in Laplace's theory, analogous to the depth of the sea in our theory, was found. Then using that constant in the fraction discussed in (455.), he finds that the proportion of the moon's effect to the sun's has been *increased*, in consequence of this value of the depth of the sea, in the ratio of 4 : 5. (In his first investigations he had found the ratio to be 10 : 11.) That is to say, $\frac{M_\vartriangle}{S_\vartriangle}$ or $\frac{M''}{S''}=\frac{5}{4}\cdot\frac{M'}{S'}$. Now $\frac{M''}{S'}$ is known from the semimenstrual inequality; therefore $\frac{M'}{S'}$ is known; and (13.)

$$\frac{M'}{S'}=\frac{M}{(\text{Moon's distance})^3}\cdot\frac{(\text{Sun's distance})^3}{S};$$

and the sun's and moon's distances and the sun's mass are known, therefore the moon's mass is known. Thus Laplace found for the moon's mass a value very nearly $=\frac{1}{75}$ of the earth's mass.

It is to be remarked that Laplace's expressions are more complicated because (as is evidently necessary for accuracy) he combines, with these considerations applying to the moon's declination, analogous considerations applying to the sun's declination. Moreover, Laplace does not ascribe the alteration of the proportion of effects to the depth of the sea, (his own theory not admitting of a solution so general as to exhibit that effect,) but to the assumed interference of waves coming by different channels. But the expressions introduced are exactly the same. We cannot sufficiently admire the union of sagacity and ingenuity in this process; the sagacity of perceiving that the effects would not be proportionate to the forces, but would also depend on the periodic times of the forces; and the ingenuity of inventing a conceived combination of circumstances which, mathematically speaking, would account for the difference of proportion and would afford means of calculating it, but which probably were not the grounds of his original conjecture, and which can scarcely be supported as applicable, locally, to Brest. We do not at all imagine that Laplace believed in his hypothesis of two canals.

Corrections of height and time for lunar parallax.

(556.) We shall not delay long on the corrections for the moon's varying parallax. It will be sufficient to state here that, in conformity with (448.) and (452.), $2.\overline{m-s}$ ought to be increased by $\frac{3n'\mathrm{b}^2 h}{n'^2\mathrm{b}^2-gk}e\sin ht$; and that instead of using $1+3e\cos ht$ as representing the factor for the effect of the moon depending on parallax, we ought to use $1+\left(3+\frac{4n'\mathrm{b}^2 h}{n'^2\mathrm{b}^2-gk}\right)e\cos ht$. The former of these terms is lost in all the investigations, for the same reason as the analogous term in the declination-corrections. In a discussion of the Liverpool tide-observations, (*Philosophical Transactions*, 1836,) Mr. Whewell found that the correction of the time of high water for parallax implied that the coefficient $3e$, instead of being increased, was to be diminished. But we attach little importance to this, as in that discussion the tides were compared with the next preceding transit of the moon, and the parallaxes, &c., were taken for that transit, instead of taking one long before it. In the *Philosophical Transactions*, 1838, the Bristol observations were discussed in reference to several transits in order to ascertain which gave the most consistent results for parallax-correction. The result was that a transit later than that which represented best the semi-menstrual inequality and the declination-correction was necessary for the parallax-correction. But this conclusion is by no means certain. It was founded upon the consideration that such a transit made it unnecessary to introduce any constant multiple of p', and therefore brought the observed inequality to a form approaching nearly to the term in (54.). But, on the other hand, an earlier epoch (the same which gave best results for semi-menstrual inequality of height and for declination-correction) exhibited more distinctly, in the magnitude of the coefficient of the variable term, the influence of the parallax-correction; and this alone is a strong proof that the earlier epoch was better. Moreover, the existence of a constant multiple may be accounted for by the supposition, that the larger tide corresponding to a larger parallax, travelling further up the small river above Bristol, does not make high water at Bristol so soon as a smaller tide which is almost stopped there. The law of the variable part of correction obtained by Mr. Whewell, as depending on the interval of the sun and moon, agreed pretty well with the theoretical law.

Difficulty of comparing them with existing results of observation.

(557.) In the same place, Mr. Whewell has stated the result of reducing the observed heights of the tides at Bristol. It appears that the same age of the tide which represents the other inequalities is proper for this. With respect to the agreement of the law of correction, as depending on the moon's distance from the sun, we can pronounce nothing precise. Mr. Whewell has however stated that this factor is, from observations, nearly constant; which agrees with theory; inasmuch as the theoretical term

$$\frac{\mathrm{M}_{\vartriangle}+\mathrm{S}_{\vartriangle}\cos 2.\overline{m-s}}{\surd\{\mathrm{M}_{\vartriangle}{}^{2}+2\mathrm{M}_{\vartriangle}\mathrm{S}_{\vartriangle}\cos 2.\overline{m-s}+\mathrm{S}_{\vartriangle}{}^{2}\}}$$

differs from 1 only by a quantity of the order of $\left(\frac{\mathrm{S}_{\vartriangle}}{\mathrm{M}_{\vartriangle}}\right)^2$, and not of the order of $\frac{\mathrm{S}_{\vartriangle}}{\mathrm{M}_{\vartriangle}}$, as Mr. Whewell has stated.

Laplace's comparison of observed parallax-inequality with theoretical corrections.

(558.) The effect of parallax on the height of the tides at Brest was discussed by Laplace in the following manner. From the discussion of the declination-observations, to which we have already alluded (555.), he took the value of the quantity $\frac{n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}$. He then substituted this, as a known quantity, in the expression $\left(3+\frac{4n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}\cdot\frac{h}{l}\right)e\cos ht$, and computed with it the ranges of tides for large parallaxes and small parallaxes, and took their differences. Thus he obtained as a computed number representing the theoretical sum of the differences for a certain number of observations, 51·52. The number actually deduced from the observations themselves was 47·27. Thus the observed difference was less than the theoretical difference. But this theoretical difference, it must be remarked, depends entirely on the circumstance that Laplace has chosen to use only the declinations for finding $\frac{n'\mathrm{b}^2l}{n'^2\mathrm{b}^2-gk}$; whereas, in our opinion, both declinations and parallaxes ought to have been used, and a value of that quantity ought to have been found from their combination.

Verification of the theoretical law depending on the magnitude of declination and parallax.

(559.) The theoretical law of the declination and parallax-corrections (that they depend on the square of the sine of declination and on the simple difference of the parallax from mean parallax) is fully proved by Mr. Whewell in the various discussions of the tides at London, Liverpool, and Bristol, to which we have so often referred. It may be interesting to the reader to remark that the whole difference in the height of high water at London, depending on declination, is about six or eight inches, and that depending on parallax is about one foot; and that at Liverpool the corresponding quantities are nearly two feet and three feet. At Bristol they are greater.

Observed inequalities depending on the sun's place.

(560.) The corrections depending on the sun's declination and parallax, being much smaller than those for the moon, (both because the absolute effect of the sun is less, and because the proportionate variations of its distance are less,) are not so easily extracted from observations. But in the *Philosophical Transactions*, 1836, pages 131 and 218, Mr. Whewell and Mr. Lubbock, taking the observations at Liverpool and London, and subtracting the effects due to semi-menstrual inequality and to corrections for the moon's declination and parallax, have obtained those which depend on the sun's place. The results appear to agree well with theory. Those at London are more completely worked out by Mr. Lubbock (the observations being referred to an earlier transit) in the *Philosophical Transactions*, 1837, p. 97, &c. In the *Ninth Report of the British Association*, p. 13, will be found some results as to solar corrections deduced by Mr. Bunt from the observations at Bristol.

(561.) The second line of (44.), which is considered at length in (46.), (47.), and (48.), contains the expression for the diurnal tide. The reader will readily understand that, upon applying to these terms the theory of (439.), modified by the theory of friction nearly as in (451.), results of the same kind but with different constants will be obtained; the relative effects of the sun and moon will be altered in a proportion different from that of the alteration in the semi-diurnal tide; the magnitude of the tide will be determined by the positions of the bodies at an epoch anterior by a different quantity; and the absolute time of diurnal high water on any day will bear no distinct relation to that of semi-diurnal high water. In fact, the diurnal tide is to be worked out with little reference to the semi-diurnal tide.

General effect of diurnal tide.

(562.) In (63.) we have pointed out the way in which the diurnal tide shows itself most obviously, by making the two semidiurnal tides appear unequal. Many notices of this kind are to be found in early records (as *Philosophical Transactions*, 1668, Colepresse on the tides at Plymouth, and Sturmy on those at Bristol: 1684, Davenport on those at Tunkin: 1776, a remarkable instance of their effect on the coast of New Holland, described by Captain Cook, &c.) In the same manner, observing the times as well as the heights of tide, it has been extracted from later observations. Thus in the *Philosophical Transactions*, 1836, page 57, &c., Mr. Lubbock has found that at Liverpool, in the month of January, when the moon's transit occurs in the afternoon at any time before 8 hours 30 minutes P.M., or in the forenoon after 8 hours 30 minutes A.M., the high tide which precedes that transit is greater than the high tide which follows it by a quantity which varies in magnitude, but which when greatest is nearly a foot. But if the moon's transit occurs at the opposite time of the day, the high tide which precedes that transit is less than that which follows it. The same rule applies to February, if, instead of 8 hours 30 minutes, we take 7 hours 30 minutes; for March, we must take 6 hours 30 minutes; for April, 3 hours 20 minutes; for May, 0 hours; for June, 9 hours in the opposite part of the day; and so on. The maximum difference is greatest in January and July, and least in April and October. From this it is plain that a considerable part of the diurnal tide depends on the sun. In the middle of January the sun's right ascension is 19 hours 30 minutes nearly; therefore the moon's right ascension, when her transit occurs at 8 hours 30 minutes P.M., is 4 hours nearly, and the diurnal tide therefore does not vanish till several days (5 or 6) after the moon has crossed the equator. The same appears from the other months. There is no sensible diurnal inequality in times. Mr. Lubbock (*Philosophical Transactions*, 1837, page 101) considers that the Liverpool diurnal tide corresponds to the position of the moon 4 days before the tide.

Diurnal tide at Liverpool.

Age of diurnal tide at several places.

(563.) In the *Philosophical Transactions*, 1836, page 289, &c., in which Mr. Whewell has given the results of the simultaneous observations of tides made at his representation on a great extent of coasts of Europe and America, he has stated the relation of the diurnal inequality to the position of the moon. On the North American coast, it corresponds to the moon's place at the same time; on the coast of Spain, Portugal, the west of France, Cornwall, the west of Ireland, the north of England and Scotland, and the Cape of Good Hope, it corresponds to the moon's place 2 days earlier;

at Cowes, Portsmouth, &c., to the place 4 days earlier. These differences suggested the notion that the diurnal wave travelled at a different rate from the semidiurnal wave. To ascertain this, a very admirable investigation was given by Mr. Whewell in the *Philosophical Transactions*, 1837, page 227, &c. The height of low water as well as high water on every day for 19 days having been observed, the effect of diurnal inequality at high waters separately and at low waters separately was found by the use of curves, as is described in (494.). Thus the effect of diurnal tide was ascertained at four times in each diurnal tide; these being used as ordinates, the curve drawn through them represented the diurnal wave; and the time of its maximum was easily found and compared with the time of semidiurnal high water. Thus from Ferrol, by the west coast of Ireland and north coast of Scotland to Lincolnshire, it precedes one of the semidiurnal tides about $3\frac{1}{2}$ hours, (with some irregularities); so also from Ferrol to Havre; but at Bridport the two tides are together, and at Lulworth and Portsmouth the diurnal tide is 4 or 5 hours later, after which it is traced with difficulty. It is to be remarked that in this paper Mr. Whewell considers the position of the moon, on which the diurnal tide depends, to be 4 days earlier than the tide (instead of 2 days for many of the places, as stated before). This seems to reconcile the phænomena much better, though it shows the great uncertainty of the conclusions from observation.

Progress of diurnal tide; it accompanies semidiurnal tide closely.

The solar diurnal tide has been omitted by investigators.

(564.) In the same volume, page 75, &c., Mr. Whewell has treated the diurnal tide at Plymouth and Singapore, and finds that the Plymouth tide corresponds to a position of the moon 4 days earlier, and that at Singapore to a position $1\frac{1}{2}$ day earlier. He remarks, however, that the agreement is best in May, June, and July, and that there are discrepancies in March, April, and August. We imagine that this arises from the circumstance that Mr. Whewell has nowhere taken account of the sun's part of the diurnal tide; and if this is sensible, (and the Liverpool observations discussed by Mr. Lubbock seem to show that in January and June it is not much inferior to the moon's part,) Mr. Whewell's determination of the corresponding position of the moon is worth little. We consider this subject as still open to investigation.

The difference of the age of the diurnal tide and the age of the semidiurnal tide theoretically explained.

(565.) That the tide should follow the sun's and moon's positions by four days is quite conceivable on the theory of canals. The investigation of (451.) and (452.), putting n for $2\,n$, applies to diurnal tide; and thus we find for the interval preceding the tide at which the moon's place is to be used,

$$\frac{f\,\mathrm{b}^2\,(n^2\,\mathrm{b}^2+4\,gk)}{(n^2\,\mathrm{b}^2-4\,gk)^2}.$$

Now this retard may be considerably greater than

$$\frac{f\,\mathrm{b}^2\,(4\,n\mathrm{b}^2+4\,gk)}{(4\,n^2\,\mathrm{b}^2-4\,gk)^2}$$

if the value of k be much nearer to $\dfrac{n^2\,\mathrm{b}^2}{4\,g}$

$$\text{than to } \frac{n^2\,\mathrm{b}^2}{g};$$

that is, if the depth of the sea be much nearer to $3\frac{1}{2}$ miles than to 14 miles.

Diurnal tide at Brest.

(566.) Laplace, in the fifth volume of the *Mécanique Céleste*, page 226, has determined the effect of diurnal tide at solstitial syzygies at Brest, to be about 7 inches at high water, (increasing one and diminishing another each by $3\frac{1}{2}$ inches,) and about 5 inches at low water. And he has found from this that the diurnal tide precedes one of the semidiurnal tides by 2 hours 17 minutes nearly, which, though not quite accordant with Mr. Whewell's determination, agrees perhaps as nearly as can be expected.

Cause of doubt in the results as to epoch of diurnal tide.

(567.) There is, however, one cause of doubt in these determinations, to which we must call the reader's attention. We have seen that, in examining the fluctuations of height of high water depending on semimenstrual fluctuations of force (537.), the effect is far less than seems properly to be expected from the cause, and in one instance at least it is smaller than at low water. It would seem probable therefore that all the high water fluctuations depending on diurnal tide ought to be multiplied in a certain degree, in order to be comparable with those of low water. This would vitiate the determination of the epoch of high diurnal tide as compared with high semidiurnal tide.

Diurnal tide insensible at London.

(568.) At London there is scarcely a sensible diurnal tide in the height of high water. This might arise from the maximum and minimum stages of diurnal tide occurring very nearly at low waters (semidiurnal), the mean stages occurring at high waters; and this is partially supported by the observation that Mr. Lubbock has discovered traces of diurnal inequality in the times of high water at London, (*Philosophical Transactions*, 1837, page 120,) not however, in our judgment, very distinct. But (as will be seen in the chart of cotidal lines) there appears to be no doubt that the tide at London is produced by the mixture of two tides, one coming from the English Channel, and the other, which has been 12 hours longer on its way, coming from the North Sea; and that while the semidiurnal fluctuations of these, being in the same phase, corroborate each other, the diurnal waves, being in opposite phases, (high diurnal tide of one corresponding to low diurnal tide of the other, &c.,) destroy each other.

Instances of very great diurnal tides.

(569.) The equilibrium-theory (47.), and Laplace's theory (101.), lead us to expect that the diurnal tide will be large in middle latitudes, and small near the equator and near the poles. The theory of canals (440.) gives the same result, supposing the canal to be a complete circle passing through the poles; if the canal is incomplete or interrupted, considerations similar to those of (296.), &c. apply, and no simple law like that of (440.) can be enunciated. The fact of observation is, that the diurnal tide is as large near the equator as in middle latitudes, at least in some very remarkable instances pointed out by Mr. Whewell, and which have been made the subject of very careful examination (although the existence of very large diurnal tide, as the general law in those parts of the earth, may be ascertained at once from Romme or any other good account of tides). Near the equator are Singapore, King George's Sound, in lat. 35° S. (*Philosophical Transactions*, 1837), Coringa Bay, Cochin, Surat Roads, Gogah, Bassadore (*Philosophical Transactions*, 1839), and others less carefully observed. In the northern seas are Petropaulofsk (Kamschatka), and Sitkhi (Norfolk Sound). In some of these localities the diurnal tide, when its range is greatest, greatly exceeds the semidiurnal. Mr. Whewell appears to imagine that the actual tides cannot be completely represented by the combination of a diurnal and a semidiurnal tide; but, so far as we can perceive, there is no difficulty at all in thus representing them.

(570.) We cannot here enter into a consideration of all the various effects produced by the mixture of these tides in various proportions. But, for a general idea, we invite the reader's attention to figures 63, 64, 65. Fig. 63 represents the course of a fortnight's tides at Plymouth; and here the diurnal inequality, though sufficiently marked, does not conspicuously disturb the semidiurnal tides. Fig. 64 represents the course of several days' tides at Singapore, and fig. 65 that at Petropaulofsk. It will easily be perceived that at Petropaulofsk there appears to be but one tide in a day, sometimes because one of the semidiurnal high waters is made so low as to be little perceptible, but more remarkably at other times, because one of the semidiurnal low waters is made so high as scarcely to disturb the appearance of a single tide.

(571.) We here close our remarks on the discussion of "the laws of the tides for varying positions of the sun and moon" as ascertained from observation. And we cannot do so without formally pointing out to the reader that absolutely the whole of these, as regards the tracing out the laws of the phænomena, and very nearly the whole as regards the determination of constants and coefficients, is due to Mr. Lubbock and Mr. Whewell. Yet while acknowledging that nearly all that we know is due to these philosophers, we cannot help expressing our wish that they had taken as guide a more complete theory than the miserable equilibrium-theory, and for this reason—we believe that so we should have known much more. Where observations are so rude and so numerous that they can be treated only in large groups, every thing depends on the assumption of the theory which is to direct the selection of the groups. We cannot, however, be too grateful for so much which has been done well, and which must suggest so clearly the critical points that will demand special attention in future discussions of observations.

Effect of the barometrical pressure on the tides.

(572.) A circumstance affecting the height of the tides, to which we have hitherto made no allusion, is the state of atmospheric pressure. It was first pointed out (we believe) by the French hydrographer, M. Daussy, (*Connaissance des Temps*, 1834,) that a low state of the barometer is accompanied with high tides. This has been carefully examined by the English investigators, and M. Daussy's result is fully supported. On this point we cannot do better than refer the reader to Mr. Lubbock's paper, *Philosophical Transactions*, 1837, page 97, &c. It appears there that in a set of observations considered by Mr. Lubbock, the unexplained fluctuations of the tide correspond precisely to those of the barometer; and Mr. Lubbock has laid it down as a rule, that a rise of 1 inch in the barometer causes a depression in the height of high water amounting to 7 inches at London, and to 11 inches at Liverpool. Mr. Bunt also, in the *Eleventh Report of the British Association*, page 31, has discussed with great skill the effect of the barometer on the tides at Bristol, and has shown that a rise of 1 inch of barometer produces a depression of 13·4 inches. He has also shown that the height of the tide, as depending on the barometer, is properly to be computed with the *contemporaneous* barometer. The explanation of this circumstance, by considering that a heavy atmosphere acts as a weight pressing down the water, is plausible enough; but as we cannot conceive air of different pressures in different parts of the earth, without supposing that there is a violent effort to restore equality of pressures, we cannot say that it is quite satisfactory.

Effect of the wind on the tides.

(573.) Connected with this is the effect of wind upon the tides (as distinguished from barometric pressure). Both M. Daussy and Mr. Lubbock, on comparing the differences between observed heights and computed heights with the directions of the wind, have come to the conclusion that the effect of the wind is insensible; see *Philosophical Transactions*, 1831. Nevertheless all practical men believe that the wind has a considerable effect, different in different localities; the rule for London is given with great clearness by Mr. Lubbock in the *Philosophical Transactions*, 1834, page 145. In fig. 41 we have shown the effect at Sheerness of a single gale (January 3, 1841), which lowered the tides in the Thames five feet, as we ascertained from examination of the registers at St. Katharine's Docks, the London Docks, Deptford, Woolwich, and Sheerness; and which produced a depression of about three feet at Hull and at Dover, and a sensible effect at Bristol. At Dublin and at Glasgow the tides were raised by it. But the want of regular observation at any great number of ports has made it impossible to trace the course of this great depression in all the seas round Britain.

Subsection 4.—On the Progress of the Tide over different parts of the Ocean.

(574.) Our remarks on this subject will consist principally of observations on figure 66, the chart of the world with cotidal lines marked on it. It is almost entirely copied from Mr. Whewell's chart in the *Philosophical Transactions*, 1833, with some modifications suggested by his papers in the *Philosophical Transactions*, 1835 and 1836. The Roman numerals upon the cotidal lines denote the hour, in Greenwich time, of high water on the day of new moon or full moon, as far as can be judged from the various considerations collected by Mr. Whewell, principally in the *Philosophical Transactions*, 1833.

The cotidal lines of the Northern Atlantic are probably very accurate.

The cotidal lines in other seas are very doubtful.

(575.) The reader's first inquiry, on looking at this chart, will probably be, what are the extent and value of the evidence upon which these lines are traced? In a matter like this, depending entirely on numerous details, we can only, as a complete answer, refer to the original discussions, but we may here state our opinion very generally. The tidal hours on the coasts of Britain, Holland, France, Spain, Portugal, and North America, are now well known, from the simultaneous observations made at a great number of stations, for which we are entirely indebted to the zeal of Mr. Whewell, and to the liberality of the various governments to whom his representations were addressed. The tidal hours at the islands of the Atlantic appear also to be well known. We conceive therefore (recognizing also the justness of the principles on which Mr. Whewell has generally drawn his curves) that the cotidal lines of the Northern Atlantic are now drawn with very great accuracy. But when we remark the importance of the alterations which those simultaneous observations have introduced in the lines on the coast of North America, (although the evidence, upon which the first draft of those lines was made, was at least as good as that upon

Sect. VIII. Comparison of Theory and Observations of Tides.

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Subsec. 4. On the Progress of the Tide over different parts of the Ocean.

which the lines in other parts of the world are drawn,) and when we see the discordance of accounts as to the time of high water at many places, we cannot imagine that they are drawn with great certainty in other parts. Even the African side of the South Atlantic is not free from serious doubts, the Indian Sea is still less known, and for the Pacific Ocean east of New Zealand we have scarcely grounds for conjecture.

Where the circumstances of depth, &c. change within a small distance, their effects may be treated as if the tides were free-tides.

Where the cotidal lines are crowded, we may expect large tides.

Instance in Bay of St. George.

(576.) Taking them, however, in those parts in which they are best known, we may find many circumstances which admit of direct and instructive comparison with theory. In all places where the circumstances of depth, &c., vary much in a small extent of sea, we may consider the alteration in the tides through that extent as following simply the laws of waves on which no force is acting (because the length of the column of water on which the Sun or Moon acts is too small to allow their attraction sensibly to modify the pressures). Suppose now that in the neighbourhood of any particular coast the bottom shelves gradually from deep sea to one comparatively shallow. This would be attended, theoretically, with two consequences. The first is, that the wave would travel more slowly (174.), and therefore the separation of the cotidal lines corresponding to successive hours would be less, or the cotidal lines would appear to be crowded together on the map. The second is, that the magnitude of the tides would be much increased (247.), &c. And these circumstances might be found in places where the change in the depth was not known from observation; for the usual limit of sounding is 200 fathoms, which is probably a small quantity compared with the depth of the ocean. We may then expect that, where the cotidal lines approach closely, the magnitude of the tides will be increased. Now this does occur. A well-marked instance is the Bay of St. George in South America, in which a close approximation of cotidal lines is accompanied with large tides. It is possible here that the tides may be still further increased by the converging form of the waves (254.).

Cotidal lines are distorted by the shoals round islands.

Instances in Azores, Bermudas, New Zealand.

Dragging of tide-wave along the shore.

(577.) Another curious effect of the same cause is the distortion of the lines produced by islands, surrounded by shoals, in the ocean. The shoals prevent the tide-wave from advancing rapidly, and the cotidal line is therefore thrown back; but, conceiving the ridge of the wave to be thus bent, it is easy to imagine that after passing the island the two lateral parts of the wave will bend round it till they unite, and will then form a straight front nearly as before coming to the island. The successive cotidal lines will have forms corresponding to the forms of the ridge of this wave at successive times. Of this there are several instances apparently beyond doubt. Thus the 1 o'clock line is thrown back by the Azores; the 11 o'clock line is bent by the Bermudas, and its lateral branches nearly meet; the 10 o'clock line, after having been interrupted, just meets behind New Zealand. A similar effect of the same cause is, the universal dragging of the wave along the shore, as we have mentioned in (359.).

Depth of the sea inferred from the progress of the tide-wave.

(578.) The velocity of the tide-wave ought, with the assistance of the table in article (174.), to give us good information as to the depth of the sea. Thus in the North Sea, figure 60, the tide-wave in 9 hours appears to describe somewhat less than 6 degrees of latitude, or, on the average, about 45 miles per hour. This, by the table in (174.), corresponds to a depth of 140 feet. We believe that the average depth along the line of deep channel is greater than this, and that at the sides less; and it is probable that the actual velocity is affected by both these. If the tide-wave of the Atlantic were purely derivative, it might be considered as describing 90 degrees of latitude, from the southern 1 o'clock line to the northern 1 o'clock line, in 12 hours, or to move about 520 miles per hour, which would imply a depth of about 18,000 feet or 3½ miles. The reader will have no difficulty in extending similar remarks to other seas.

These remarks do not apply to large seas.

The tides of the Atlantic cannot be treated as derivative tides.

(579.) But these remarks apply *solely* on the supposition that the seas which are the subject of remark are so small, that the action of the Sun or Moon on a column of water as long as the sea will produce a pressure which is insignificant in proportion to the height of the tides or the changes in the heights. And this consideration, of which the justice is obvious, does, in our opinion, make it impossible to receive one of Mr. Whewell's speculations as to the tides of the Atlantic. The general course of the tide-wave there being from S. to N., or rather from SSE. to NNW., Mr. Whewell has inferred from this that the tides of the Atlantic are mainly of a derivative character, produced by the tides of the Southern Ocean and transmitted up the Atlantic in the same manner in which the tides of the Atlantic are transmitted up the English channel. We doubt this entirely. A channel as large as the Atlantic, if it ran due north, would have large tide-waves of a stationary character (440.) and (443.); but being in some degree inclined, the tide-waves would travel towards the NW. (439.) and (441.), but with a very irregular velocity, not dependent on the depth of the water (supposed uniform.) And even if the waves were stopped by a barrier (such as we may conceive the coast of North America to be, omitting the comparatively narrow channel between Britain and Greenland), yet the investigation of (338.) shows that the effect of friction is to give nearly the same formula for the tides in mid-sea as if no barrier existed. Indeed we may remark as the general effect of friction, that the tides at a distance from shore, or at a distance from a communicating channel, will depend more, so far as regards their form and succession, on the acting external forces than on reflection from barriers or communication from other seas, their magnitudes, however, being diminished and their epochs being altered. Without, therefore, denying entirely the effect of the Southern Sea, we conceive it to be extremely small.

Difficulty of explaining theoretically the transmission of tide-wave in the Atlantic.

(580.) But, whatever supposition we make as to effects of friction, &c., the power of mathematics fails totally in the attempt to express the transmission of the tide-wave or waves through the Atlantic Ocean. It has some analogy to a canal, (an analogy which, though distant, gives us confidence in applying some of the results deduced from canal investigations, as for instance the effect of the depth on the magnitude of the tide, and the effect of friction in producing the age of the tide.) But it has not so much as to enable us to predict what will become of the tide-wave in its general progress; and of course the analogy fails totally as regards the effect of those very important features which have no existence in a canal; huge promontories projecting into its sides, or vast bays opening large lateral expanses for the spread of the tide-wave.

(581.) There is moreover another consideration which must not be left out of sight in estimating the difficulty of reconciling our observations of the tides of an ocean with such a theory as we are able to give even for the simplest cases. It is that, supposing the cotidal lines to be accurately what they profess to be, namely, the lines connecting all the points at which high water is simultaneous, we have nevertheless, in (366.), &c., given reasons for supposing that they may not at all represent the ridges of the tide-waves that run over the ocean. Thus, an eye at a great distance, capable of observing the swells of the tide-waves, might see one huge longitudinal ridge extending from the mouth of the Amazons to the sea beyond Iceland, making high water at one time from Cape de Verde to the North Cape, and at another time from Florida to Greenland; and another ridge, transversal to the former, travelling from the coast of Guiana to the northern sea; and the cotidal lines which we have traced may depend simply on the combination of these waves. It does not appear likely that we can ever ascertain whether it is so or not; but it is certainly possible that the original waves may have these or similar forms; and if so, it is vain for us to attempt entirely to explain the tides of the Atlantic.

The cotidal lines may not represent the ridges of the tide-waves.

The interference of wave-ridges may explain the cotidal lines west of America.

(582.) To the Pacific Ocean the same remark applies still more strongly. There is however one part in which we seem to discover, from the considerations mentioned above, some explanation of the singular form of the curves as traced by Mr. Whewell; namely, the western coast of South and Central America. It seems not unlikely that the east-and-west action of the tidal forces, acting as if on the water in canals stretching east-and-west, would produce waves running east-and-west, (the length of their ridges being north-and-south,) according to the laws of (296.) and (337.). It seems probable also that the north-and-south action of the tidal forces would produce another wave, either running north or south, or of stationary character. The effect of the composition of these, as in (366.), &c., would be to produce cotidal lines, having considerable resemblance to those on the American shore of the Pacific.

Difficulty of explaining the diurnal tides in the Pacific.

(583.) If we look to the Pacific Ocean with reference to diurnal tides, and consider the southern sea as a part of it, we seem to have a case which possesses considerable analogy with Laplace's assumption. Yet it is remarkable that here we appear to find a more complete failure than anywhere else, of Laplace's celebrated result as to the non-existence of diurnal tide, (all the large diurnal tides being in that sea.) It is true that this failure might be explained by supposing the depth of the sea to be extremely unequal.

(584.) Upon the whole, therefore, we are driven to the conclusion, that we cannot at all explain the cause of the form of the cotidal lines in the ocean, so far as they have been traced with any probability. And, supposing us to know with tolerable certainty those corresponding to the semidiurnal tide, we cannot at all predict those which should hold for the diurnal tide.

Laws which nevertheless will apply everywhere.

(585.) Nevertheless we are able to lay down one law of vast theoretical importance, (or rather two laws, one applying to the group of semidiurnal tides and the other to the group of diurnal tides.) It is simply that assumed by Laplace, and which also applies to, or includes the effect of, our theories of "the relation of the magnitude of the tide to the depth of the water" and "friction." Whatever be the number of communications from other seas, or reflections from barriers, whatever the depth of the water or the amount of friction, still the result of a given periodical force will be a periodical effect whose period is the same, but whose coefficient is represented by the coefficient of the force multiplied by a function of the period, and whose argument also contains a term which is a function of the period. This law completely brings under our management all terms depending on the slow variation of forces (as those which depend on ellipticity, declination, &c.) For $(1+3e\cos ht)\cos nt$ is

$$=\cos nt+\frac{3e}{2}\cos(n+h)\,t+\frac{3e}{2}\cos(n-h)\,t,$$

and, h being small, either of the functions of n mentioned above, and which we will call N, is changed for the additional terms to $\mathrm{N}+\frac{d\mathrm{N}}{dn}h$ and $\mathrm{N}-\frac{d\mathrm{N}}{dn}h$; and, $\frac{d\mathrm{N}}{dn}$ being discovered in any one case, (as, for instance, when Laplace inferred it from the observed declination-correction,) is applicable to the other cases, (as, for instance, when Laplace applied the value so found to correct the first value of the mass of the Moon.) And this holds for any port under any circumstances whatever; it even holds, so far as we can see, for the heights of tide as distinguished from the time of tide; and from these two classes of phænomena (supposing observation to be perfect) the correct mass of the Moon ought equally to be inferred, though the laws of the two classes are, in the first inspection, discordant. But this law does not in the same manner bring under our management the rules of diurnal tides to be inferred from semidiurnal, or *vice versâ*; the difference between n and $2n$ being too great to permit us to confine ourselves to the two first terms of Taylor's theorem. A similar law however would connect the diurnal tide with the slow variations of diurnal tide and the alteration to be made in the Moon's mass as applying to diurnal tide; and thus, theoretically speaking, the Moon's mass might be inferred from the parallax-corrections of diurnal tides in the same manner in which Laplace has found it from declination-corrections of semidiurnal tide.

One law for the semidiurnal tides, and another for the diurnal tides.

CONCLUSION.—ON THE PRESENT DESIDERATA IN THE THEORY AND OBSERVATIONS OF TIDES.

(586.) If we advert to the different parts of this extensive subject in the same order in which we have treated them in the preceding Essay, we shall find the following to be the most important points requiring attention.

Extension of Laplace's theory desirable, to take in the effects of limits.

(587.) A subject of very great importance is, the more general solution of Laplace's equation in (96.), to the extent mentioned in (119.), so as to enable us to apply the general theory to seas limited by shore-boundaries. A great point would be gained if this could be done on any assumption whatever as to the depth of the sea.

To take in the effects of difference of angular velocities.

(588.) The value of Laplace's theory is lost in a great measure, because the solutions hitherto attempted apply only to those cases in which $i=n$ exactly (100.), or $i=2n$ exactly (107.). Thus the difference in the specific action of the Sun and Moon, depending on the difference of their angular velocities, is entirely lost. It would be highly important to obtain solutions on the supposition, that i differs from n or from $2n$ by a small quantity whose square may be neglected.

Conclusion. Present Desiderata in the Theory and Observations of Tides.

(589.) Laplace's theory would be much more valuable if it were extended so far as to include the effects of friction. The methods of (315.), &c., would probably apply with sufficient accuracy.

(590.) The additions to the theory, indicated in the three last articles, would be exceedingly valuable, even if quite independent. If they could be treated in combination, their value would be very greatly increased.

Extension of the theory of river-tides desirable.

(591.) In the theory of waves, the most important point by far is the theory of river-tides, in which the extent of vertical oscillation bears a sensible proportion to the depth; which we have partially treated in (192.), &c. The following extensions would very much increase its value; it is apprehended that they would introduce more of labour than of difficulty.

To include the variable form of section of channel.

(592.) The investigations of (218.) and (260.) must be extended so as to include the terms depending on the vertical oscillation, at least to the third order; observations showing that those terms are not only sensible but important. This investigation, if properly conducted, will include the extension of the investigation of (309.).

To include friction.

(593.) The effect of friction must be introduced in combination with these investigations.

Extension required for the cases where the tide is very great.

(594.) The investigation must, if possible, be effected for the case where the vertical oscillation is very great; as, for instance, where the low water leaves very small depth on the bottom. In this case it is hopeless to attempt a converging series, and an independent and finite method must be tried. Much would be gained if this could be effected in the simplest case, as for a rectangular channel, of uniform section, without friction.

Simultaneous observations on various points of rivers and estuaries desirable.

(595.) In regard to observations, it is very desirable that simultaneous observations at short intervals should be made on different points of some long tidal river, or some deep estuary, for examination of the change of the wave. These observations should be discussed as is mentioned in (479.). It is particularly desirable that these observations should be made in very high spring tides and in very low neap tides, to discover the laws of alteration of the various constants as depending on the range of the tide.

Observations near the Isle of Wight desirable.

(596.) As a special locality, we may point out the various channels near the Isle of Wight as most particularly requiring attention. It would be very useful that simultaneous observations of a few tides should be made at two or more points on Southampton water, two or more on the Solent, one or two on the eastern side, as at and beyond Portsmouth, and one or two on the west side of Hurst Point.

(597.) In regard to the reduction of long series of tide-observations as applicable to particular ports, we shall only call the reader's attention to the following points.

Methods to be adopted in reducing long series of tide-observations.

(598.) It will probably be found, from the inquiries, theoretical and experimental, to which we have alluded above, that the elevation of high water in rivers or bays requires a certain multiplier to make its fluctuation of range comparable proportionably with the fluctuation of range on the coast, and that the depression of low water in like manner requires a multiplier different from the former. Much confusion would be removed by ascertaining these multipliers and applying them at once to the observations.

Correction of time, depending on range of tide.

(599.) It will probably also be found that the time of high water requires a correction depending on the whole vertical range, to make it comparable with that of the sea; and that the time of low water requires a different correction. These should be ascertained, if possible, and applied. It would, perhaps, be best to assume that such a correction is needed, and to determine its quantity from the observations themselves in such a manner that the epochs of highest tides and mean lunitidal intervals shall synchronize. And in like manner, for the corrections to the heights, it might be best to determine the factors, so that the mean height shall be uniform, and that the first proportion of the Moon's mass to the Sun's, inferred from the semimenstrual inequality of heights, shall be the same as that given by the semimenstrual inequality of times.

Correction of height.

Methods for diurnal tide.

(600.) In the places where the diurnal tide, though sensible, is small, its effect in height at the time of high or low water may be considered independent of its effect on the time of the high or low water, and *vice versâ*. The best way of disengaging it, numerically, would probably be, to calculate a small approximate table of second differences of the heights, and, subtracting from each observation of height the mean of the preceding and following heights, to apply that computed second difference. But where the diurnal tide is very large, the effect on height is not independent of the effect on time. In this case we see no method so clear and easy as to calculate beforehand a few tables of the values of $\cos\theta + a \cos(2\theta + b)$ with different values of a and b; the result will enable the experimenter to judge how much the real epoch of high semidiurnal tide differs from the time of highest water, and what is the real epoch of diurnal tide. The same will apply to the times.

Methods for general inequalities.

(601.) The whole of the inequalities should then be treated with reference to the theory of (451.), and so as to include the terms pointed out in (550.) and (554.). For these it will not be sufficient to class together all observations at which the declination was the same; before this is done, the whole must be divided into two categories, namely, those of declinations increasing and of declinations diminishing, which are to be afterwards subdivided by absolute declinations. In like manner, as regards parallax, all the observations must be divided into the two categories of parallax increasing and parallax diminishing; which are to be afterwards sub divided by absolute parallax. The whole of these elements are to be taken for an epoch anterior by a quantity equal to the age of the tide. From the discussion of these inequalities in time as well as in height, the mass of the Moon is to be inferred by the process sketched in (555.); and the agreement of the different values of the mass will be the proof of agreement of theory and observation.

(602.) The best method of starting in these reductions cannot be the subject of general rule; the age of the tide however should be determined as early as possible. When the lunar parallax correction is ascertained, that part of it which applies uniformly in the same age of the Moon (depending on variation) should be subtracted from all the observations, or rather from the means of the groups, and then only can the semimenstrual inequality be found exactly. Each inequality, when determined, should be subtracted from the observations before investigating a new one.

(603.) The same methods should be used for diurnal inequality, where it is conspicuous. But, as north declinations and south declinations are not now to be confounded, it will be advantageous to express the places of the Sun and Moon by north-polar-distances instead of declinations.

Points to be noticed in the tide-observations in the Pacific Ocean.

(604.) In regard to the tracing of cotidal lines, the principal defect, in marine localities which otherwise are well known, is in the Pacific Ocean generally. The attention of those who are interested in defining these lines should be particularly directed to the devising of means for rendering the tide even coarsely sensible, in places where its range is small, where it is partly masked by day-breezes and night-breezes, and where the water is held in a state somewhat different from that of the open sea by the rings of coral reef which surround so many of the islands.

Conclusion.

Reference to the simplest form of tide-function, for drawing cotidal lines.

(605.) In some smaller seas scrupulous attention should also be given to the distinction between the actual time of high water and the time when the simple sine expressing the sea-tide reaches its maximum. It is not unlikely that in this way some part of an apparent anomaly which Mr. Whewell has remarked to the east of the Isle of Wight may be removed.

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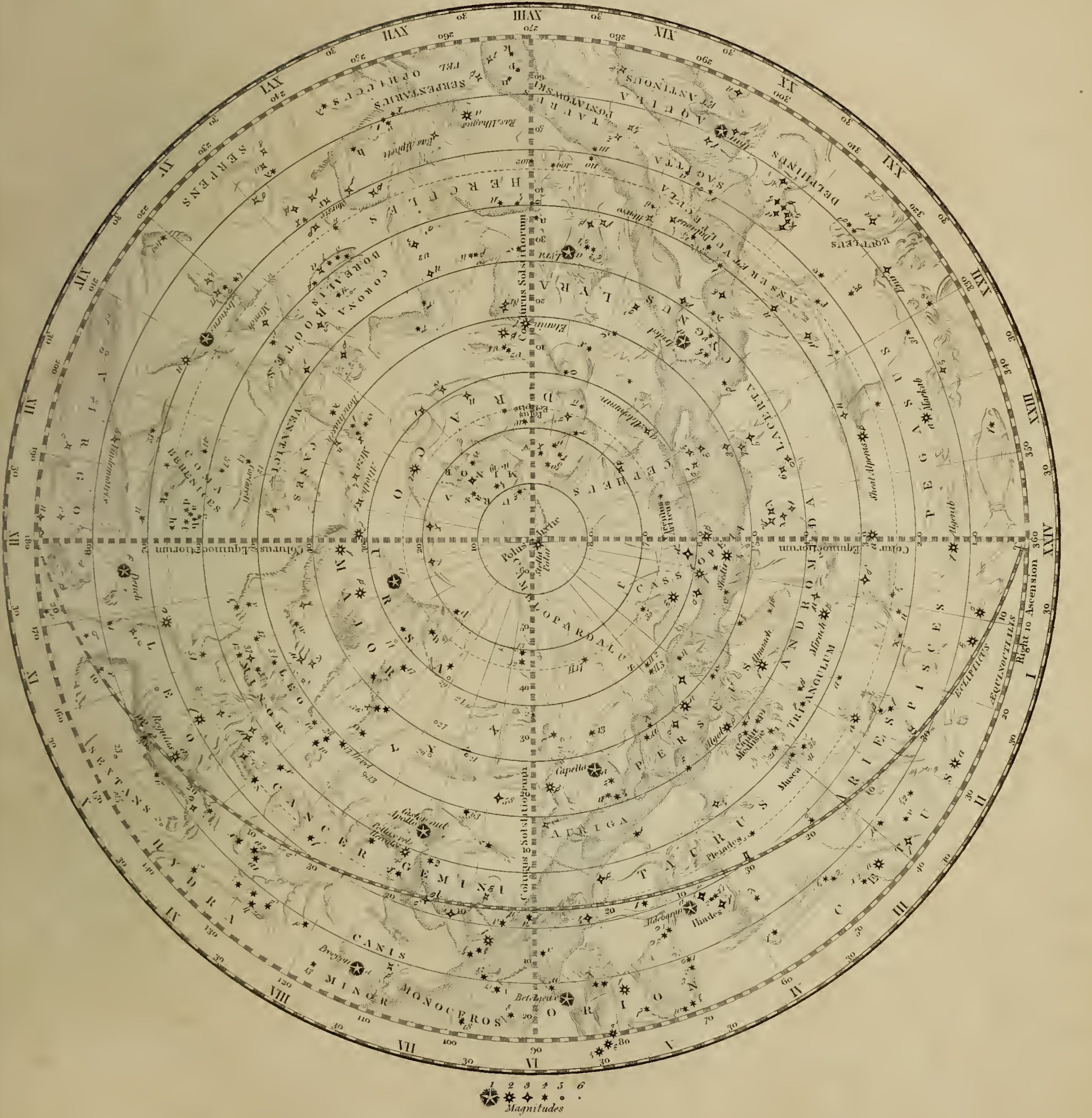

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Southern Hemisphere.

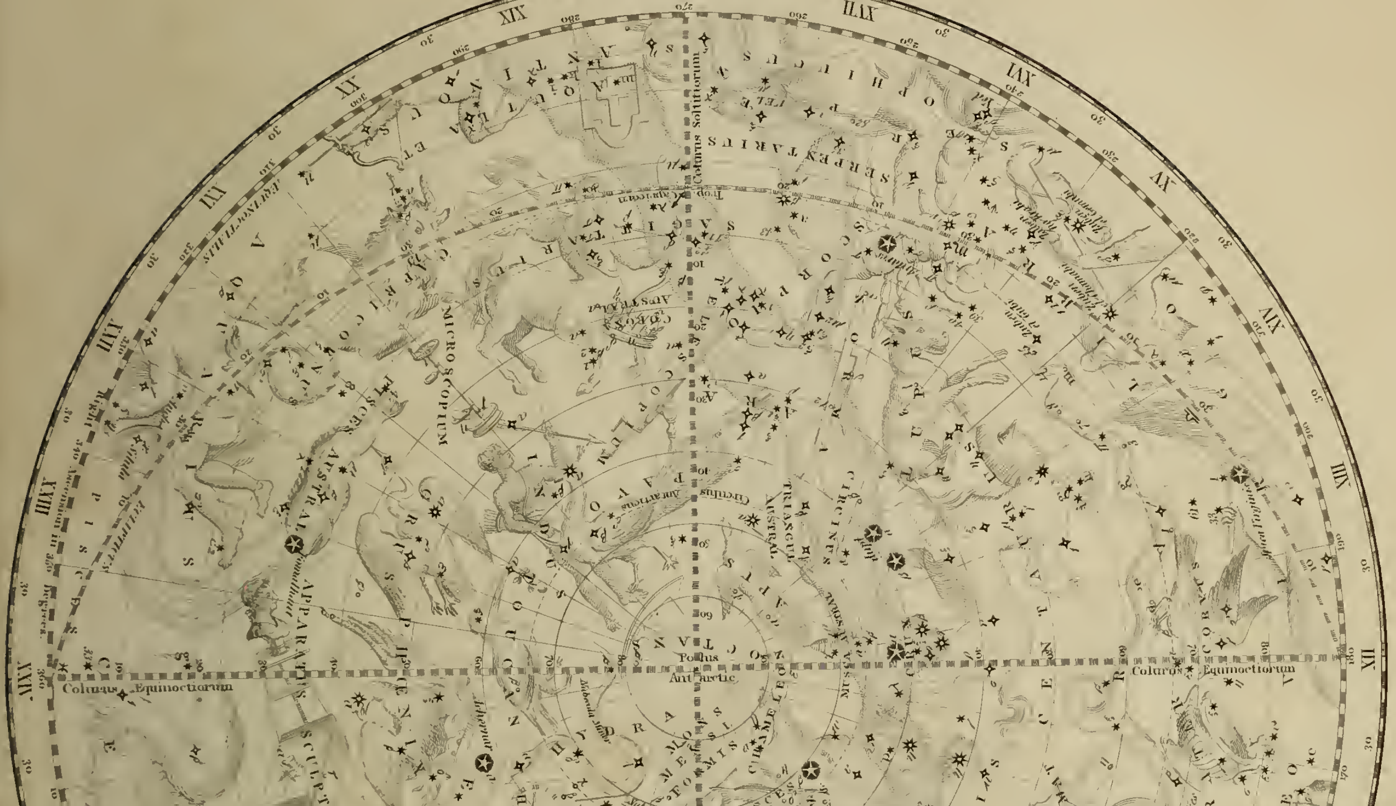

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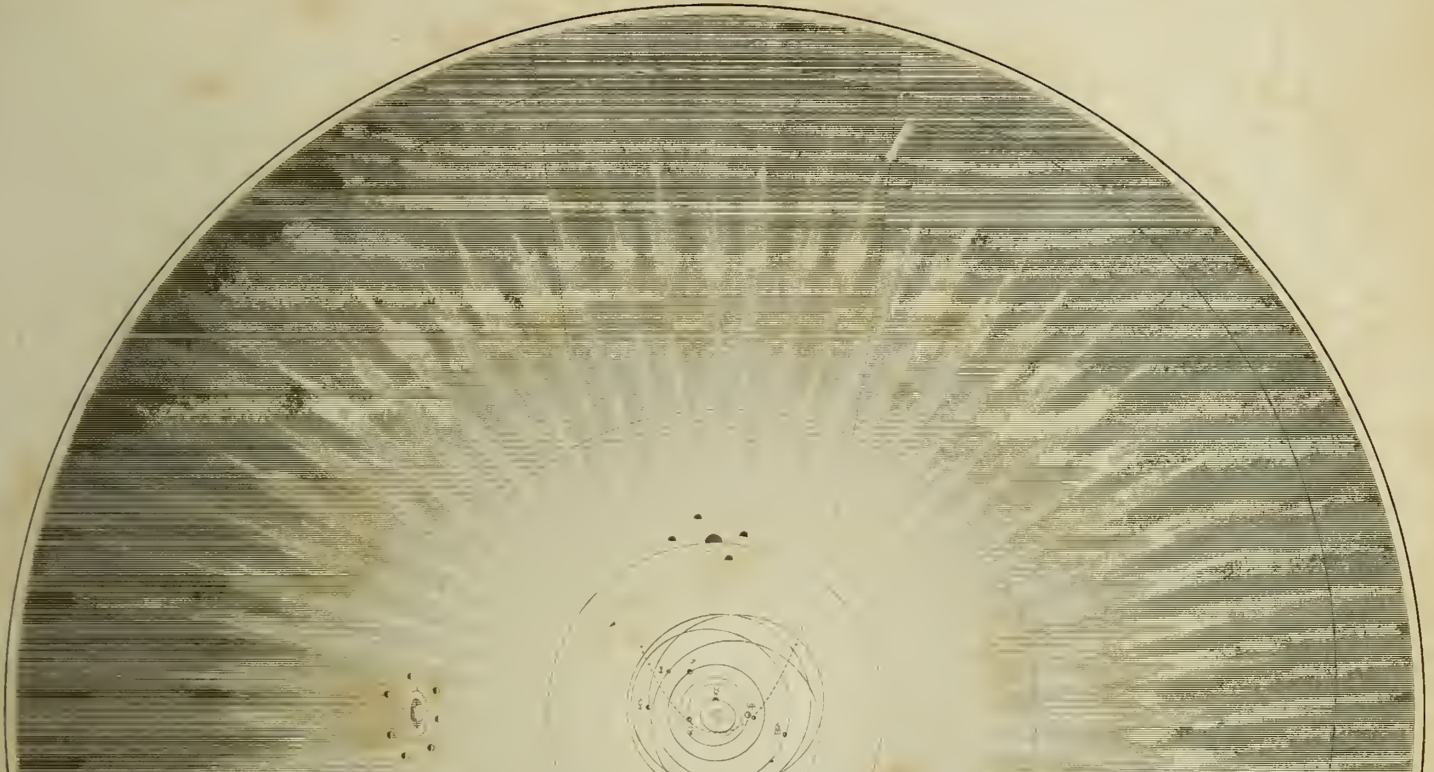
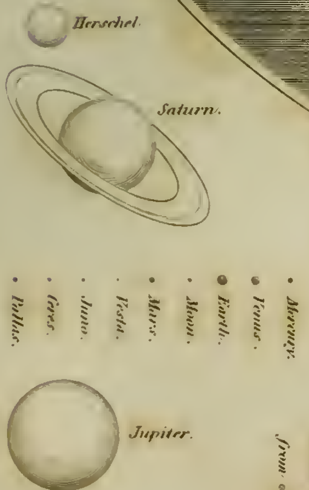
Plate 3.

The Solar System.



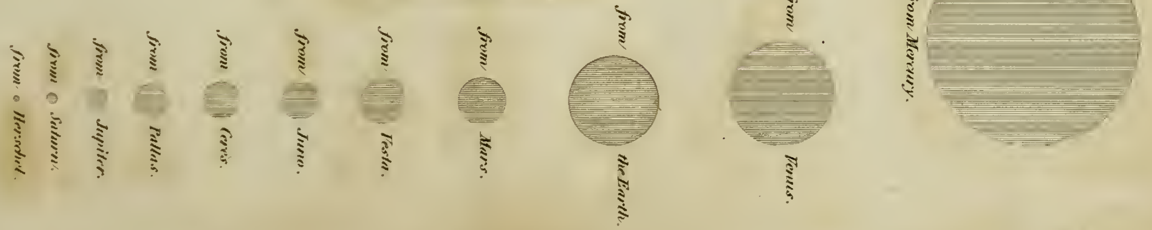
Comparative Magnitudes of the Planets.



Signs of the Planets.

- ☿ Mercury.
- ♀ Venus.
- ☽ ⊕ Earth & Moon.
- ♂ Mars.
- ⚶ Vesta.
- ⚵ Juno.
- ⚳ Ceres.
- ⚴ Pallas.
- ♃ Jupiter.
- ♄ Saturn.
- ♅ Herschel.

Comparative Magnitude of the Sun seen from each Planet.



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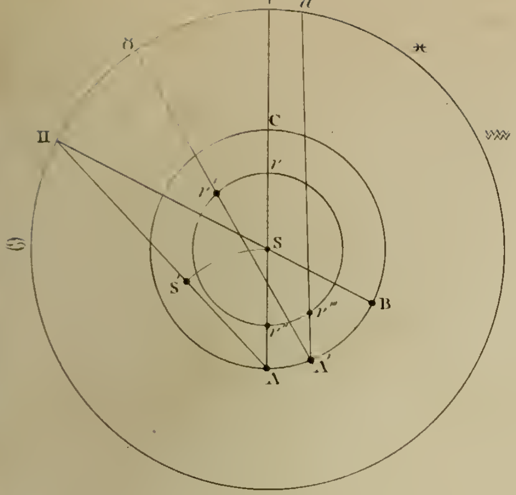

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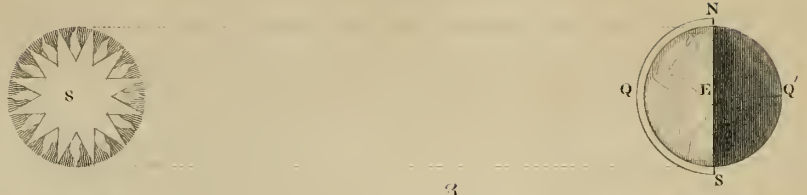

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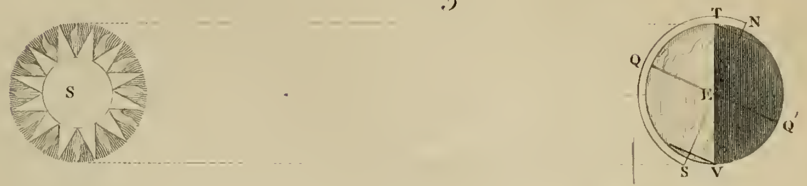

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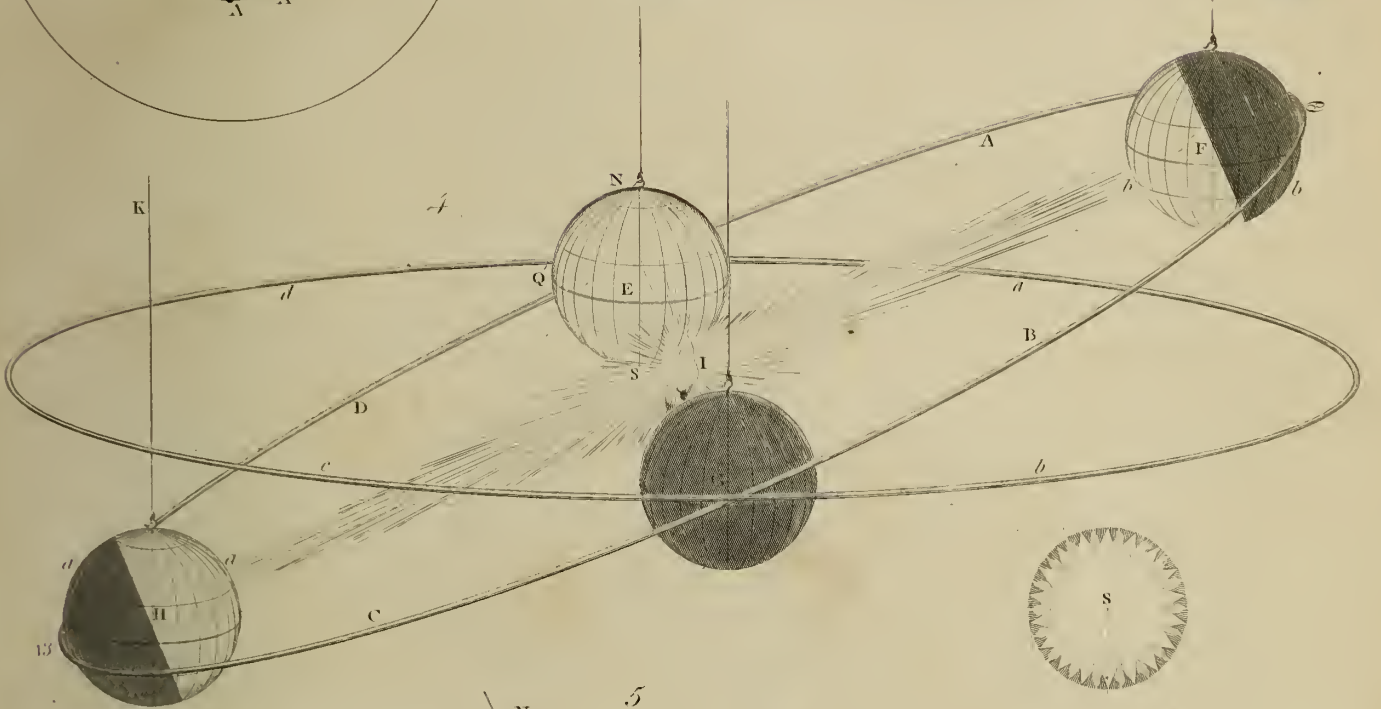

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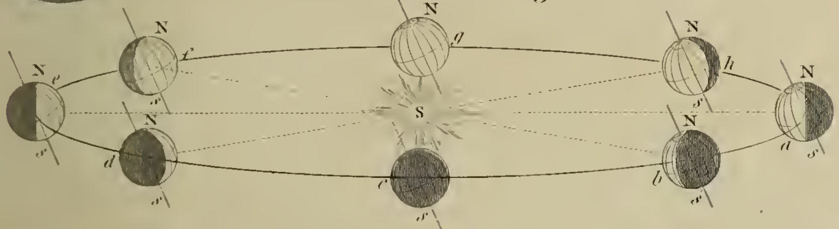

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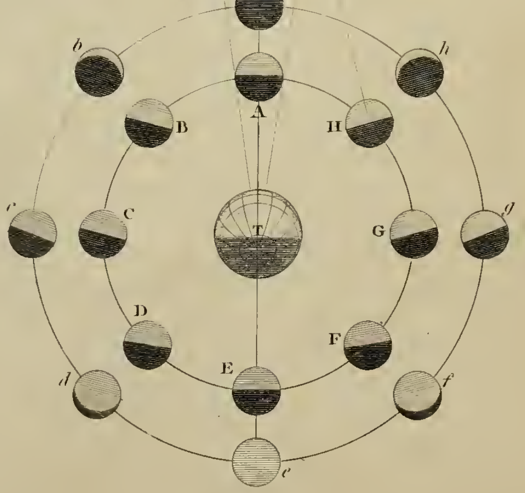

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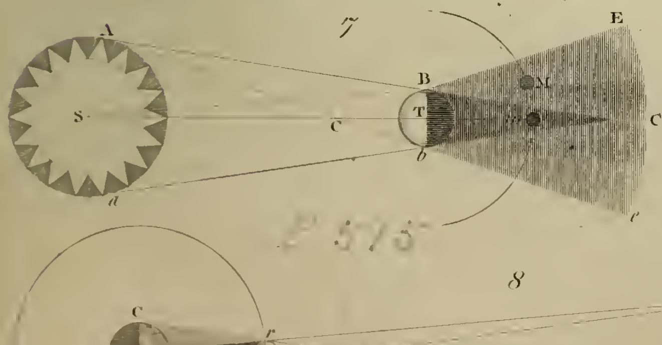

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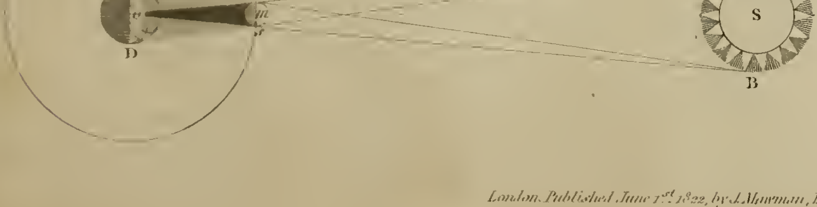

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Planetarium

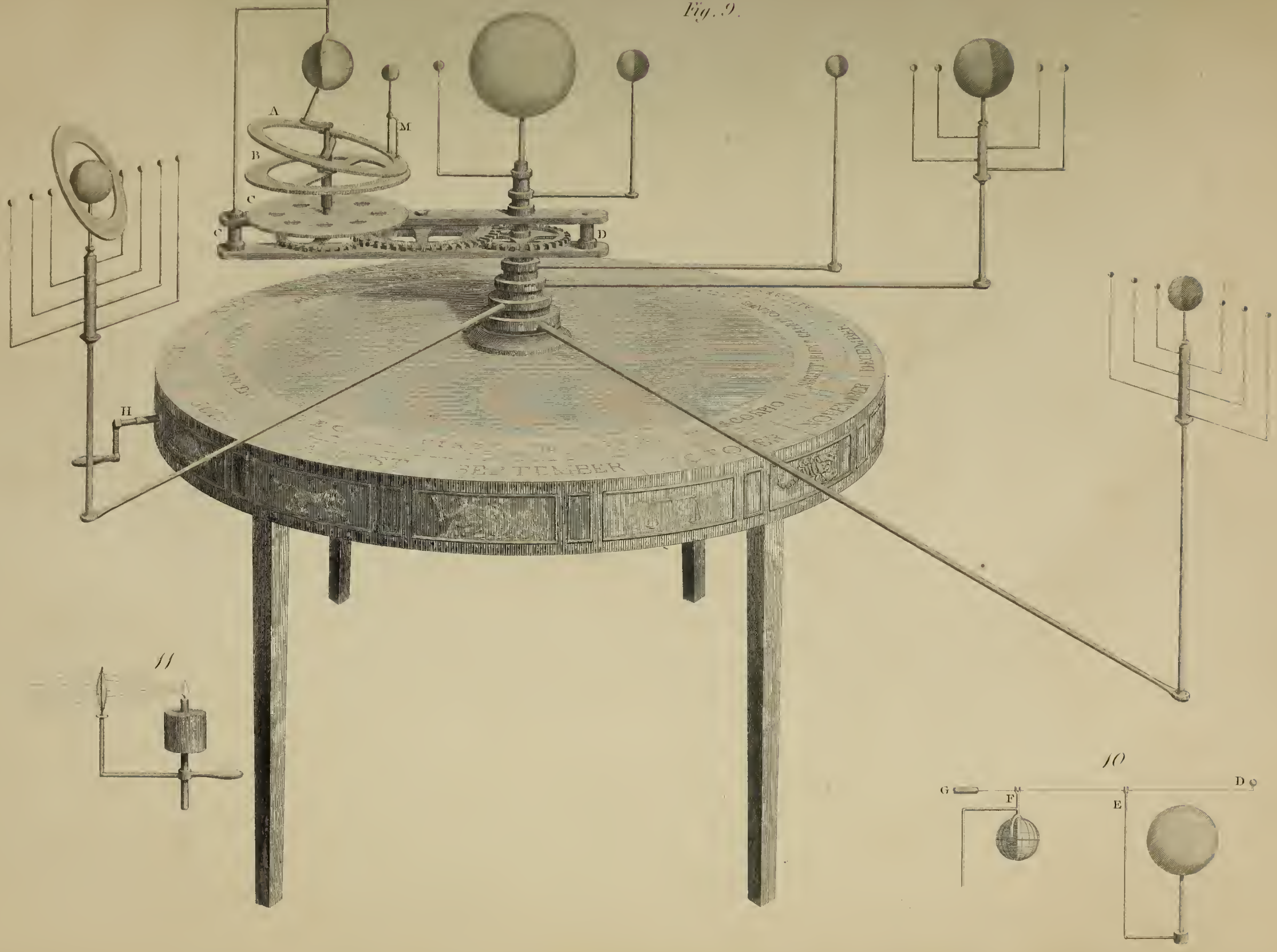
Fig. 9.

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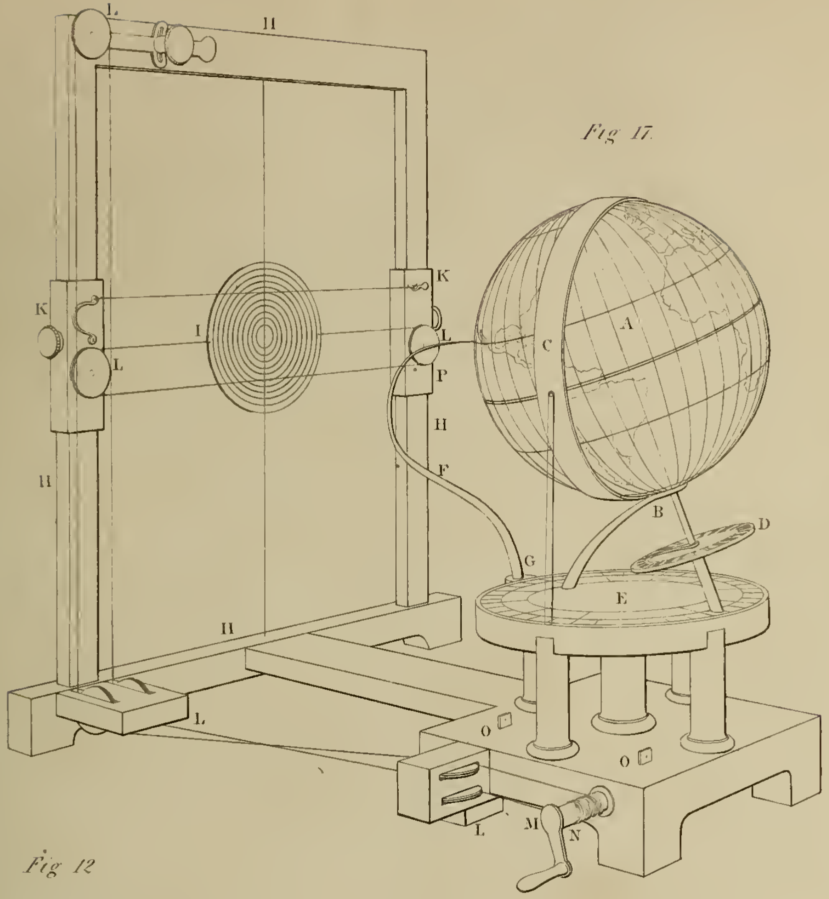
Fig. 16.

Fig. 17.



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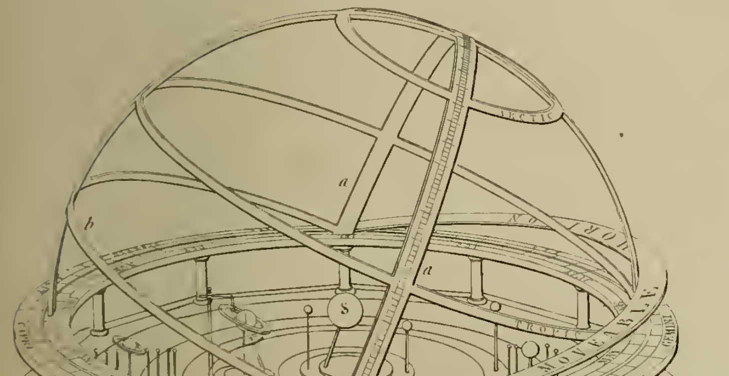

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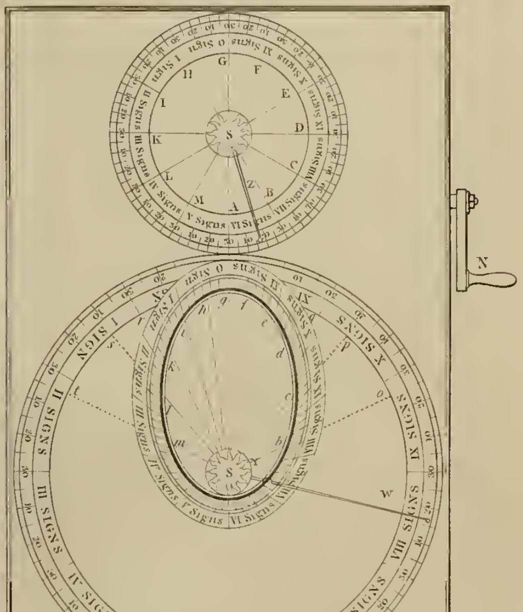

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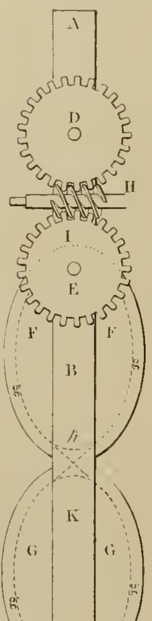

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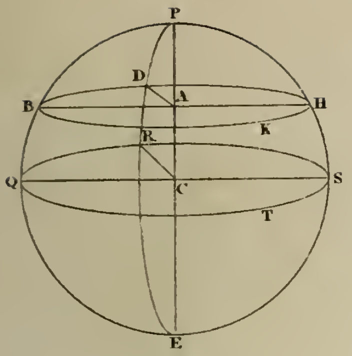

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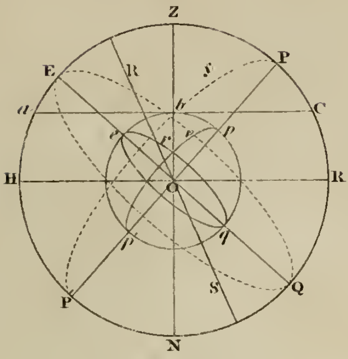

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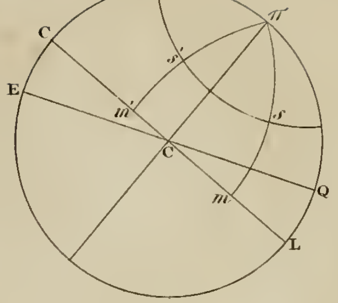

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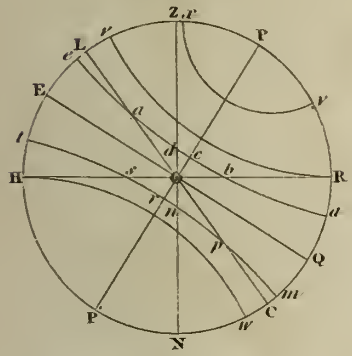

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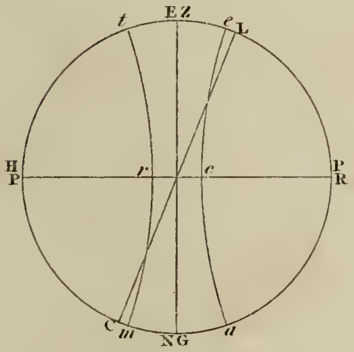

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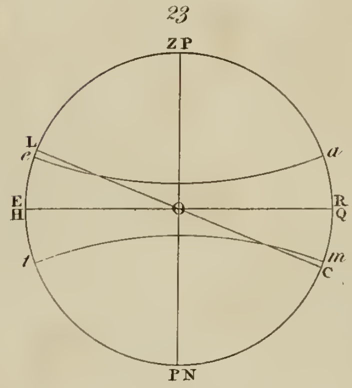

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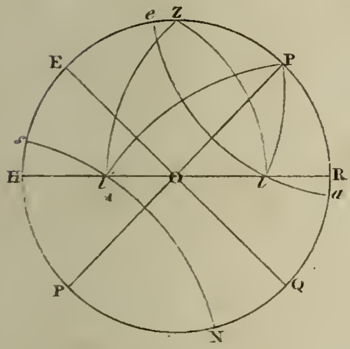

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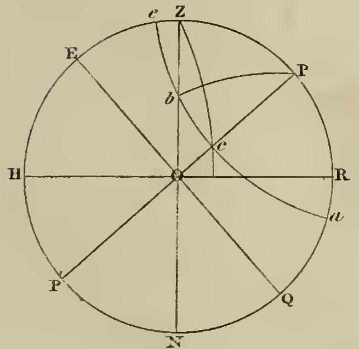

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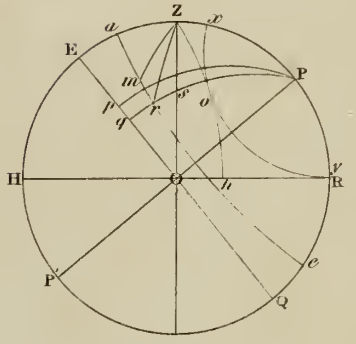

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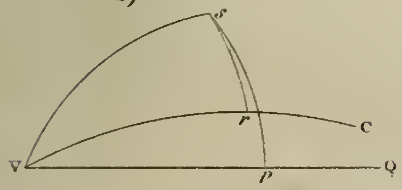

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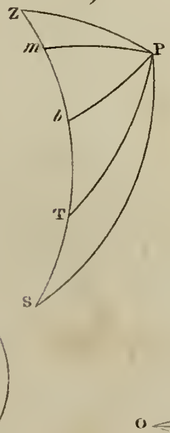

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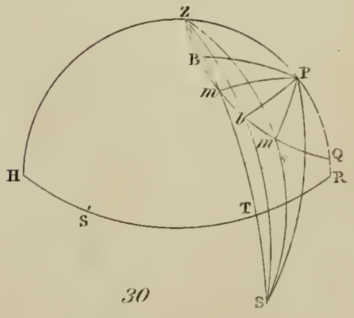

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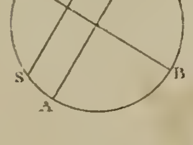

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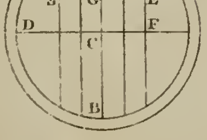

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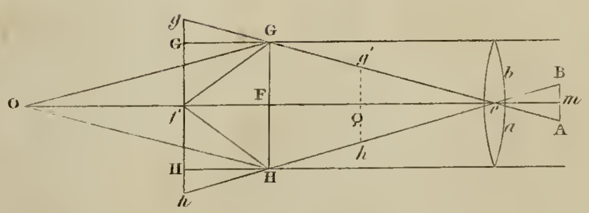

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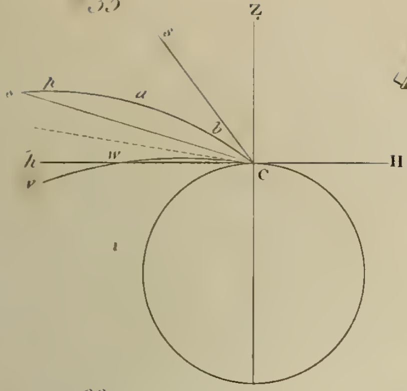

Transit Instrument

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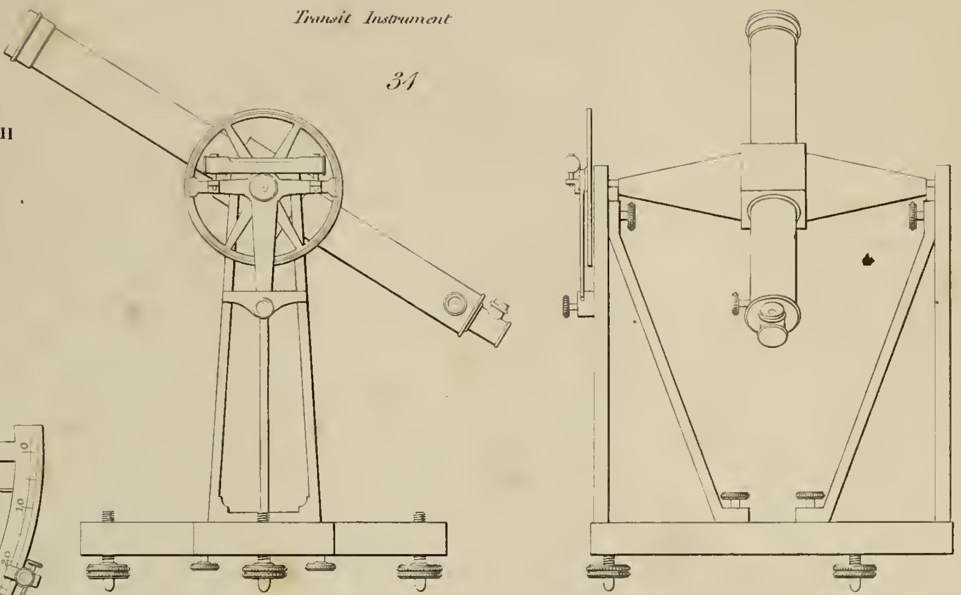

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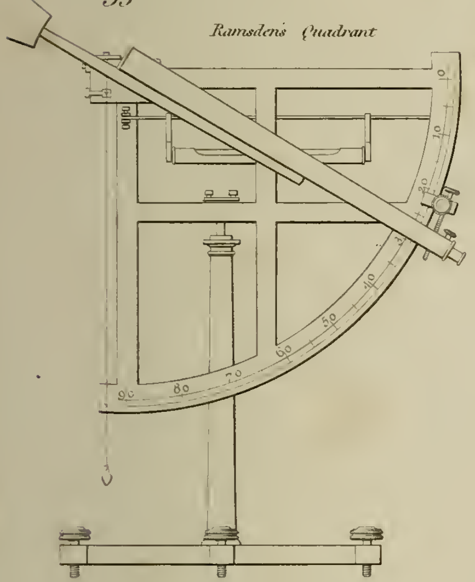
Ramsden's Quadrant



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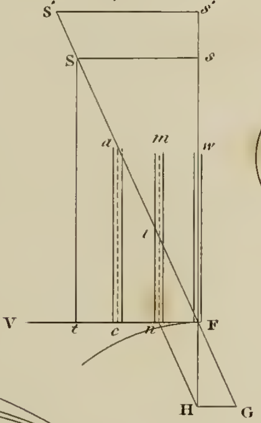

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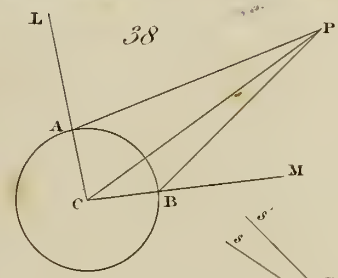

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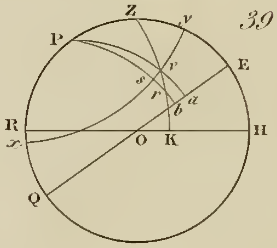

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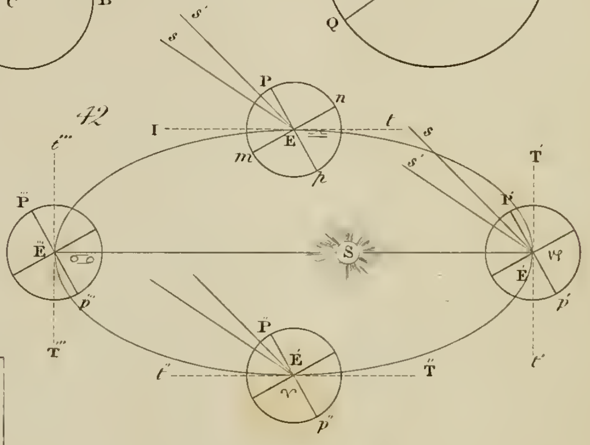

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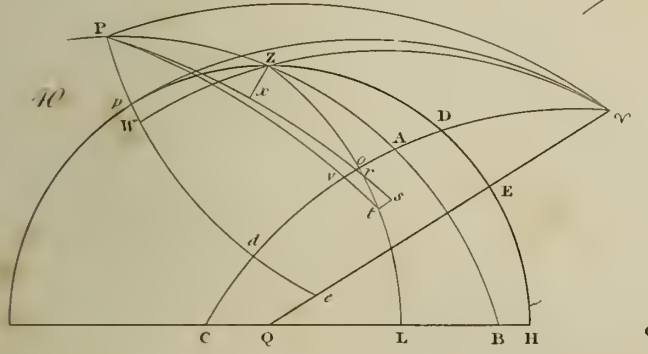

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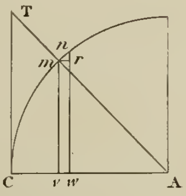

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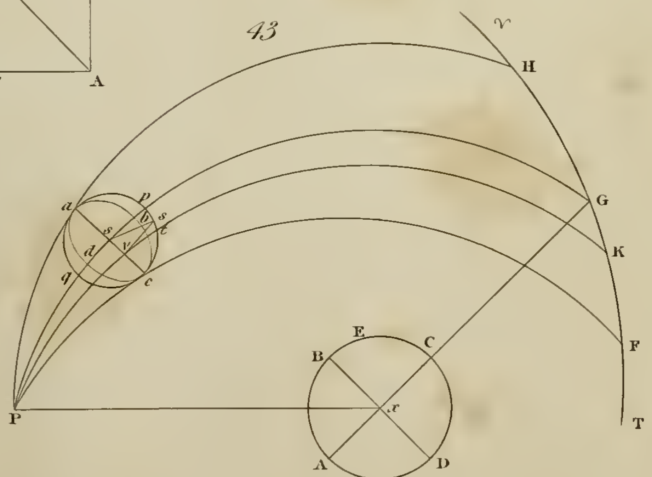

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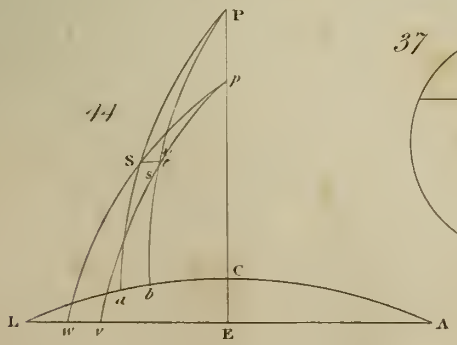

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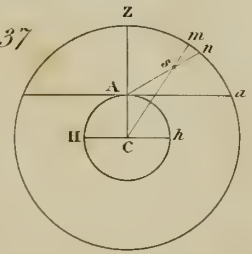

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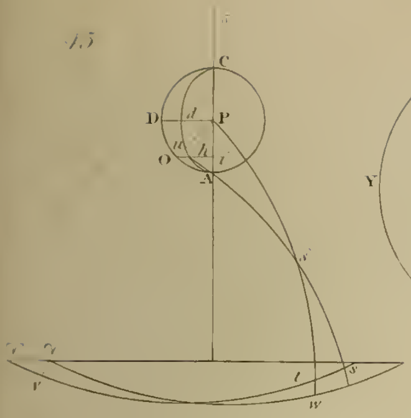

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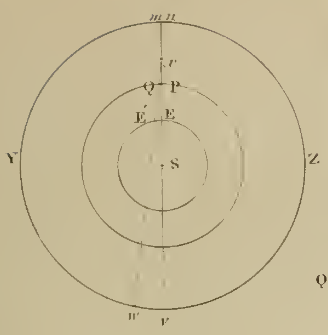

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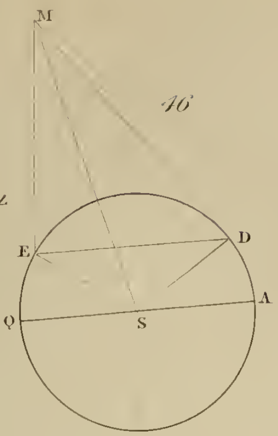

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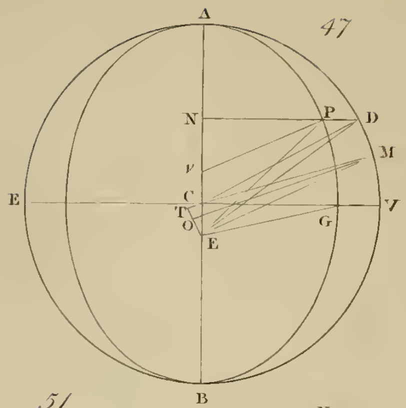

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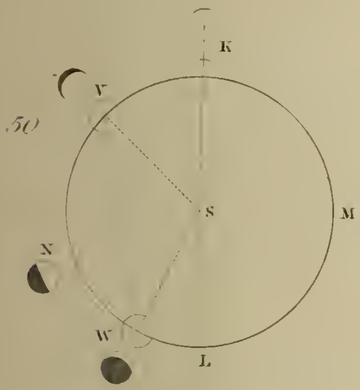

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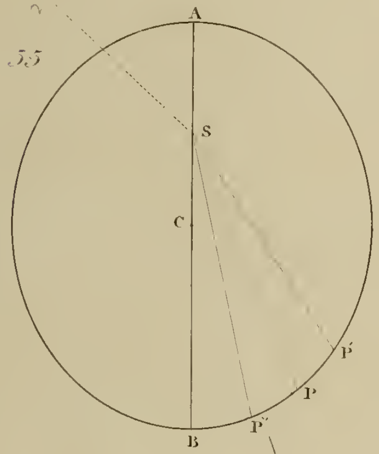

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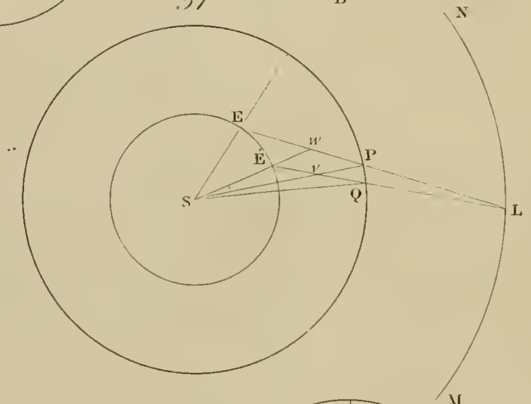

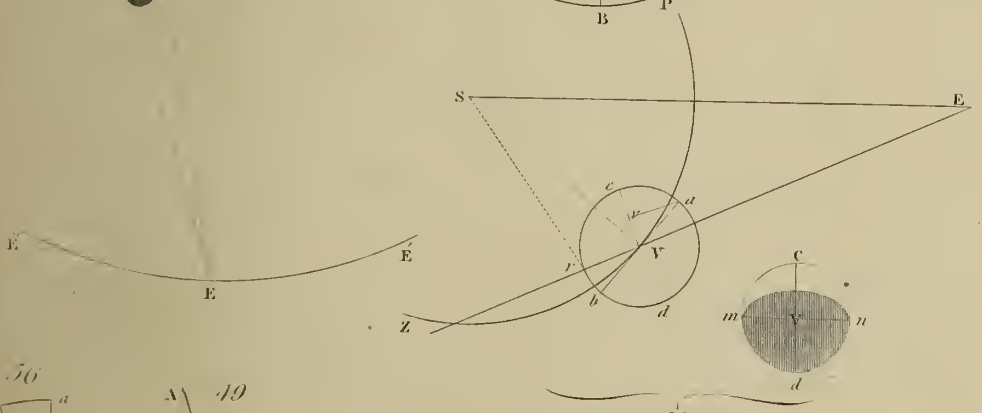

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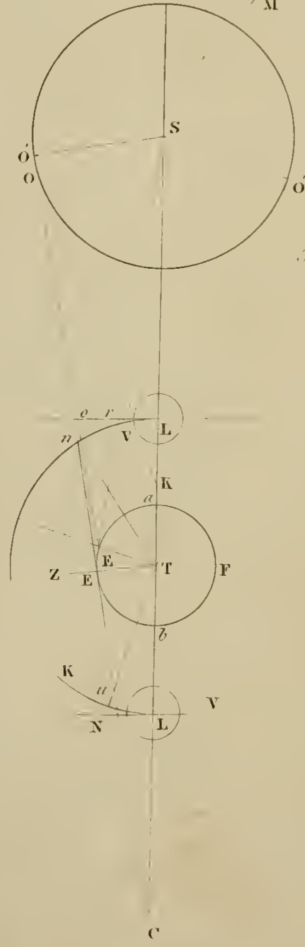

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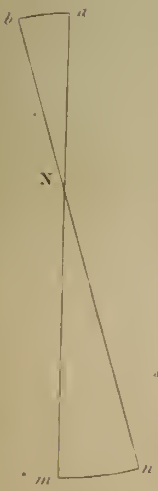

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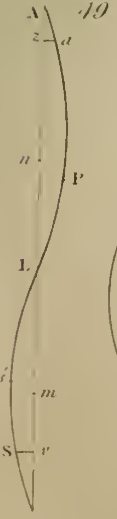

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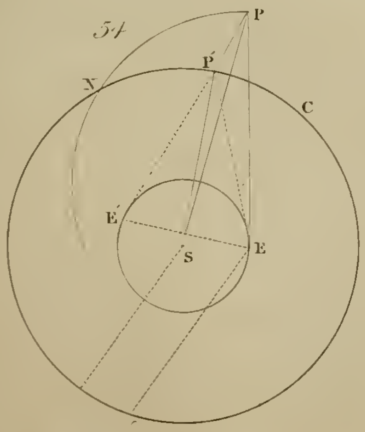

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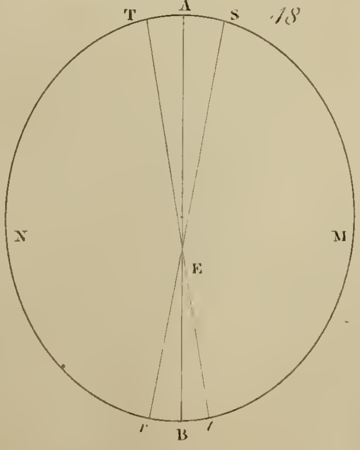
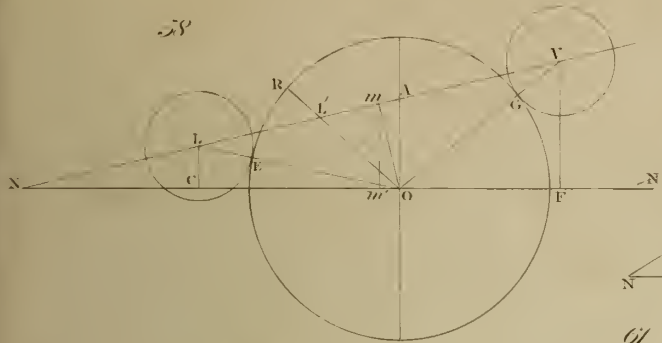


London, Published Oct.r 1 1822 by J. Mawman Ludgate Street.

Engraved by W.L

58



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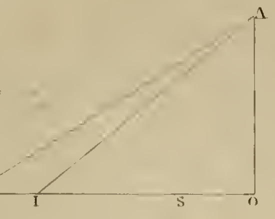

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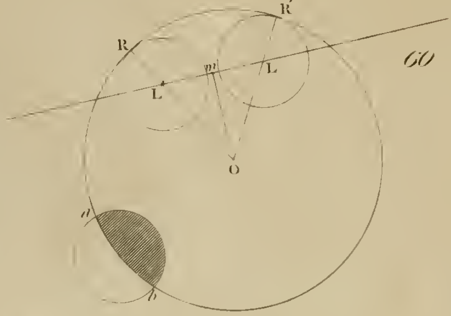

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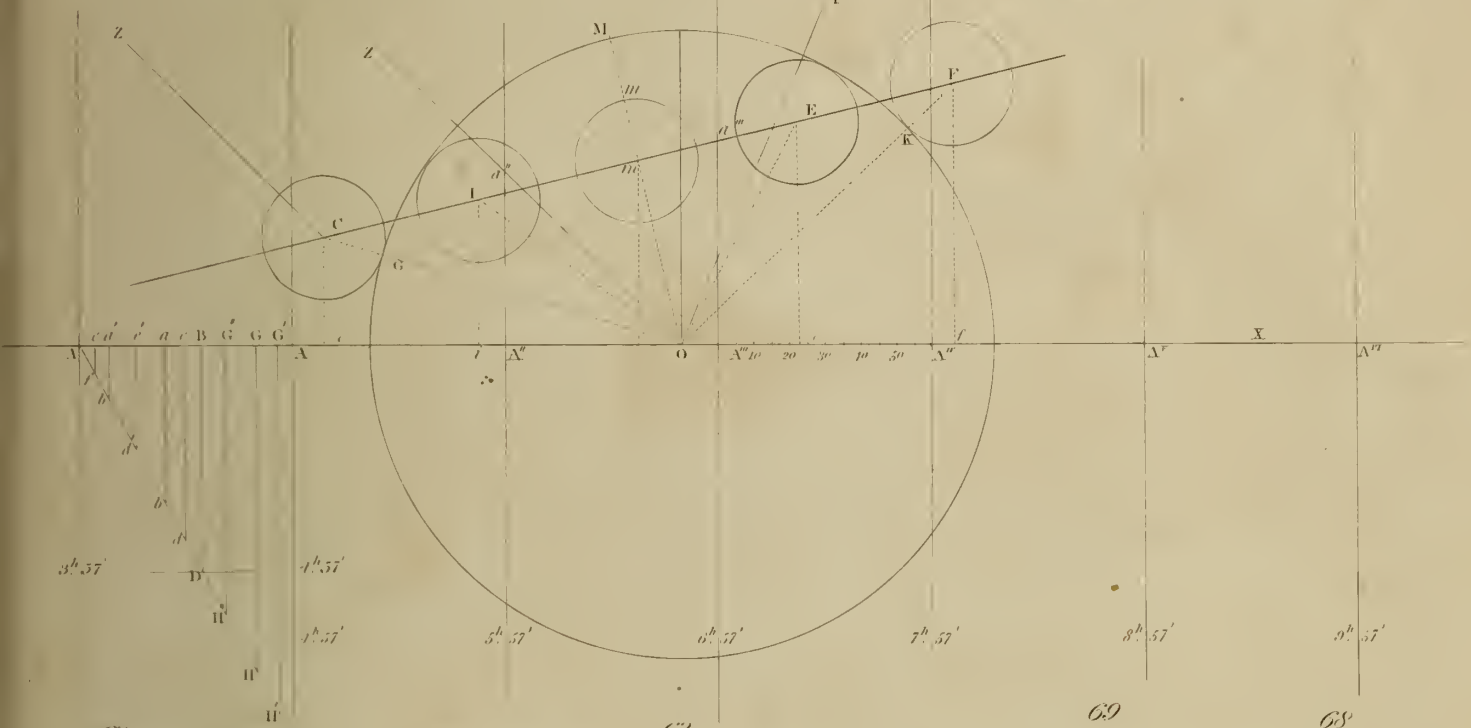

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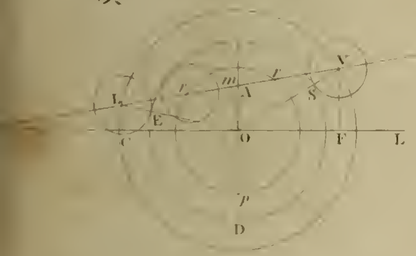

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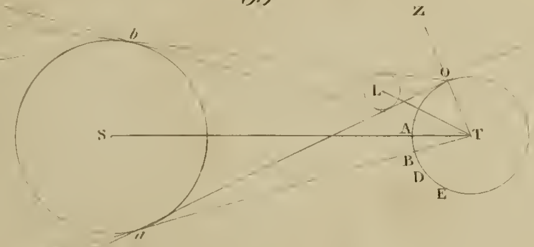

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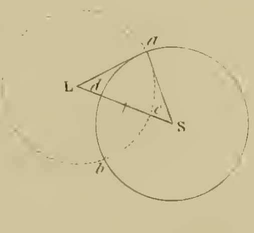

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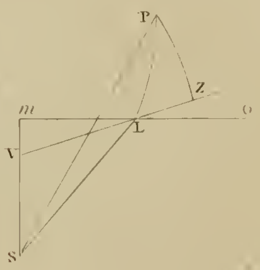

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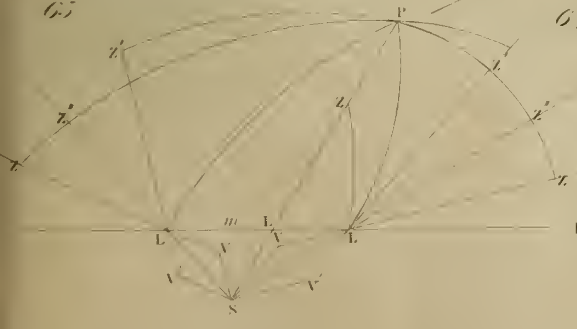

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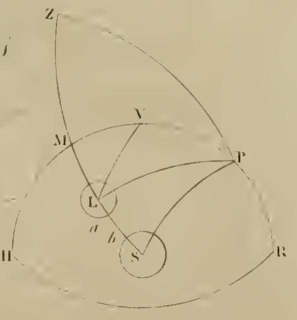

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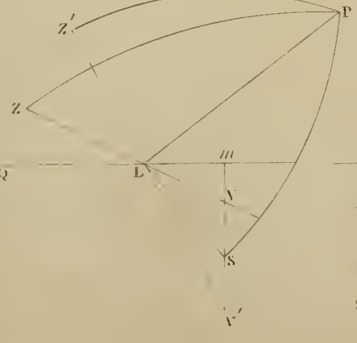

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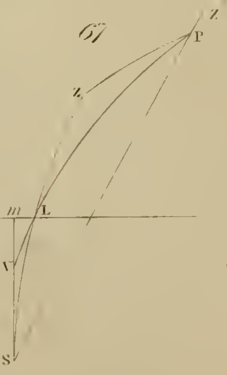


Published as the Act directs, [illegible] by J. Mawman, Ludgate Street

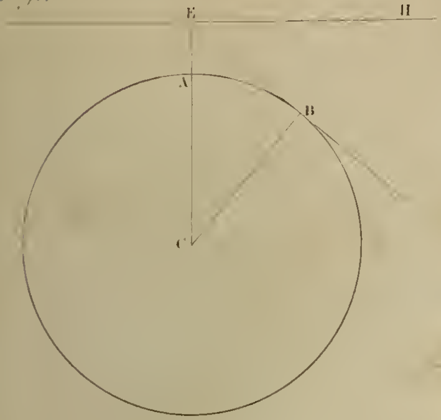
W. Lowry sc.

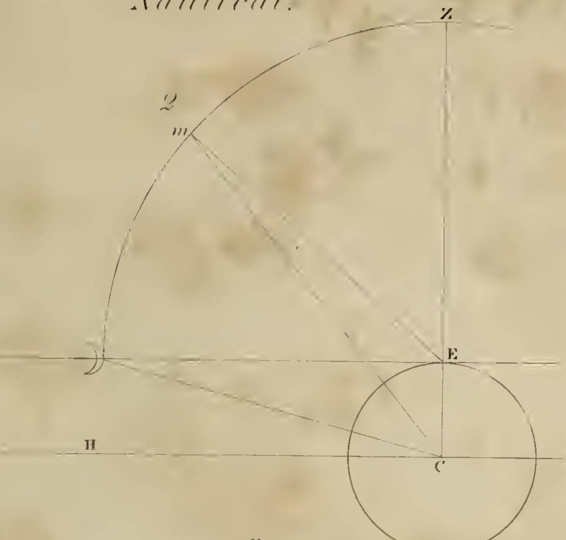
P. 612

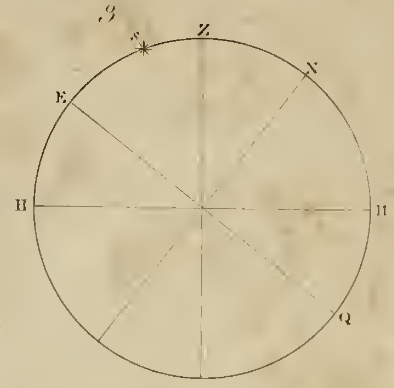
ASTRONOMY,
Nautical.

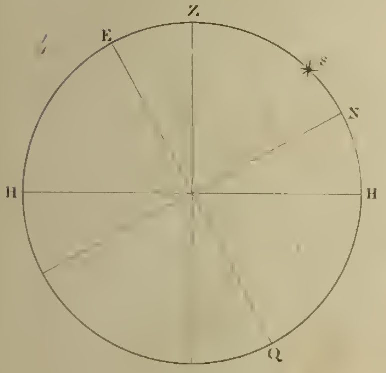
Plate I.

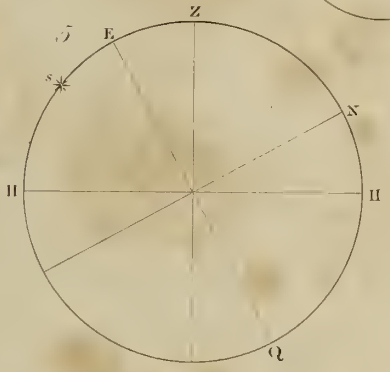
Fig. 1

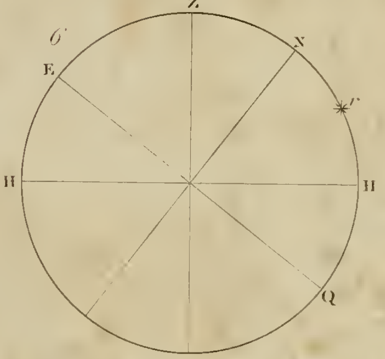


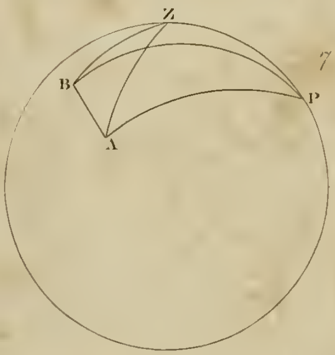


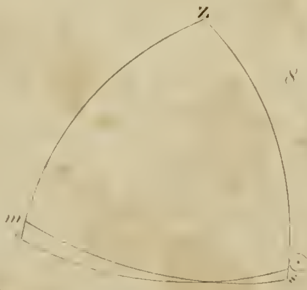




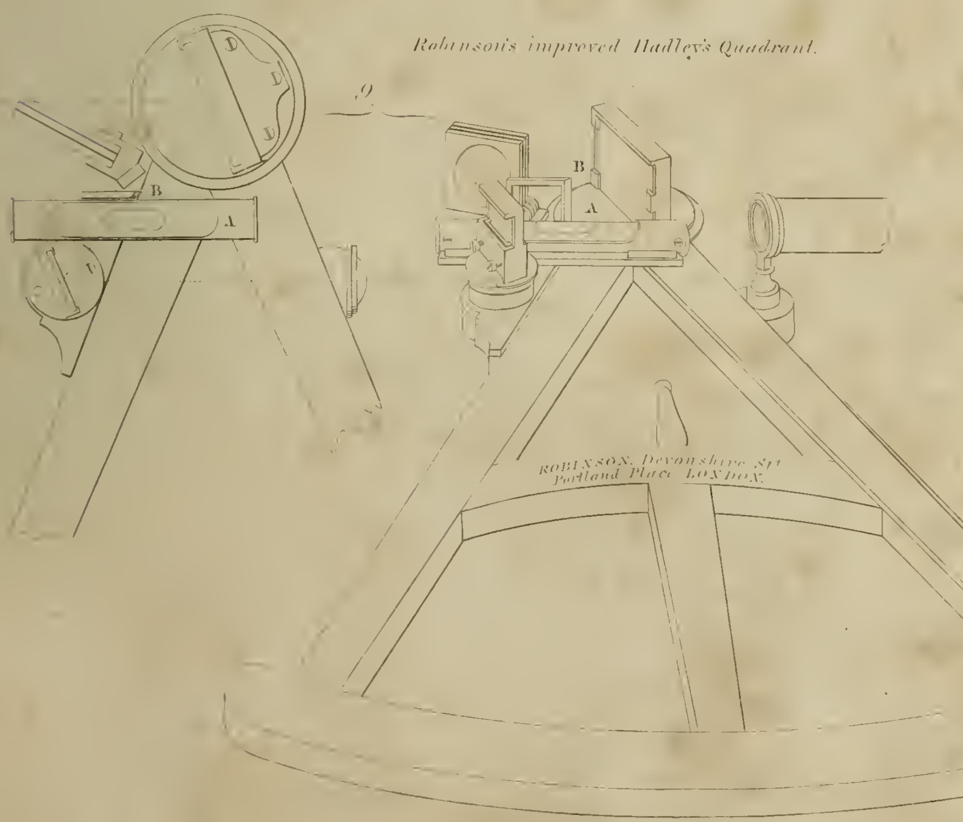








Robinson's improved Hadley's Quadrant.



Publish'd as the Act directs Sep 1 1800, by J. Mawman, Ludgate Str.t London

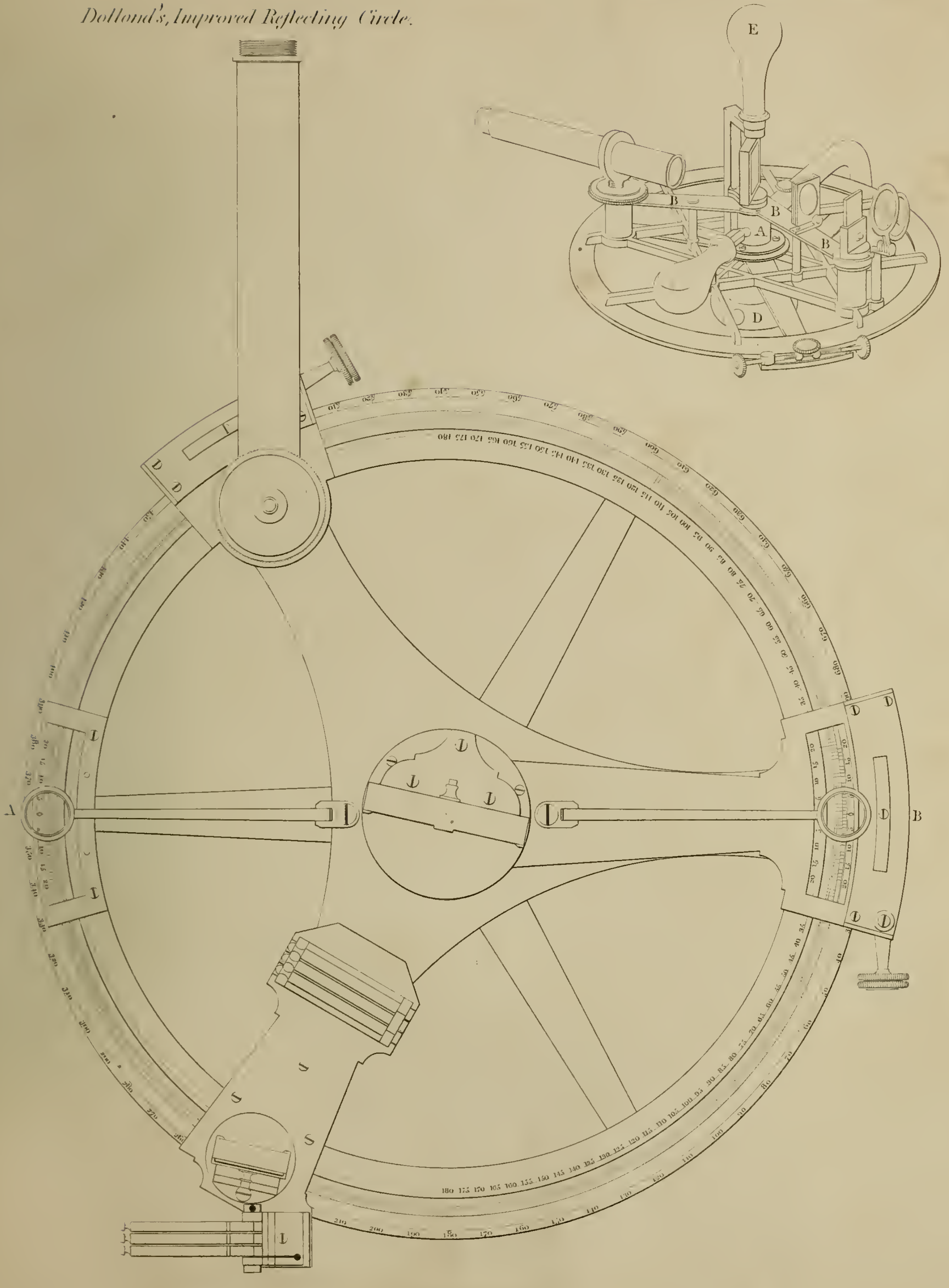
[illegible]

ASTRONOMY.

Nautical.

Troughton's, Reflecting Circle.

Dollond's, Improved Reflecting Circle.



Drawn by Miss Lowry. *Published as the Act directs, Sepr. 1. 1822 by J. Mawman, Ludgate Strt. London.* *W. Lowry sculp.*

ASTRONOMY.

Physical.

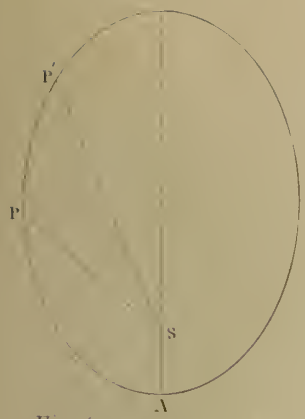
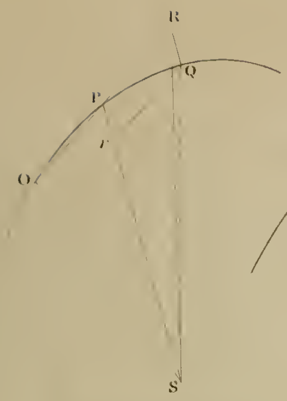
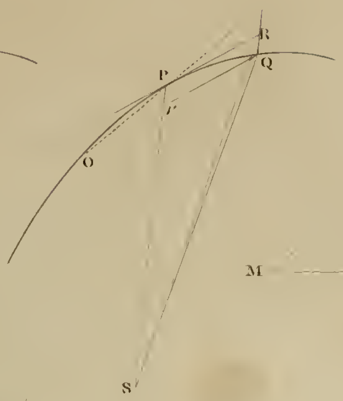


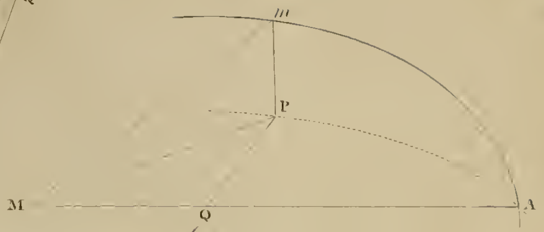
Fig. 1. P. 649 Col 1 line 34.



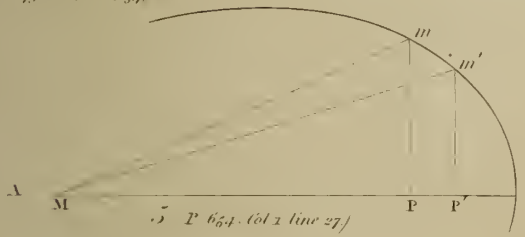
2 No. 1. (P. 649 Col. 2. line 32.)



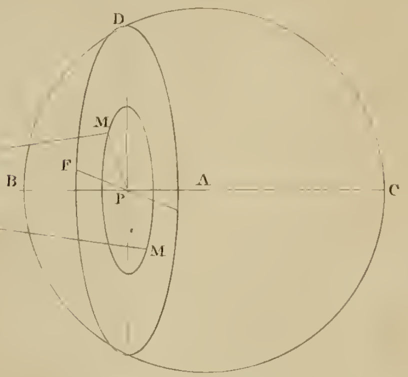
2 No. 2 (P. 656 Col 1 line 27.)



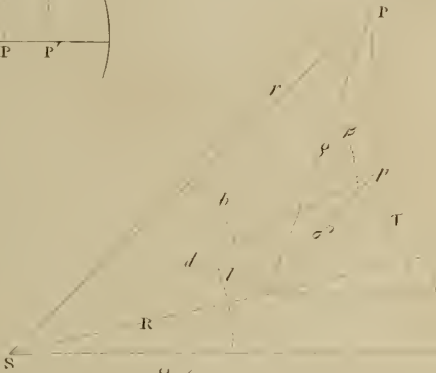
4 (P. 653 Col. 2 line 1.)



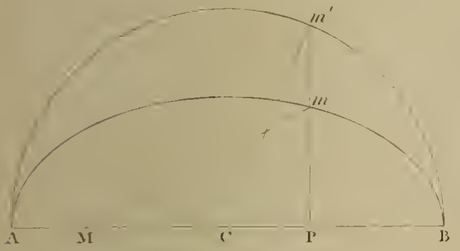
5 P. 654. Col 1 line 27.



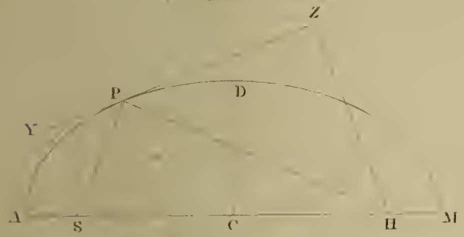
3 (P. 651. Col. 2. line 33.)



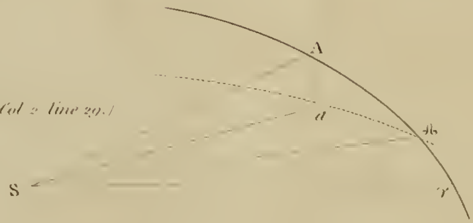
8 (P. 664. Col. 1. line 11.)



6 (P. 655 Col. 2. line 21.)



7 (P. 659 Col 2 line 15.)



9 (P. 661. Col. 2 line 29.)

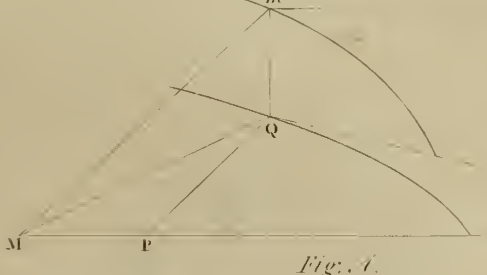
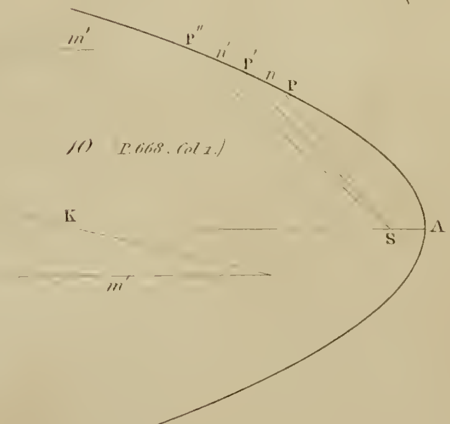
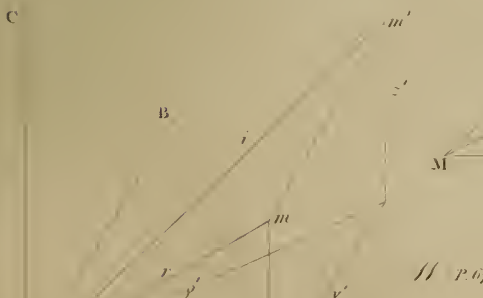


Fig. A.



10 (P. 668. Col. 1.)

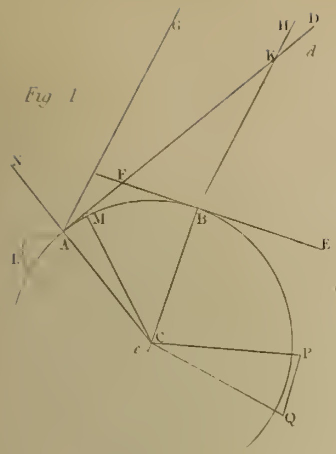


11 (P. 674 line 7.)

J F W Herschel, del.
Published as the Act directs, May 1. 1823, by J. Mawman, Ludgate Str.
W. Lowry, sc.

FIGURE OF THE EARTH,

Fig. 1



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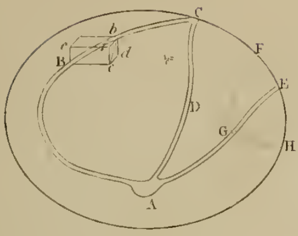
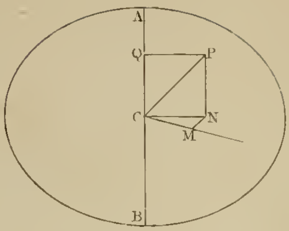


Fig. 3.



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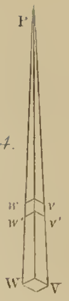
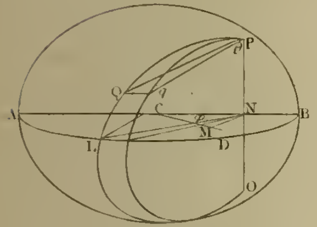


Fig. 5.



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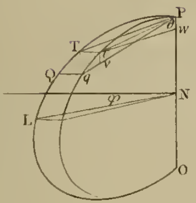
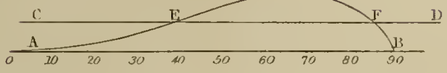


Fig. 7.



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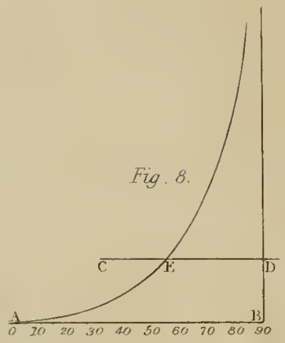

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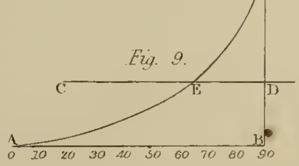

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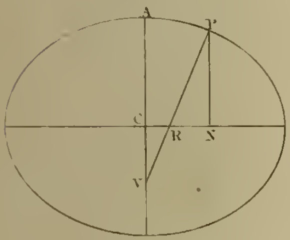

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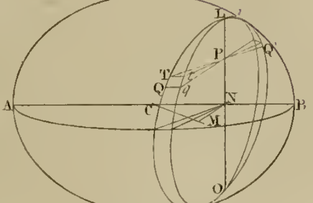

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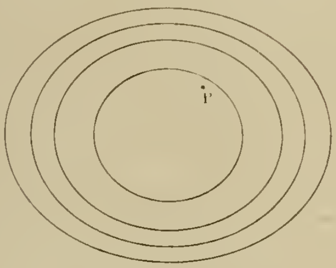


Fig 13.

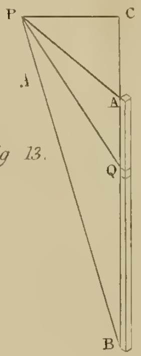
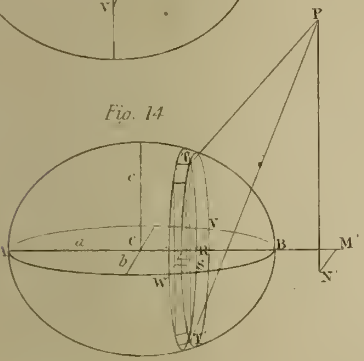


Fig. 14



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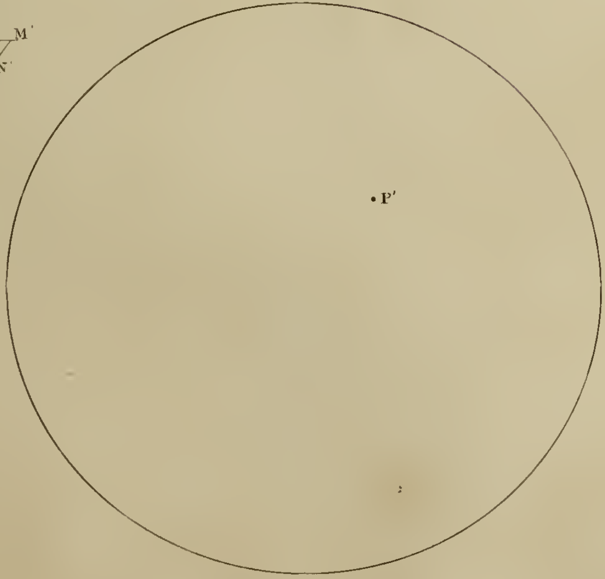

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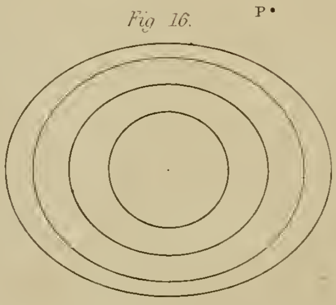


Fig 17

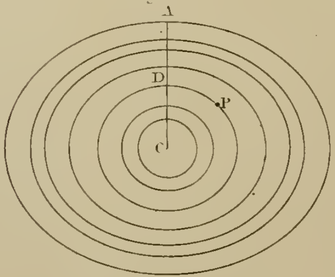
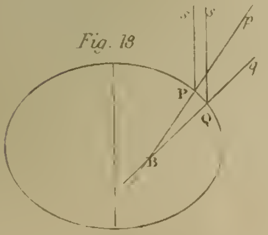


Fig. 18



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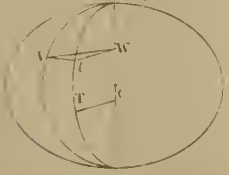
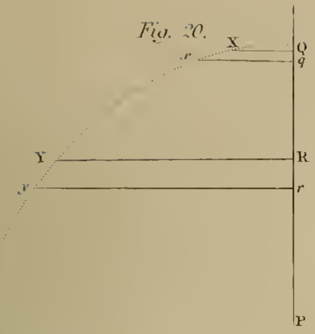


Published as the Act directs, June 1830 by Baldwin & Cradock Paternoster Row London

FIGURE OF THE EARTH

Fig. 20.



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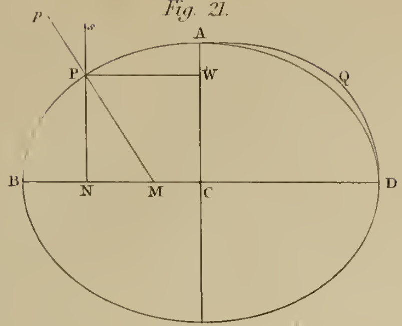
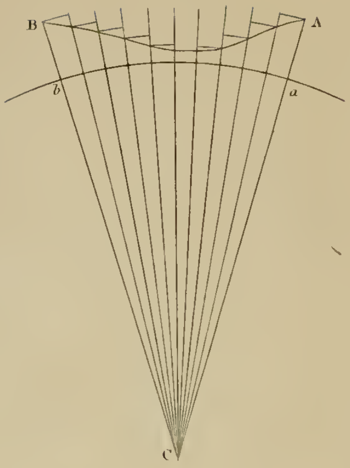


Fig. 22.



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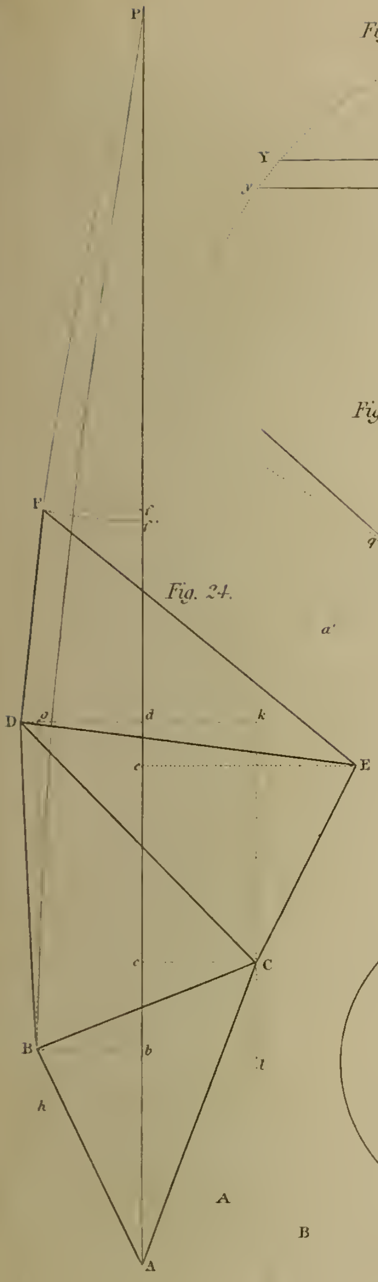

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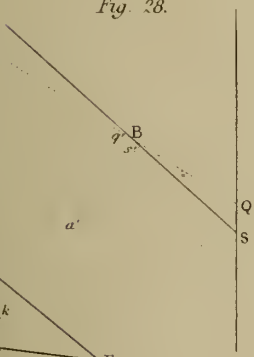
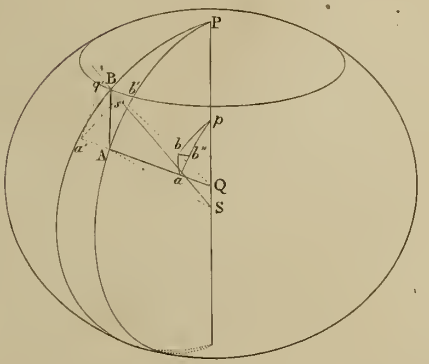


Fig. 27.



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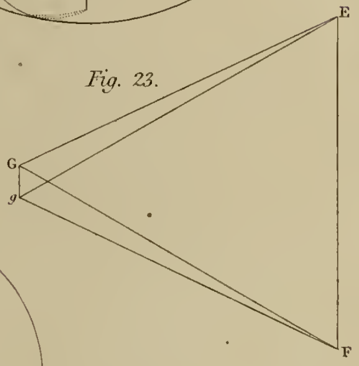

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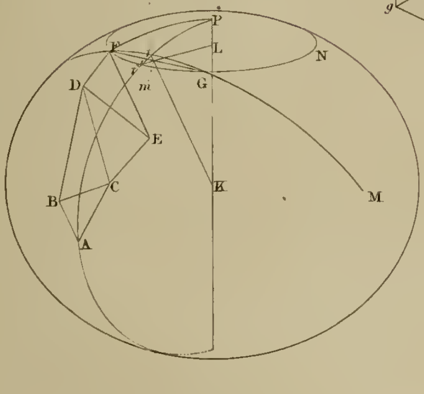

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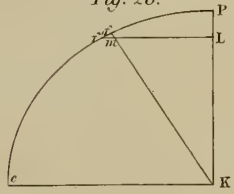

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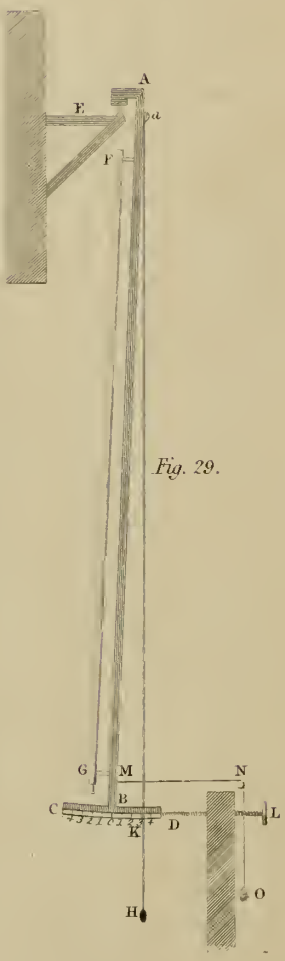

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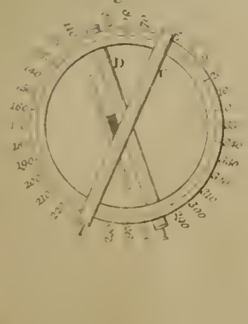

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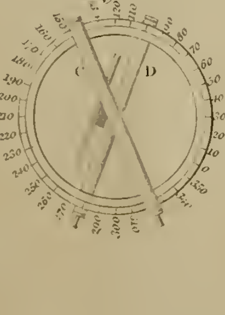

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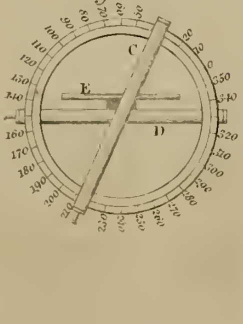

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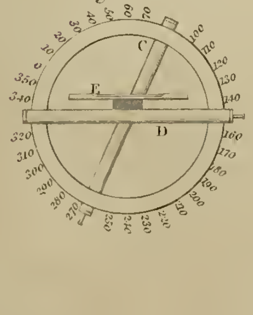

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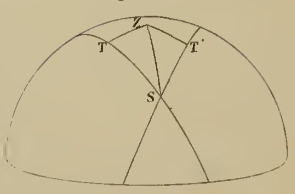

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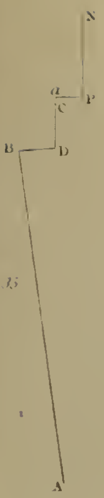
Fig. 35.



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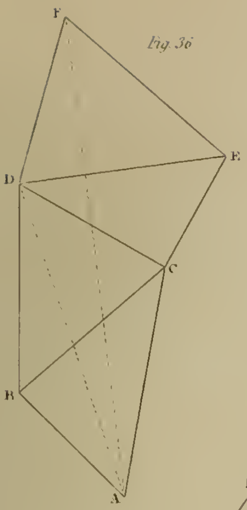

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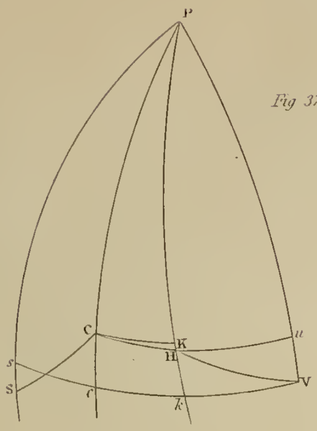

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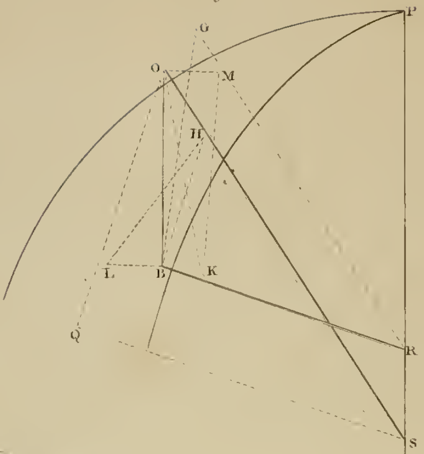


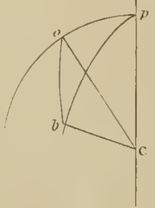
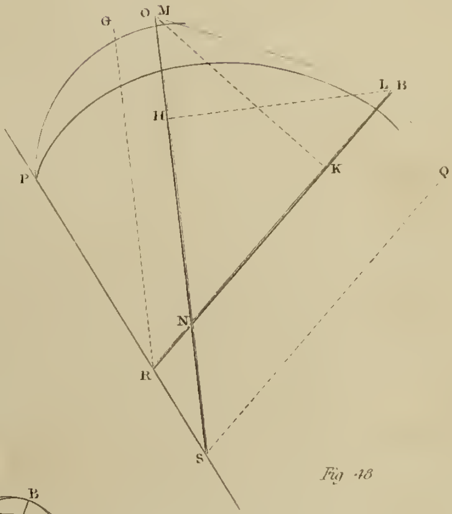


Fig. 39.



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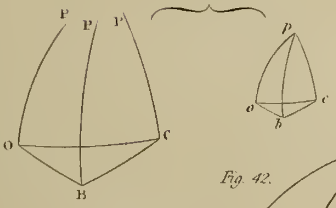

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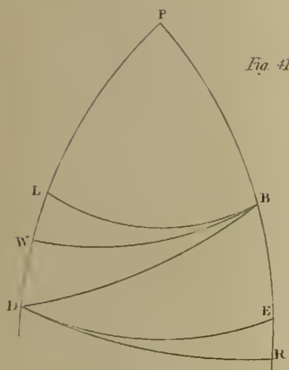


Fig. 42.

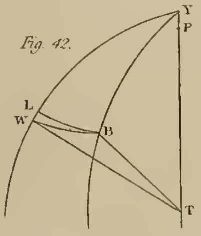


Fig. 43.

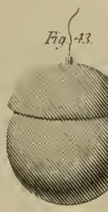


Fig. 44.

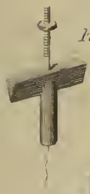


Fig. 46.

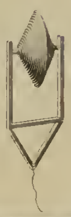


Fig. 48.

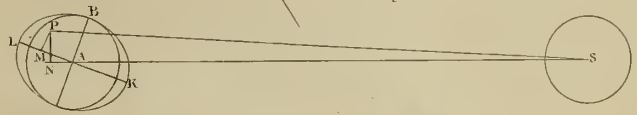


Fig. 47.

Fig. 51. P. 238. Col. 1.

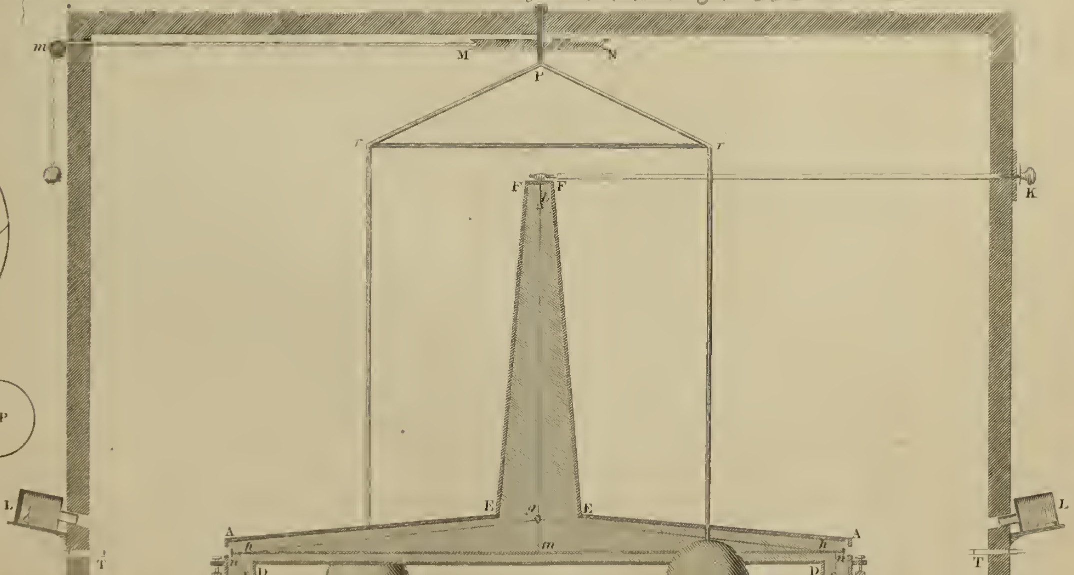


Fig. 49.

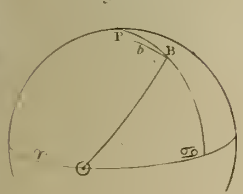


Fig. 45.

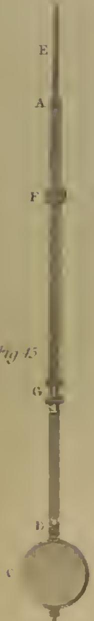
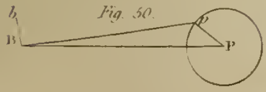
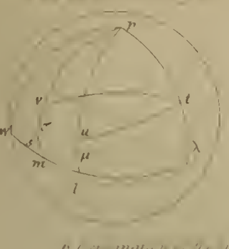


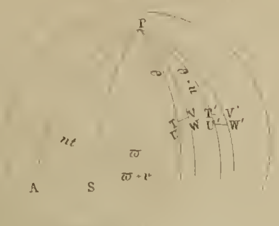
Fig. 50.

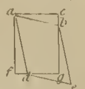


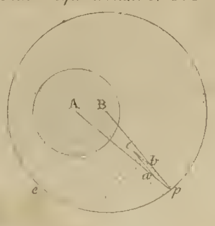
TIDES AND WAVES.

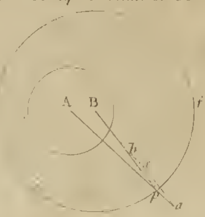
Plate I

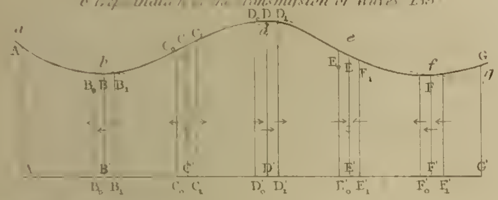

1 Equilibrium theory (71)

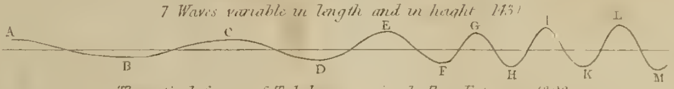

2 Laplace's theory (71)


3 Equation of continuity (72)

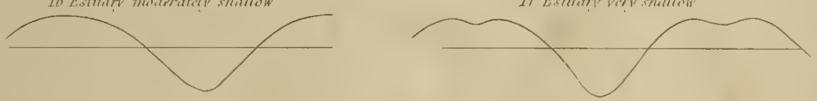

4 Stable equilibrium of Sea (123)


5 Unstable equilibrium of Sea (124)

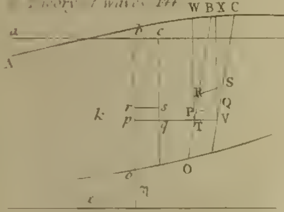

6 Explanation of the transmission of Waves (135)

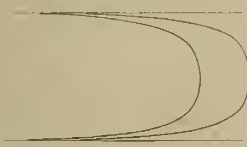

7 Waves variable in length and in height (143)

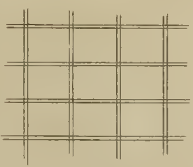
Theoretical forms of Tidal curves in shallow Estuaries (309)


16 Estuary moderately shallow

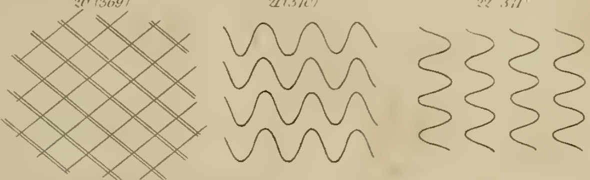
17 Estuary very shallow


8 Theory of waves (144)


18 Plan of tide waves in channels deepest in the middle (359)


19 Plan of intersecting waves (367)

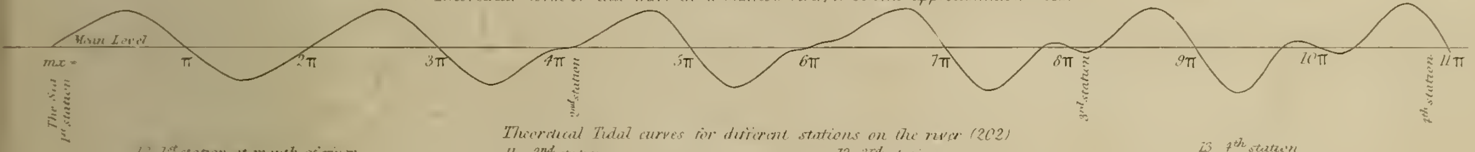
Corresponding forms of cotidal lines


20 (369)

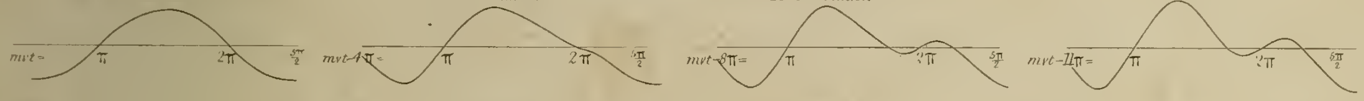
21 (370)

22 (371)

9 Theoretical form of tide-wave in a shallow river, to second approximation (201)



Theoretical Tidal curves for different stations on the river (202)

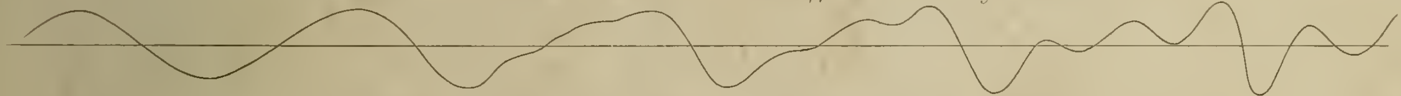

10 1st station at mouth of river

11 2nd station

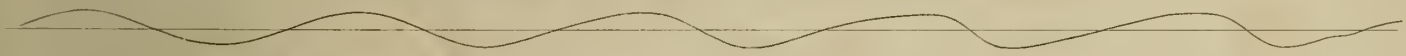
12 3rd station

13 4th station

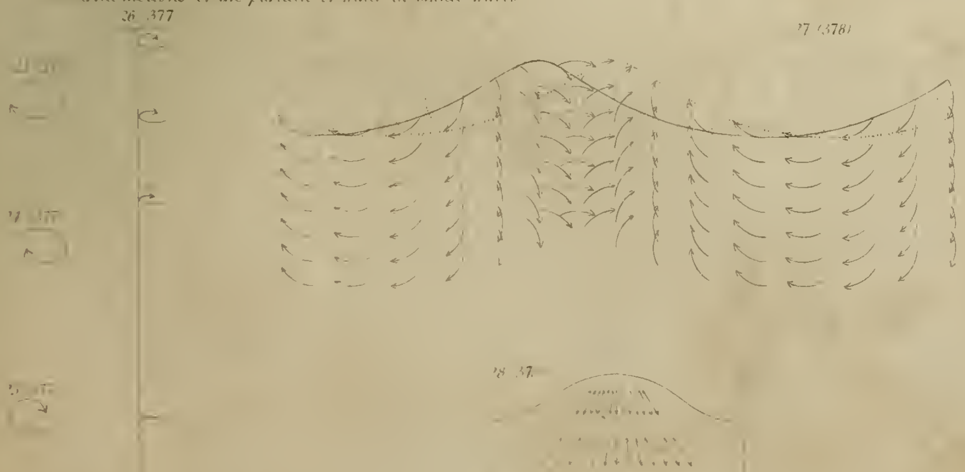
14 Theoretical form of tide-wave in a shallow river, to third approximation, with large tide (210)



15 Theoretical form of tide-wave in a shallow river, to third approximation, with small tide (210)



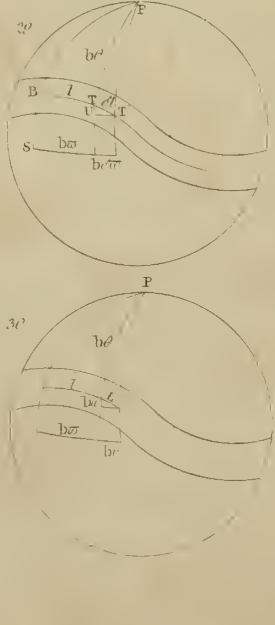
Curved motions of the particles of water in small waves


26 (377)

27 (378)

28 (37

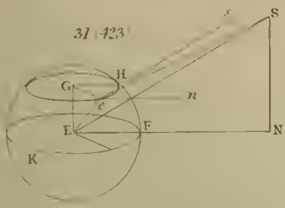
Theory of canals upon the earth (420)


29

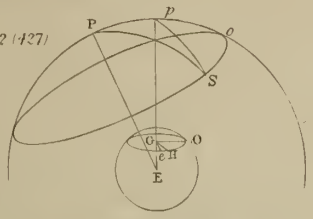
30

Theory of canals on the Earths surface

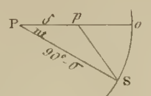
31 (423)



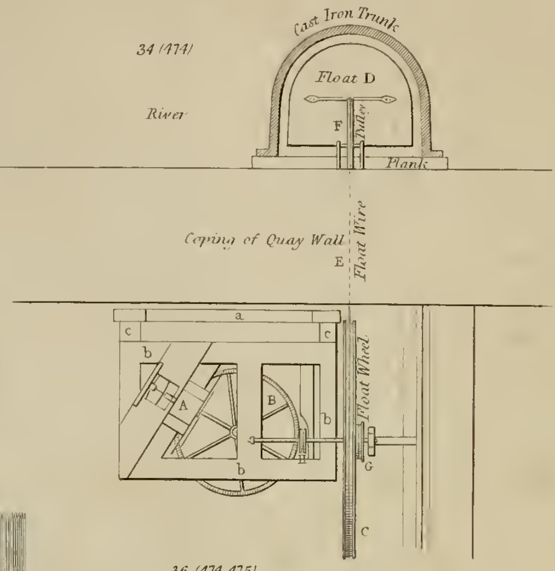
32 (427)



33 (428)

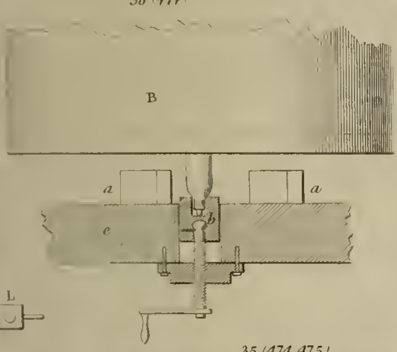


34 (474)

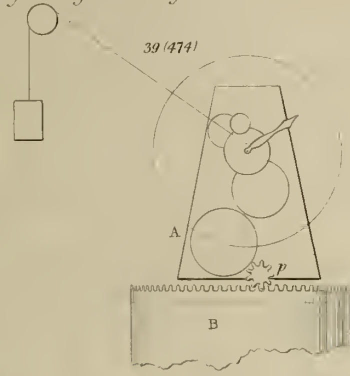


Bunt's self-registering Tide Guage

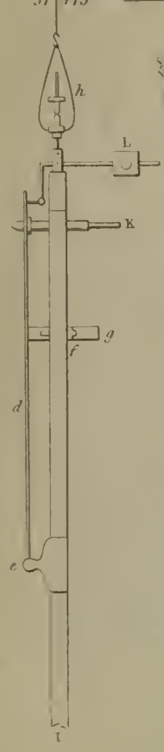
38 (474)



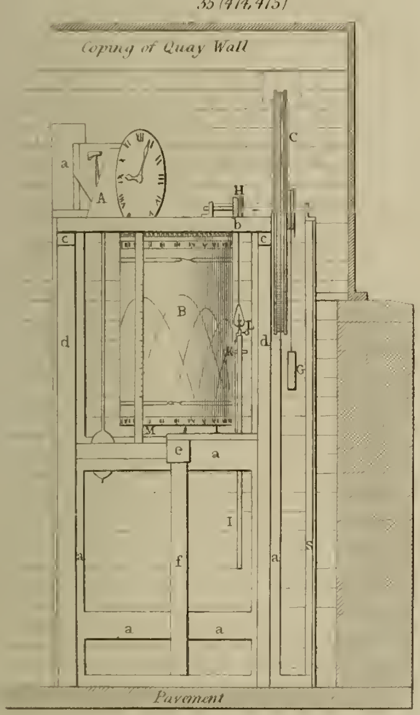
39 (474)



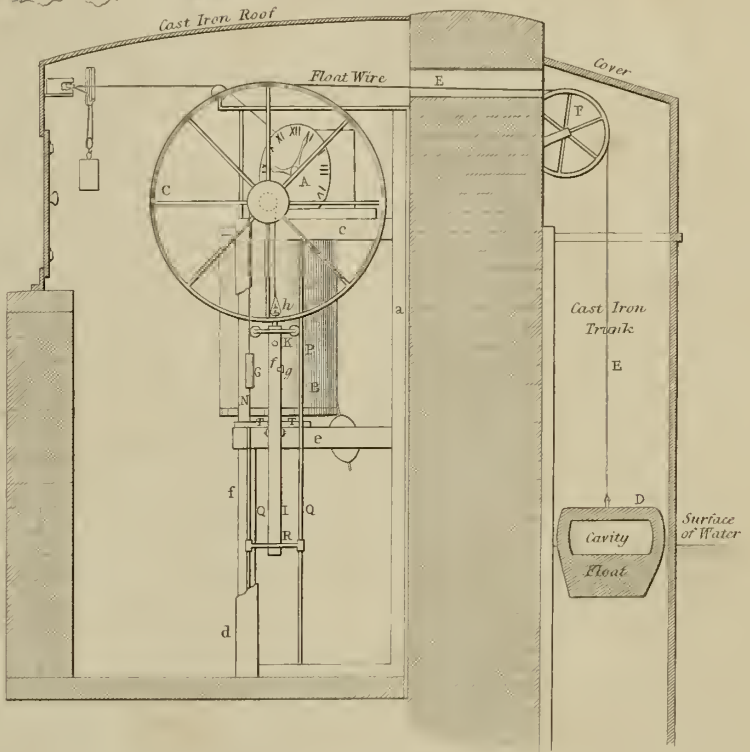
37 (475)



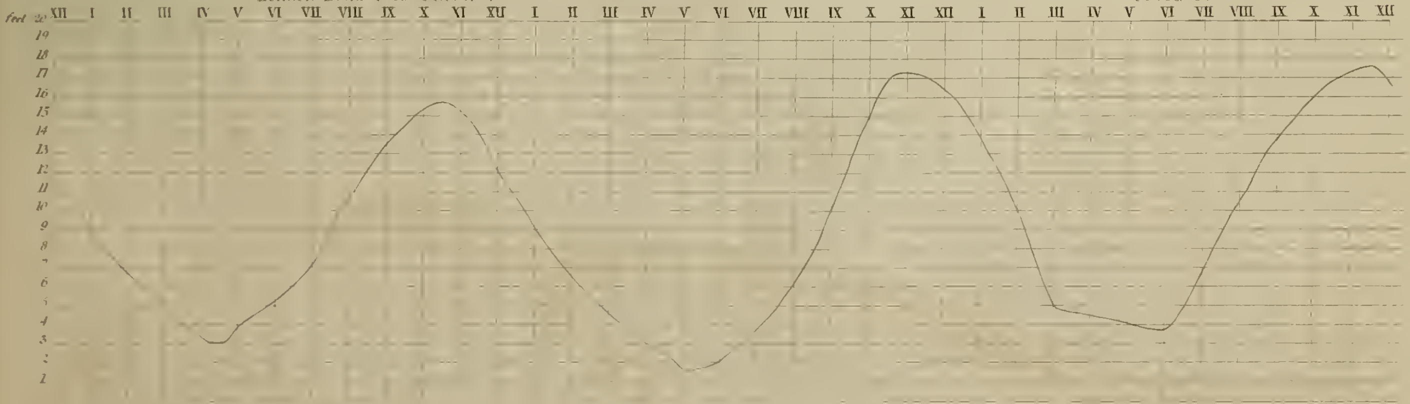
35 (474, 475)



36 (474, 475)



40. Exhibition of Tides in continued sequence, by Palmer's self-registering Tide Guage (478.)



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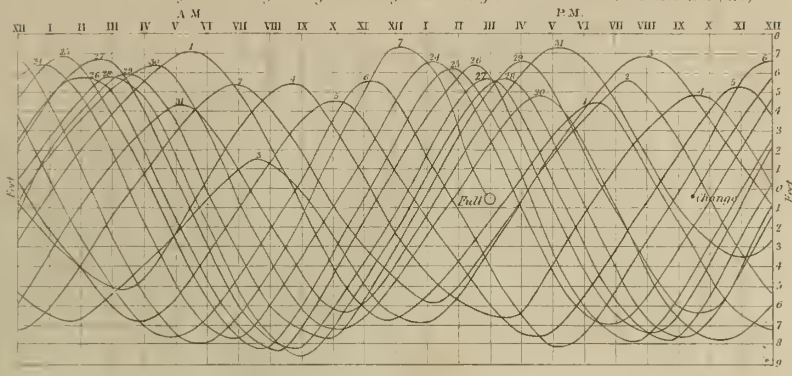
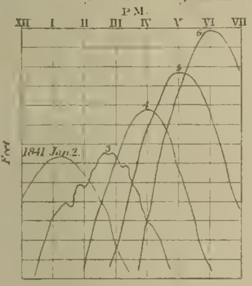
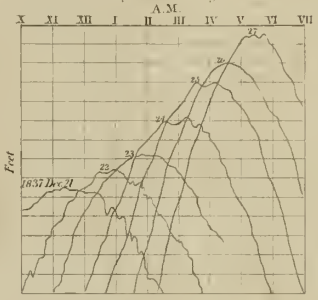
TIDES AND WAVES.



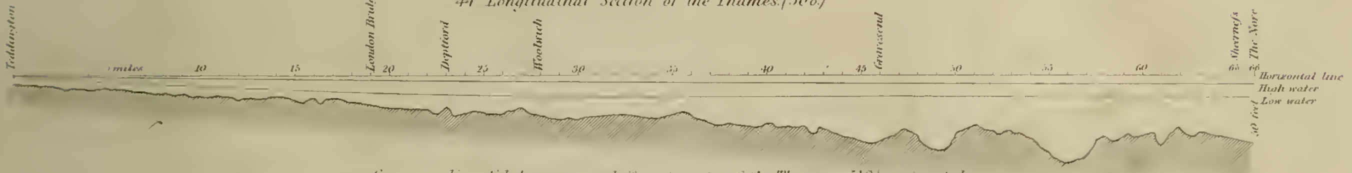
41 Curves traced by the self-registering Tide-Guage at Sheerneſs, from 1840 Decr 23; to 1841. Jany 7 (478.)



42 Curves traced by the Tide Guage at Bristol (478.)

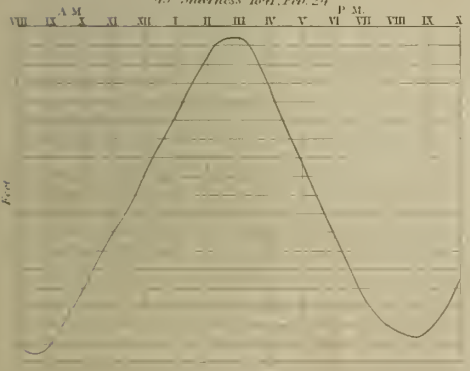


43. Curves traced by the Tide Guage at Swansea. (478.)

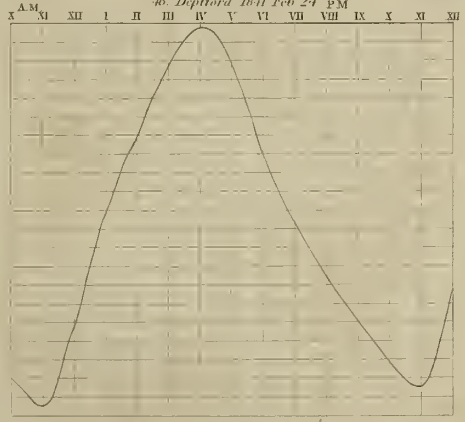


44 Longitudinal Section of the Thames. (508.)

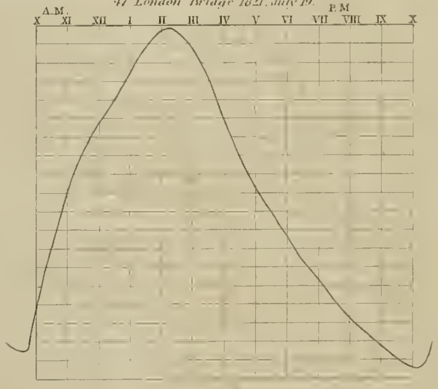
Corresponding tidal curves on different parts of the Thames (510); spring tides.



45 Sheerness 1841, Feb. 24

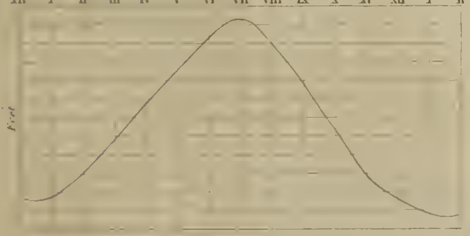


46. Deptford 1841 Feb 24

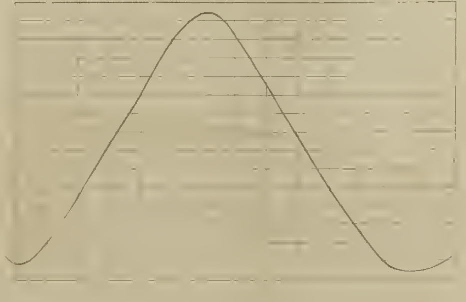


47 London Bridge 1821, July 19.

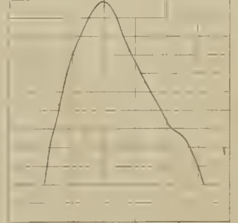
Corresponding tidal curves, on different parts of the Thames (510); neap tides



48 Sheerness, 1841 March 2



49. Deptford 1841, March 2



50. Tidal curve on the Dee. (511.)

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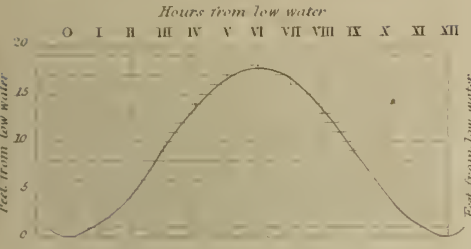
TIDES AND WAVES.

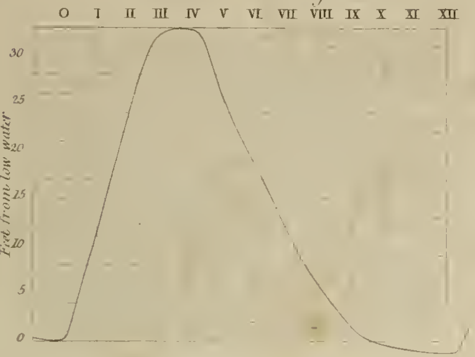
Plate 4.

Corresponding tidal curves on different parts of the Severn (513)

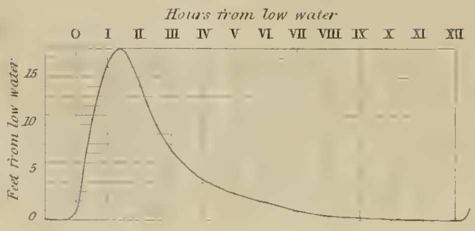
51. Tidal curve at the mouth of the Severn



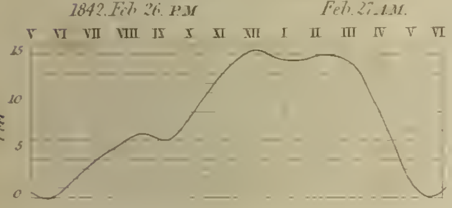
52. Tidal curve at Hung Road



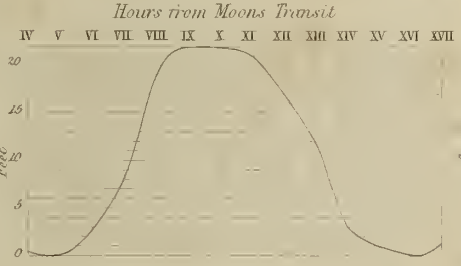
53. Tidal curve at Newnham



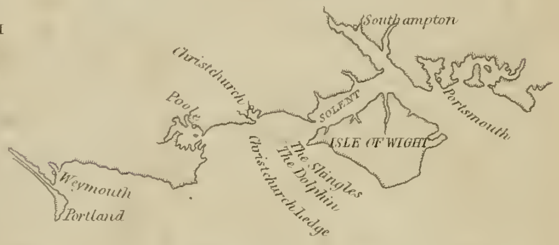
54. Tidal curve at Southampton 517



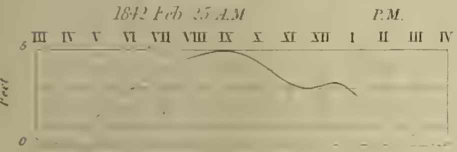
55. Tidal curve at Havre (518)



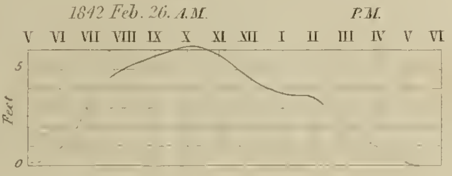
56. Chart of the coast from Weymouth to Portsmouth (519)



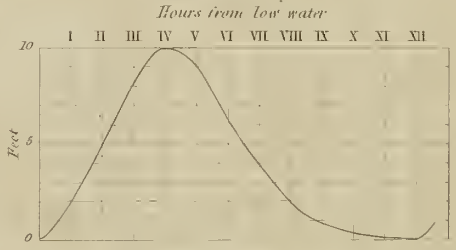
57. Tidal curve at Christchurch 519



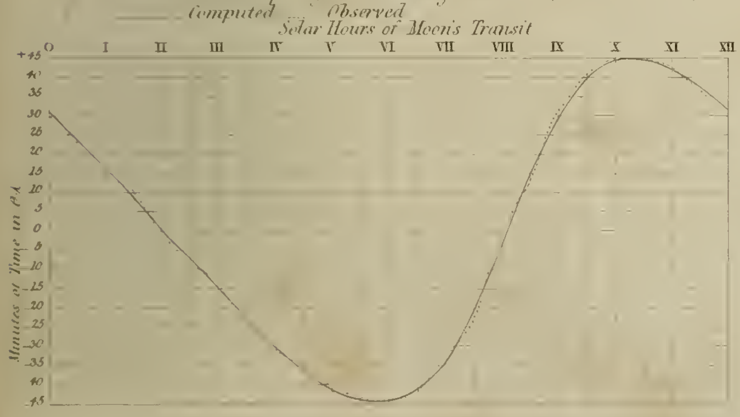
58. Tidal curve at Poole (519)



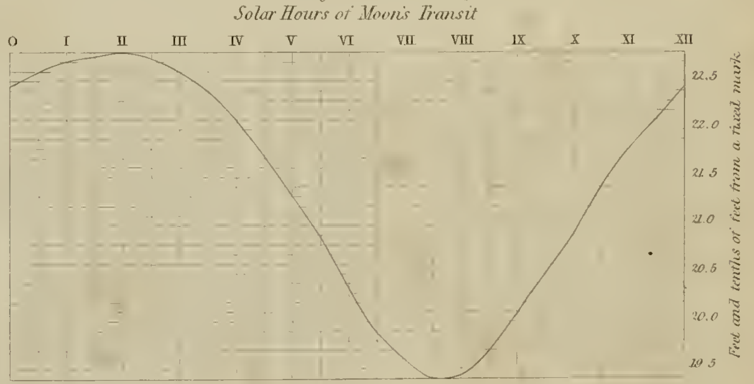
59. Tidal curve at Weymouth (519)



61. Semimenstrual inequality of Time of High Water (θ-λ) for London (547)

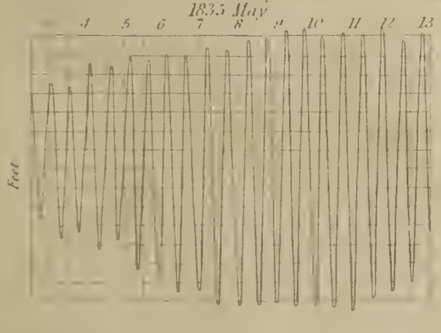


62. Observed semimenstrual inequality of Height of High Water for London (549)

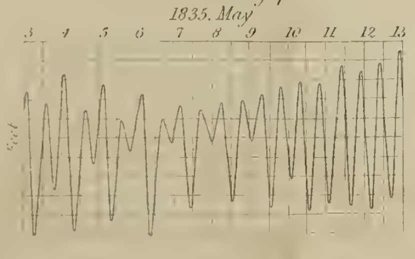


Tidal curves exhibiting the mixture of diurnal and semidiurnal tides (570)

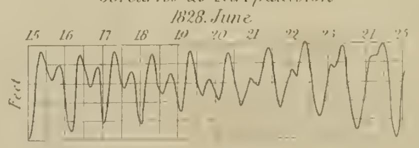
63. Curves at Plymouth



64. Curves at Singapore



65. Curves at Petropaulofsk



J.W. Lowry sculp

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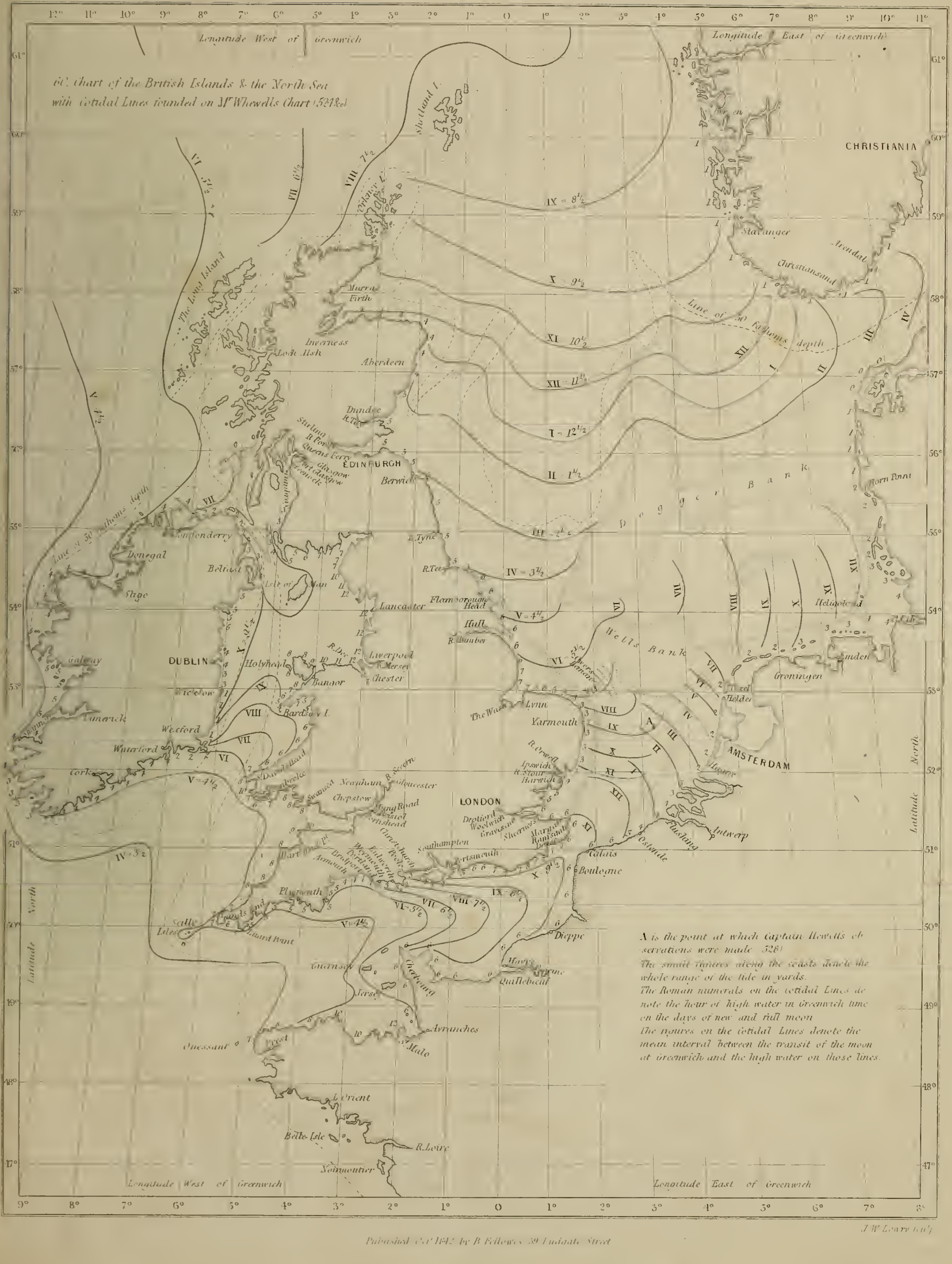

6C. Chart of the British Islands & the North Sea with Cotidal Lines founded on Mr Whewells Chart (524 &c.)

A is the point at which Captain Hewett's observations were made (528)
The small figures along the coasts denote the whole range of the tide in yards.
The Roman numerals on the cotidal Lines denote the hour of high water in Greenwich time on the days of new and full moon
The figures on the cotidal Lines denote the mean interval between the transit of the moon at Greenwich and the high water on those lines.

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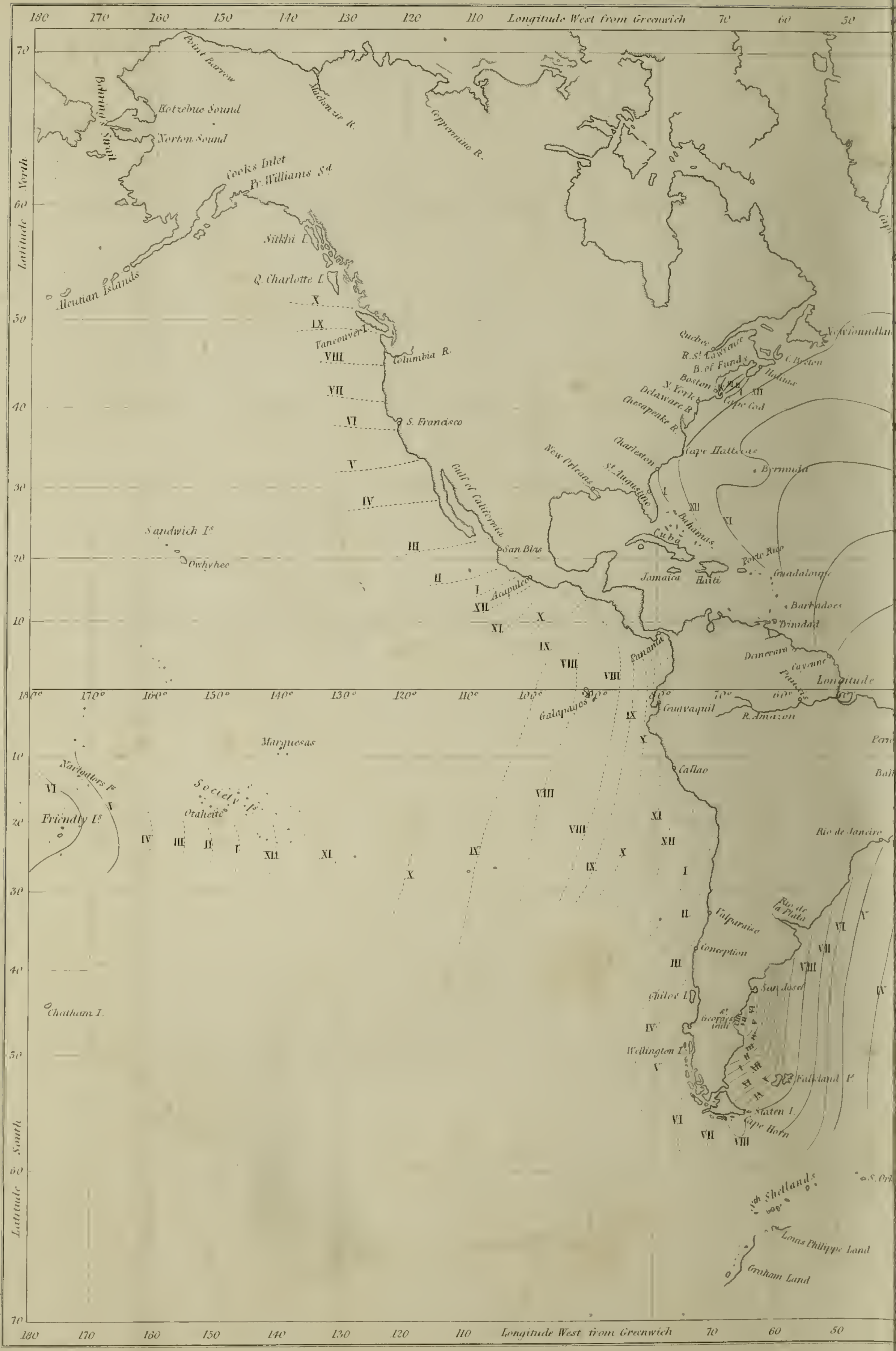
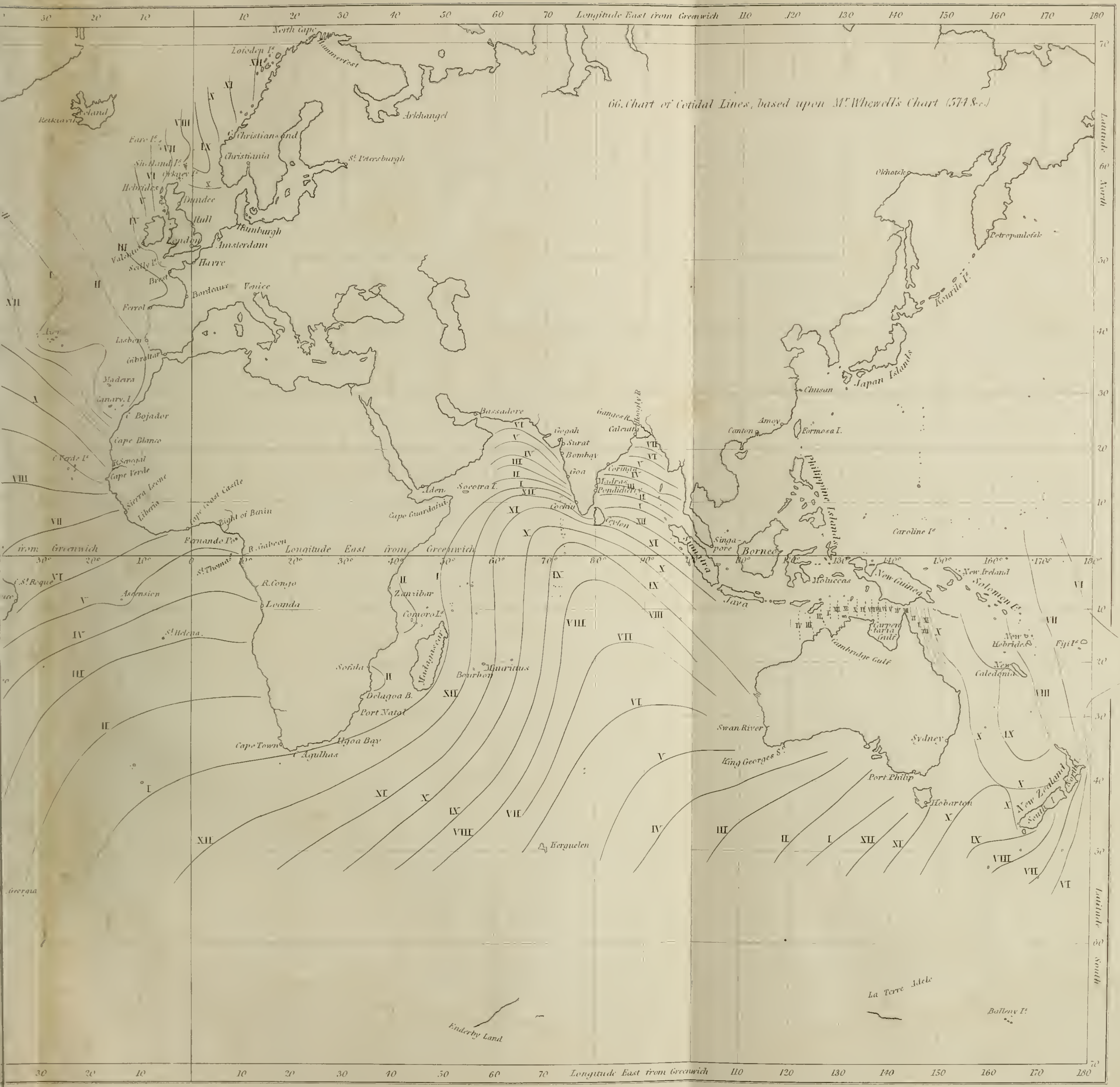

Longitude West from Greenwich
180
170
160
150
140
130
120
110
70
60
50
Latitude North
Latitude South
Point Barrow
Behring's Strait
Kotzebue Sound
Norton Sound
Mackenzie R.
Coppermine R.
Cooks Inlet
Pr. Williams Sd.
Sitkhi I.
Q. Charlotte I.
Aleutian Islands
Vancouver I.
Columbia R.
S. Francisco
Gulf of California
San Blas
Acapulco
Sandwich Is.
Owhyhee
Quebec
R. St. Lawrence
B. of Fundy
Boston
N. York
Delaware B.
Chesapeake B.
Cape Cod
Halifax
C. Breton
Newfoundland
Charleston
New Orleans
St. Augustine
Cape Hatteras
Bermuda
Bahamas
Cuba
Jamaica
Haiti
Porto Rico
Guadaloupe
Barbadoes
Trinidad
Demerara
Cayenne
Para
Panama
Galapagos
Guayaquil
R. Amazon
Longitude
Callao
Pern
Bahi
Rio de Janeiro
Marquesas
Navigators Is.
Friendly Is.
Society Is.
Otaheite
Valparaiso
Conception
Rio de la Plata
San Josef
Chiloe I.
St. Georges Bay
Wellington I.
Falkland Is.
Staten I.
Cape Horn
Chatham I.
Sth. Shetlands
S. Ork
Louis Philippe Land
Graham Land
I
II
III
IV
V
VI
VII
VIII
IX
X
XI
XII



66. Chart of Cotidal Lines, based upon Mr Whewell's Chart (574 &c.)

J. W. Lowry fadp.

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